Lecture 3: Compactness, EVT, Correspondences

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Contents

Notation

- $\forall$ translates to "for all"
- $\exists$ translates to "there exists"
- $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers
- $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ is the set of integers
- $\mathbb{Q} = \{p/q : p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}\backslash\{0\}\}$ is the set of rational numbers
- $\mathbb{R}$ is the set of real numbers
- If S is a set and $n \in \mathbb{N}$, then S^n is the n^{th} order Cartesian product of S. E.g., $S^2 = S \times S$
- For any $\varepsilon > 0$, $B_\varepsilon(x)$ is the Euclidean ball around x with radius ε
- Unless otherwise specified, $d(x, y)$ is a metric on the contextual set x, y belong to
- $\text{cl}(S)$ is the closure of the set S

1 Compactness

So, straight up, the first time I encountered compactness (back in undergrad real analysis) it seemed like an inscrutable concept. If you get it right away, awesome! But if you don't, you're in good company. It can take a little bit for this to sink in.

1.1 Introduction

Definition 1. A class $\mathcal{F} = \{A_\omega\}_{\omega\in\Omega}$ is said to ***cover*** a set S if $S \subseteq \cup_{\omega\in\Omega} A_\omega$. If all members of the class $\mathcal{F}$ are open, we say it is an ***open cover***.

Definition 2. A set S is ***compact*** if every open cover of S has a ***finite sub-cover*** of S.

Some examples of sets that are and are not compact:

- $S = (0,1)$ is not compact. $\mathcal{F} = \{(1/n, 1) : n \in \mathbb{N}\}$ covers S. However, there is no finite sub-cover: Any finite sub-cover gives the interval $(1/N, 1)$ for some $N \in \mathbb{N}$. But we can take $z = 1/(2N)$. Then $z \in (0,1)$ and $z \notin (1/N, 1)$.
- $S = [0,\infty)$ is not compact. $\mathcal{F} = \{(-1, n) : n \in \mathbb{N}\}$ covers S. However, there is no finite sub-cover: Any finite sub-cover gives the interval $(-1, N)$. Take $z = N + 1$. Then $z \in [0,\infty)$ but $z \notin (-1, N)$.
- $[0,1]$ *is* compact. Compactness is really trying to get to a notion of "finiteness," and there is a sense in which intervals that are open or not bounded are not finite. Of course, compactness is more general than that, but at least in $\mathbb{R}^N$ we will get a more intuitive definition of compactness.

Remark 1. You can prove the set $[0,1]$ is compact by following the same steps of the proof for Bolzano-Weierstrass—the ideas are related. Suppose by contradiction that there is an open cover with no finite sub-cover. You can split the set in halves so that at least one half has no finite sub-cover; then you can iterate on this idea and, just like in Bolzano-Weierstrass, the half-intervals will converge to a single point and give you a contradiction. Can you see what the contradiction will be? If you can then you've basically proven Heine-Borel in $\mathbb{R}$!

Remark 2. Any finite set S is compact. Take any open cover of S, call it $\mathcal{F} = \{A_\omega\}_{\omega\in\Omega}$. For $x \in S$, $x \in A_\omega$ for some $\omega \in \Omega$ (there may be several ω's, and a single A_ω may contain many $x \in S$). Name this ω_x for each x; since S is finite, $\{\omega_x\}_{x\in S}$ is finite. Hence

$$\mathcal{F}_A = \{A_{\omega_x}\}_{x\in S}$$

is a finite sub-cover of S.

Is $\mathbb{Q} \cap [0,1]$ compact (the rational numbers between 0 and 1, inclusive)?

Definition 3. A set S ***sequentially compact*** if every sequence in S has a sub-sequence that converges to a point in S ($\forall (x_m) \in S \ \ \exists (x_{m_k})$ s.t. $x_{m_k} \to x \in S$).

Theorem 1. *A set S is compact $\iff$ S is sequentially compact.*

Since compactness is a pretty crucial concept in analysis, Theorem 1 is an important characterization to know, but you will have to wait for a proof until ECON 2010.

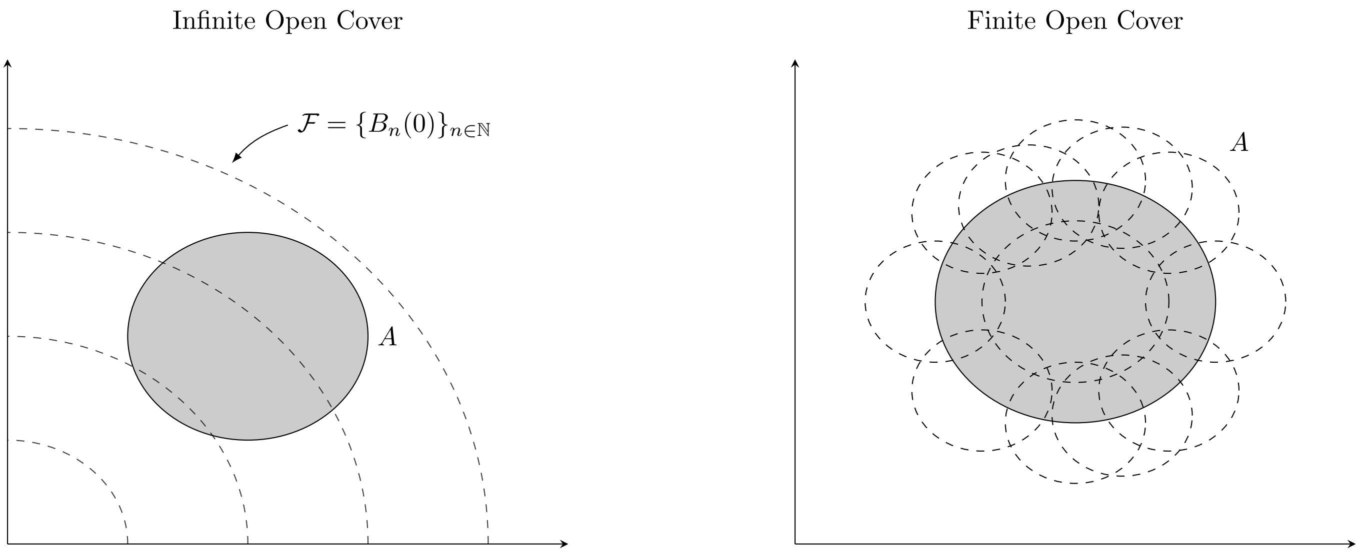


Figure 1: Examples of Open Covers in $\mathbb{R}^2$

1.2 Heine-Borel and Other Theorems

Theorem 2 (Heine-Borel)**.** *For any finite N, $S \subseteq \mathbb{R}^N$ is compact iff S is closed and bounded.*

Proof ($\Rightarrow$)*.* We show that compactness $\implies$ closed and bounded. The converse is a bit more involved, and in the interest of brevity, we only walk through how to prove it. Let S be a compact set; first we show S is bounded. Fix $s \in S$ and take

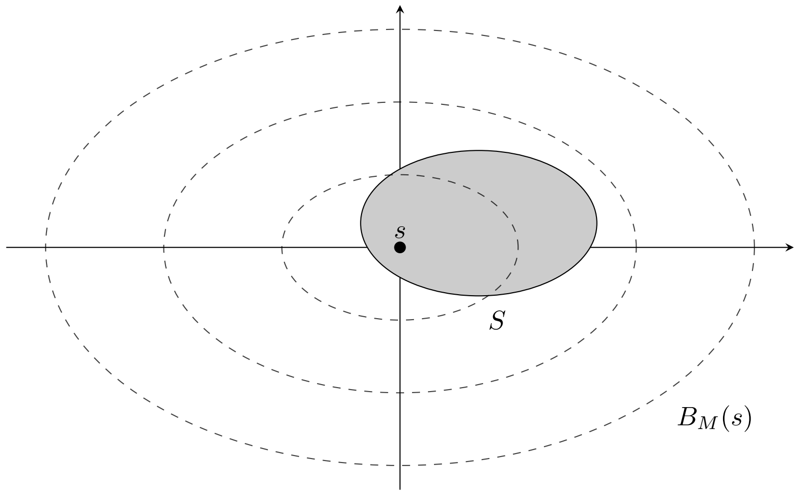
$$\mathcal{F} = \{B_m(s)\}_{m\in\mathbb{N}}$$

We know that $\mathcal{F}$ is an open cover of $\mathbb{R}^N$, and since $S \subseteq \mathbb{R}^N$, $\mathcal{F}$ is also an open cover of S. Since S is compact, there exists a finite subset of $\mathcal{F}$ that covers S; that is, there exists an $M \in \mathbb{N}$ such that

$$S \subseteq \bigcap_{m\leq M} B_m(s) = B_M(s)$$

where the equality holds because $B_m(s) \subseteq B_{m+1}(s)$ for any $m \in \mathbb{N}$.

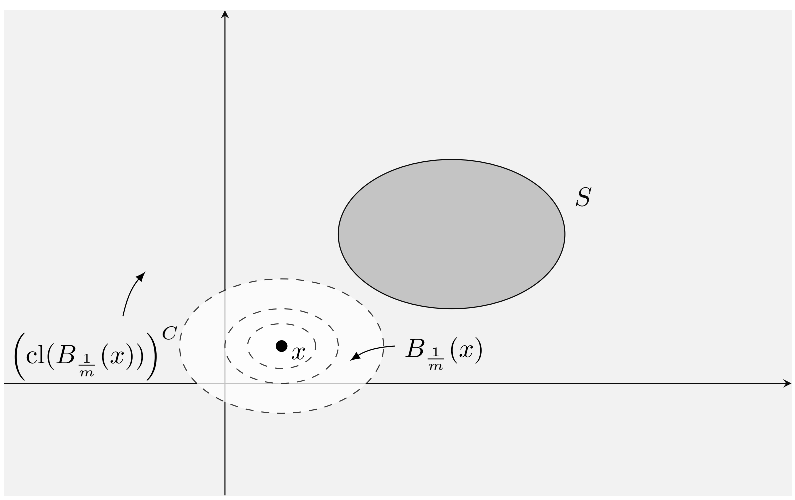
Hence, $x \in S \implies x \in B_M(s)$ for some $M > 0$ and any $s \in S$, which is the definition of boundedness. Visually,



Now we show that S is closed. Take any $x \in S^c = \mathbb{R}^N \setminus S$, the ***complement*** of S in $\mathbb{R}^N$. Define the collection

$$\mathcal{F} = \left\{ \left(\mathrm{cl}(B_{\frac{1}{n}}(x)) \right)^c \right\}_{n \in \mathbb{N}}$$

so $\mathcal{F}$ is the complement of the *closure* of all the balls[1] of radius $\frac{1}{m}$ around x. A graphical example in $\mathbb{R}^2$:



Note we need the complements of the closure of the balls because we want the sets in the collection to be open. Now $\mathrm{cl}(B_{\frac{1}{m}}(x)) \to \{x\}$ by the nested interval theorem, so

$$\bigcup_{m \in \mathbb{N}} \left(\mathrm{cl}(B_{\frac{1}{m}}(x)) \right)^c = \mathbb{R}^N \setminus \{x\}$$

That is, the union of the complements converges to the entire space *except* for x. Since $S \subseteq \mathbb{R}^N \setminus \{x\}$ ($S \subseteq \mathbb{R}^N$ and $x \in S^c \implies x \notin S$), $\mathcal{F}$ is an open cover of S. By compactness of S, we know that it admits a

[1] I vividly remember making a mistake on an analysis exam in undergrad because I confused the concept of a closed ball with the closure of a ball. A closed ball is

$$\bar{B}_\varepsilon(x) = \{y : d(x, y) \leq \varepsilon\}$$

(note the $\leq$ as opposed to $<$) which, in a general metric space, does not necessarily equal $\mathrm{cl}(B_\varepsilon(x))$. These subtle intricacies are simultaneously some of the most frustrating and beautiful parts of mathematics.

finite sub-cover. Note

$$B_{\frac{1}{m+1}}(x) \subseteq B_{\frac{1}{m}}(x) \implies \left(\mathrm{cl}(B_{\frac{1}{m}}(x))\right)^c \subseteq \left(\mathrm{cl}(B_{\frac{1}{m+1}}(x))\right)^c$$

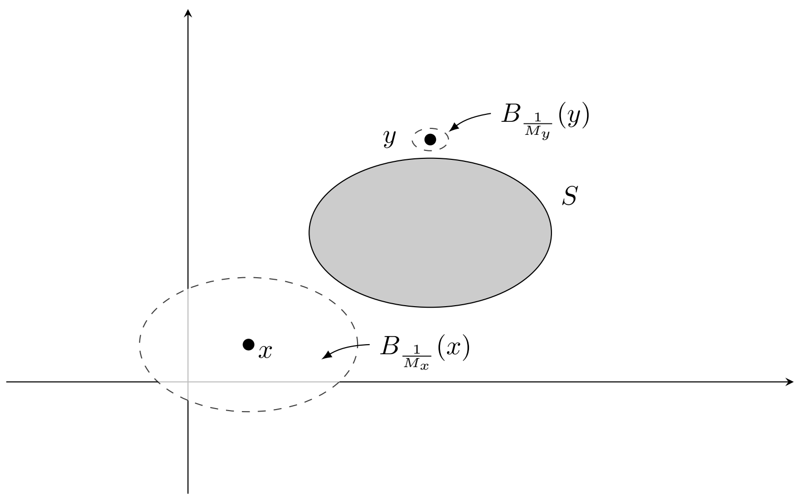
(why?) Hence any finite union gives

$$\bigcup_{m \leq M} \mathrm{cl}\left(B_{\frac{1}{m}}(x)\right)^c = \left(\mathrm{cl}(B_{\frac{1}{M}}(x))\right)^c$$

Since $S \subseteq \left(\mathrm{cl}(B_{\frac{1}{M}}(x))\right)^c$, it must be that $B_{\frac{1}{M}}(x) \subseteq S^c$. Finally, we can say $\forall x \in S^c \;\; \exists \varepsilon > 0$ (any $\varepsilon \leq 1/M$) s.t.

$$B_\varepsilon(x) \subseteq S^c$$

which is the definition of an open set. This shows S^c is open, so S is closed. Graphically, we see that at any point x outside of the set we can construct a ball of radius $1/M_x$ for some M_x that is entirely outside of S:



□

The other direction follows from Theorems 3 and 4 below. If a set S is bounded in $\mathbb{R}^N$ then it is the subset of some N-dimensional cube. Once we show the cube is compact, Theorem 4 gives that S is compact (the closed subset of a compact set is compact). I sketch the proof in the Appendix, but I will omit the details from the lecture; you will probably see it during your math class this fall and it's not worth going through it now unless you're very curious.

Theorem 3. *$\forall -\infty < a < b < \infty$, the N-dimensional cube $[a, b]^N$ is compact.*

Proof. See a sketch in Section A. □

Theorem 4. *Any closed subset of a compact set is compact.*

Proof. X be compact and S be a closed subset of X. Let $\mathcal{F}$ be any open cover of S and consider

$$\mathcal{G} = \mathcal{F} \cup \{S^c\}$$

Since S is closed, S^c is open. Since $\mathcal{F}$ is an open cover of S, $\mathcal{G}$ is an open cover of $S \cup S^c = \mathbb{R}^N \supseteq X$. Since X is compact, $\mathcal{G}$ has a finite sub-cover $\{G_m : m = 1, 2, \ldots, M\}$ s.t.

$$S \subseteq X \subseteq \bigcup_{m=1}^{M} G_m$$

The only set in $\mathcal{G}$ that is not in $\mathcal{F}$ is S^c, but by definition $S \cap S^c = \emptyset$. Hence, $\{G_m : m = 1, 2, \ldots, M\} \setminus \{S^c\} \subseteq \mathcal{F}$ is a finite sub-cover of S. $\square$

1.2.1 Weierstrass Extreme Value Theorem (EVT)

Theorem 5. *Let $S \subseteq \mathbb{R}$ be non-empty and compact. Then S has a minimum and a maximum.*

Proof. Since S is compact, it is closed and bounded. Since it is bounded, $\sup S \equiv s$ exists. For the sake of contradiction, suppose $s \notin S$. Since S is closed, the complement is open, and we can find some $\varepsilon > 0$ s.t. $B_\varepsilon(s) \cap S = \varnothing$. We know that $x \in S \implies x \leq s$, but since $x \notin B_\varepsilon(s) = (s - \varepsilon, s + \varepsilon)$ we also have

$$x < s - \varepsilon < s$$

so $s - \varepsilon$ is an upper bound of S that is smaller than s, a contradiction. Thus, $\max S$ exists and equals $\sup S$.

The proof for $\min S = \inf S$ is analogous. $\square$

Theorem 6. *Let $f : S \to T$ be a continuous function. If S is compact, then $f(S)$ is compact.*

Proof. Take any open cover of $f(S)$:

$$\mathcal{F} = \{F_\omega : \omega \in \Omega\} \quad \text{with} \quad f(S) \subseteq \bigcup_{\omega \in \Omega} F_\omega$$

Consider the inverse-image of each set in the open cover:

$$f^{-1}(\mathcal{F}) = \{f^{-1}(F_\omega) : \omega \in \Omega\}$$

For each $s \in S$, we know $f(s) \in f(S)$, and in turn for each $f(s) \in S$ there is some ω s.t. $f(s) \in F_\omega \implies s \in f^{-1}(F_\omega)$. In other words $f^{-1}(\mathcal{F})$ covers S. Since f is continuous, we know the pre-image of open sets is open, meaning $f^{-1}(\mathcal{F})$ is an open cover. Since S is compact, it admits a finite sub-cover:

$$\mathcal{G} = \{f^{-1}(F_{\omega_i}) : i = 1, \ldots, N\} \quad \text{with} \quad S \subseteq \bigcup_{i=1}^{N} f^{-1}(F_{\omega_i})$$

The image of a finite union of sets is just the union of their individual images. Hence

$$f(S) \subseteq f\left(\bigcup_{i=1}^{N} f^{-1}(F_{\omega_i})\right) = \bigcup_{i=1}^{N} f\left(f^{-1}(F_{\omega_i})\right) = \bigcup_{i=1}^{N} (F_{\omega_i} \cap f(S)) \subseteq \bigcup_{i=1}^{N} F_{\omega_i}$$

$\mathcal{F}$ was arbitrary and we found a finite sub-cover $\{F_{\omega_i} : i = 1, \dots, N\}$. By definition $f(S)$ is compact. (We remark that we need to write $F_{\omega_i} \cap f(S)$ because the image of the pre-image of an arbitrary set need not be the set itself! For example, let $f(x) = x$ with $S = [0, 3]$. Note $f(f^{-1}([0, 4])) = f([0, 3]) = [0, 3] \neq [0, 4]$.) $\square$

Theorem 7 (Weierstrass' EVT)**.** *If S is a compact set and $\varphi : S \to \mathbb{R}$ is continuous then $\exists x, y$ s.t. $\varphi(x) = \sup \varphi(S)$ and $\varphi(y) = \inf \varphi(S)$.*

Proof. The ***EVT*** follows directly from other theorems in this section. Since S is compact and φ continuous, $\varphi(S)$ is compact. Since $\varphi(S) \subseteq \mathbb{R}$ is compact, it has a minimum and a maximum. $\square$

Application to Economics Consider a standard utility maximization problem

$$\max_{x \in B(p,w)} u(x)$$

with $B(p, w) = \{x : p \cdot x \leq w\}^2$, $w \in \mathbb{R}^+$, and $x, p \in \mathbb{R}^N_+$. $B(p, w)$ is closed and bounded, so if $u(x)$ is continuous the maximum exist and the problem has a solution at some $x^* \in B(p, w)$. You will see different variations of this problem repeatedly throughout your career as an economist.

1.3 Using Sequential Definitions

The idea here is to show examples of how to construct sequences in a way that helps when doing proofs. We saw these proofs already without using sequences; however, we have seen that various definitions often have a sequential version, so let us see how they might help.

1. Let us show if $S \subseteq X$ is closed and X compact then S is compact.

   Proof.
   - Take any sequence $(x_m) \in S \subseteq X$.
   - X is compact, so it is sequentially compact; that is, $\exists x_{m_k} \to x$ for some $x \in X$.
   - S is closed, so $x \in S$. Hence any sequence in S has a convergent subsequence in S.

   By definition, S is sequentially compact, which means it is compact. $\square$

2. Let us show if $f : S \to T$ is continuous function, then S is compact implies $f(S)$ compact.

   Proof.
   - Take any sequence $y_m \in f(S)$; we know $\forall m \;\; \exists x_m \in S$ s.t. $f(x_m) = y_m$.
   - S is compact, so it is sequentially compact; that is, $\exists x_{m_k} \to x$ for some $x \in S$.
   - f is continuous, so $y_{m_k} = f(x_{m_k}) \to f(x) \in f(S)$. Let $y \equiv f(x)$.
   - Hence $\forall y_m \in f(S) \;\; \exists y_{m_k} \to y \in S$.

   By definition, $f(S)$ is sequentially compact, which means it is compact. $\square$

[2] The (Euclidean) inner product of the price vector p and the consumption bundle x, denoted $p \cdot x$, is defined as

$$p \cdot x = \sum_{\ell=1}^{N} p_\ell x_\ell$$

3. Let us show Theorem 7:

Proof.
- Since S is compact and φ is continuous, $\varphi(S)$ is compact.
- $\varphi(S)$ is compact, so it is closed and bounded.
- $\varphi(S)$ bounded means $-\infty < \inf \varphi(S) \leq \sup \varphi(S) < \infty$.
- By definition of $\sup \varphi(S)$, $\forall \varepsilon_m = 1/m \ \ \exists z_m \in \varphi(S)$ s.t. $\sup \varphi(S) - \varepsilon_m < z_m \leq \sup \varphi(S)$. Note $z_m \to \sup \varphi(S)$.
- $\varphi(S)$ closed means it has all its limits, so $\sup \varphi(S) \in \varphi(S)$. Hence $\exists x \in S$ s.t. $\varphi(x) = \sup \varphi(S)$.
- For the inf, construct a sequence $z_m \in \varphi(S)$ s.t. $\inf \varphi(S) \leq z_m < \inf \varphi(S) + \varepsilon_m$. $z_m \to \inf \varphi(S)$ so $\inf \varphi(S) \in \varphi(S)$, and $\exists y \in S$ s.t. $\varphi(y) = \inf \varphi(S)$.

Therefore φ attains its sup and its inf. □

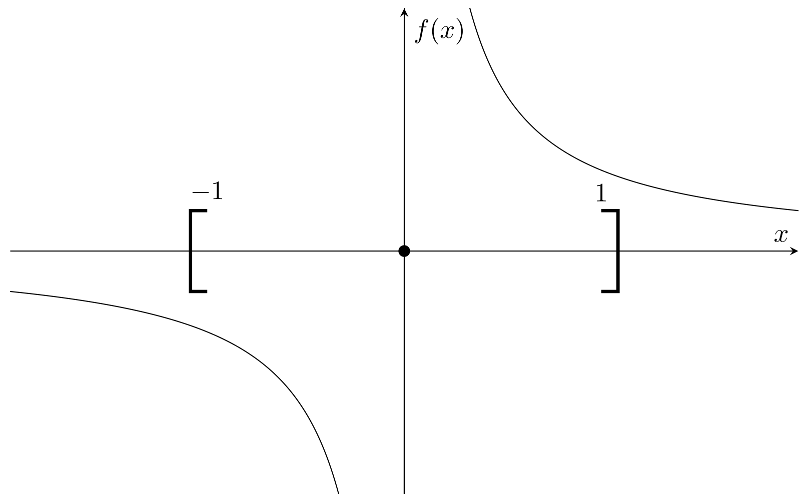
I think focusing on the *properties* of compactness can be more important than all the proofs above. Further, since we'll typically work with the reals, I think the intuition of compactness as equivalent to closed and bounded is fine (certainly for this course).

Table 1: Compactness! What is it good for? Actually, quite a bit.

S is compact:	
Definition	For any ***open cover*** there exists a ***finite sub-cover***.
	$\forall \mathcal{O} = \{O_\omega : \omega \in \Omega\}$ open cover $\exists W \subseteq \Omega$ s.t. W finite and $S = \bigcup_{\omega \in W} O_\omega$
Characterization	$\iff$ ***sequentially compact***: Any sequence has a convergent subsequence.
	$\forall (x_m) \in S \ \ \exists x \in S$ and (x_{m_k}) s.t. $x_{m_k} \to x$.
Implications	$\implies$ S is ***closed and bounded***.
	$\implies$ any ***closed subset*** of S is compact.
	$\implies$ $f(S)$ is compact for any continuous f.
	$\implies$ $f(S)$ has a min and a max for any continuous f (***EVT***).
Heine-Borel	***In Euclidean space only*** ($\mathbb{R}^N$): $\iff$ S is ***closed and bounded***.

- If S is compact, then I can construct an ***arbitrary collection of open sets*** that contains S, and I know I will get ***something finite*** out of it.
- If S is compact, then I can construct an ***arbitrary sequence*** in S, and I know I will get something ***convergent*** out of it.

Finally, I wanted to make a note about why continuity is additionally required to get maxima and minima. It's easiest to visualize with real functions: Consider $f(x) = 1/x$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. This is not continuous, and does not have a min or a max on, say, $[-1, 1]$, which is a compact set. Visually:



The set is compact, but the function diverges to ∞ as it approaches 0 from the right, and to $-\infty$ as it approaches 0 from the left.

2 Correspondences

2.1 "Set-Valued Functions"

A correspondence, denoted $\Gamma : X \rightrightarrows Y$, assigns points in X to non-empty subsets of Y. In a sense, a correspondence is a "set-valued function" with "input" $x \in X$ and "output" is $\Gamma(x) \subseteq Y$.[3] Some terminology is completely analogous relative to when we were working with functions:

- X is the ***domain*** and Y is the ***co-domain***.
- $\forall S \subseteq X$ let $\Gamma(S) \equiv \bigcup_{x \in S} \Gamma(x)$ be the ***image*** of S.
- $\Gamma(X)$ is the ***range***, and if $\Gamma(X) = Y$ we say Γ is ***surjective***.

Here's the first roadblock: What would it mean for a correspondence to be *injective*? For functions, we want to capture the idea of *one-to-one*. A correspondence, however, starts from the premise that a mapping can be one to many. Is there an analogous idea that we *should* try to capture? We leave this question unanswered as an example of why we need to be careful when dealing with correspondences.

Example 1. Consider the choice correspondence from utility maximization:

$$\arg\max_{\mathbb{R}^N_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq w$$

We can go a step further and also define the problem over correspondences. Let

$$\Gamma(p, w) = \left\{ x \in \mathbb{R}^+ : p \cdot x \leq w \right\}$$

[3]Conversely, functions are "singleton-valued correspondences," where $f(x)$ is equivalent to the correspondence $\Gamma(x) = \{f(x)\}$.

be the budget correspondence. Then we can define the arg max correspondence as

$$\arg\max u(x) \quad \text{s.t.} \quad x \in \Gamma(p, w)$$

Why go through the trouble? If we can prove enough theorems and properties of correspondences, then re-expressing some problems we're familiar with in terms of correspondences might make solution methods and properties of solutions more transparent.

Remark 3. Since correspondences map points to sets, it is typical to refer to correspondences as [property]-valued, where [property] is any property of a set. For example, they can be closed-valued, compact-valued, convex-valued, and so on.

2.2 Inverse Images

With a function f, we had a useful characterizations of continuity that was independent of distance: the inverse image maps open sets to open sets. What is the analogue for correspondences? For a function, we can write

- $f^{-1}(O) = \{x \in X : f(x) \in O\} = \{x \in X : \{f(x)\} \subseteq O\}$.
- $f^{-1}(O) = \{x \in X : \{f(x)\} \cap O \neq \varnothing\}$.

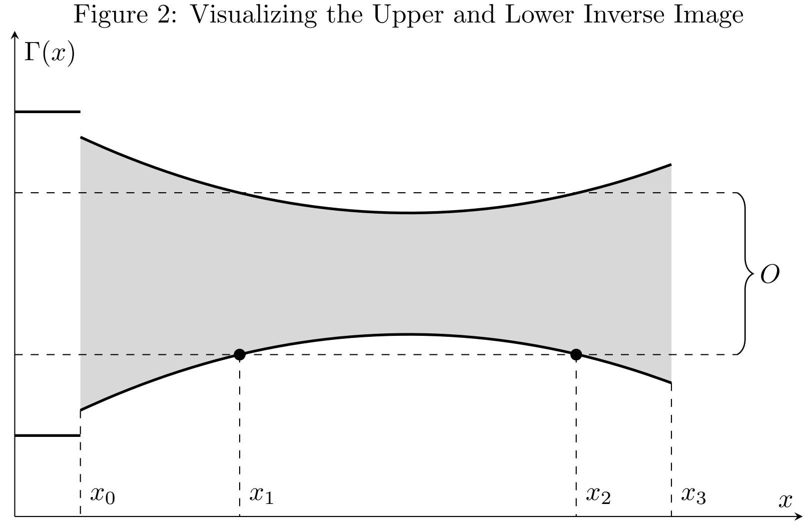
The sets above are equal for functions (even if the second bullet point isn't how we typically think of a pre-image), but for correspondences it defines two distinct sets, the *upper* inverse image and the *lower* inverse image, which will give rise to two different notions of continuity:

Definition 4. Given a correspondence $\Gamma : X \rightrightarrows Y$

- $\Gamma^{-1}(O) \equiv \{x \in X : \Gamma(x) \subseteq O\}$ is the ***upper inverse image***.
- $\Gamma_{-1}(O) \equiv \{x \in X : \Gamma(x) \cap O \neq \varnothing\}$ is the ***lower inverse image***.

Note that $\Gamma(x) \neq \varnothing$ and $\Gamma(x) \subseteq O \implies \Gamma(x) \cap O \neq \varnothing$. So necessarily $\Gamma^{-1}(O) \subseteq \Gamma_{-1}(O)$.

Figure 2: Visualizing the Upper and Lower Inverse Image



In Figure 2, *every* point $x \in [x_0, x_3]$ is s.t. $\Gamma(x) \cap O \neq \varnothing$, so $\Gamma_{-1}(O) = [x_0, x_3]$; however, not every point is s.t. $\Gamma(x) \subseteq O$. Assuming O is open, only points $y \in (x_1, x_2)$ are s.t. $\Gamma(y) \subseteq O$, so $\Gamma^{-1}(O) = (x_1, x_2)$. Last, if $z \in [0, x_0)$ then $\Gamma(z)$ is neither contained in nor intersects with 0.

2.3 Hemicontinuity

We present two distinct definitions of continuity. If we use the *upper* inverse image:

Definition 5. $\Gamma : X \rightrightarrows Y$ is ***upper hemi-continuous*** (uhc) if whenever $O \subseteq Y$ is open, $\Gamma^{-1}(O)$ is also open.

If $\Gamma(x) \subseteq O$ then $x \in \Gamma^{-1}(O)$; if $\Gamma^{-1}(O)$ is open $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq \Gamma^{-1}(O)$, so $\Gamma(B_\delta(x)) \subseteq O$. Therefore we have the following equivalent definition of uhc.

Definition 6. $\Gamma : X \rightrightarrows Y$ is uhc iff for any open $O \subseteq Y$ with $\Gamma(x) \subseteq O \;\; \exists \delta > 0$ s.t. $\Gamma(B_\delta(x)) \subseteq O$.

We can similarly define continuity in terms of the *lower* inverse image instead:

Definition 7. $\Gamma : X \rightrightarrows Y$ is ***lower hemi-continuous*** (lhc) if whenever $O \subseteq Y$ is open, $\Gamma_{-1}(O)$ is also open.

If $\Gamma(x) \cap O \neq \varnothing$ then $x \in \Gamma_{-1}(O)$; if $\Gamma_{-1}(O)$ is open $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq \Gamma_{-1}(O)$, so $z \in B_\delta(x) \implies \Gamma(z) \cap O \neq \varnothing$. Therefore we can equivalently write the following definition:

Definition 8. $\Gamma : X \rightrightarrows Y$ is lhc iff for any open $O \subseteq Y$ with $\Gamma(x) \cap O \neq \varnothing \;\; \exists \delta > 0$ s.t. $\Gamma(z) \cap O \neq \varnothing \;\; \forall z \in B_\delta(x)$.

- Intuitively, if Γ is uhc at x and z is "close" to x, every point in $\Gamma(z)$ will be "close" to some point in $\Gamma(x)$.

  If there is some neighborhood around x s.t. every open set *containing* $\Gamma(x)$ also contains $\Gamma(z)$ for z in the neighborhood, then nothing in $\Gamma(z)$ can be suddenly "far" from the all values of x.

- By contrast, if Γ is lhc at x and z "close" to x, each point in $\Gamma(x)$ will be "close" to some point in $\Gamma(z)$.

  Intersections, unlike containment, can happen at *any* point. Hence lhc *does not* require every point in $\Gamma(z)$ to always be close to $\Gamma(x)$; rather, it requires *every* point in $\Gamma(x)$ to be close to *some* point in $\Gamma(z)$.

We will formalize the intuition above when we discuss the sequential definition of uhc and lhc.

Let $\Gamma : [x_0, x_2] \rightrightarrows \mathbb{R}$ be as depicted in Figure 3:

- In the left figure, $\Gamma(x_1) = \{y_1\} \subseteq O$; however, no matter how small the δ, $y_2 \in \Gamma(x_1 + \delta) \notin O$, meaning the set values of points near x_1 always have elements far away from $\Gamma(x_1)$. Hence it cannot be uhc. Given the depicted O, $\Gamma^{-1}(O) = [x_0, x_1]$, which is not open.

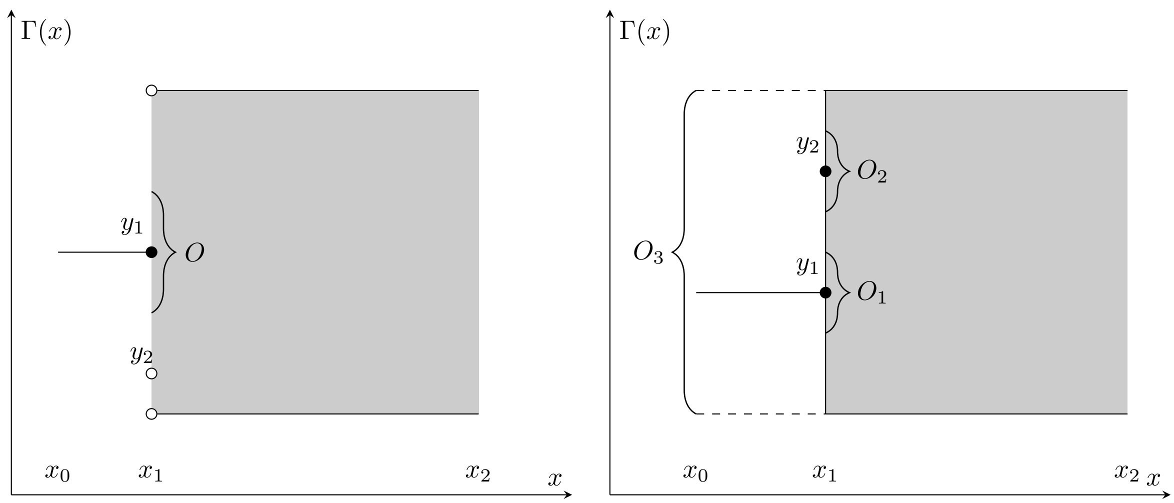
  But it is lhc: The set values of every point near x_1 will intersect O, and generally every set intersecting $\Gamma(x_1)$. For the depicted O, $\Gamma_{-1}(O) = [x_0, x_2]$ (note $X = [x_0, x_2]$, and X is open relative to X).

- In the right figure, $y_1 \in \Gamma(x_1) \cap O_1 \neq \varnothing$, and every point around x_1 will also intersect O_1. *However*, $y_2 \in \Gamma(x_1) \cap O_2 \neq \varnothing$, but no matter how small the δ, $\Gamma(x_1 - \delta) \cap O_2 = \varnothing$, meaning not every element

Figure 3: Upper and Lower Hemicontinuity (Inverse Images)

Lower but not upper hemicontinuous

Upper but not lower hemicontinuous



in $\Gamma(x_1)$ is near the set-values of points near x_1. As drawn, $\Gamma_{-1}(O_2) = [x_1, x_2]$, which is not open in $[x_0, x_2]$.

But it is uhc: O_3, and generally any set containing all of x_1, will also contain the set value of every point around x_1. As drawn, $\Gamma^{-1}(O_3)$, or any such set, is $[x_0, x_2]$, the space itself, which is open. Notably, $\Gamma^{-1}(O_1) = [x_0, x_1)$ and $\Gamma^{-1}(O_2) = \varnothing$ which are both open.

2.4 Sequential Characterization of Hemicontinuity

Remark 4. It not uncommon to encounter the sequential characterizations as the definition (in fact the very first time I learned what a correspondence was, I only encountered the sequential characterization of hemicontinuity).

Theorem 8. *Suppose* $\Gamma : X \rightrightarrows Y$. *If* $\forall (x_m) \in X$ *and* $\forall (y_m) \in Y$ *s.t.* $x_m \to x, y_m \in \Gamma(x_m)$ *for all* $m \in \mathbb{N}$, *and* $\exists y_{m_k} \to y \in \Gamma(x)$, *then* Γ *is uhc at* x. *If* Γ *is compact-valued, the converse is also true.*

Some remarks:

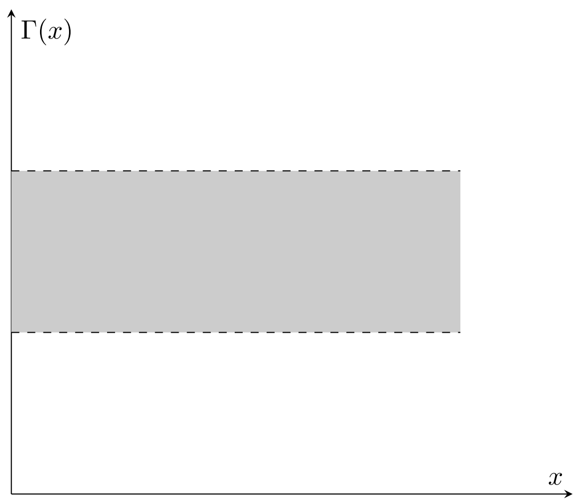
- The definition says that for every sequence converging to x and every sequence in the set-values of x_m, $y_m \in \Gamma(x_m)$, there is a convergent sub-sequence to an element $y \in \Gamma(x)$.

  Recall our intuition for uhc: Every point in $\Gamma(z)$, for z sufficiently "close" to x, is also "close" to *some* point of $\Gamma(x)$. This closely mirrors the sequential definition: If every time we get arbitrarily close to x (that is, $x_m \to x$) *every* point in those set values (arbitrary $y_m \in \Gamma(x_m)$) will be arbitrarily close to *some* point in $\Gamma(x)$ (there is some sub-sequence $y_{m_k} \to y \in \Gamma(x)$, which means that *every* sequence (y_m) such that $y_m \in \Gamma(x_m)$ has infinitely many points near *some* value of $\Gamma(x)$).

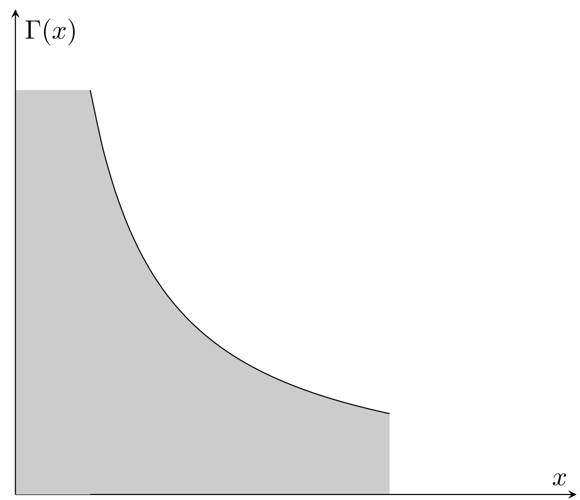
- So why doesn't the converse hold? The above sequential definition requires the function to be uhc, but uhc correspondences don't have to be closed *or* bounded:

Figure 4:

$\Gamma(x) = (1, 2)$ on $\mathbb{R}_{++}$



$\Gamma(x) = [0, 1/x]$ on $\mathbb{R}_{++}$



- The figure on the left is uhc: Since the correspondence is constant, if $\Gamma(x) \subseteq O$ for any O, then $\Gamma(y) = \Gamma(x) \subseteq O$ for every $y \in [0, \infty)$. Thus $\Gamma^{-1}(O) = \mathbb{R}_{++}$. However, we know $1/(n+1) \to 0 \implies 1 + 1/(n+1) \to 1$. But $1 + 1/(n+1) \in \Gamma(1/(n+1))$ and $1 \notin \Gamma(0) = (1, 2)$.
- The figure on the right is also uhc: Take any open $O \subseteq \mathbb{R}$ that is bounded above, $\sup O > 0$, and either unbounded below or $\inf O < 0$ (the last 2 condition ensure $\Gamma^{-1}(O) \neq \varnothing$). Define $x_0 \in (0, \infty)$ such that

$$\frac{1}{x_0} = \sup O$$

Then $\Gamma^{-1}(O) = (x_0, \infty)$ because $1/x_0 \notin O$ (why?). Alternatively, if O is open, unbounded above, and either unbounded below or $\inf O < 0$, then $\Gamma^{-1}(O) = \mathbb{R}_{++} = (0, \infty)$ which is open. However, take $\frac{1}{n} \to 0$ and $n \in \Gamma(1/n)$. We know that $n \to \infty$ and has no convergent subsequences.

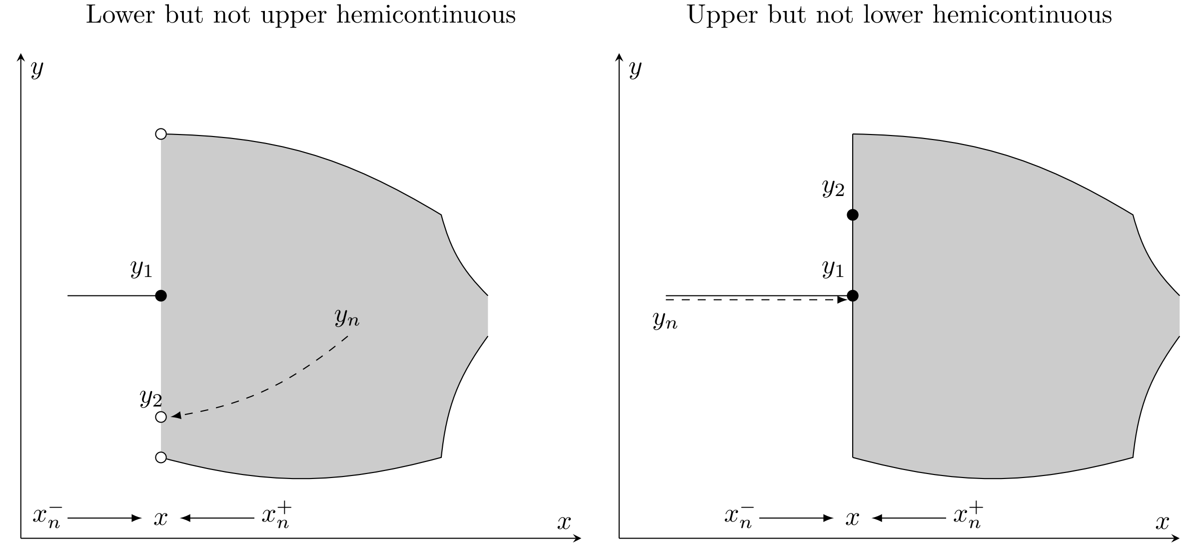
- Therefore uhc is not enough to guarantee that a sequence always exists. More precisely, it's not so much that we need the set-values of Γ to be closed and bounded: We need them to be compact, and that will guarantee the existence of a convergent subsequence. (Recall here the link between compactness and sequential compactness, which would show up in a proof of Theorem 8.)

Theorem 9. *$\Gamma : X \rightrightarrows Y$ is lhc at $x \in X \iff \forall (x_m) \in X$ s.t. $x_m \to x \in X$ and $\forall y \in \Gamma(x)$, $\exists (y_m) \in Y$ s.t. $y_m \to y$ and $y_m \in \Gamma(x_m)$ for all $m \in \mathbb{N}$.*

Given both sequential definitions of uhc and lhc, we repeat our intuition:

1. If Γ is uhc at x then every point $y \in \Gamma(z)$, for z arbitrarily close to x, is itself close to some point in $\Gamma(x)$. The limit definition follows this closely: As $x_m \to x$, every sequence $y_m \in \Gamma(x_m)$ will have infinitely many elements arbitrarily close to *some* point in $\Gamma(x)$, so there will be *some* subsequence $y_{m_k} \to y \in \Gamma(x)$ (with the caveat that Γ is compact-valued).
2. If Γ is lhc at x and z is arbitrarily close to x, then every point $y \in \Gamma(x)$ must be arbitrarily close to *some* point in $\Gamma(z)$. Again, the limit definition follows this: As $x_m \to x$, every point $y \in \Gamma(x)$ will be arbitrarily close to *some* point in $\Gamma(x_m)$, so there will exist a sequence $y_m \in \Gamma(x_m)$ s.t. $y_m \to y$.

Figure 5: Upper and Lower Hemicontinuity (Sequences)



2.5 Closed Graph

Definition 9. The ***graph*** of a correspondence $\Gamma : X \rightrightarrows Y$, denoted $Gr(\Gamma)$, is

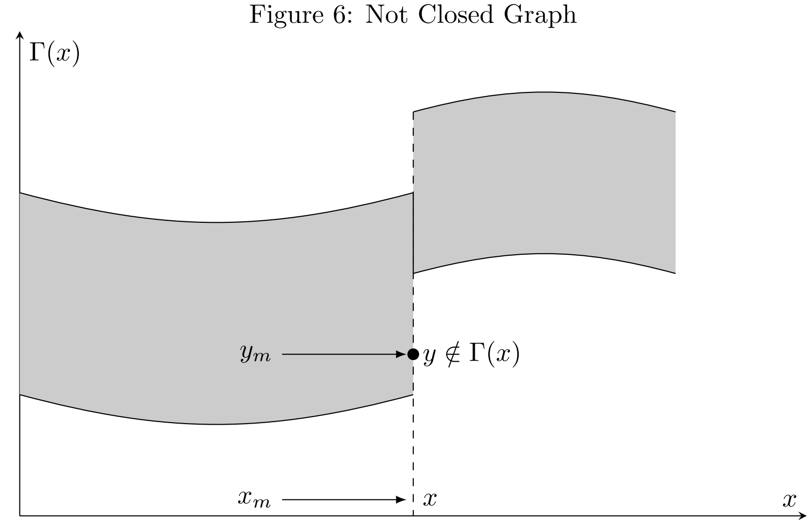
$$Gr(\Gamma) \equiv \{(x, y) \in X \times Y : y \in \Gamma(x)\}$$

Γ has a closed graph if its graph is closed in $X \times Y$.

Definition 10. Γ is ***closed*** at $x \in X$ if $\forall (x_m) \in X, (y_m) \in Y$ s.t. $x_m \to x \in X, y_m \to y \in Y$, and $y_m \in \Gamma(x_m)$ we also have $y \in \Gamma(x)$. Γ has a ***closed graph*** if it is closed at every $x \in X$.

This definition might seem complicated but if you look closely, it is just the sequential characterization of what it means for a set to be closed (it has all its limits).

Figure 6: Not Closed Graph



The idea is that *every* sequence with values in $\Gamma(x_m)$ that converges will converge to a point in $\Gamma(x)$ (if $x_m \to x$).

Remark 5. Closed graph is **not** the same as closed-valued!

The example in Figure 6 above is closed-valued but not closed graph, so the former need not imply the latter. Further, a closed graph is not the same as uhc: A uhc correspondence doesn't have to be closed-valued, which would mean that it would not have a closed graph (if $\Gamma(x)$ is not closed-valued, then there is some sequence in $\Gamma(x)$ that converges to a point outside of $\Gamma(x)$, contradicting the definition of closed graph).

Conversely, a correspondence can have a closed graph with a discontinuity. $\Gamma(x) = \{1/x\}$ if $x > 0$ and $\Gamma(x) = \{0\}$ if $x = 0$. Note that any sequence in $\Gamma(x)$ as $x \to 0^+$ will diverge, so there is no contradiction of the closed graph property. However, there is clearly a discontinuity at 0.

Can you tell, by the way, whether the correspondence in Figure 6 is uhc, lhc, both, or neither?

What we *can* say is that closed graph, closed-valued, and uhc are related:

Claim 1.
1. *If $\Gamma : X \rightrightarrows Y$ has a closed graph and Y is compact then Γ is uhc.*
2. *If Γ is uhc and closed-valued then it has a closed-graph.*

2.5.1 Berge's Theorem of the Maximum

Theorem 10 (Berge's Maximum Theorem)**.** *Let $\Gamma : \Theta \rightrightarrows X$ be compact-valued, $\varphi : X \times \Theta \to \mathbb{R}$ be continuous,*

$$\sigma(\theta) \equiv \arg\max_{x \in \Gamma(\theta)} \varphi(x, \theta)$$
$$\varphi^*(\theta) \equiv \max_{x \in \Gamma(\theta)} \varphi(x, \theta)$$

If Γ is both upper and lower hemi-continuous at some $\theta_0 \in \Theta$ then

1. *$\sigma : \Theta \rightrightarrows X$ is compact-valued everywhere, uhc at θ_0, and closed at θ_0.*
2. *$\varphi^* : \Theta \to \mathbb{R}$ is continuous at θ_0.*

Application to Economics Recall the utility maximization problem with parameters $(p, w) \in \mathbb{R}_+^{N+1}$:

$$v(p, w) \equiv \max u(x) \quad \text{s.t.} \quad p \cdot x \leq w$$
$$x(p, w) \equiv \arg\max u(x) \quad \text{s.t.} \quad p \cdot x \leq w$$

Recall $B(p, w) = \{x : p \cdot x \leq w\}$, the budget correspondence, is compact. It turns out it is also uhc and lhc, so we not only know that a maximum exists (if $u(x)$ continuous), but in particular the indirect utility function, $v(p, w)$, is continuous and the demand correspondence, $x(p, w)$, is compact-valued, uhc, and closed.

Remark 6. For the curious, proofs that the BC is compact, uhc, and lhc can be found in Appendix B.

2.6 Fixed Point Theorems

Definition 11. A self-map $f : S \to S$ has a ***fixed point*** if $\exists x^* \in S$ s.t. $x^* = f(x^*)$.

Definition 12. A set[4] S is ***convex*** if $\forall \alpha \in [0,1]$ and $\forall s, s' \in S$, $\alpha s + (1-\alpha)s' \in S$

Theorem 11 (Brouwer's FPT)**.** *Take any $S \subseteq \mathbb{R}$ compact, convex, and non-empty. If $f : S \to S$ is continuous then it has a fixed point.*

Definition 13. A self-map correspondence $\Gamma : S \rightrightarrows S$ has a ***fixed point*** if $\exists x^* \in S$ s.t. $x^* \in \Gamma(x^*)$.

Theorem 12 (Kakutani's FPT)**.** *Take any $S \subseteq \mathbb{R}^N$ compact, convex, and non-empty. If a correspondence $\Gamma : S \rightrightarrows S$ is upper hemicontinuous, convex-valued, and closed-valued (alternatively, convex-valued and has a closed graph) then it has a fixed point.*

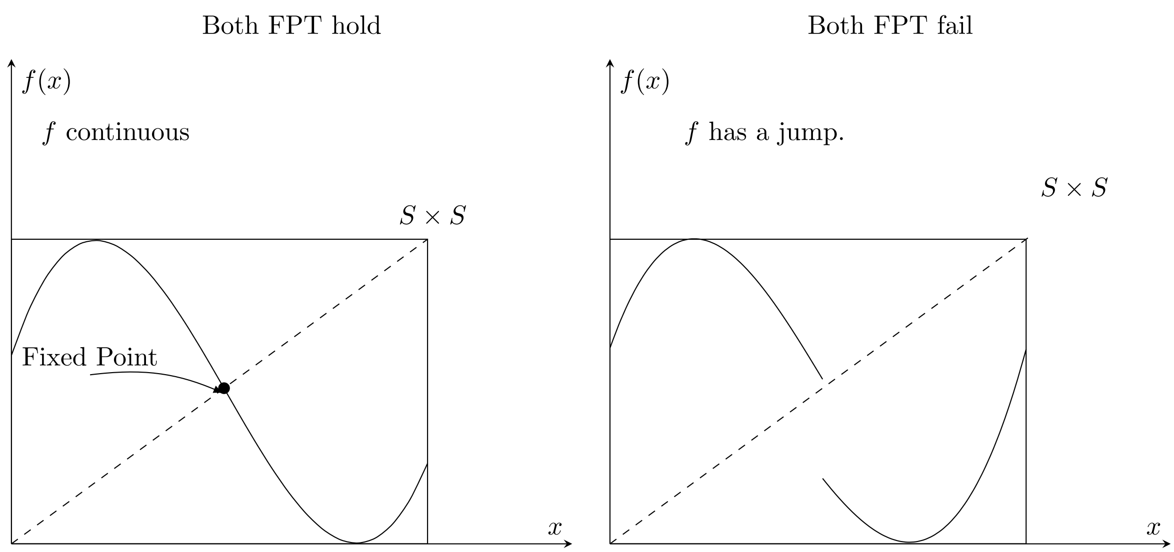


Figure 7: Examples of when f does or not have a fixed point

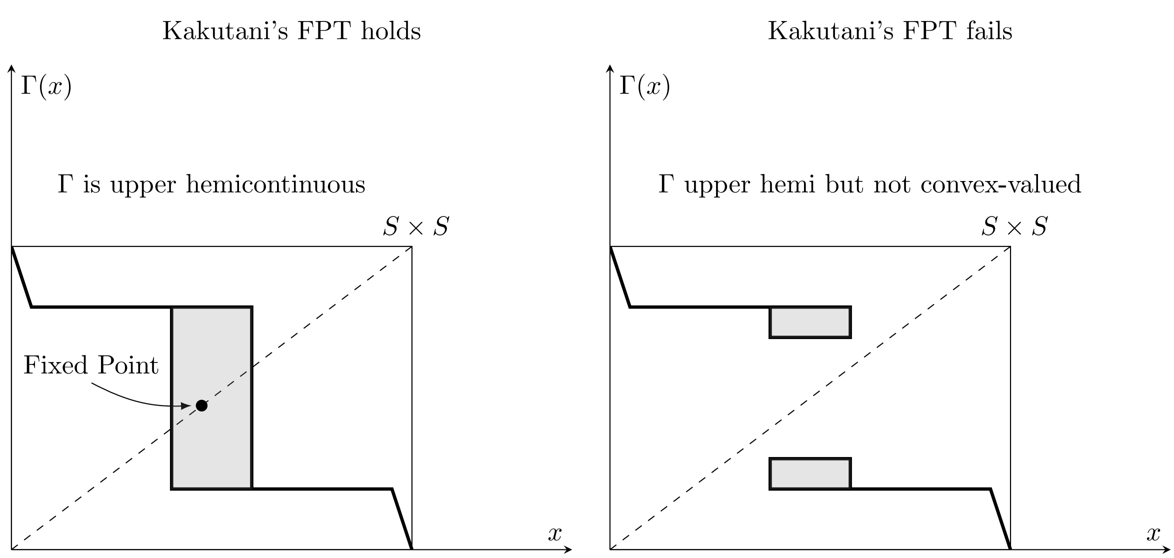


Figure 8: Examples of when Γ does or not have a fixed point

[4]Technically, S needs to be the subset of a real vector space (or real affine space). We define vector spaces in Lecture 6 notes (and affine spaces are defined near the end of ECON 2010).

Application to Economics The proof of the existence of a Nash Equilibrium in game theory is an application of Kakutani's fixed point theorem.

- We have N players, $1, 2, \dots, N$ and the corresponding strategy sets, $S_1, \dots, S_N$.
- Let $s = (s_1, \dots, s_N)$ be any collection of strategies from all players, with $s_i \in S_i$.
- Let $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ the collection of strategies from all players other than i.
- For each player i we can define a *best-response* correspondence to the strategies of other players,

$$b_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i; s_{-i})$$

  the utility-maximizing strategy for i given the other player's strategies.
- Let $b(s) = (b_1(s_{-1}), \dots, b_N(s_{-N}))$ be the collection of best-response strategies from all players.

A Nash Equilibrium is then defined as a set of strategies such that no player has an incentive to deviate. That is, the strategy s_i^* chosen by player i is in the set of best responses to all the other strategies, s_{-i}^*, or $s_i^* \in b_i(s_{-i}^*) \ \ \forall i$. We can hence express a Nash Equilibrium s^* as a fixed point of b,

$$s^* \in b(s^*)$$

If S_i are compact, non-empty, and convex-valued, and u_i are continuous and quasiconcave (this gives convexity), then we will be able to apply Kakutani's fixed point theorem to show that $s^* \in b(s^*)$ for some $s^* \in S = \prod_{i=1}^N S_i$ (noting $b : S \rightrightarrows S$ is upper hemicontinuous by Theorem 10).

3 Fun Remarks

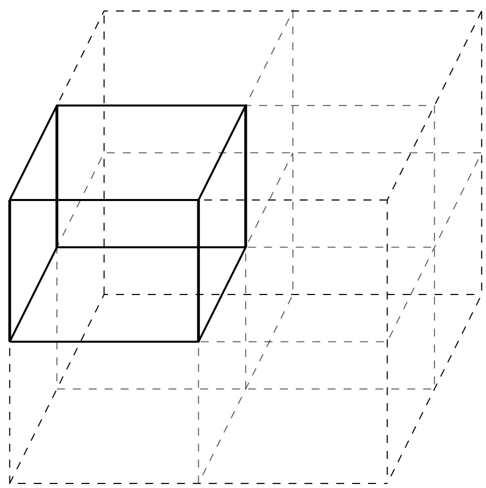
- It is rumored than when John Nash came to John Von Neumann to discuss his ideas and his proof of the existence of a Nash Equilibrium (though of course he probably just called it "Equilibrium"), Von Neumann interrupted to dismiss the result as trivial: "That's just a fixed point theorem," he said. I always found this anecdote to be quite fascinating, in particular given the extent to which game theory plays a role in modern economics.
- The other famous application of Kakutani's fixed point theorem in economics is in the proof of the existence of general equilibrium. A perhaps lesser known but no less fun example of course is the fair cake-cutting theorem, where Kakutani guarantees that there exists a division of a cake (which is a non-uniform resource) among n agents that is not only Pareto efficient but envy-free (i.e. no agent prefers someone else's allocation). The cake, of course, is a lie, and the theorem refers to an allocation as a disjoint n-partition of a set among n agents with heterogeneous preferences over the set.
- Speaking of general equilibrium, I very briefly met Arrow once after he gave a short talk; I basically said hello and that was that. However, a classmate of mine had the, let's say, very fun idea to ask Arrow to autograph his class notes on general equilibrium, which I always remember as one of the more endearing things I've seen someone do.

- I first encountered the definition of compactness several years ago during my undergraduate real analysis course. As I've mentioned, I thought the definition was rather disconcerting[5]; I heavily relied on the "closed and bounded" intuition and the sequential compactness characterization to get through that part of the course. I repeat this point here because you shouldn't be too concerned if you don't find compactness terribly easy at first. It is a difficult and deep concept to wrap your head around (at least I think so). I am sure you'll get there (:

[5]To be candid, I legit thought "the fuck is this?"

A Proof of Theorem 3

Proof. Suppose $-\infty < a < b < \infty$ and let $S = [a, b]^N$ be such that for some open cover $\mathcal{F} = \{F_\omega\}_{\omega\in\Omega}$ of S there does not exist a finite sub-cover. Bisect S into 2^N equal closed hypercubes with planes parallel to the faces of S (by the way, this is why I call this proof a "sketch," as I have not defined hypercube, plane, parallel, or face). While that sounds fairly complicated, we can visualize it in 3-dimensional space:



At least one cube has no finite sub-cover (maybe none do, but we only need one); call this C_1. Recursively, partition C_m into 2^N equal closed cubes and let C_{m+1} be one such cube with no finite sub-cover.

1. C_m are closed.
2. C_m are non-empty (otherwise C_{m-1} is empty, contradiction).
3. $C_{m+1} \subset C_m$
4. Let δ be the maximum distance between any two points in S. The maximum distance in C_m is $\dfrac{\delta}{2^m}$.
5. C_m is not covered by any finite sub-cover of $\mathcal{F}$ by construction.

For each m, let x_m be any element of C_m. This sequence is Cauchy: For any $\varepsilon > 0$ [6],

$$k, l > M > \frac{\log(\delta) - \log(\varepsilon)}{\log(2)} \implies d(x_k, x_l) < \frac{\delta}{2^M} < \varepsilon$$

Because $\mathbb{R}^N$ is complete, we know that $x_m \to x$ for some $x \in \mathbb{R}^N$. Therefore,[7]

$$x \in C_m \quad \forall m$$

Since $\mathcal{F}$ has an infinite sub-cover of C_m (which might be comprised of all the sets in $\mathcal{F}$), $x \in F_\omega$ for some set in that infinite sub-cover. F_ω is open, so for some $\varepsilon_0 > 0$,

$$d(y, x) < \varepsilon_0 \implies y \in F_\omega$$

[6] Recall that $\log(x) - \log(y) = \log(x/y)$ and that $\log_b(a) = \log_c(a)/\log_c(b)$

[7] This uses the fact that each C_m is closed and $x_{m+n} \in C_m$ for all n; hence the limit is also in C_m.

But $x \in C_m$ for any m, and the maximum distance between any two points in C_m is $\delta/2^m$. Hence for M s.t.

$$M > \frac{\log(\delta) - \log(\varepsilon_0)}{\log(2)} \implies \forall y \in C_M, \quad d(x,y) < \frac{\delta}{2^M} < \varepsilon_0 \implies y \in F_\omega \implies C_M \subseteq F_\omega$$

$\{F_\omega\}$ is a finite sub-cover of C_M, a contradiction. Therefore $[a,b]^N$ is compact. □

B Budget Correspondence Properties

Here are some formal proofs about some properties I claimed for the BC. Only for fun!

B.1 Compact

We invoke Heine-Borel. First, let us show $B(p,w)$ is closed:

Fix a $p \in \mathbb{R}^N_{++}$ and a $w \in \mathbb{R}_+$. By definition

$$B(p,w) = \left\{x \in \mathbb{R}^N_+ : p \cdot x \leq w\right\} \implies B(p,w)^c = \left\{x \in \mathbb{R}^N_+ : p \cdot x > w\right\}$$

Let $z \in B(p,w)^c$. We want to show that there exists an $\varepsilon > 0$ such that $N_\varepsilon(z) \subseteq B(p,w)^c$ (where $N_\varepsilon(z)$ is the ball/neighborhood of radius ε centered at z). Let

$$\varepsilon = \frac{p \cdot z - w}{\sum_{n=1}^N p_n} > 0$$

and let $y \in N_\varepsilon(z)$. We want to show $p \cdot y > w$ (as this will mean that $y \in B(p,w)^c \implies N_\varepsilon(z) \subseteq B(p,w)^c$). Noting that

$$\|z - y\| < \varepsilon \implies \forall n \in \{1, \ldots, N\} \ \ |z_n - y_n| < \varepsilon$$

(why?) and letting

$$n_0 = \underset{n \in \{1,\ldots,N\}}{\arg\max} \ |z_n - y_n|$$

we can write

$$\begin{aligned} p \cdot z - p \cdot y &= p \cdot (z - y) \\ &= \sum_{n=1}^N p_n (z_n - y_n) \\ &\leq \sum_{n=1}^N p_n |z_n - y_n| \\ &\leq |z_{n_0} - y_{n_0}| \sum_{n=1}^N p_n \\ &< \varepsilon \sum_{n=1}^N p_n \\ &= p \cdot z - w \\ \implies w &< p \cdot y \end{aligned}$$

Therefore, $B(p,w)^c$ is open implying that $B(p,w)$ is closed.

To see that $B(p,w)$ bounded,

$$w \geq p \cdot x \geq \sum x_k \cdot \min p_k$$

$$\frac{w}{\min_k p_k} \equiv M \geq \sum x_k \overset{!}{\geq} x_k \qquad \forall k$$

(to see why $\overset{!}{\geq}$ is true, recall $x \in \mathbb{R}^N_+ \implies x_k \geq 0 \;\; \forall k$). Hence $x \in B(p,w) \implies 0 \leq x_k \leq M \;\; \forall k$, meaning $B(p,w)$ is bounded. Since $B(p,w)$ is a closed and bounded subset of $\mathbb{R}^N_+$, it is also compact.

B.2 Upper hemi-continuous

Fix any $(p,w) \in \mathbb{R}^N_{++} \times \mathbb{R}_+$ and let $(p_m, w_m) \to (p,w)$. Further, let $(x_m) \in \mathbb{R}^N_+$ be defined such that $x_m \in B(p_m, w_m)$ for all $m \in \mathbb{N}$. We want to show that there exists a subsequence of (x_m), call it (x_{m_k}), such that $x_{m_k} \to x \in B(p,w)$ (then we will have that $B(p,w)$ is uhc by Theorem 8). We will accomplish this in two steps:

1. We first establish that the tail of the sequence (x_m) is contained in a compact set, so we can exploit the sequential characterization of compactness.
2. Then we utilize the continuity of inner products to establish that the limit of the convergent subsequence found in part 1 is contained in $B(p,w)$.

Let

$$0 < \varepsilon < \min_{n : p_n > 0} p_n$$

where p_n is the n component of the vector p (why do we define ε this way?). Note that the upper bound is well-defined because $p \in \mathbb{R}^N_{++}$. By assumption, there exists an $M \in \mathbb{N}$ such that for any $m \geq M$,

$$\|(p_m, w_m) - (p,w)\| < \varepsilon \implies \forall n \in \{1, \dots, N\}, \; |p_{m,n} - p_n| < \varepsilon \quad \text{and} \quad |w_m - w| < \varepsilon$$

where $p_{m,n}$ is the n^{th} component of the vector p_m (you were asked to prove this implication in Appendix B.1). Thus, for any $n = 1, \dots, N$

$$p_n - \varepsilon < p_{m,n} < p_n + \varepsilon \implies (p_n - \varepsilon) x_{m,n} \leq p_{m,n} x_{m,n} \leq (p_n + \varepsilon) x_{m,n}$$

where the implication holds because $x_m \in \mathbb{R}^N_+$. Putting all the pieces together, we have shown that

$$(p - \varepsilon\iota) \cdot x_m \leq p_m \cdot x_m \leq w_, < w + \varepsilon$$

where $\iota \in \mathbb{R}^N$ is a vector of ones. Put differently, we have shown that $x_m \in B(p - \varepsilon\iota, w + \varepsilon)$ whenever $m \geq M$.

In Appendix B.1, we showed that $B(p - \varepsilon, w + \varepsilon)$ is compact. By Theorem 1, we can then say that there exists a subsequence $(x_{m_k}) \in B(p - \varepsilon\iota, w + \varepsilon)$ such that $x_{m_k} \to x \in B(p - \varepsilon\iota, w + \varepsilon)$. For clarity in limit notation, we will write this subsequence simply as (x_k). Since the Euclidean inner product is continuous, we can write

$$p \cdot x = \lim_{k\to\infty} p_k \cdot x_k \le \lim_{k\to\infty} w_k = w$$

implying that $x \in B(p, w)$, as desired.

B.3 Lower hemi-continuous

Fix any (p, w) and consider an arbitrary sequence $(p_m, w_m) \to (p, w)$ and any point $x \in B(p, w)$. If $\exists x_m \in B(p_m, w_m)$ s.t. $x_m \to x$ then $B(p, w)$ is lhc by Theorem 9.

1. Here we are restricted in that every point in the correspondence must have a sequence that converges to it. However, the degree of freedom we have is that we can pick the sequence. Our strategy, then, is to construct a sequence that will be contained in $B(p_m, w_m)$.
2. Let $\widetilde{x}_{k,n} \equiv \max\{x_n - 1/k, 0\}$ be the n^{th} component of the vector $\widetilde{x}_k$ (where $\iota \in \mathbb{R}^N$ is a vector of ones). We claim $\exists K \in \mathbb{N}$ s.t. $\forall k \ge K,\ \exists M_k$ s.t.

$$m \ge M_k \implies \widetilde{x}_k \in B(p_m, w_m) \quad \text{or, equivalently} \quad p_m \cdot \widetilde{x}_k \le w_m$$

3. Here we use the fact $(p_m, w_m) \to (p, w)$. By definition $\forall \delta_k > 0,\ \exists M_k \in \mathbb{N}$ s.t.

$$m \ge M_k \implies \|(p_m, w_m) - (p, w)\| < \delta_k \implies \forall n \ \ |p_{m,n} - p_n| < \delta_k \quad \text{and} \quad |w_m - w| < \delta_k$$

   where $p_{m,n}$ is the n^{th} component of the vector p_m.
4. Therefore, we want to find $\delta_k > 0$ s.t. $(p + \delta_k \iota) \cdot \widetilde{x}_k \le (w - \delta_k)$. The corresponding M_k would give

$$m \ge M_k \implies p_m \cdot \widetilde{x}_k \le (p + \delta_k \iota) \cdot \widetilde{x}_k \le (w - \delta_k) \le w_m$$

   Some arithmetic on the desired inequality yields

$$\begin{aligned}
&(p + \delta_k \iota) \cdot \widetilde{x}_k \le w - \delta_k \\
\iff& p \cdot \widetilde{x}_k + \delta_k(\iota \cdot \widetilde{x}_k) \le w - \delta_k \\
\implies& p \cdot x - \frac{1}{k}(p \cdot \iota) + \delta_k \left(\iota \cdot x - \frac{N}{k}\right) \le p \cdot \widetilde{x}_k + \delta_k(\iota \cdot \widetilde{x}_k) \le w - \delta_k \\
\implies& \delta_k \left[\sum_{n=1}^{N} x_n - \frac{N}{k}\right] + \delta_k \le w - p \cdot x + \frac{1}{k}\sum_{n=1}^{N} p_n \\
\implies& \delta_k \le \frac{k(w - p \cdot x) + \sum_{n=1}^{N} p_n}{k\sum_{n=1}^{N} x_n - N + k}
\end{aligned}$$

   where we can simply define $k \ge K \ge N$ to ensure the denominator is positive consequently ensuring that $\delta_k > 0$.

   Therefore we can see that

$$\delta_n \equiv \frac{k(w - p \cdot x) + \sum_{n=1}^{N} p_n}{k\sum_{n=1}^{N} x_n - N + k} \implies (p + \delta_n) \cdot \widetilde{x}_n \le (w - \delta_n) \implies p_m \cdot \widetilde{x}_n \le w_m \quad \forall m \ge M_n$$

5. Last, we define the sequence that gives us the result:

   - Let $x_m = 0$ for $m < M_K$ (0 is in every budget correspondence).
   - Otherwise, let $x_m = \widetilde{x}_k$ for $m \geq M_K : M_k \leq m < M_{k+1}$.

   since $1/k \to 0$ we have $\widetilde{x}_k \to \max\{x, 0\} = x$ (recall $x \geq 0$ because $x \in \mathbb{R}_+^N$).

Hence for any $x \in B(p, w)$ we can construct a sequence $x_m \to x$ s.t. $x_m \in B(p_m, w_m)$ for each m, meaning $B(p, w)$ is lhc.

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