Advanced Computational Methods in Geotechnical Engineering

Lecture

March 26, 2009

2. FD Solution of Terzaghi's Consolidation Equation

- The governing 1D Consolidation equation is:

$$\frac{\partial u}{\partial t} = C_V \frac{\partial^2 u}{\partial z^2}$$

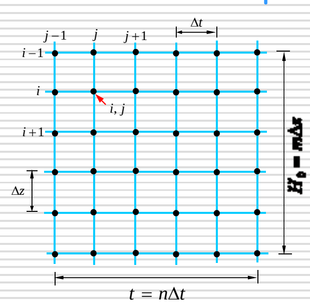
Where, u is the excess pore water pressure, C_V is the coefficient of consolidation, z is the depth and t is time.

- The above equation allows us to predict the change in excess porewater pressure at various depths within the soil with time.

$$C_V = \frac{k}{m_v \gamma_w}$$

- ✓ We need to know the excess porewater pressure at a desired time because we have to determine the vertical effective stress to calculate the consolidation settlement.
- ✓ Solution of the 1D consolidation equation can be found in two ways:
  - Based on Fourier series expansion
  - Based on Finite Difference Scheme.

- We will use Taylor's Expansion Theorem.
  - First divide the domain into depth – time grid.



- Columns: time divisions & Rows: soil depth divisions.

- Applying Taylor's series expansion to Terzaghi's eqn.

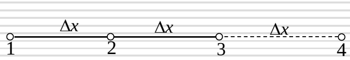
$$\frac{\partial u}{\partial t} = \frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) \qquad \frac{\partial^2 u}{\partial z} = \frac{1}{(\Delta z)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$\Rightarrow \quad u_{i,j+1} = u_{i,j} + \frac{C_V \Delta t}{(\Delta z)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \quad \textbf{(2.1)}$$

- Equation (1) is valid for non-boundary nodes.
- Special conditions apply to boundary nodes.

- For example, at an impermeable boundary, no flow across it can occur and, consequently:

$$\frac{\partial u}{\partial z} = 0 \quad \Rightarrow \quad \textbf{Neumann Boundary}$$



Install an imaginary node

Dirichlet
Neumann

$$\begin{cases} u = 0 & at\ x = 0 \\ \frac{du}{dx} = 1 & at\ x = 1 \end{cases}$$

-FDE at node 3:

$$\frac{du}{dx} = 1 \quad \Rightarrow \quad \frac{u_4 - u_2}{2\Delta x} = 1$$

- The FDE at an impermeable boundary is, thus:

$$\frac{\partial u}{\partial z} = 0 \Rightarrow \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta z} = 0 \quad \Rightarrow \quad u_{i+1,j} = u_{i-1,j}$$

-And the governing consolidation equation becomes:

$$u_{i,j+1} = u_{i,j} + \frac{C_V \Delta t}{(\Delta z)^2}\left(2u_{i-1,j} - 2u_{i,j}\right) \quad \textbf{(2.2)}$$

※ Initial *u* distribution:- required to be determined first.

- For a layered soil with different *k* and C_V, Scott (1963) proposed the following FDE equation:

$$u_{i,j+1} = u_{i,j} + \frac{k_T + k_B}{k_B + k_T\left({C_{VT}}/{C_{VB}}\right)} \frac{\Delta t}{(\Delta z)^2}\left(\frac{2k_T}{k_T + k_B}u_{i+1,j} - 2u_{i,j} + \frac{2k_B}{k_T + k_B}u_{i-1,j}\right)$$

where suffix *T* and *B* stand for top and bottom layers.

2.1 Procedure to apply FDM to 1D consolidation problem

1. Divide the soil layer into a depth-time grid. Care must be taken in selecting Δt and Δz. For numerical convergence, researchers have proposed the following:

$$\alpha = \frac{C_V \Delta t}{(\Delta z)^2} < \frac{1}{2}$$

α = 0.25 usually ensures convergence

2. Identify the boundary conditions. If the boundary is a drainage boundary, u is zero. If, however, the boundary is an impervious boundary, then no flow can occur across it and Eqn (2.2) has to be used.

3. Estimate the distribution of initial u.

4. Calculate u at interior nodes using eqn. (2.1) and at impermeable boundary nodes using eqn. (2.2). If the boundary is permeable, then u is zero.

EXAMPLE 2.1

Table E2.1

Column	1	2	3	4	5	6
Depth	Time (yr)					
	0.00	0.1	0.2	0.3	0.4	0.5
0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	78.0	57.0	※			
2.0	72.0	※	※			
3.0	62.0	※	※			
4.0	48.0	※	※			
5.0	30.0	39.0	※			

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0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	78.0	57.0	46.3	39.4	34.5	30.7
2.0	72.0	71.0	65.0	59.1	54.0	49.7
3.0	62.0	61.0	60.0	58.4	56.5	54.5
4.0	48.0	47.0	48.5	50.0	51.0	51.6
5.0	30.0	39.0	43.0	45.8	47.9	49.5

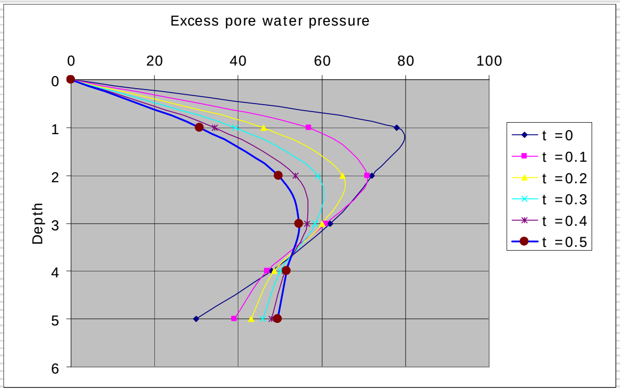

Figure E2.1

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Assignment #2:

1. Redo example 2.1, assuming that the initial excess pore water pressure is distributed according to:

$$\Delta u_0 = \frac{180}{\sqrt{z}}$$

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