School of international Liberal Studies
Waseda University
Advanced Course: Virtual Earth
Spring 2025

Problem Set 5

1. Find the eigenvalues and normalized eigenvectors of the following matrices and verify that the trace of the matrix (the sum of its diagonal elements) is equal to the sum of the eigenvalues.

   (a) $\begin{pmatrix} 13 & 5 \\ 2 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 5 & 2 & 4 \\ -3 & 6 & 2 \\ 3 & -3 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$

2. A symmetric matrix $\mathbf{A}$ is one satisfying the condition: $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$. Symmetric matrices are an extremely important class of matrices as they arise frequently in many applications. Two fundamental properties of symmetric matrices are that: (1) they have real eigenvalues, and (2) they are diagonalizable, i.e., they have a complete set of linearly independent eigenvectors. Putting these together gives rise to the *Spectral Theorem* which states that every symmetric matrix has the factorization $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}}$, with real eigenvalues in $\mathbf{\Lambda}$ and normalized (unit length) eigenvectors in $\mathbf{Q}$. The proof of this theorem involves showing that: (1) the eigenvalues are real, (2) eigenvectors corresponding to *distinct* eigenvalues are orthogonal to each other, and (3) if an eigenvalue is repeated k times (the "algebraic multiplicity" of that eigenvalue) then the number of independent eigenvectors corresponding to it (the "geometric multiplicity" of that eigenvalue) is also k. Note that (3) doesn't guarantee that the computed eigenvectors corresponding to any eigenvalue will be orthogonal. However, they can always be made orthogonal. This is for two reasons. First, since the eigenvectors are independent (as per (3)) they form a basis for the subspace spanned by those eigenvectors. Second, any linear combination of eigenvectors belonging to an eigenvalue is also an eigenvector with the same corresponding eigenvalue. Thus, one can always take appropriate linear combinations of the eigenvectors belonging to the same eigenvalue to define a new basis for that space made up of orthogonal (eigen)vectors. (This latter procedure of finding an orthogonal basis from a set of independent vectors is known as *Gram-Schmidt* orthogonalization (`https://en.wikipedia.org/wiki/Gram-Schmidt_process`).) The bottom-line is that every $n \times n$ symmetric matrix has n eigenvectors that are orthogonal to each other. These (normalized) eigenvectors can be put into the columns of an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, implying that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. Whew!

With that background consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) What are the algebraic and geometric multiplicities of the different eigenvalues?

(c) Construct $\mathbf{Q}$ and verify that it is an orthogonal matrix.

(d) Verify the spectral theorem, i.e., that $\mathbf{A}$ has the factorization $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}}$.

3. One of the most famous differential equations in all of science (and with special significance in Earth Sciences) is the Lorenz System (`https://en.wikipedia.org/wiki/Lorenz_system`), named after Edward Lorenz, a meteorologist at MIT who discovered the phenomenon of chaos when studying a simplified model of weather (described by these equations). Chaos is what fundamentally limits our ability to predict weather to about 5 days ahead (popularly known as the "butterfly effect"). Incidentally, Lorenz's (accidental) discovery had much to do with the finite precision–and round-off error–of the computers of that era. The Lorenz System is described by the following set of coupled ODEs:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned}$$

where ρ, σ and β are constants. Defining the vector $\mathbf{u} = (x, y, z)$, we can write this system more compactly as $\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u})$, with $\mathbf{F} = (\sigma(y-x), x(\rho-z)-y, xy-\beta z)$. For a certain set of values of these three parameters (ρ, σ and β), the system exhibits chaotic behavior.

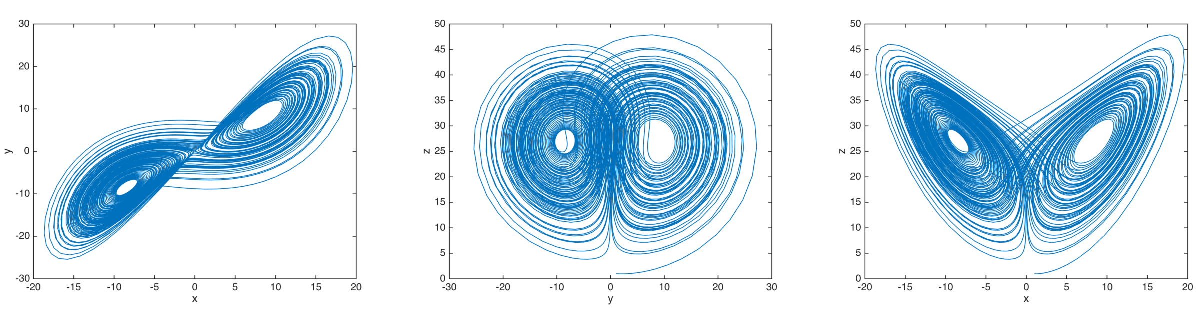

Figure 1: Numerical solution of the Lorenz system with parameter values of $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$. Shown are three 2-d views of the 3-d "phase space" of the system in which one independent variable is plotted against another. From left to right: $x-y$, $y-z$ and $x-z$.

Our main interest here is not in integrating this system of ODEs (although you will in the first problem) but in finding their steady-state solution ("fixed points"), i.e., $\frac{d\mathbf{u}}{dt} = \mathbf{0}$, and analyzing their *stability*, i.e., what happens if we perturb the system from a fixed point. In mathematical terms, the first problem involves finding the roots of $\mathbf{F}(\mathbf{u}) = \mathbf{0}$, whereas the second involves calculating the eigenvalues of the Jacobian matrix of $\mathbf{F}$. One can go further with the analysis, for example, calculate how the fixed points vary as a function of the different parameters of the system. In the Lorenz system, for instance, there is only one fixed point for $\rho < 1$, but this "splits" into two fixed points for $\rho > 1$ in what is known as a "pitchfork bifurcation" (`https://en.wikipedia.org/wiki/Pitchfork_bifurcation`). Such a fundamental change in behavior when a parameter passes some critical threshold is characteristic of many natural systems. The ocean thermohaline circulation is believed to exhibit such behavior. Its not something we really want to find out! (`http://www.imdb.com/title/tt0319262/`.))

(a) Write code to numerically integrate the Lorenz system of equations (you can use your own forward/backward Euler or Runge-Kutta solver or ODE solver library). Use the following parameter values $\sigma = 10$ and $\beta = 8/3$. For ρ, try a few different values starting at 0 and increasing to 30 to see how the trajectories change. Use any initial condition you like.

(b) Now integrate the system with $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$, and using two sets of initial conditions very very slightly apart (for example (0,0,0) and (1.e-12,0,0)). See how the trajectories diverge even starting from *almost* the same initial state.

(c) Use your Newton solver to compute the fixed points (denoted by $\mathbf{u}^* = (x^*, y^*, z^*)$). (This problem is simple enough that you can even do it analytically by hand. But we'll do it numerically.) Using $\sigma = 10$ and $\beta = 8/3$, calculate the fixed points by varying ρ between 0 and 30 in increments of, e.g., 0.2. Make a plot of x^* as a function of ρ.

(d) Next, calculate the stability of these fixed points. To do so we write out the linearization, $\mathbf{F}^{(0)}_{\text{lin}}(\mathbf{x})$, of $\mathbf{F}(\mathbf{x})$ about a point $\mathbf{x}^{(0)}$ by Taylor-expanding $\mathbf{F}(\mathbf{x})$. The linearization requires the partial derivatives of each equation $F_i(\mathbf{x})$ with respect to x, y and z. These partial derivatives go into the i^{th}-row of the Jacobian matrix. With $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$, use your favorite eigenvalue solver (e.g., `eig` in Matlab or scipy) to compute the eigenvalues and eigenvectors of the Jacobian matrix for the Lorenz system evaluated at the fixed points you found above. What can you say about the stability of these fixed points and the behavior of small perturbations about the fixed points?

4. Consider the 1-d advection equation $\partial u/\partial t + c\partial u/\partial x = 0$ on the domain $0 \leq x \leq L$ ($L = 1$) with periodic boundary conditions and $c = 0.1$.

(a) Solve this equation using: (1) forward time, centered space (FTCS), (2) upwind, (3) Crank-Nicolson, (4) backward Euler, centered space and (5) Lax schemes with initial conditions given by: (a) a top hat function ($u(x, 0) = 1$ for $0.4 \leq x \leq 0.6$ and zero otherwise) and (b) a Gaussian ($u(x, t) = \exp(-100(x - 0.5)^2)$). Pick a convenient grid spacing and time increment, and time-step the equation long enough to advect the initial condition at least once "around the domain". Compare with the analytical solution.

(b) Bonus For each scheme above perform a von Neumann stability analysis to investigate whether the method is stable or not, and under what conditions.

5. Consider a thin layer of fluid (representing the ocean or atmosphere) on a rotating planet. We will model a rectangular region that is small enough in the latitudinal (north-south) direction to ignore the curvature of the planet. The (linearized) dynamics of this system are governed by the so-called shallow water equations:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) = 0,$$

where η is the surface displacement, H the "equivalent depth" of the undisturbed fluid, u the velocity in the x (east-west) direction and v the velocity in the y (north-south) direction. g is the acceleration due to gravity and f the Coriolis parameter. There are two situations of interest. The first is when the box is centered about the equator. In that case $f = \beta y$, with $\beta = 2.3 \times 10^{-11}\text{m}^{-1}\text{s}^{-1}$. This is known as the equatorial β-plane approximation. The other is when the box is centered about $\sim$45° (north or south) latitude, for which f is a constant $f_o = 10^{-1}\text{s}^{-1}$. This is known as the mid-latitude f-plane approximation.

Discretize the equations in space using central differences on a "C-grid", in which η is defined at the center of the grid box and the velocity field at the center of the edges (u on the west and east edges and v on the south and north edges). For boundary conditions, use no-flux southern and northern boundaries (i.e., solid walls) and either periodic or no-flux boundary conditions in x depending on whether your interest is atmospheric or oceanographic, respectively.

Use your favorite time-stepping method (or canned ODE solver) to compute the time-dependent solution to this problem for the two situations given above. At time $t = 0$, take η to be a "Gaussian bump" (in both x and y) centered about the middle of the domain, and $u = v = 0$. (You can play around with other initial conditions as well.) To simulate a tsunami, use a depth $H = 4000m$. To simulate the "first baroclinic mode" of the ocean use an equivalent depth of $H = 1$ m; for the atmosphere use $H = 370$ m. Note that the speed of gravity waves is given by $\sqrt{gH}$, so that should give you an idea of what time step to use based on the CFL condition. Make an animation of the surface displacement and velocity field as the fluid adjusts.