Problem Set 3

August 20, 2024

1. Take a collection of functions $f_n \Omega \to \mathbb{R}^N$, $\Omega \subseteq \mathbb{R}^M$, $n \in \mathbb{N}$. The collection $\{f_n\}_{n\in\mathbb{N}}$ define a ***sequence of functions***, and for each $x \in \Omega$, we have a possibly different sequence $\{f_n(x)\}$ in $\mathbb{R}^N$.

   - Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions with $f_n : \Omega \to \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_n\}_{n\in\mathbb{N}}$ ***converges point-wise*** to $f : \Omega \to \mathbb{R}^N$ if $x \in \Omega \implies f_n(x) \to f(x)$.
   - Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions with $f_n : \Omega \to \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_n\}_{n\in\mathbb{N}}$ ***converges uniformly*** to $f : \Omega \to \mathbb{R}^N$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

     $$\|f_n(x) - f(x)\| < \varepsilon$$

     when $n \geq N$ and $x \in \Omega$.[1]

   (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined such that $f_n(x) = x/n$ and $f(x) = 0$. Check that f_n converges point-wise to f.

   (b) Show f_n defined above does not converge uniformly to f.

   (c) Show that uniform convergence implies point-wise convergence.

2. Let $A \subseteq \mathbb{R}^N$ be a convex set. We say that $f : A \to \mathbb{R}^N$ is ***quasi-concave*** if for any $x, y \in A$ and for any $\alpha \in [0, 1]$, we have

   $$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$$

   and ***strictly quasi-concave*** is the above inequality holds strictly for any $\alpha \in (0, 1)$. Show that if f is quasi-concave, then $\arg\max_{x\in A} f(x)$ is a convex set (recall the empty set is vacuously convex). Further show that if f is strictly quasi-concave, then $\arg\max_{x\in A} f(x)$ is a singleton or empty.

3. Consider a continuous function $f : \mathbb{R}^N \to \mathbb{R}$. Show

   (a) If f is differentiable and $x^* \in \mathbb{R}^N$ is a local maximizer or minimizer of f, then $\nabla f(x^*) = 0$.

   (b) If f is three times continuously differentiable and $x^* \in \mathbb{R}^N$ is such that $\nabla f(x^*) = 0$, then if x^* is a local maximizer, the symmetric $N \times N$ Hessian $D^2 f(x^*)$ is negative semi-definite. *Optional*: Prove that if $D^2 f(x^*)$ is negative definite, then x^* is a unique global maximizer (*Hint*: For the first part, you could potentially use a Taylor expansion formula. For the second part, you could leverage the fact that a matrix is ND iff it has all strictly negative eigenvalues)

[1] Note the difference between this definition and the definition for point-wise convergence is that the $N \in \mathbb{N}$ in the definition for point-wise convergence can potentially depend on x whereas the $N \in \mathbb{N}$ in the uniform convergence definition can only depend on ε. This is a subtle but important distinction.

(c) If f is concave, then $f(x+z) \leq f(x) + Df(x)z$ for any x, z.

(d) If f is concave, then any critical point (i.e. x such that $Df(x) = 0$) is a global maximizer.

4. Define the set $\Delta = \left\{p \in \mathbb{R}^L_+ : \sum_\ell p_\ell = 1\right\}$ and the functions z^+ on Δ as $z^+_\ell(p) = \max\{z_\ell(p), 0\}$, where $z(p) = \{z_1(p), z_2(p), \ldots, z_L(p)\}$ is a continuous homogeneous function of degree 0 and satisfying $p \cdot z(p) = 0$ for all $p \in \mathbb{R}^L$. Denote $\alpha(p) = \sum_\ell \left[p_\ell + z^+_\ell\right]$.

   (a) Show that Δ is a non-empty compact and convex set.

   (b) Show that $f : \Delta \to \Delta$ is continuous in p where

   $$f(p) = \frac{1}{\alpha(p)}\left(p + z^+(p)\right)$$

   (c) Prove that f has a fixed point. (*Hint:* use some existing theorems!)

   (d) Use the fact that f has a fixed point and the properties of z to argue that $\exists p^*$ such that $z^+(p^*) \cdot z(p^*) = 0$. (*Hint*: Use the fact that $p^* \cdot z(p^*) = 0$).

   (e) Conclude that $z(p^*) \leq 0$

Remark 1. For consumer i, we define the excess demand function $z_i(p) = x_i(p, \omega_i) - \omega_i$ for wealth ω_i and prices p. One way to define general equilibrium is a vector of prices such that $\sum_i z_i(p) \leq 0$ for all i (i.e., there is no aggregate excess demand). You have just shown that under some conditions such a price vector always exists.