Affine Hecke Algebras and Quantum Symmetric Pairs

Chun-Ju Lai University of Virginia

Abstract

One breakthrough in the theory of quantum groups is the construction of the canonical bases for quantum groups by Lusztig and Kashiwara. For type A, there is a geometric construction for (idempotented) quantum group together with a canonical basis due to Beilinson, Lusztig and MacPherson (BLM) using a stabilization procedure on a family of quantum Schur algebras of type A. In this work we realize the affine q-Schur algebras of type C as an endomorphism algebra of certain module of affine Hecke algebras, and then establish a multiplication formula on the Schur algebra level. We provide a direct construction of monomial bases for Schur algebras. Via a BLM-type stabilization on the Schur algebras, we construct an algebra $\dot{\mathbf{K}}^{\mathrm{affC}}_n$ admitting canonical basis. We obtain that $(\mathbf{K}^{\mathrm{affA}}, \mathbf{K}^{\mathrm{affC}}_n)$ forms a quantum symmetric pair, where $\mathbf{K}^{\mathrm{affA}} \simeq \mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is a quantum group of affine type A. The affine type C construction above is associated to an involution on Dynkin diagrams of affine type A.

This is a joint work with Z. Fan, Y. Li, L. Luo, W. Wang.

Background

The BLM Construction

Beilinson, Lusztig and MacPherson [1] developed a geometric construction for (idempotented) quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ together with canonical basis. The essential steps in their work are to obtain:

(1) Multiplication formulas on $\mathbf{S}_{n,d}$ with Chevalley generators.

(2) Monomial basis for $\mathbf{S}_{n,d}$.

As a consequence, by taking stabilization on $\{\mathbf{S}_{n,d}\}_{d\in\mathbb{N}}$, one obtains the idempotented quantum group admitting canonical bases as below:

$$\underset{q\text{-Schur algebra } (d\geq 1)}{\mathbf{S}_{n,d}} \underset{d\to\infty}{\overset{\text{stabilization}}{\Longrightarrow}} \underset{\text{stabilization algebra}}{\mathrm{Stab}(\mathbf{S}_{n,d}) := \dot{\mathbf{K}}_n} \simeq \underset{\text{idempotented quantum group}}{\dot{\mathbf{U}}(\mathfrak{gl}_n)}$$

BLM-type Constructions for Affine Type A

Via a geometric realization of affine q-Schur algebras $\mathbf{S}^{\mathrm{affA}}_{n,d}$, Ginzburg-Vasserot ('93) and Lusztig ('99) obtain generalization to affine type A partially, due to a new phenomenon in affine types – the Schur algebra is not generated by Chevalley generators. Here one needs a larger generating set consisting of the *bidiagonal generators*.

By realizing $\mathbf{S}^{\mathrm{affA}}_{n,d}$ via Hecke algebras, Du and Fu [3] obtained

(1) Multiplication formulas with bidiagonal generators on $\mathbf{S}^{\mathrm{affA}}_{n,d}$.

(2) Monomial bases for $\mathbf{S}^{\mathrm{affA}}_{n,d}$ from Hall algebras of the cyclic quiver due to Deng-Du-Xiao ('07).

It is then shown that the stabilization algebra is isomorphic to $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$.

In a work [7] joint with Luo, we provide a direct construction (without Hall algebras) of monomial bases by multiplying suitable bidiagonal generators.

BLM-type Constructions and Quantum Symmetric Pairs

Bao, Kujawa, Li and Wang [2] provided generalization for finite type B/C by taking a BLM-type stabilization on the q-Schur algebras of type B/C. The non-idempotented quantum algebras $\mathbf{K}^{\mathrm{finBC}}_n \simeq \mathbf{iU}(\mathfrak{gl}_n)$ are not the Drinfel'd-Jimbo type quantum groups of type B/C, they arise from the quantum symmetric pair $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{iU}(\mathfrak{gl}_n))$, whose theory is developed and studied by Letzter ('02) and Kolb ('14). One crucial property here is that $\mathbf{iU}(\mathfrak{gl}_n)$ is a coideal subalgebra of $\mathbf{U}(\mathfrak{gl}_n)$ – the comultiplication $\Delta : \mathbf{U}(\mathfrak{gl}_n) \to \mathbf{U}(\mathfrak{gl}_n) \otimes \mathbf{U}(\mathfrak{gl}_n)$ sends $\mathbf{iU}(\mathfrak{gl}_n)$ to $\mathbf{iU}(\mathfrak{gl}_n) \otimes \mathbf{U}(\mathfrak{gl}_n)$.

Main Results

We provide BLM-type constructions for affine type C, for which a geometric approach has been developed in [1]. Here we focus on the algebraic approach by realizing the affine Schur algebra $\mathbf{S}^{\mathrm{affC}}_{n,d}$ of type C as an endomorphism algebra of certain permutation module over affine Hecke algebra of type C.

Multiplication formulas on $\mathbf{S}^{\mathrm{affC}}_{n,d}$

The standard basis $\{[A]\}_A$ of $\mathbf{S}^{\mathrm{affC}}_{n,d}$ is parametrized by

$$\Xi_{n,d} := \{A \in \mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{N}) \mid \text{(X1) – (X3)}\},$$

(X1) (periodicity) $a_{ij} = a_{i+n,j+n}$ for $i,j \in \mathbb{Z}$;

(X2) (centro-symmetry) $a_{-i,-j} = a_{ij}$ for $i,j \in \mathbb{Z}$;

(X3) (size) $\sum a_{ij} = d$ over any "half period".

We prove that $\mathbf{S}^{\mathrm{affC}}_{n,d}$ is generated by the *tridiagonal generators* $[A]$ where $A = (a_{ij})$ satisfying that $a_{ij} = 0$ unless $|i-j| \leq 1$.

Theorem A: Multiplication Formula

For $A, B \in \Xi_{n,d}$ with B being tridiagonal, we establish a multiplication formula for $[B] * [A] \in \mathbf{S}^{\mathrm{affC}}_{n,d}$ with explicit coefficients. Moreover, the set $\{[A] \mid A \in \Xi_{n,d} \text{ is tridiagonal}\}$ is a generating set for $\mathbf{S}^{\mathrm{affC}}_{n,d}$.

Monomial Bases

By multiplying the tridiagonal generators in a suitable order, we construct a semi-monomial basis $\{m'_A\}_{A\in\Xi_{n,d}}$. Another new phenomenon for affine type C is that the tridiagonal generators (and hence m_A) are not necessarily bar-invariant. Nevertheless, the semi-monomial basis can be adapted to a monomial basis $\{m_A\}_{A\in\Xi_{n,d}}$.

Theorem B

The Schur algebra $\mathbf{S}^{\mathrm{affC}}_{n,d}$ admits both monomial and canonical bases.

Affine Coideal Subalgebras

Let $\dot{\mathbf{K}}^{\mathrm{affC}}_n$ be the free $\mathbb{Z}[v, v^{-1}]$-module generated by $\{[A]\}_{A\in\tilde{\Xi}}$, where $\tilde{\Xi}$ is adapted from $\bigcup_{d\in\mathbb{N}} \Xi_{n,d}$ by allowing diagonal entries to be negative integers. We show that $\dot{\mathbf{K}}^{\mathrm{affC}}_n$ has a unique associative algebra structure in the sense that for any $B, A \in \tilde{\Xi}$, the structure constants for $[B] * [A] \in \dot{\mathbf{K}}^{\mathrm{affC}}_n$ are compatible with the structure constants for $[B + pI] * [A + pI]$ for all even p that is large enough. We can lift the monomial basis for $\mathbf{S}^{\mathrm{affC}}_{n,d}$ to the stabilization algebra level to construct canonical bases for $\dot{\mathbf{K}}^{\mathrm{affC}}_n$.

Theorem C

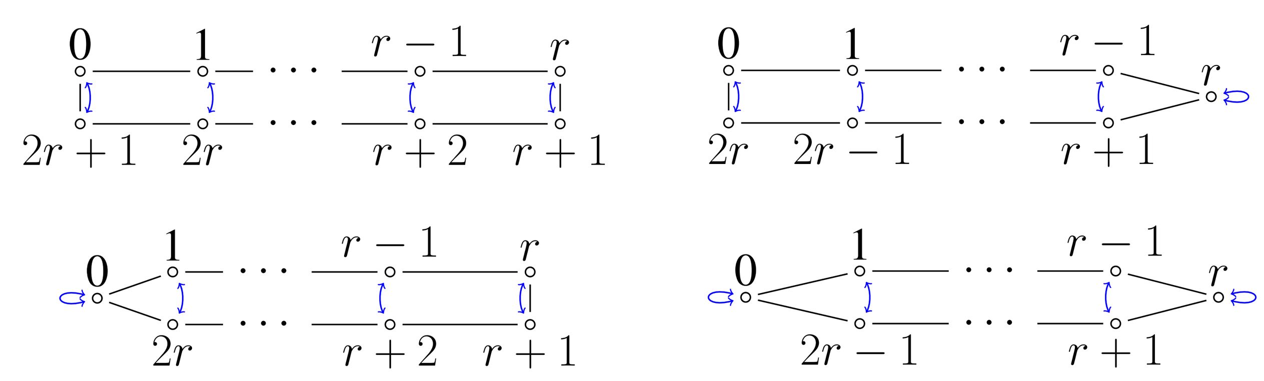
The algebra $\dot{\mathbf{K}}^{\mathrm{affC}}_n$ admits both monomial and canonical bases. Moreover, there is a surjective map $\dot{\mathbf{K}}^{\mathrm{affC}}_n \to \mathbf{S}^{\mathrm{affC}}_{n,d}$ preserving canonical bases.

Theorem D

The algebra $\mathbf{K}^{\mathrm{affC}}_n$ is a coideal subalgebra of $\mathbf{K}^{\mathrm{affC}}_n$.

Variants

For all four types of involutions for Dynkin diagrams of affine type A as depicted below, we provide similar constructions for different affine coideal subalgebras, with compatible canonical bases.



References

[1] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of* GL_n, Duke Math. J., **61** (1990), 655–677.

[2] H. Bao, J. Kujawa, Y. Li, and W. Wang, *Geometric Schur duality of classical type*, arXiv:1404.4000v3.

[3] J. Du and Q. Fu, *The integral quantum loop algebra of* $\mathfrak{gl}_n$, arXiv:1404.5679 (2014).

[4] Z. Fan, C. Lai, Y. Li, L. Luo, W. Wang, *Affine flag varieties and quantum symmetric pairs*, arXiv: 1602.04383.

[5] Z. Fan, C. Lai, Y. Li, L. Luo, W. Wang, *Affine Hecke algebras and quantum symmetric pairs*, preprint.

[6] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. in Math. **267** (2014), 395–469.

[7] C. Lai, L. Luo, *An elementary construction of monomial bases of quantum affine* $\mathfrak{gl}_n$, arXiv:1506.07263.