Affine Hecke algebras and Quantum Symmetric Pairs

Chun-Ju Lai

University of Virginia

joint with Fan, Li, Luo and Wang

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Beilinson, Lusztig and MacPherson ('90) developed a geometric construction for (idempotented) quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ together with canonical basis.

An overview:

Background

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- $oldsymbol{0}$ Convolution algebra $\mathbf{S}_{n,d}$ via partial flag varieties
- 2 Multiplication formulas on $S_{n,d}$ with Chevalley generators
- **3** Monomial basis for $S_{n,d}$
- **4** Stabilization algebra $\dot{\mathbf{K}}_n = \operatorname{Stab} \mathbf{S}_{n,d}$ as $d \to \infty$
- \Rightarrow $\dot{\mathbf{K}}_n \simeq \dot{\mathbf{U}}(\mathfrak{gl}_n)$ admits canonical bases

- Fix $n, d \in \mathbb{N}$, let $X_{n,d} = \{n \text{-step flags in } \mathbb{F}_q^d\}$: partial flag variety $GL_d = GL_d(\mathbb{F}_q)$: general linear group
- $GL_d \sim \chi_{n,d}$

Background

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- $\Rightarrow GL_d \curvearrowright X_{n,d} \times X_{n,d}$ diagonally
- ⇒ Convolution algebra

 $\mathbf{S}_{n,d} := \{GL_d \text{-invariant functions on } X_{n,d} \times X_{n,d}\}$

Background 00000000

• The bases for $S_{n,d}$ are parametrized by

$$\begin{array}{cccc} \{GL_d\text{-orbits on } X_{n,d} \times X_{n,d}\} & \stackrel{1:1}{\longleftrightarrow} & \Theta_{n,d}^{\text{fin}} = \left\{A \in \mathsf{Mat}_{n \times n}(\mathbb{N}) \; \middle| \; \sum a_{ij} = d\right\} \\ O_A & \longleftrightarrow & A \end{array}$$

- $\{[A] \mid A \in \Theta_{n,d}^{fin}\}$: standard basis, i.e. [A] = normalized characteristic function on orbit O_A
- A (divided power of) Chevalley generator $e_i^{(R)} \leftrightarrow$

$$B = \begin{bmatrix} * & & & & \\ & \ddots & & & \\ & & * & R & \\ & & \ddots & \\ & & & * \end{bmatrix}$$

Multiplication formulas (finite A)

Background 000000000

Lemma (Beilinson-Lusztig-MacPherson 90)

Assume that $B \leftrightarrow$ Chevalley generator $\mathbf{e}_i^{(R)}.$ Then

$$[B] * [A] = \sum_{\substack{T \le A \\ (+ \text{ cond.})}} (\text{coeff}) [A - T + \widehat{T}],$$

where $A - T + \widehat{T} \leftarrow$ shifting T up by 1 row in A.

- (There is also a counterpart for $\mathbf{f}_i^{(R)}$)
- ullet The Chevalley generators form a generating set of $\mathbf{S}_{n,d}$

Example

Background 000000000

$$B = \begin{bmatrix} * & & \\ & * & 3 \\ & & * \end{bmatrix} \leftrightarrow \mathbf{e}_2^{(3)}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

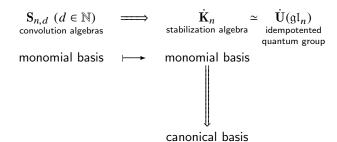
We have $[B] * [A] = (coeff)[A_1] + (coeff)[A_2]$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4 + 3 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 + 1 & 1 & 4 + 2 \\ 5 & 0 & 0 & 1 \end{bmatrix}$$

Facts

Background 000000000

- [BLM 90] Multiplying Chevalley generators in a suitable order ⇒ monomial basis $\{m_A\}_A$ for $\mathbf{S}_{n,d}$.
- monomial basis ⇒ canonical basis



Background

• $S_{n,d}$ admits a Schur-type duality

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{gl}_n) \\ & & \\ \mathbf{S}_{n,d} & \curvearrowright & \mathbb{V}^{\otimes d} & \backsim & \mathbf{H} = \mathbf{H}(\mathfrak{S}_d) \\ & & & \\ \mathbf{Hecke\ algebra} \end{array}$$

• In other words, $S_{n,d} = q$ -Schur algebra as an endomorphism algebra

$$\mathbf{S}_{n,d} = \mathsf{End}_{\mathbf{H}}(\mathbb{V}^{\otimes d})$$

where $\mathbb{V}^{\otimes d} \simeq \bigoplus_{\lambda} x_{\lambda} \mathbf{H}$: permutation module

- ⇒ The BLM construction can be done in two approaches:
 - 1 (Geometric) partial flags and counting over finite fields
 - 2 (Hecke algebraic) permutation modules and combinatorics

BLM-type constructions

• There are generalizations to other types

Realization of	Partial flag varieties	Hecke algebras	
Schur algebras	(dimension counting)	(combinatorics)	
Affine A	Ginzburg-Vasserot ¹ ('93)	Du-Fu ('14)	
	Lusztig ¹ ('99)		
Finite B/C	Bao-Kujawa-Li-Wang ('14)		
Finite D	Fan-Li ('14)		

¹partially generalized

BLM-type constructions

• [Fan-L.-Li-Luo-Wang '16] We provide a BLM-type construction for affine type C in both approaches

Realization of	Partial flag varieties	Hecke algebras	
Schur algebras	(dimension counting)	(combinatorics)	
Affine A	Ginzburg-Vasserot ¹ ('93)	Du-Fu ('14)	
	Lusztig ¹ ('99)		
Finite B/C	Bao-Kujawa-Li-Wang ('14)		
Finite D	Fan-Li ('14)		
Affine C	FLLLW1 ('16)	FLLLW2 ('16)	

¹partially generalized

Difficulties in affine type A

- An overview for affine type A (Du and Fu):
 - 1 Hecke algebraic realization of affine Schur algebra SaffA
 - 2 Multiplication formulas on $S_{n,d}^{affA}$
 - 3 Monomial basis for $S_{n,d}^{affA}$
 - 4 Stabilization algebra $\mathbf{K}_n^{\mathsf{affA}} = \mathsf{Stab} \, \mathbf{S}_{n,d}^{\mathsf{affA}}$ as $d \to \infty$
- Chevalley generators do not generate affine Schur algebra
- Du and Fu establish 2 with bidiagonal generators instead.
- 3 is non-trivial
- ⇒ Du and Fu use monomial bases from Hall algebras of the cyclic quiver due to Deng-Du-Xiao

Lemma (Du-Fu 15)

Let B be upper bidiagonal. Then

$$[B] * [A] = \sum_{\substack{T \le A \\ (+\mathsf{cond.})}} (\mathsf{coeff}) \ [A - T + \widehat{T}].$$

- (There is also a counterpart for lower bidiagonal generators)
- ullet The bidiagonal generators form a generating set of $\mathbf{S}_{n,d}^{\mathsf{affA}}$

Theorem A (L.-Luo 15)

Multiplying bidiagonal generators in a suitable order

 \Rightarrow monomial basis for $\mathbf{S}_{n,d}^{\mathsf{affA}}$.

- An overview for affine type C:
 - Hecke algebraic realization of affine Schur algebra S^{affC}
 - 2 Multiplication formulas on $S_{n,d}^{affC}$

 - $\textbf{3} \ \, \text{Monomial basis for } \mathbf{S}_{n,d}^{\text{affC}} \\ \textbf{4} \ \, \text{Stabilization algebra } \dot{\mathbf{K}}_n^{\text{affC}} = \underbrace{\mathsf{Stab}}_{n,d} \mathbf{S}_{n,d}^{\text{affC}} \ \, \text{as } d \to \infty$
- Chevalley/bidiagonal generators do not generate the entire Schur algebra
- ⇒ We establish ② with tridiagonal generators instead
- Du-Fu's monomial bases do not generalize to since no Hall algebra approach is known in affine type C
- ⇒ We adapt Theorem A

Finite type B/C

- We review first finite type B/C
- Fix n = 2r + 1. The bases for $\mathbf{S}_{n,d}^{\mathsf{finBC}}$ are parametrized by

$$\Xi_{n,d}^{\text{fin}} = \left\{ A \in \mathsf{Mat}_{[-r,r]\times[-r,r]}(\mathbb{N}) \mid (1), (2) \right\}$$

- (1) (centro-symmetry) $a_{ij} = a_{-i,-j} \quad (i, j \in \mathbb{Z})$
- (2) (size) $\sum a_{ij} = d$ over "half index set"
- A (divided power of) Chevalley generator $e_i^{(R)} \leftrightarrow$

Multiplication formula (finite B/C)

Let $T^{\theta} = (t_{ij}^{\theta})$ is the centro-symmetrization of T, i.e., $t_{ij}^{\theta} = t_{ij} + t_{-i,-j}$

Lemma (Bao-Kujawa-Li-Wang 14)

Assume that $B \leftrightarrow$ Chevalley generator. Then

$$[B] * [A] = \sum_{\substack{T \text{ s.t.} \\ T^{\theta} \leq A \\ (+\text{cond.})}} (\text{coeff}) \ [A - T^{\theta} + (\widehat{T})^{\theta}],$$

Affine type C

where $A - T^{\theta} + (\widehat{T})^{\theta} \sim$ shifting T up by 1 row, and its counterpart down by 1 row in A.

Multiplication formula (finite B/C)

Example

$$B = \begin{bmatrix} * \\ 3 & * & 3 \\ & & * \end{bmatrix} \leftrightarrow \mathbf{e}_{1}^{(3)}, \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

Affine type C 000000000

We have $[B] * [A] = (coeff)[A_1] + (coeff)[A_2]$, where

$$A_{1} = \begin{bmatrix} 0 & 0 & 1 \\ 2+3 & 1 & 2+3 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 2+2 & & 2+1 \\ 2 & 1 & 2 \\ 1 & & & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Affine Schur algebra of type C

- Let's switch back to affine type C. The affine Schur algebra $\mathbf{S}_{n,d}^{\mathsf{affC}}$ can be defined:
- (geometric)

$$Sp_d = Sp_d(K)$$
 = symplectic group over K : formal Laurent series $/\mathbb{F}_q$
 $X_{n,d} = \{n\text{-step affine partial flags of type C}\} \curvearrowleft Sp_d$

$$\mathbf{S}_{n,d}^{\mathsf{affC}} = \{Sp_d\text{-invariant functions on } \mathcal{X}_{n,d} \times \mathcal{X}_{n,d}\}$$

Affine type C

(Hecke algebraic)

 $\mathbf{H}_d^{\mathsf{affC}} = \mathsf{non}\text{-extended}$ affine Hecke algebra of type C_d tensor space $\mathbb{V}^{\otimes d} \simeq \bigoplus_{i} x_{\lambda} \mathbf{H}_{d}^{\mathsf{affC}}$: permutation module

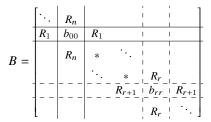
$$\mathbf{S}_{n,d}^{\mathsf{affC}} = \mathsf{End}_{\mathbf{H}_d^{\mathsf{affC}}}(\mathbb{V}^{\otimes d})$$

Tridiagonal generators

• Fix n = 2r + 1. The bases for $\mathbf{S}_{n,d}^{\mathsf{affC}}$ are parametrized by

$$\Xi_{n,d} := \{ A \in \mathsf{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{N}) \mid (1) - (3) \},$$

- (1) (periodicity) $a_{ij} = a_{i+n,j+n}$
- (2) (centro-symmetry) $a_{-i,-j} = a_{ij}$
- (3) (size) $\sum a_{ij} = d$ over any "half period"
- A tridiagonal generator ↔



Multiplication formula (affine C)

Theorem B

Let B be tridiagonal. Then

$$[B] * [A] = \sum_{\substack{T^{\theta} \leq A \\ (+\text{cond.})}} \sum_{\substack{S \leq T \\ (+\text{cond.})}} (\text{coeff}) \left[A - (T - S)^{\theta} + (\widehat{T - S})^{\theta} \right],$$

Affine type C

where $A - (T - S)^{\theta} + (\widehat{T - S})^{\theta} \sim \text{shifting } T - S \text{ up by } 1 \text{ row, and its counterpart}$ down by 1 row in A.

• The tridiagonal generators form a generating set of $\mathbf{S}_{n,d}^{\mathsf{affC}}$

Monomial basis (affine C)

Theorem C

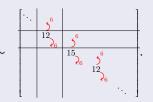
The Schur algebra $\mathbf{S}_{n,d}^{\mathsf{affC}}$ admits both monomial and canonical bases.

Example

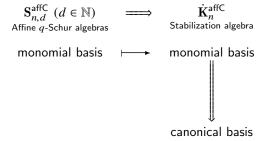
Monomial basis element m_A is obtained in the following sense

$$A = \begin{bmatrix} \ddots & 4 & 5 & & & \\ 1 & 0 & 1 & 2 & & & \\ 5 & 4 & 3 & 4 & 5 & & \\ 2 & 1 & 0 & 1 & 2 & & \\ & & 5 & 4 & \ddots & & \end{bmatrix} \quad \boldsymbol{\sim}$$

	·	6	5 ⁵			
Ī	$\binom{6}{2_5}$	0	6	52		
Ī		$\binom{6}{2}_2$	3	6	5 ⁵	
			$^{6}_{\downarrow_{5}}$	0	6	52
				6	٠.	



Stabilization algebra (affine C)



Theorem D

We have an algebra $\dot{\mathbf{K}}_n^{\mathrm{affC}}$ from stabilization on $\mathbf{S}_{n.d}^{\mathrm{affC}}$. Moreover, $\dot{\mathbf{K}}_n^{\mathrm{affC}}$ admits both monomial and canonical bases.

Quantum symmetric pairs

• A quantum symmetric pair (U, B) (cf. Letzter('02), Kolb('14)) is a q-deformation of the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\theta})$ where

g: Lie algebra

 θ : involution on \mathfrak{q}

 \mathfrak{g}^{θ} : fixed-point subalgebra

ullet One essential property of quantum symmetric pairs is that ${f B}$ is a coideal subalgebra of U, i.e., the comultiplication $\Delta: U \to U \otimes U$ satisfies that

$$\Delta(\mathbf{B}) \subset \mathbf{B} \otimes \mathbf{U}$$

Quantum symmetric pairs

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Non-idempotented quantum algebras

By taking certain infinite sum, one can obtain the non-idempotented stabilization algebras:

$$\dot{\mathbf{K}}_n \Rightarrow \mathbf{K}_n \simeq \mathbf{U}(\mathfrak{gl}_n)$$
 (finA)

$$\dot{\mathbf{K}}_n^{\mathrm{affA}} \Rightarrow \mathbf{K}_n^{\mathrm{affA}} \simeq \mathbf{U}(\widehat{\mathfrak{gl}}_n)$$
 (affA)

$$\dot{\mathbf{K}}_n^{\mathsf{finBC}} \Rightarrow \mathbf{K}_n^{\mathsf{finBC}} \simeq \mathbf{i}\mathbf{U}(\mathfrak{gl}_n)$$
 (finBC)

 $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{i}\mathbf{U}(\mathfrak{gl}_n))$ forms a quantum symmetric pair

$$\dot{\mathbf{K}}_n^{\mathsf{affC}} \Rightarrow \mathbf{K}_n^{\mathsf{affC}}$$
 (affC)

Proposition

The pair $(U(\widehat{\mathfrak{gl}}_n), K_n^{\mathsf{affC}})$ forms a quantum symmetric pair.

Thank you for your attention