An elementary approach to monomial and canonical bases of quantum affine $\mathfrak{gl}_n$

Chun-Ju Lai

University of Virginia

March 7, 2015

(joint work with Li LUO (Shanghai))

Outline

1. Finite type A [Beilinson-Lusztig-MacPherson 90]

   Construct: quantum $\mathfrak{gl}_n$ and canonical basis
   Essential: construction of a monomial basis

2. Affine type A

   Construct: quantum affine $\mathfrak{gl}_n$ and canonical basis
   (new) Construct a monomial basis in the BLM sense
   $\Rightarrow$ easy construction of canonical basis of quantum affine $\mathfrak{gl}_n$

Finite type A

Fix $n \geq 1$, for each $d \geq 1$:

- $GL_d = GL_d(\mathbb{F}_q)$
- $X_d = \{n\text{-step flags in } \mathbb{F}_q^d\}$

Fact

$$\{GL_d\text{-orbits on } X_d \times X_d\} \overset{1:1}{\longleftrightarrow} \Theta_d := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{N}) \mid \Sigma a_{ij} = d\}$$

Define q-Schur algebra over $\mathbb{Z}[v, v^{-1}]$ by

$$\mathcal{S}_d := \{GL_d\text{-invariant functions on } X_d \times X_d\}$$

$\mathcal{S}_d$ *has basis* $\{\text{characteristic function on } A \mid A \in \Theta_d\}$

Finite type A

$\mathcal{S}_d$ also has following bases:

1. standard basis $\{[A] \mid A \in \Theta_d\}$,
   each $[A]$: normalized characteristic function on A

2. monomial basis $\{m_A \mid A \in \Theta_d\}$ satisfying
   - $\overline{m}_A = m_A$
   - $m_A = [A] +$ lower terms

3. canonical basis: follows readily from monomial basis

Finite type A

Proposition/definition [BLM]

For each $A \in \Theta_d$, there are "divided power" matrices B_i such that

$$m_A := \prod_i [B_i] = [A] + \text{ lower terms},$$

each $B_i = \text{diag} + rE_{j,j+1}$ or $\text{diag} + rE_{j+1,j}$, $r \geq 1$, i.e.

$$B_i = \begin{bmatrix} * & & & \\ & * & r & \\ & & * & \\ & & & * \end{bmatrix} \text{ or } \begin{bmatrix} * & & & \\ & * & & \\ & r & * & \\ & & & * \end{bmatrix},$$

(B_i corresponds to divided power $e_j^{(r)}$ or $f_j^{(r)}$)

Finite type A, example

- $n = 3$,

$$A = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ \color{red}{3} & 2 & 0 \end{bmatrix} \in \Theta_6$$

We have $[B_1][A_1] = [A] +$ lower terms, where

$$B_1 = \underset{\updownarrow}{\begin{bmatrix} * & & \\ & * & \\ & 3 & * \end{bmatrix}}, \quad A_1 = \begin{bmatrix} 0 & & \\ 1 + \color{red}{3} & 0 & \\ \color{red}{\uparrow} & 2 & 0 \end{bmatrix}$$

divided power $f_2^{(3)}$

Finite type A, example

$$A_1 = \begin{bmatrix} 0 & & \\ 4 & 0 & \\ & 2 & 0 \end{bmatrix}$$

We have $[B_2][A_2] = [A_1] +$ lower terms, where

$$B_2 = \begin{bmatrix} * & & \\ 4 & * & \\ & & * \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0+4 & & \\ \uparrow & 0 & \\ & 2 & 0 \end{bmatrix} = B_3$$

$$\begin{array}{cc} \updownarrow & \updownarrow \\ \text{divided power } f_1^{(4)} & \text{divided power } f_2^{(2)} \end{array}$$

Therefore

$$\begin{array}{cccccc} m_A = & [B_1] & [B_2] & [B_3] & = [A] + \text{ lower terms} \\ & \updownarrow & \updownarrow & \updownarrow & \\ & f_2^{(3)} & f_1^{(4)} & f_2^{(2)} & \end{array}$$

Finite type A, stabilization

q-Schur algebra $\mathcal{S}_d, d \geq 1$ $\overset{\text{stabilization}}{\Longrightarrow}$ BLM algebra $\mathcal{K} \simeq \dot{\mathbf{U}}(\mathfrak{gl}_n)$

monomial basis	monomial basis
$\rightsquigarrow$	$\rightsquigarrow$
canonical basis	canonical basis

Outline

1. Finite type A
   - q-Schur algebras
   - A monomial basis
   - Quantum $\mathfrak{gl}_n$
2. Affine type A
   - Quantum affine $\mathfrak{sl}_n$ v.s. Quantum affine $\mathfrak{gl}_n$
   - Affine q-Schur algebra
   - A monomial basis
   - Quantum affine $\mathfrak{gl}_n$

Affine type A

Remark

1. For finite type A, divided powers generate quantum $\mathfrak{sl}_n$ ($\simeq$ quantum $\mathfrak{gl}_n$).
2. For affine type A, divided powers only generate quantum affine $\mathfrak{sl}_n$ ($\not\simeq$ quantum affine $\mathfrak{gl}_n$). In order to generate the entire quantum affine $\mathfrak{gl}_n$, we need semisimple elements, i.e.,
   those corresponding to semisimple representations in the presentation via Double Hall algebra of cyclic quiver.

Affine type A

Fix $n \geq 0$, for each $d \geq 0$:

- $\Theta_d^{\text{aff}} := \{A \in \text{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{N}) \mid (T1), (T2)\}$

(T1) (periodicity)

$a_{ij} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$.

$\Rightarrow$ each $A \in \Theta_d^{\text{aff}}$ is uniquely characterized by any $n \times \mathbb{Z}$ submatrix

Example ($n = 2$)

$$A = \left[\begin{array}{cc:cc:cc} & & & & & \\ \hdashline & 3 & 0 & 4 & & \\ & & 0 & 0 & 1 & 2 \\ \hdashline & & & & & \end{array}\right] \rightsquigarrow A = \left[\begin{array}{cc:cc:cc} & 4 & & & & \\ & & 1 & 2 & & \\ \hdashline & 3 & 0 & 4 & & \\ & & 0 & 0 & 1 & 2 \\ \hdashline & & & 3 & & 4 \end{array}\right]$$

(T2) $\displaystyle\sum_{1\leq i\leq n} \sum_{j\in\mathbb{Z}} a_{ij} = d.$

Affine type A

$\mathcal{H}_d$: extended affine Hecke algebra of type A
$\mathcal{S}_d^{\text{aff}}$: affine q-Schur algebra
:= Endomorphism algebra of a sum of certain permutation modules for $\mathcal{H}_d$

Remark

This definition via Hecke algebras is equivalent to the geometric one, and there is also a natural way to define

1. standard basis $\{[A] \mid A \in \Theta_d^{\text{aff}}\}$.
   each $[A]$: normalized characteristic function on A

Affine type A

Theorem/definition [Lai-Luo]

For each $A \in \Theta_d^{\text{aff}}$, we have an explicit algorithm producing matrices B_i such that

$$m_A := \prod_i [B_i] = [A] + \text{ lower terms},$$

each $B_i = \text{diag} + \sum \alpha_j E_{j,j+1}$ or $\text{diag} + \sum \alpha_j E_{j+1,j}$, $R \geq 1$, i.e.,

$$B_i = \left[\begin{array}{c|cccccc|c} \ddots & \alpha_n & & & & & & \\ \hline & * & \alpha_1 & & & & & \\ & & * & \ddots & & & & \\ & & & \ddots & \alpha_{n-1} & & & \\ & & & & * & & & \alpha_n \\ \hline & & & & & & & \ddots \end{array}\right] \text{ or its transpose}$$

Affine type A, example

- $n = 2$

$$A = \left[\begin{array}{cc:cc:cc} \ddots & & & & & \\ & 0 & 1 & {\color{red}2} & & \\ \hdashline & 3 & 0 & 4 & {\color{red}0} & \\ & & & 0 & 1 & {\color{red}2} \\ \hdashline & & & & \ddots & \end{array}\right].$$

We have $[A] = [B_1][A_1] +$ lower terms, where

$$B_1 = \left[\begin{array}{cc:cc:c} \ddots & & & & \\ & * & 2 & & \\ \hdashline & & * & 0 & \\ & & & * & 2 \\ \hdashline & & & & \ddots \end{array}\right], \quad A_1 = \left[\begin{array}{cc:cc:c} \ddots & & & & \\ & 0 & 1 + \overset{\color{red}\downarrow}{\color{red}0} & & \\ \hdashline & 3 & 0 & 4 + \overset{\color{red}\downarrow}{\color{red}2} & \\ & & & 0 & 1 + \overset{\color{red}\downarrow}{\color{red}0} \\ \hdashline & & & & \ddots \end{array}\right]$$

$\updownarrow$

divided power $e_0^{(2)}$

Affine type A, example

$$A_1 = \left[\begin{array}{c:cc:c} \ddots & 1 & & \\ \hdashline 3 & 0 & 6 & \\ & & 0 & 1 \\ \hdashline & & & \ddots \end{array}\right].$$

We have $[A_1] = [B_2][A_2] +$ lower terms, where

$$B_2 = \left[\begin{array}{c:cc:c} \ddots & 1 & & \\ \hdashline & * & 6 & \\ & & * & 1 \\ \hdashline & & & \ddots \end{array}\right], \qquad A_2 = \left[\begin{array}{c:cc:c} \ddots & \downarrow & & \\ \hdashline 3 & 0+1 & \downarrow & \\ & & 0+6 & \\ \hdashline & & & \ddots \end{array}\right] = B_3$$

$\updownarrow$ semisimple element $S_{(1,6)}$ $\qquad$ $\updownarrow$ divided power $f_0^{(3)}$

Therefore

$$m_A = \underset{\substack{\updownarrow \\ e_0^{(2)}}}{[B_1]} \quad \underset{\substack{\updownarrow \\ S_{(1,6)}}}{[B_2]} \quad \underset{\substack{\updownarrow \\ f_0^{(3)}}}{[B_3]} = [A] + \text{ lower terms}$$

Affine type A

Remark

The (new) construction is based on an observation on certain "admissible" pairs (B, A) of matrices such that

- $[B]$ corresponds to a semisimple element
- $\overline{[B]} = [B]$
- $[B][A] = [M] +$ lower terms

Therefore, we have

1. $\overline{m}_A = m_A$
2. $m_A = [A] +$ lower terms

which leads to the canonical basis for $\mathcal{S}_d^{\text{aff}}$

Affine type A

Remark

Our algorithm of constructing monomial basis can be adapted from affine q-Schur algebra to quantum affine $\mathfrak{gl}_n$.

$$\begin{array}{ccc} \text{affine } q\text{-Schur algebra } \mathcal{S}_d^{\text{aff}} & \overset{\text{stabilization}}{\Longrightarrow} & \text{BLM algebra } \mathcal{K}^{\text{aff}} \simeq \dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \\ \text{monomial basis} & & \text{monomial basis} \\ \rightsquigarrow & & \rightsquigarrow \\ \text{canonical basis} & & \text{canonical basis} \end{array}$$

Theorem

The canonical basis for quantum affine $\mathfrak{gl}_n$ exists.

Affine type A

Remark

In [Du-Fu 2014], there is another construction of a monomial basis for $\mathcal{S}_d^{\text{aff}}$:

1. a monomial basis of Ringle-Hall algebra of the cyclic quiver due to Deng-Du-Xiao (difficult)
2. a monomial basis for Double Hall algebra $\mathfrak{D}$.
3. a monomial basis for $\mathcal{S}_d^{\text{aff}}$ via a surjection $\mathfrak{D} \twoheadrightarrow \mathcal{S}_d^{\text{aff}}$

Thank you for your attention