Schur duality, canonical bases, and quantum symmetric pairs of affine type

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- This is a joint work [FLLLW] with Zhaobing Fan, Yiqiang Li, Li Luo, and Weiqiang Wang, manuscript, 100+ pages
- In this talk I will present some key features, with details omitted
- It'll be made public on arXiv/my website (`people.virginia.edu/~cl8ah/`) presumably in November

Review: finite A

[Beilinson-Lusztig-MacPherson 90]

- Partial flag varieties
  $\rightsquigarrow$ (geometric) realization of q-Schur algebras $\mathbf{S}_d^A(N), d \geq 1$
  $\rightsquigarrow$ canonical basis of $\mathbf{S}_d^A(N)$
- $$\underset{q\text{-Schur algebra}}{\mathbf{S}_d^A(N)} \qquad (d \geq 1) \quad \underset{d\to\infty}{\overset{\text{stabilization}}{\Longrightarrow}} \quad \underset{\text{modified quantum group}}{\dot{\mathbf{U}}(\mathfrak{gl}_N)}$$

  $\rightsquigarrow$ canonical basis of $\dot{\mathbf{U}}(\mathfrak{gl}_N)$

Stabilization relies on two key lemmas

1. Multiplication formula between an arbitrary element and a divided power of a Chevalley generator
2. Explicit algorithm that constructs a monomial basis with favorable properties

Review: finite B/C

- [Bao-Kujawa-Li-Wang 14]
  1. Quantum symmetric pairs $(\mathbf{U}(\mathfrak{gl}_N), \mathbf{U}^\jmath)$ ($\mathbf{U}^\jmath \leq \mathbf{U}(\mathfrak{gl}_N)$: coideal subalg.)
     or, $\Delta(\mathbf{U}^\jmath) \subset \mathbf{U}(\mathfrak{gl}_N) \otimes \mathbf{U}^\jmath$
  2. Partial flag varieties of type B/C
     $\rightsquigarrow$ Modified quantum algebras $\dot{\mathbf{U}}^\jmath$ with canonical bases
  3. Schur-type duality

$$\underset{\text{quantum alg.}}{\mathbf{U}^\jmath} \quad \curvearrowright \quad \underset{\text{tensor space}}{\mathbb{V}^{\otimes d}} \quad \curvearrowleft \quad \underset{\text{Hecke alg.}}{\mathcal{H}_d^C}$$

- In [Bao-Wang 13], 3 is used to give a new approach to Kazhdan-Lusztig theory for category $\mathcal{O}$ of classical type $\rightsquigarrow$ solve the irreducible character problem for type B Lie superalgebras

Affine C

Our work [FLLLW] is about affine type C

The affine q-Schur algebra $\mathbf{S}_d^{\widetilde{C}}(N)$ of type C can be defined:

1. (geometric)
   K = formal Laurent series $/\mathbb{F}_q$
   $Sp_d(K)$ = sympletic group over K
   $X_d(N) = \{$ "N-step" affine partial flags of type C $\} \curvearrowleft Sp_d(K)$

$$\mathbf{S}_d^{\widetilde{C}}(N) = \{Sp_d(K)\text{-invariant functions on } X_d(N) \times X_d(N)\}$$

2. (Hecke algebraic)
   $\mathcal{H}_d^{\widetilde{C}}$ = non-extended affine Hecke algebra of type C_d

$$\mathbf{S}_d^{\widetilde{C}}(N) = \mathrm{End}_{\mathcal{H}_d^{\widetilde{C}}}(\widetilde{\mathbb{V}}^{\otimes d}),$$

   tensor space $\widetilde{\mathbb{V}}^{\otimes d} \simeq$ sum of certain permutation modules

Index set of bases

Define $\Xi_d := \{A \in \mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{N}) \mid (X1) - (X4)\}$, where

(X1) (periodic) $a_{ij} = a_{i+2n,j+2n} \quad (i,j \in \mathbb{Z})$

(X2) (centrosymmetric) $a_{-i,-j} = a_{ij} \quad (i,j \in \mathbb{Z})$

(X3) a_{00} and a_{nn} are odd

(X4) $\sum\limits_{(i,j)\in I^+} a_{ij} = d$, $I^+ =$ "core" that determines A uniquely

Fact

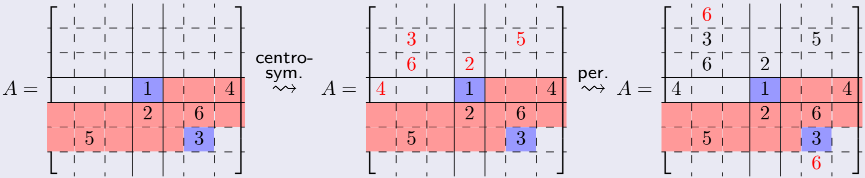
$\{Sp_d(K)\text{-orbits on } X_d \times X_d\} \xleftrightarrow{1:1} \Xi_d$

$\Rightarrow \mathbf{S}_d^{\widetilde{C}}$ has basis $\{e_A \mid A \in \Xi_d\}$,
where $e_A =$ characteristic function on the orbit $\leftrightarrow A$

Index set of bases

Each $A \in \Xi_d$ is uniquely characterized by its entries in I^+

Example ($n = 2$)



- $I^+ \leftrightarrow$ red region, in which $a_{00} = 1$ and $a_{22} = 3$ are colored in blue
- A is also centrosym. about (n, n) (i.e., $a_{n+i,n+j} = a_{n-i,n-j}$ for $i, j \in \mathbb{Z}$)

Constructing affine coideal subalgebra

Next step is to generalize the stabilization:

$$\underset{\text{affine } q\text{-Schur alg.}}{\mathbf{S}_d^{\widetilde{C}}(N)} \quad (d \geq 1) \quad \xrightarrow[d\to\infty]{\text{stabilization}} \quad \underset{\text{limit alg.}}{\mathcal{K}^{\widetilde{C}}(N)} \quad \simeq \quad \underset{\text{affine coideal subalg.}}{\dot{\mathbf{U}}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)}$$

Remark

- This is significantly harder in affine type – $\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N) \supsetneq \mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{sl}}_N)$, the algebra generated by divided powers of Chevalley generators only; while in finite type there is no such a difference
- $\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)$ is generated by "semisimple elements" $\rightsquigarrow$ need a "stronger" multiplication formula.

Multiplication formula (finite A)

For matrix T, let $\widehat{T}=$ matrix obtained from T by shifting every entry up by 1 row.

Lemma (Beilinson-Lusztig-MacPherson 90)

Let B be such that $e_B \leftrightarrow$ divided power $e_j^{(r)}$, i.e.,

$$B = \begin{bmatrix} * & & & & \\ & \ddots & & & \\ & & * & r & \\ & & & \ddots & \\ & & & & * \end{bmatrix}$$

Then

$$e_B * e_A = \sum_{\substack{T \leq A \\ (+\text{ cond.})}} q^{(\star)} \cdot (q\text{-binomial}) \cdot e_{A-T+\widehat{T}}$$

There is also a counterpart for $f_j^{(r)}$

Multiplication formula (finite A)

Example

$$B = \begin{bmatrix} * & & \\ & * & 3 \\ & & * \end{bmatrix} \leftrightarrow e_2^{(3)}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

We have $e_B * e_A = (*)e_{A_1} + (*)e_{A_2}$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4+3 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2+1 & 1 & 4+2 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication formula (affine A)

In affine A we already see that $\mathbf{U}(\widehat{\mathfrak{sl}}) \subsetneq \mathbf{U}(\widehat{\mathfrak{gl}})$ (cf. [Lusztig 99])

Lemma (Du-Fu 13)

Let B be upper bidiagonal ($e_B \leftrightarrow$ semisimple representations of Hall algebra of the cyclic quiver), i.e.

$$B = \left[\begin{array}{c|ccccc|c} \ddots & \alpha_n & & & & & \\ \hline & * & \alpha_1 & & & & \\ & & * & \ddots & & & \\ & & & \ddots & \alpha_{n-1} & & \\ & & & & * & & \alpha_n \\ \hline & & & & & & \ddots \end{array}\right].$$

Then

$$e_B * e_A = \sum_{\substack{T \le A \\ (+\text{cond.})}} q^{(\star)} \cdot (q\text{-binomial}) \cdot e_{A - T + \widehat{T}}$$

Multiplication formula (finite B/C)

For matrix $T = (t_{ij})$, let $T^\theta = (t^\theta_{ij})$ be the centro-symmetrization of T, i.e., $t^\theta_{ij} = t_{ij} + t_{-i,-j}$

Lemma (Bao-Kujawa-Li-Wang 14)

Let B be such that $e_B \leftrightarrow$ divided power $e_j^{(r)}$, i.e.,

$$B = \left[\begin{array}{ccc|c|ccc} * & & & & & & \\ & \ddots & & & & & \\ r & & & & & & \\ \hline & & & * & & & \\ \hline & & & & & \ddots & r \\ & & & & & & * \end{array}\right]$$

Then

$$e_B * e_A = \sum_{\substack{T \text{ s.t.} \\ T^\theta \leq A \\ (+\text{cond.})}} q^{(\star)} \cdot (q\text{-binomial}) \cdot e_{A - T^\theta + (\widehat{T})^\theta}$$

Multiplication formula (finite B/C)

Example

$$B = \begin{bmatrix} * & & \\ 3 & * & 3 \\ & & * \end{bmatrix} \leftrightarrow e_2^{(3)}, \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

We have $e_B * e_A = (*)e_{A_1} + (*)e_{A_2}$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 2{\color{red}+3} & 1 & 2{\color{red}+3} \\ 1 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2{\color{red}+3} & 1 & 2{\color{red}+3} \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication formula (affine C)

Theorem 1 (FLLLW 15)

Let B be tridiagonal. Then

$$e_B * e_A = \sum_{\substack{T^\theta \leq A \\ (+\text{cond.})}} \sum_{\substack{S \leq T \\ (+\text{cond.})}} (q-1)^{(\star)} q^{(\star\star)} \cdot \prod (q\text{-binomial}) \cdot e_{A-(T-S)^\theta + (\widehat{T-S})^\theta}$$

- It covers the previous ones (finite A/B/C, affine A) as special cases
- This formula, although complicated, is explicit enough to construct a monomial basis and hence derive stabilization
- It is obtained in both geometric and Hecke-algebraic approaches

Multiplication formula (affine C)

Multiplication formula $\rightsquigarrow$ explicit algorithm constructing a monomial basis for $\mathbf{S}_d^{\widetilde{C}}(N)$, with favorable properties

Theorem 2 (FLLLW 15)

1. $\mathbf{S}_d^{\widetilde{C}}(N)$ has a monomial and hence a canonical basis (with positivity)
2. Stabilization gives a monomial basis and a "stably-canonical" basis for the affine coideal subalgebra $\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)$

- Surprisingly, the stably-canonical basis for $\dot{\mathbf{U}}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)$ does NOT have positivity (this is observed in [Li-Wang 15] for finite and affine $\mathfrak{gl}_N$)
- Meanwhile, the canonical basis for $\dot{\mathbf{U}}^{\widetilde{C}}(\widehat{\mathfrak{sl}}_N)$ has positivity (wrt multiplication and comultiplication), see [Fan-Li 15] for positivity comult. in finite type C

Schur type duality

$$
\begin{array}{ccccccc}
\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{sl}}_N) \subsetneq & \underset{\text{affine coideal subalg.}}{\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)} & & & & \\
& \downarrow & & & & \\
& \underset{\text{affine } q\text{-Schur alg.}}{\mathbf{S}_d^{\widetilde{C}}} & \overset{\psi}{\curvearrowright} & \underset{\text{tensor space}}{\mathbb{V}^{\otimes d}} & \overset{\varphi}{\curvearrowleft} & \underset{\text{affine Hecke alg.}}{\mathcal{H}_d^{\widetilde{C}}}
\end{array}
$$

Geometrically, $\mathbf{S}_d^{\widetilde{C}}$ and $\mathcal{H}_d^{\widetilde{C}}$ can be realized as convolution algebras of partial/complete flags.

Proposition (FLLLW 15)

The algebras $\mathbf{S}_d^{\widetilde{C}}$ and $\mathcal{H}_d^{\widetilde{C}}$ satisfy double centralizer property. That is,

$$
\begin{aligned}
\mathrm{End}_{\psi(\mathbf{S}_d^{\widetilde{C}})}(\mathbb{V}^{\otimes d}) &= \varphi(\mathcal{H}_d^{\widetilde{C}}) \\
\psi(\mathbf{S}_d^{\widetilde{C}}) &= \mathrm{End}_{\varphi(\mathcal{H}_d^{\widetilde{C}})}(\mathbb{V}^{\otimes d})
\end{aligned}
$$

Quantum symmetric pairs

Proposition (FLLLW 15)

1. $(\mathbf{U}(\widehat{\mathfrak{gl}}_N), \mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N))$ form a quantum symmetric pair
2. $(\mathbf{U}(\widehat{\mathfrak{sl}}_N), \mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{sl}}_N))$ form a quantum symmetric pair

Namely, $\Delta(\mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)) \subset \mathbf{U}(\widehat{\mathfrak{gl}}_N) \otimes \mathbf{U}^{\widetilde{C}}(\widehat{\mathfrak{gl}}_N)$, similar for $\widehat{sl}_N$

Remark

Similar to the fact that there are two versions (i.e., $\mathbf{U}^{\imath}, \mathbf{U}^{\jmath}$) for finite B/C, the affinization above has four versions (i.e., $\mathbf{U}^{\jmath\jmath}, \mathbf{U}^{\jmath\imath} \simeq \mathbf{U}^{\imath\jmath}, \mathbf{U}^{\imath\imath}$) depending on "fixed points of involution". In this talk we demonstrated the $\jmath\jmath$ version.

Thank you for your attention