Advanced Computational Methods in Geotechnical Engineering

Lecture

March 20, 2009

References:

1. Bathe, K.J., Finite Element Procedures (Available in pdf format).
2. Rao, S.S., The Finite Element Method in Engineering (Available in pdf format).
3. Zenkiewicz O.C. & Taylor, R.L. (2000), The Finite Element Method, Vol. I & II (Available in pdf format).
4. Budhu, M. (2007): Soil Mechanics and Foundations, Wiley & Sons (a CD-ROM available).
5. Wood, D. M. (1990): Soil behavior & CSSM, Cambridge (available in pdf format).
6. Any other FDM or FEM reference (Internet, etc…)

General Introduction

Many problems in engineering are governed by differential or integral equations.

The solutions to these equations would provide an exact, closed-form solution to the particular problem being studied.

However, complexities in the geometry, loadings, properties and boundary conditions that are seen in most real-world problems usually mean that an exact solution cannot be obtained or obtained in a reasonable amount of time.

Therefore, it is necessary to seek and rely on a computational solution.

This fundamental necessity and the steadfast improvement in the speed and memory size of computers since the 1950s have led to the emergence of computational Mechanics.

Consequently, several computational methods have been developed to obtain approximate solutions that can be readily obtained in a reasonable time frame, and with reasonable effort.

The computational techniques replace the governing differential or integral equations with systems of simultaneous algebraic equations, so that a computer can be used to obtain the solution.

The computational techniques yield approximate values of the unknowns at discrete number of points in the continuum.

Where This Course Fits?

- Start from general mechanics and specialize our interest toward the field of computational mechanics.

The field of Mechanics can be subdivided into 3 major areas:

Mechanics
- *Theoretical*
- *Applied*
- *Computational*

Theoretical Mechanics deals with fundamental laws and principles of mechanics.

Applied Mechanics transfers this theoretical knowledge to scientific and engineering applications, through the formulation of mathematical models of physical phenomena.

Computational Mechanics solves specific problems by simulation through numerical methods implemented on digital computers.

An old joke about mathematicians: one may define computational mechanician as a person who searches for solutions to a given problem, an applied mechanician as a person who searches for problems that fit given solutions, a theoretical mechanician as a person who can prove the existence of problems & solutions.

Computational Mechanics

- Several branches according to the *physical scale* of the focus of attention:

Computational Mechanics
- *Nanomechanics and micromechanics*
- *Continuum mechanics*
  - *Solids & Structures*
  - *Fluids*
  - *Multiphysics*
- *Systems*

Computational Solid and Structural Mechanics

CSM
- *Statics*
- *Dynamics*

CSM Statics

CSM Statics
- *Linear*
- *Nonlinear*

CSM Simulation on the Computer

CSM Approximation Methods
- *Finite Difference Method*
- *Finite Element Method*
- *Boundary Element Method*
- *Finite Volume Method*
- *Spectral Method*
- *Mesh-Free Method*

1.Finite Difference Method (FDM)

- **An Introduction to FDM**
- FD Solution of Terzaghi's Consolidation Equation.
- FD Solution of Two-dimensional Flow in Soils.

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1.Finite Difference Method (FDM) : Introduction

● One Dimensional Computations by FDM

In FDM derivatives in the governing equations are written in finite difference equations.

To illustrate, let us consider the second-order, one-dimensional differential equation:

$$\frac{d^2u}{dx^2} - 2 = 0, \quad 0 < x < 1$$

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With Dirichlet boundary conditions (values of variable u specified at boundaries):

$$\begin{cases} u = 0 & at\ x = 0 \\ u = 0 & at\ x = 1 \end{cases}$$

Exact Solution: $\boxed{u = x^2 - x}$

●In FDM, solutions are computed at specific – discrete – points called Nodes.

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●The process of subdividing the domain into discrete points is called descretization.

●Starting from x=0 the domain is subdivided in steps, until x=1. The chosen step is called step size and is denoted by Δx.

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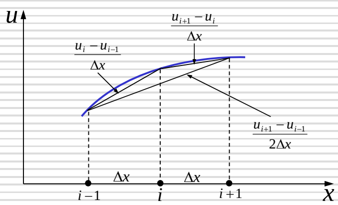


Finite Difference Equations:

$$\left(\frac{du}{dx}\right)_i \approx \frac{u_{i+1}-u_i}{\Delta x} \quad \textit{Forward difference}$$

$$\left(\frac{du}{dx}\right)_i \approx \frac{u_i-u_{i-1}}{\Delta x} \quad \textit{Backward difference}$$

$$\left(\frac{du}{dx}\right)_i \approx \frac{u_{i+1}-u_{i-1}}{2\Delta x} \quad \textit{Central difference}$$

Hence,

$$\left(\frac{d^2u}{dx^2}\right)_i = \frac{d}{dx}\left(\frac{du}{dx}\right)_i \cong \frac{1}{\Delta x}\left[\left(\frac{du}{dx}\right)_{i+1} - \left(\frac{du}{dx}\right)_i\right]$$

$$= \frac{1}{\Delta x}\left(\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i-u_{i-1}}{\Delta x}\right) = \left(\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}\right)$$

Therefore:

$$\boxed{\frac{d^2u}{dx^2} = \frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2} = 2}$$

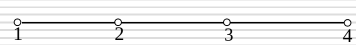
Given Boundary condition: u_{i-1}=0, u_{i+1}=0; and at point *i*, Δx = 1/2

$$\Rightarrow \frac{0-2u_i+0}{(0.5)^2} = 2 \Rightarrow u_i = -0.25$$

From Exact Solution: $u = (x^2-x)_{x=0.5} = -0.25$

Example Problem

Consider the three-element system shown below.



And solve the differential equation shown below.

$$\frac{d^2u}{dx^2} - 2u = f(x) \quad 0<x<1 \qquad \text{where } f(x) = 4x^2-2x-4$$

Subject to the Dirichlet boundary condition:

$$\begin{cases} u = 0 & at\ x = 0 \\ u = -1 & at\ x = 1 \end{cases}$$

Whose exact solution: $u = -2x^2 + x$

Solution

Write FDE at nodes 2 and 3:

Node 2:

$$\frac{u_3 - 2u_2 + u_1}{\Delta x^2} - 2u_2 = f_2$$

$$\Rightarrow \frac{u_3 - 2u_2 + 0}{(1/3)^2} - 2u_2 = 4\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - 4 = \frac{-38}{9}$$

$$\Rightarrow 9(u_3 - 2u_2) - 2u_2 = \frac{-38}{9}$$

$$\Rightarrow 9u_3 - 20u_2 = \frac{-38}{9}$$

Node 3:

$$\frac{u_4 - 2u_3 + u_2}{\Delta x^2} - 2u_3 = f_3$$

$$\Rightarrow \frac{-1 - 2u_3 + u_2}{(1/3)^2} - 2u_3 = 4\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right) - 4 = \frac{-32}{9}$$

$$\Rightarrow 9(-1 - 2u_3 + u_2) - 2u_3 = \frac{-32}{9}$$

$$\Rightarrow 9u_2 - 20u_3 = \frac{49}{9}$$

Combining, we have:

$$\begin{bmatrix} -20 & 9 \\ 9 & -20 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\frac{38}{9} \\ \frac{49}{9} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.111 \\ -0.222 \end{bmatrix}}$$

Exact Solution!!

1.1 Derivation of Finite Difference Equations

- **Consider a function *u(x)* and its derivative:**

$$\frac{\partial u(x)}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

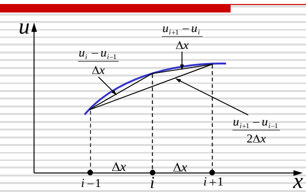
- **Using Taylor's series expansion we get:**

$$u(x+\Delta x) = u(x) + \Delta x \frac{\partial u(x)}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u(x)}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u(x)}{\partial x^3} + \cdots$$

$$\Rightarrow \frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{\partial u(x)}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u(x)}{\partial x^2} + \cdots$$

$$= \frac{\partial u(x)}{\partial x} + O(\Delta x)$$

- **Taylor's series expansion:**



$$u_{i+1} = u_i + \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{(\Delta x)^4}{4!}\left(\frac{\partial^4 u}{\partial x^3}\right)_i \cdots \textbf{(1)}$$

$$u_{i-1} = u_i - \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{(\Delta x)^4}{4!}\left(\frac{\partial^4 u}{\partial x^3}\right)_i \cdots \textbf{(2)}$$

- **Rearranging eqn. (1) we have:**

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)$$

Forward difference

Trunction error

- **Similarly, rearranging eqn. (2) we have:**

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$

Backward difference

Trunction error

- **Subtracting eqn. (2) from (1) we get:**

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

Central Difference

Trunction error

- **Finally adding eqns. (1) and (2) gives:**

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Trunction error

Assignment #1

1. Derive Terzhagi's 1-D Consolidation Equation.

2. Derive Laplace's equation for 2-D flow of water in soils.

Submission date: March 26, 2009