On Weyl modules over affine Lie algebras in char p

§0 Motivation

<u>Hope</u>: Study modular repn of affine Lie alg.

[OrdFin] Repn of ss Lie alg$/\mathbb{C}$ (Weyl group) $\rightsquigarrow$ Repn of ss Lie alg in char p (Affine Weyl group) [ModFin]

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[OrdAff] Repn of affine Lie alg$/\mathbb{C}$ (Affine Weyl group) $\rightsquigarrow$ Repn of affine Lie alg. in char p (Double affine Weyl group) [ModAff]

<u>Ultimate problem</u>: irreducible characters.

[OrdFin]

· (1926) Weyl char formula: for $\lambda \in X_+$ (dominant int):

$$\underbrace{ch\, L(\lambda)}_{\text{irred. mod.}} = \underbrace{\sum_{w\in W}}_{\text{Weyl group}} (-1)^{\ell(w)}\, ch\, \underbrace{M(w\cdot\lambda)}_{\text{Verma mod.}}$$

dot action.

Example

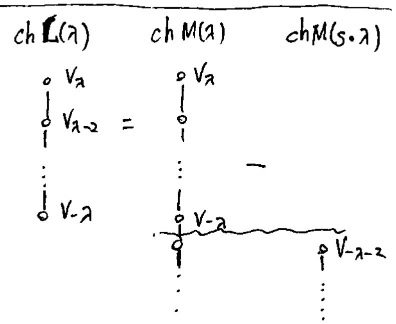
· $\mathfrak{g} = sl_2(\mathbb{C})$, $W = \{1, s\}$

$\mathfrak{h}^* \cong \mathbb{C}$ where $s(\lambda) = -\lambda$

· $M(\lambda) = \mathrm{Span}\{v_\lambda, v_{\lambda-2}, \dots\}$ ← inf. dim.

· $L(\lambda) = \begin{cases} \mathrm{Span}\{v_\lambda, v_{\lambda-2}, \dots, v_{-\lambda}\} & \text{if } \lambda \in X_+ \cong \mathbb{Z}_{\geq 0} \\ M(\lambda) & \text{otw} \end{cases}$

· $s\cdot\lambda = s(\lambda+1) - 1 = -\lambda - 2$



<u>Ultimate solution</u>: [OrdFin] Kazhdan-Lusztig conj (1979) [Beilinson-Bernstein, Brylinski-Kashiwara 81]

$$ch\, L(w\cdot\lambda) = \underbrace{\sum_{x\leq w}}_{\text{Bruhat order}} (-1)^{\ell(x)-\ell(w)} \underbrace{P_{x,w}(1)}_{\text{KL polyn.}}\, ch\, M(x\cdot\lambda)$$

$\Updownarrow$

$$ch\, M(w\cdot\lambda) = \sum_{x\leq w} P_{\underbrace{w_0}_{\text{longest element}} w, w_0 x}(1)\, ch\, L(x\cdot\lambda)$$

⇐ describes the comp. factors of Verma

<u>Remark</u>: $[M(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \in W\cdot\lambda$

Why Weyl mod?

In char p, the role of Verma is replaced by Weyl mod.

[ModFin] [Lusztig conj, 1980]

If $p \gg 0$ then

$$ch\, L(w\cdot\lambda) = \sum_{x\leq w} (-1)^{\ell(x)-\ell(w)} P_{w_0 x, w_0 w}(1)\ \text{“}ch\text{”}\ \underbrace{V(x\cdot\lambda)}_{\text{Weyl mod.}}$$

extend ch using Euler char.

The very first step: Strong linkage principle

· [OrdFin] [Verma 66 + BGG 71]

$[M(\lambda) : L(\mu)] \neq 0 \Leftrightarrow \mu \uparrow\uparrow \lambda$ (i.e. $\mu = \lambda$ or $\mu \uparrow \dots \uparrow \lambda$)

where $\mu \uparrow \lambda \Leftrightarrow \exists \beta \in \Phi^+$ s.t. $\mu = s_\beta \cdot \lambda < \lambda$

$\left(\Leftrightarrow \exists \begin{cases} \beta \in \Phi^+ \\ n \in \mathbb{Z}_{>0} \end{cases} \text{s.t.} \begin{cases} \lambda - \mu = n\beta \\ n = \langle \lambda+\rho, \beta^\vee \rangle \end{cases} \right)$

· [OrdAff] [Kac-Kazhdan 79]

$[M(\lambda) : L(\mu)] \neq 0 \Leftrightarrow \mu \uparrow\uparrow \lambda$

where $\mu \uparrow \lambda \Leftrightarrow \exists \begin{cases} \beta \in \Phi^+ \\ n \in \mathbb{Z}_{>0} \end{cases} \text{s.t.} \begin{cases} \lambda - \mu = n\beta \\ n(\beta,\beta) = 2(\lambda+\rho, \beta) \end{cases}$) recover

· [ModFin] [Jantzen 77, for $p \geq h$; Andersen 80, Jantzen 80 for all p]

$[V(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \uparrow\uparrow \lambda$

where $\mu \uparrow \lambda \Leftrightarrow \exists \begin{cases} \beta \in \Phi^+ \\ m \in \mathbb{Z} \end{cases} \text{s.t. } \mu = s_{\beta, mp} \cdot \lambda < \lambda$

$\Leftrightarrow \exists \begin{cases} \beta \in \Phi^+ \\ m \in \mathbb{Z} \\ n \in \mathbb{Z}_{>0} \end{cases} \text{s.t.} \begin{cases} \lambda - \mu = (n - mp)\beta \\ n = \langle \lambda+\rho, \beta^\vee \rangle \end{cases}$

· [ModAff] $\boxed{\text{Conj}}$ [L.-Wang 2013]

If $p > h$,

$[V(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \uparrow\uparrow \lambda$

where $\mu \uparrow \lambda \Leftrightarrow \exists \begin{cases} \beta \in \Phi^+ \\ m \in \mathbb{Z} \\ n \in \mathbb{Z}_{>0} \end{cases} \text{s.t.} \begin{cases} \lambda - \mu = (n - mp)\beta \\ n(\beta,\beta) = 2(\lambda+\rho, \beta) \end{cases}$

recover by intepreting $p = 0$, ..etc

§1 Modular repn of affine Lie algebras

- $\mathfrak{g}_{\mathbb{C}}$: affine KM Lie alg. / $\mathbb{C}$ assoc. to symmetrizable GCM A
- $\mathbb{K} = \overline{\mathbb{K}}$ of char $p > 0$
- For $\lambda \in X^+$, we define the Weyl module $V(\lambda)$ as the reduction mod p of the irred. module in char 0.
  - $L(\lambda)_{\mathbb{C}}$: irred. highest weight $U(\mathfrak{g}_{\mathbb{C}})$-module with hwt λ
  - $L(\lambda)_{\mathbb{Z}}$: $\mathbb{Z}$-form of $L(\lambda)_{\mathbb{C}}$
  - $V(\lambda) = L(\lambda)_{\mathbb{Z}} \otimes \mathbb{K}$
- [ModAff] is quite different from [ModFin]

  e.g. ◇ [Mathieu 96] The Steinberg mod. $V((p-1)\rho)$ is reducible.

  ◇ [DKK 89] For untwisted affine ADE,
  basic repn $V(\omega_0)$ is irred. $\Longleftrightarrow$ $p \nmid \det(A_0)$ (finite part of A)

  ◇ [Chari-Jing 01] For untwisted affine ADE,
  basic repn $V(\omega_0)$ is irred. $\Longleftrightarrow$ $\gcd(p,h) = 1$

  ◇ [BK 02] For poss. twisted affine ADE,
  if $p > h$ then $V(\omega_0)$ is reducible.

Remark: We know NOTHING on the reducibility of Weyl mod. beyond level 1.

- Main difficulty:
  - No longest element.

  eg. In [ModFin], the current proofs for SLP:

  (i) Andersen's result on $V(\lambda) \cong H^{\ell(w_0)}(w_0 \cdot \lambda)$.

  (ii) Jantzen sum formula $\sum_{i>0} \mathrm{ch}\, V(\lambda)^i = \sum_{\beta \in \Phi^+} \sum_{0 < mp < \langle \lambda+\rho, \alpha^\vee \rangle} \nu_p(mp)\, \chi(s_{\beta,mp} \cdot \lambda)$

  Both rely on w_0.

- What can we do?

  The opposite direction of conj – constructing nonzero hom. between Weyl mod:

§2 Main thm

- In [ModFin], $\mu \uparrow \lambda \not\Rightarrow \mathrm{Hom}(V(\mu), V(\lambda)) \neq 0$
- [Franklin, 81] [ModFin]

  $\mu \uparrow_\gamma^e \lambda \Rightarrow \exists\, V(\mu) \hookrightarrow V(\lambda)$, where $\uparrow_\gamma^e$ = nearest lower p^e-reflection.

  (except 5 roots in G_2, F_4, E_8)

(Idea)

(1) $\exists\, M(\mu)_{\mathbb{C}} \longrightarrow M(\lambda)_{\mathbb{C}}$ if $\mu = s_\gamma \cdot \lambda < \lambda$ for $\gamma \in \Phi^+$

$v_\mu^+ \mapsto S v_\lambda^+$ (Shapovalov element)

+$\mathbb{Z}$-form

(2) $\exists\, M(\mu)_{\mathbb{Z}} \to M(\lambda)_{\mathbb{Z}}$ if $\mu = s_\gamma \cdot \lambda < \lambda$ for $\gamma \in \Phi^+$

$v_\mu^+ \mapsto Z v_\lambda^+$ (int. shap. elt.)

+ affine refl'n

(3) $\exists\, M(\mu)_{\mathbb{Z}} \longrightarrow M(\lambda)_{\mathbb{Z}}$ if $\begin{cases} \mu = s_{\gamma, Mp^e+D} \cdot \lambda < \lambda \\ D g_\eta < p^e \text{ (close enough)} \end{cases}$ for $\gamma \in \Phi^+$, $M, e \in \mathbb{Z}_{>0}$, $0 < D < p^e$, γ: "η-good"

$v_\mu^+ \mapsto Z_\eta v_\lambda^+$ (η-good int. shap. elt.)

red mod p

(4) $\exists\, L(\mu)_{\mathbb{Z}} \longrightarrow L(\lambda)_{\mathbb{Z}}$ if $\begin{cases} \mu \uparrow_\gamma^e \lambda \\ D g_\eta < p^e \end{cases}$ for $\gamma \in \Phi^+$, γ: "η-good"

$\overline{v}_\mu^+ \mapsto \underline{Z_\eta \overline{v}_\lambda^+}$

nonzero : non-trivial

req: contrav. form, Shap. factor lemma

tensoring $\mathbb{K}$

(5) $\exists\, V(\mu) \longrightarrow V(\lambda)$ if $\begin{cases} \mu \uparrow_\gamma^e \lambda \\ D g_\eta < p^e \\ D g_\eta < \langle \lambda+\rho, \alpha_\eta^\vee \rangle \end{cases}$ for $\gamma \in \Phi^+$, γ: "η-good"

$\overline{v}_\mu^+ \otimes 1 \mapsto \dfrac{Z_\eta \overline{v}_\lambda^+}{p^g} \otimes 1$

nonzero: dividing ht p-power p^g

hom: BGG resl'n

Thm [L. 13] [ModAff]

> Assume $\mu \uparrow_\gamma^e \lambda$, if
>
> D small enough, $\gamma \in \Phi^{re}$ is good enough ——— $(\star)$
>
> then $\exists$ nonzero $V(\mu) \hookrightarrow V(\lambda)$.

- Gen. int. Shap. elts. is not trivial. It involves detailed calculation on the Kostant-Garland type basis ([Gar 80] for untwisted, [Mit 85] for twisted affine). The idea is to rewrite $\frac{Z_\eta \overline{v}_\lambda^+}{p^g}$ in terms of th basis, and compute $e_i^{(n)} \frac{Z_\eta \overline{v}_\lambda^+}{p^g} = \sum_{m=1}^{D g_\eta} (*) \binom{\langle \lambda+\rho, \gamma^\vee \rangle - D}{m} (*)$

§ Application

Cor If $\lambda \in \Upsilon^+ := \{\lambda \in X^+ \mid \exists\, \mu \in X^+ \text{ s.t. } \mu \uparrow_\gamma^e \lambda\}$ then $(\star) \Rightarrow V(\lambda)$ is reducible.

- $(\star)$ is mild:

  e.g. For $\widehat{A}_1$, $p = 5$, among wts in Υ^+ of level < 150, only 11 of them disobey $(\star)$

  e.g. lowest level $ll(p)$ of reducible $V(\lambda)$ detected by our thm.

p	2	3	5	7	11	13	17	19	23	29	31	37
$ll(p)$	2	2	4	3	4	3	5	5	5	6	7	6

← grow much slower

Remark

1. For poss. twisted aff ADE, $w_0 \notin \Upsilon^+$.
   However, $\exists\, p \leq h$ s.t. $V(w_0)$ is reducible by [BK 02] $\rightsquigarrow$ $p \leq h$ in our conj
2. $\mu \uparrow_\gamma^e \lambda$ is the building block of repn theory of G_nT, insight?
   (i.e. $\mu \uparrow \lambda \Longleftrightarrow \mu \uparrow_\gamma^{e_m} \cdots \uparrow_\gamma^{e_1} \lambda$ where $n \geq e_1 > \cdots > e_m > 0$)