School of international Liberal Studies
Waseda University
Advanced Course: Virtual Earth
Fall 2025

Problem Set 2

1. *Transient climate change*: Earth's radiation budget is generally in balance, especially over long time scales. But an imbalance could occur if the incoming solar radiation changes and/or there are internal changes in earth's climate (e.g., more greenhouse gas in the atmosphere), leading to a change in the outgoing long wave radiation. Such an imbalance would lead to changes in temperature, i.e., climate would not be in a steady state as we assumed in formulating our energy balance equations. Eventually, however, earth's climate will adjust so that it reaches an equilibrium. It is easy to incorporate time-dependence by writing the energy balance equation as:
$$C_{\mathrm{eff}}\frac{dT_{\mathrm{s}}}{dt} = F_{\mathrm{SW}} - F_{\mathrm{LW}}. \tag{1}$$
Here, T_{s} is the global mean surface temperature, C_{eff} an "effective" heat capacity per unit area for the planet, F_{SW} the incoming shortwave radiation flux per unit area, and F_{LW} the outgoing longwave radiation flux per unit area.

   To calculate the radiative fluxes we will use the 1-layer "grey body" model of the greenhouse effect. In that model, the *equilibrium* "top of atmosphere" energy balance is given by:
$$\underbrace{(1-\alpha)S/4}_{F_{\mathrm{SW}}} = \underbrace{(1-\epsilon)\sigma T_{\mathrm{s}}^4 + \epsilon\sigma T_{\mathrm{a}}^4}_{F_{\mathrm{LW}}}, \tag{2}$$
where we have equated the incoming shortwave radiation (on the left) to the outgoing longwave radiation (on the right). Similarly, the *equilibrium* energy balance for the atmosphere is:
$$\epsilon\sigma T_{\mathrm{s}}^4 = 2\epsilon\sigma T_{\mathrm{a}}^4, \tag{3}$$
which allows us to write the longwave flux as:
$$F_{\mathrm{LW}} = \underbrace{(1-\epsilon/2)}_{\epsilon_{\mathrm{LW}}}\sigma T_{\mathrm{s}}^4. \tag{4}$$
The term in brackets, called (confusingly) the "longwave emissivity" (ϵ_{LW}) or "greenhouse factor", in effect reduces the radiation emitted by the surface (σT_{s}^4). A larger value of ϵ_{LW} thus corresponds to a *weaker* greenhouse effect (the opposite of ϵ). Plugging in a value for ϵ of ~ 0.77 (which we saw in the lecture gives T_{s} ∼288 K), results in ϵ_{LW} ∼0.61.

   We now substitute the above expressions for F_{SW} and F_{LW} into eq. 1:
$$C_{\mathrm{eff}}\frac{dT_{\mathrm{s}}}{dt} = (1-\alpha)S/4 - \epsilon_{\mathrm{LW}}\sigma T_{\mathrm{s}}^4. \tag{5}$$

This is an ordinary differential equation for T_s that we can integrate (analytically or numerically) given an initial temperature and suitable parameter values.

What about C_{eff}? Most of earth's heat capacity is due to the ocean, particularly the upper 70-100 m or so that interacts directly with the atmosphere. The heat capacity per unit area is thus $C_{\text{eff}} = \rho c_p h_{\text{ml}}$, where ρ is the density of water (1000 kg m^{-3}), c_p the specific heat of water (4218 J K^{-1}kg^{-1}), and h_{ml} the mixed layer depth ($\sim$70 m).

In the follow, we will denote the current solar "constant" as $S_o = 1367$ W m^{-2}.

(a) Write a computer program to numerically integrate eq. 5 forward in time given an initial temperature $T_s(t = 0)$, and appropriate values for S, α, ϵ_{LW} and C_{eff}. You can use any of the integration methods you have already written code for (forward or backward Euler, RK2, RK4, etc). Play around with the time step. My guess is that a step size of 10 days should be short enough for numerical accuracy and stability, but long enough that you can run the model for many time steps without getting bored. Or you can use an ODE solver from a numerical library. Alternatively, since what we're interested in is the equilibrium or steady-state solution (when $dT_s/dt = 0$), you can use Newton's method to directly calculate that solution without performing a transient solution.

(b) Integrate eq. 5 for 50 years with an initial condition $T_s(t = 0) = 0$°C and $S = S_o$. Save the solution every 1 year. Make a plot of T_s as a function of time.

(c) Now rerun the model for different values of S, starting from, say, twice S_o to 40% of S_o, decreasing in 10% increments . For each experiment, store the final equilibrium temperature. When you've finished doing all the runs, plot the final equilibrium temperature for each experiment against the corresponding value of S (divided by S_o).

2. *Albedo feedback*: In the previous problem we used a fixed albedo. In our 0-dimensional Energy Balance Model (EBM) world, this albedo represents the average reflectivity of the planet. But we know that different materials have different reflectivities. For example, sand is $\sim$0.2-0.3, while snow is closer to 0.8. Moreover, the albedo depends to some extent on the temperature. Thus, if the temperature falls below freezing (or some threshold) we expect more snow and hence a higher albedo. But this leads to more of the incoming solar radiation being reflected cooling the planet further, and so on. This is known as the ice-albedo feedback. In practice, the relationship between albedo, geographical location and temperature is a complex one but observations suggest it looks something like the figure below:

(a) Modify your EBM model code to incorporate a temperature-dependent albedo according to the figure shown above (with $a_1 = 0.3$, $a_2 = 0.7$, $T_U = 10$°C, $T_L = -10$°C).

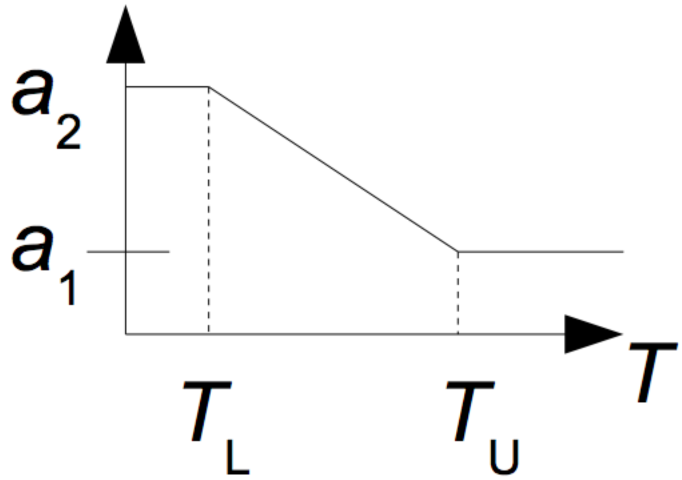

Figure 1: Simplified relationship between surface temperature and albedo (α). For $T_\mathrm{s} > T_U$, $\alpha = a_1$, and for $T_\mathrm{s} < T_L$, $\alpha = a_2$. In-between, α is a linear function of T_s.

(b) Now repeat the previous sequence of experiments in which you varied S. For the first experiment (with $S = 2S_o$), initialize the temperature at 0°C. For subsequent experiments in which S is decreased, use the final temperature from the previous experiment as the initial value. The idea is to mimic a situation in which S is decreasing slowly on time scales much longer than that for the EBM to reach a steady state. You can of course modify your code to replicate that situation by making S a slowly-varying function of time. But it is easier to do a sequence of experiments in which S undergoes a step change and the model is run to steady state. As before, store the final equilibrium temperature for each experiment. Plot T_s versus S/S_o. You will notice that at some value of $S/S_o < 1$, there is a precipitous drop in temperature. Congratulations! You've succeeded in glaciating the entire planet.

(c) A glaciated world is not especially pleasant so lets try to get out of it. How do you do that? Increasing S should do the trick, no? So lets try that. Once you've completed the above sequence of runs and reached $S = 0.4S_o$, start *increasing* S in 10% increments, once again initializing the temperature from the equilibrium value of the previous run. Do this until you've reached $S = 2S_o$. On the same figure plot T_s versus S/S_o for this "reverse" set of runs. (You will need to do a `hold on` in Matlab so as to not erase the previous plot. And use a different color for the second plot.) Do you notice something odd? The model trajectory (the equilibrium temperature) as S is increased does not follow the same path as when S was decreased! It depends on the *history* of S. This is a phenomenon known as hysteresis and is common in many non-linear systems displaying multiple equilibria or steady states for the same parameter values (here T_s can take on two possible values for a given S).

3. One of the most basic models used to study the spread of infectious diseases is known as "SIR". This is a system of ODEs for the number of susceptible (S), infected (I) and recovered (R) individuals in a population (e.g., the UK as a whole). You can read about the theoretical basis for this and other epidemiological models in this excellent article (in particular Sec. 2.3): `https://epubs.siam.org/doi/10.1137/S0036144500371907`. Wikipedia is another good place: `https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology`.

   The governing equations are:

$$\begin{aligned}\frac{dS}{dt} &= -\beta IS/N \\ \frac{dI}{dt} &= \beta IS/N - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

   Here, $N = S + I + R$ is the total (constant) size of the population; β is the contact or transmission rate, i.e., the average number of contacts sufficient for transmission (so-called "adequate contacts") a person makes per unit time; and γ the average rate at which an infectious person "recovers" (rather gruesomely, this includes dying from the disease!). You can think of $1/\gamma$ as the average period for which a person remains infectious. Thus, if there are I infected people in the population, γI is the number of people per unit time going from the infected to the recovered compartment. (Mathematically, this is identical to how radioactive decay works. The time τ it takes for a radionuclide to decay is a random variable governed by an exponential probability distribution function given by $p(\tau) = \lambda \exp(-\lambda\tau)$. The probability that a radionuclide decays between times τ and $\tau + d\tau$ is then $p(\tau)d\tau$, and the average time to decay is given by $\int_0^\infty \tau' p(\tau')d\tau'$, which is simply $1/\lambda$. Here, the same probability distribution, with λ replaced by γ, applies to the time τ a person remains infected. The average time to recovery is thus $1/\gamma$.)

   The product of the contact rate per unit time β and the average infectious period $1/\gamma$ is the (dimensionless) parameter $\sigma = \beta/\gamma$, the average number of adequate contacts a typical infectious person makes during the period she/he is infectious. For the SIR model (and SEIR one below) σ is equal to R_0, the "basic reproduction number", defined as the average number of new ("secondary") infections arising from the introduction of a single infected person into a population in which everyone is susceptible. (See: `https://en.wikipedia.org/wiki/Basic_reproduction_number`.) For the SIR and many other models R_0 must be greater than 1 for an infection to spread. You can decrease R_0 by reducing the number of contacts β. This is the basis for the "social distancing" measures we're all so familiar with now (and those of us who are introverts have always known about!). So much for the theory. Now on to the actual problem.

   The SIR system of ODEs can be generically written as $d\mathbf{u}/dt = \mathbf{f}(\mathbf{u}, t)$, where $\mathbf{u} = (S, I, R)$

and $\mathbf{f}$ is a vector of the right hand side (RHS) of the above equation. $\mathbf{f}$ can depend explicitly on time t if, for example, the parameters in the equation vary in time. For instance, the contact rate β might be higher during peak commuting times or change as social distancing and other measures are put into effect (see this article for an example of how you can incorporate such effects: `https://www.ijidonline.com/article/S1201-9712(20)30117-X/fulltext`). We'll ignore that complexity here and drop the dependency in what follows below.

(a) Write a Matlab program to numerically integrate this system (for 60 days) using the forward Euler method. To remind you, discretizing the LHS gives: $(\mathbf{u}^{n+1} - \mathbf{u}^n)/\Delta t \approx \mathbf{f}(\mathbf{u}^n)$. Rearranging, we have an explicit time-stepping formula: $\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \mathbf{f}(\mathbf{u}^n)$. Use a total population $N = 1000$. Play around with the time step, initial conditions (start with $\mathbf{u}(0) = (995, 5, 0)$), and values of β and γ (start with $\beta = 1$ per day and $1/\gamma = 3$ days, implying a R_0 of 3). Make plots of S, I and R as a function of time. See how the shape of these curves changes as you vary β (decreasing it will "flatten the curve", a notion you will have heard a lot about over the past few years). In case you're wondering what your plots should look like, see Fig. 3 of the article reference above.

(b) One of Matlab's core strengths is the range of highly sophisticated algorithms it puts at your disposal to solve systems of ODEs. We need not get into the details but the various functions it provides are typically called `odeXX`, where `XX` is a number that (loosely) represents the order of accuracy of the underlying algorithm (e.g., forward Euler is first order in Δt because the error incurred from truncating the Taylor expansion is O(Δt)). `ode23` and `ode45` are probably the two most useful solvers but there are many others (type `help ode23` to see the different options available and have a look at this page for more information: `https://www.mathworks.com/help/matlab/math/choose-an-ode-solver.html`). If you're feeling ambitious, solve the above system of equations using one of Matlab's solvers (I suggest starting with `ode23`).

4. It can be shown that in the SIR model, I will eventually go to zero, i.e., the infection disappears from the population. Many infectious diseases however remain endemic in the population. This is modeled by adding so-called vital effects (births and deaths) to the basic SIR model. A further enhancement is to add an incubation period during which individuals who have been exposed and infected are not yet infectious themselves. This is done through an extra variable E of the number of exposed individuals. The equations for this SEIR model

are:

$$\begin{aligned}
\frac{dS}{dt} &= -\beta IS/N + \mu N - \mu S \\
\frac{dE}{dt} &= \beta IS/N - \epsilon E - \mu E \\
\frac{dI}{dt} &= \epsilon E - \gamma I - \mu I \\
\frac{dR}{dt} &= \gamma I - \mu R,
\end{aligned}$$

where, μ is the (equal) birth and death rate and $1/\epsilon$ is the average incubation period (about 5-7 days for SARS-CoV-2: `https://www.ncbi.nlm.nih.gov/pmc/articles/PMC7014672/`). Because we have assumed that the birth and death rates are equal, the population size $N = S + E + I + R$ remains constant.

(a) Implement code to numerically integrate the SEIR equations using forward Euler (you know the drill) or use one of Matlab's solvers. You can work out a value for μ by looking up the number or births (or deaths) in a country and the total population. In the UK for instance, in 2018 there were 731,213 live births in a total population of 66,436,000. (Most countries are not in steady state. In the UK, birth rates exceed death rates and there is net migration into the country as well. But we'll ignore that wrinkle.) Epidemiological parameters are very difficult to estimate and there is a very wide range in published values for the current outbreak. Based on data I've found (`https://github.com/alsnhll/SEIR_COVID19/blob/master/COVID19seir/www/Parameters.Rmd`), I suggest plugging in a transmission rate $\beta \sim 0.5$, an average incubation period $1/\epsilon$ of 5 days, and an average infection period $1/\gamma$ of 6 days. But by all means explore different values to see how they affect the solution. If you're feeling ambitious you could even try making β a function of time (starting at a high value and gradually decreasing as government interventions go into effect; see: `https://www.ijidonline.com/article/S1201-9712(20)30117-X/fulltext`).