Day 1

Representation theory

Goal (A) PBW thm
(B) Ultimate problem of repn theory
(C) Classification thm of simple Lie algebras / $\mathbb{C}$

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§1 What is a representation

(1) A representation of a group G over a field K is a grp hom

$$\rho: G \to GL(V) \text{ for some } K\text{-vec space } V$$

$$\left(\text{i.e. } \rho(gh) = \rho(g)\rho(h) \ \forall g, h \in G\right)$$

(2) A (unital) associative algebra A/F is a F-vector space with an ….. assoc. bilinear multiplication

- A repn of a unital assoc. alg A/K is an alg hom.

$$\rho: A \to \mathrm{End}(V) \text{ for some } K\text{-vec space } V$$

$$\left(\text{i.e. } \rho(x+y) = \rho(x)+\rho(y),\ \rho(xy) = \rho(x)\rho(y),\ \forall x, y \in A\right)$$

(3) A Lie algera $\mathfrak{g}/F$ is a F-vec space with a Lie bracket s.t. ~~~

- A repn of a Lie alg $\mathfrak{g}/K$ is a Lie alg hom.

$$\rho: \mathfrak{g} \to GL(V)$$

(4) A repn of a Lie group, algebraic grp, Hecke algebra, Quantum grp, Lie superalg, ……

Answer:

A representation of $\square$ X/K is a $\square$-morphism

$$\rho: X \to \mathrm{End}(V) \text{ for some } K\text{-vector space } V$$

if it makes sense

Question:

1. Why in (1), it's $GL(V)$ instead of $\mathrm{End}(V)$?
2. Does it make sense to talk about repn of a field?
3. ——————————————— ring?

§2 Module

(1) A G-module is a K-vector space V w/ G-action

$$gv = \rho(g)(v) \text{ for some repn } \rho \text{ of } G.$$

(2) An A-module is a K-vector space V w/ A-action

$$xv = \rho(x)(v) \text{ ——————— of } A$$

(3) A Lie alg module is a K-vec sp V w/ $\mathfrak{g}$-action

$$gv = \rho(g)(v) \text{ ——————— } \mathfrak{g}$$

$\boxed{\text{Fact}}$

$$\left\{\begin{matrix}\text{representation} \\ \text{of } \square\end{matrix}\right\} \xrightarrow{\sim} \{\square\text{-modules}\}$$

- A <u>submodule</u> W of V is an invariant subspace $W \subseteq V$
  (i.e. $GW \subseteq W$, $AW \subseteq W$, $\mathfrak{g}W \subseteq W$)

- A module is <u>simple</u> if it has no submodules other than 0 and itself.

<u>Ultimate problem of repn theory</u>:

Construct & classify simple modules.

Examples

(1) Repn of sym grp:
Simple modules$^?$ = Specht modules $\longleftrightarrow$ partitions

(2) Repn of simple Lie alg/$\mathbb{C}$
simple modules = ?, parametrized by ?

§3 Lie algebras

- A <u>Lie alg</u> / K is a K-vector space $\mathfrak{g}$ with a bilinear map (called <u>Lie bracket</u>)

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow K \qquad \text{s.t.}$$
$$(x, y) \mapsto [x,y]$$

(L1) $[x,x] = 0 \quad \forall x \in \mathfrak{g}$

(L2) Jacobi identity.

Examples

(1) <u>general linear Lie alg.</u>

$\mathfrak{gl}_n(K) = \{n \times n \text{ mat.} / k\}$ with $[A,B] = AB - BA$

(2) <u>special linear Lie alg</u>

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{gl}_n(K) \mid tr(A) = 0\}$$

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\underset{e}{\underset{\uparrow}{\begin{bmatrix} & 1 \\ 0 & \end{bmatrix}}} \oplus \mathbb{C}\underset{f}{\underset{\uparrow}{\begin{bmatrix} & 0 \\ 1 & \end{bmatrix}}} \oplus \mathbb{C}\underset{h}{\underset{\nwarrow}{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}}$$

$[e,f] = ?$ $\qquad [h,h] = ?$

$[h,e] = ?$

$[h,f] = ?$

- An <u>ideal</u> I of a Lie alg $\mathfrak{g}$ is a subspace s.t. $[\mathfrak{g}, I] \subseteq I$.
- A Lie alg $\mathfrak{g}$ is <u>simple</u> if it has no ideals other than 0 and $\mathfrak{g}$.

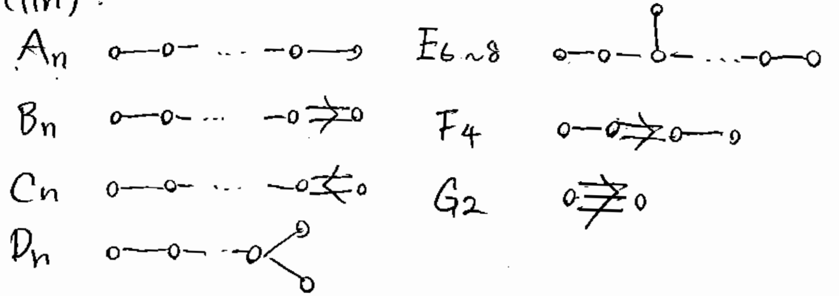
<u>Goal</u>: Understand simple Lie alg / $\mathbb{C}$

Thm

(1) $\{\text{simple Lie alg}/\mathbb{C}\} \xrightarrow{\sim} \{\text{connected Dynkin diagrams}\}$

In particular, $\mathfrak{g}$-action is clear via ↗

List(fin):



(Outline of the proof)

$$\begin{tikzcd}
\left\{\begin{matrix}(\text{simple})\\ \text{Lie alg}\end{matrix}\right\} \arrow[r, "①"] & \left\{\begin{matrix}(\text{Irred.})\\ \text{root sys}\end{matrix}\right\} \arrow[r, "②"] & \left\{\begin{matrix}(\text{connected})\\ \text{Dynkin diag}\end{matrix}\right\} \arrow[ll, bend left, "③"]
\end{tikzcd}$$

① : Given $\mathfrak{g}$: simple ~~Lie~~ Lie alg $/\mathbb{C}$

$\rightsquigarrow$ Cartan decomp. $\mathfrak{g} = \underset{\substack{\uparrow \\ \text{Cartan} \\ \text{subalg}}}{\mathfrak{h}} \oplus \left(\bigoplus_{\alpha \in \underset{\substack{\uparrow \\ \text{root} \\ \text{sys}}}{\Phi}} \underset{\substack{\uparrow \\ \text{root space}}}{\mathfrak{g}_\alpha} \right)$

② : Given Φ : root sys

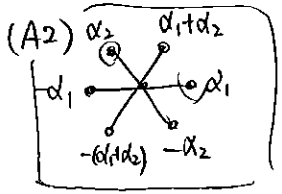
$\rightsquigarrow \Phi \supseteq \Delta = \{\alpha_i\}_{i\in I}$ $\leftarrow$ simple roots

$\rightsquigarrow$ Cartan matrix $(a_{ij})_{ij}$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$, $\overset{\substack{\text{coroot} \\ \downarrow}}{\alpha^\vee} = \frac{2\alpha}{(\alpha,\alpha)}$

$\rightsquigarrow$

$\boxed{\text{Example}}$ $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ is a root sys

(A2)

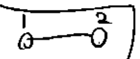


with $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2$

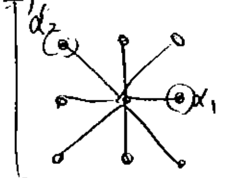
$\alpha_1^\vee = \alpha_1$ $\rightsquigarrow$ $(\alpha_1^\vee, \alpha_1) = 2$, $(\alpha_1^\vee, \alpha_2) = -1$

$\alpha_2^\vee = \alpha_2$ $\quad (\alpha_2^\vee, \alpha_2) = 2$ $\quad (\alpha_2^\vee, \alpha_1) = -1$

$\rightsquigarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \rightsquigarrow$



(C2)



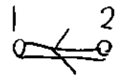
$(\alpha_1,\alpha_1) = 2$

$(\alpha_2,\alpha_2) = 4$

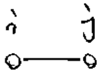
$\alpha_1^\vee = \alpha_1$ $\rightsquigarrow$ $(\alpha_1^\vee, \alpha_1) = 2$, $(\alpha_1^\vee, \alpha_2) = -2$

$\alpha_2^\vee = \frac{\alpha_2}{2}$ $\quad (\alpha_2^\vee, \alpha_2) = 2$ $\quad (\alpha_2^\vee, \alpha_1) = -1$

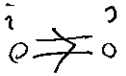
$\rightsquigarrow \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \rightsquigarrow$



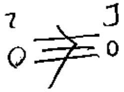
$\rightsquigarrow$ Dynkin diag (I, E) where



$\in E$ if $a_{ij} = a_{ji} = -1$



$\in E$ if $\begin{cases} a_{ij} = -1 \\ a_{ji} = -2 \end{cases}$



$\in E$ if $\begin{cases} a_{ij} = -1 \\ a_{ji} = -3 \end{cases}$

③ Given a Dynkin diag $\rightsquigarrow$ cartan matrix A

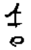
$\rightsquigarrow$ simple Lie alg $\mathfrak{g}(A) = \langle e_i, f_i, h_i \mid i \in I \rangle / \sim$

Chevalley relations
- $[h_i, h_j] = 0$,
- $[e_i, f_j] = \begin{cases} h_i & \text{if } i = j \\ 0 & \text{otw,} \end{cases}$
- $[h_i, e_j] = a_{ij} e_j$;
  $[h_i, f_j] = -a_{ij} f_j$;

Serre relations
- $\underbrace{[e_i[e_i[\cdots[e_i}_{1-a_{ij}}, e_j]\cdots] = 0 \quad i \neq j$
- $\underbrace{[f_i[f_i[\cdots[f_i}_{1-a_{ij}}, f_j]\cdots] = 0 \quad i \neq j$

$\boxed{\text{Example}}$

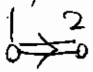
(A1)



$\rightarrow A = [2] \rightsquigarrow \mathfrak{g}(A) = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ s.t.

- $[h, h] = 0$
- $[e, f] = h$
- $[h, e] = 2e$
- $[h, f] = -2f$

(B2)



$\rightsquigarrow A = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$

$\rightsquigarrow \mathfrak{g}(A) = \mathbb{C}e_1 \oplus \mathbb{C}f_1 \oplus \mathbb{C}h_1 \oplus \mathbb{C}[e_1,e_2] \oplus \mathbb{C}[e_2,e_1] \oplus \mathbb{C}[e_2[e_2,e_1]]$
$\mathbb{C}e_2 \oplus \mathbb{C}f_2 \oplus \mathbb{C}h_2 \oplus \mathbb{C}$ ~~$[f_1,f_2]$~~ $\oplus$ —— $\oplus$ ——

- Chevalley relations
- $[e_1[e_1e_2]]=0$; ——
  $[e_2[e_2[e_2e_1]]]=0$ ——

§4 PBW thm

- $U(\mathfrak{g}) = \mathrm{Span}_{\mathbb{C}}\{x_1 \cdots x_N \mid x_i \in \mathfrak{g}, N\} \Big/ \left(\begin{matrix} xy-yx \sim [x,y] \\ \forall x,y \in \mathfrak{g}\end{matrix}\right)$
  is the univ. env. alg of $\mathfrak{g}$ ^ (it's an assoc. alg, not a Lie alg.)

Example

(A1) $\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$

- $2fef + 3hfe \in U(\mathfrak{g})$
- $100[h,e] \in U(\mathfrak{g})$
  $\quad \|$
  $100he - 100eh$

Claim: $U(\mathfrak{g})$ has a basis $\{f^a h^b e^c \mid a,b,c \in \mathbb{Z}_{\geq 0}\}$

- $hfe = ([h,f] + fh)e = -2fe + fhe$
- $fef = f([e,f]+fe) = fh + f^2e$
- $100[h,e] = 200e$.

Thm (PBW)

Fix an ordered basis $\{x_1 < x_2 < \dots < x_n\}$ of $\mathfrak{g}$

Then $\underbrace{\{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid m_i \in \mathbb{Z}_{\geq 0}\} \text{ is a basis of } U(\mathfrak{g})}_{\text{PBW}}$

§5 Weyl group

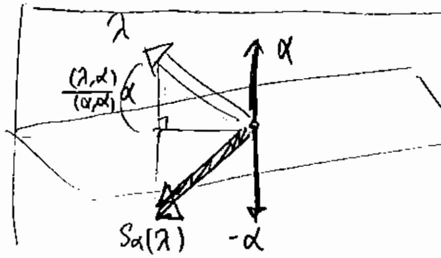
- $\underset{\substack{\uparrow \\ \text{CSA}}}{\mathfrak{g} = \mathfrak{h}} \oplus \sum_{\underset{\substack{\uparrow \\ \text{root sys}}}{\alpha \in \Phi}} \mathfrak{g}_\alpha$

Fact

(1) $\Phi \subseteq \mathfrak{h}^*$ (i.e. each root is a map $\alpha : \mathfrak{h} \to \mathbb{C}$)

(2) For each $\alpha \in \Phi$, we define a reflection

$$S_\alpha : \mathfrak{h}^* \longrightarrow \mathfrak{h}^* \qquad \text{where } \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$
$$\lambda \longmapsto \lambda - (\lambda, \alpha^\vee)\alpha$$



$\mathcal{H}_\alpha = \{\lambda \in \mathfrak{h}^* \mid (\alpha,\lambda)=0\}$

(3) $\Phi \supseteq \Delta = \{\alpha_i\}_{i\in I}$, simple sys s.t.

if $\beta \in \Phi$ then $\beta = \sum \underbrace{C_i}\alpha_i$ all pos or all neg.

$\rightsquigarrow \Phi = \Phi^+ \sqcup \Phi^-$ wrt Δ

(4) $W := \langle S_\alpha \mid \alpha \in \Phi \rangle$ is the Weyl group

$\rightsquigarrow$ length fcn $\ell(w) = |\Phi^+ \cap w^{-1}\Phi^-|$

(5) For $\alpha \in \Delta$, $S_\alpha(\alpha) = -\alpha$ and S_α permutes $\Phi^+ \setminus \{\pm\alpha\}$

(pf) Assume $\alpha = \alpha_j$; $\beta \neq \pm\alpha_j \rightsquigarrow \beta = \sum c_i \alpha_i$ where $c_i \neq 0$ for some $i \neq j$

$\rightsquigarrow S_\alpha(\beta) = \sum_i C_i S_{\alpha_j}(\alpha_i) = \sum C_i (\alpha_i - \underset{a_{ji} \leftarrow \text{ neg except } i=j}{(\alpha_i, \alpha_j^\vee)}\alpha_j)$

$\rightsquigarrow S_\alpha(\beta)$ has sme coeff pos $\rightsquigarrow S_\alpha(\beta) \in \Phi^+ \setminus \{\pm\alpha\}$.

Day 2

Character formulas

Goal:

(A) "Simple" modules $L(\lambda)$

(i) construction via Verma modules

(ii) Weyl character formula ("if $\dim L(\lambda) < \infty$")

(iii) Kazhdan-Lusztig conj (in general)

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§1 Verma modules

- $\lambda \in \mathfrak{h}^*$, define Verma mod. $M(\lambda) = \mathcal{U}(\mathfrak{g})/I(\lambda)$, where

$$I(\lambda) = \mathcal{U}(\mathfrak{g})\mathfrak{n} + \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})\left(h - \lambda(h)\mathbf{1}\right) \trianglelefteq \mathcal{U}(\mathfrak{g})$$

In words, $\exists$ highest weight vector $v_\lambda^+ = \mathbf{1} + I(\lambda) \in M(\lambda)$ s.t.

(a) $M(\lambda) = \mathcal{U}(\mathfrak{n}^-)v_\lambda^+ \left(= \mathrm{Span}_{\mathbb{C}}\{ f_{i_1}^{m_1} \cdots f_{i_N}^{m_N} v_\lambda^+ \}\right)$

(b) $h \cdot v_\lambda^+ = \lambda(h) v_\lambda^+ \quad \forall \ h \in \mathfrak{h}$ (i.e. $v_\lambda^+ \in M(\lambda)_\lambda$)

Example

(A$_1$) $\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$, $\mathfrak{h} = \mathbb{C}h$

$\mathfrak{h}^* = \{\lambda : \mathfrak{h} \to \mathbb{C}\} \cong \mathbb{C}$ (identifying λ with $\lambda(h)$)

$\Delta = \{\pm\alpha\}$ where $\alpha \mapsto 2$ ($\because [h,e] = \alpha(h)e$)

$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha = \lambda - \lambda(h)\alpha = \lambda - 2\lambda = -\lambda$

$M(\lambda) = \mathrm{Span}_{\mathbb{C}}\{ v_\lambda^+, f v_\lambda^+, f^2 v_\lambda^+, \dots \}$

Q: $\mathfrak{g}$-action on $M(\lambda)$:

$h(v_\lambda^+) = \lambda v_\lambda^+$

$h(f v_\lambda^+) = ([h,f] + fh)v_\lambda^+ = -2f v_\lambda^+ + f\lambda v_\lambda^+ = (\lambda-2) f v_\lambda^+$

$\vdots$

$\boxed{h(f^k v_\lambda^+) = (\lambda - 2k) f^k v_\lambda^+}$

$e v_\lambda^+ = 0$

$ef v_\lambda^+ = ([e,f] + fe) v_\lambda^+ = \lambda v_\lambda^+$

$ef^2 v_\lambda^+ = ([e,f] + fe) f v_\lambda^+ = hf v_\lambda^+ + f([e,f] + fe) v_\lambda^+ = (\lambda-2) f v_\lambda^+ + \lambda f v_\lambda^+ = 2(\lambda-1) f v_\lambda^+$

$\vdots$

$\boxed{ef^k v_\lambda^+ = k(\lambda - (k-1)) f^{k-1} v_\lambda^+}$

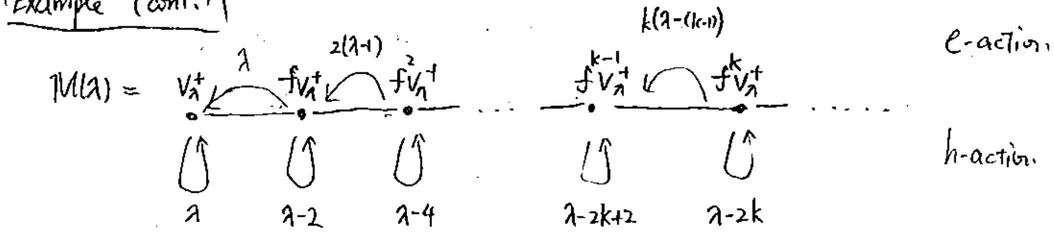
Fact

(1) $M(\lambda)$ has a uniq maximal submodule $N(\lambda)$,

$\rightsquigarrow$ a uniq simple quotient $L(\lambda) := M(\lambda)/N(\lambda)$

(2) If L is simple, then $L \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$

Example (cont.)



Case 1: $\lambda = k-1$ for some $k \in \mathbb{Z}_{\geq 0}$ (i.e. $\lambda \in \mathbb{Z}_{\geq -1}$)

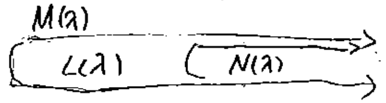
$\rightsquigarrow ef^k v_\lambda^+ = 0$

$\rightsquigarrow \mathcal{U}(\mathfrak{g}) f^k v_\lambda^+ = \mathcal{U}(\mathfrak{n}^-) f^k v_\lambda^+ = \mathrm{Span}_{\mathbb{C}}\{ f^k v_\lambda^+, f^{k+1} v_\lambda^+, \dots \}$ is a proper submod.

$= N(\lambda) \cong M(-2\lambda-2)$

$\rightsquigarrow L(\lambda) = M(\lambda)/N(\lambda) = \mathrm{Span}_{\mathbb{C}}\{ \overline{v_\lambda^+}, \overline{f v_\lambda^+}, \dots, \overline{f^k v_\lambda^+} \}$

e.g.



Case 2: $\lambda \notin \mathbb{Z}_{\geq -1}$

$\rightsquigarrow N(\lambda) = 0, \ M(\lambda) \cong L(\lambda)$

§ 2 formal character

- Given a weight module M $\left(\text{i.e. } M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \text{, where } M_\lambda = \{ m \in M \mid h \cdot m = \lambda(h) m \ \forall h \in \mathfrak{h} \} \right)$

$\rightsquigarrow$ formal character $\mathrm{ch} M = \sum_{\lambda \in \mathfrak{h}^*} (\dim M_\lambda)\, e(\lambda)$ ↖ formal symbol.

Example

(A1) $M(\lambda) = \mathrm{Span}_{\mathbb{C}} \{ v_\lambda^+ , f v_\lambda^+ , \dots \}$

$\mathrm{ch} M(\lambda) = e(\lambda) + e(\lambda-2) + e(\lambda-4) + \cdots$

$\mathrm{ch} L(0) = e(0) \qquad = \mathrm{ch} M(0) - \mathrm{ch} M(2)$

$\mathrm{ch} L(1) = e(1) + e(-1) \qquad = \mathrm{ch} M(1) - \mathrm{ch} M(-3)$

$\mathrm{ch} L(2) = e(2) + e(0) + e(-2) \qquad = \mathrm{ch} M(2) - \mathrm{ch} M(-4)$

$\mathrm{ch} L(-2) = ?$

- Convolution $f * g$ is given by $(f*g)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu) g(\nu) \quad \forall \lambda \in \mathfrak{h}^*$
- Kostant fcn $P(\mu) = \# \{ (c_\alpha)_{\alpha \in \Phi^+} \mid c_\alpha \in \mathbb{Z}_{\geq 0} , \ \Sigma c_\alpha \alpha = \mu \}$

Example

(A1) $\alpha \mapsto 2$, $P(\mu) = \begin{cases} 1 & \text{if } \mu = 0, 2, 4, \dots \\ 0 & \text{otw} \end{cases}$

$(P * e)(\lambda) = \sum_{\mu+\nu=\lambda} P(\mu) e(\nu) = e(\lambda) + e(\lambda-2) + e(\lambda-4) + \cdots$

$= \mathrm{ch} M(\lambda)$

e^{μ}

$\forall \lambda \in \mathfrak{h}^*$, view $e(\lambda)$ as the function s.t. $e(\lambda) * e(\mu) = e(\lambda+\mu)$

$\left(\text{i.e. } e(\lambda)(\mu) = \begin{cases} 1 & \text{if } \lambda + \mu = 0 \\ 0 & \text{otw} \end{cases} \right)$

Fact

(1) $P = \prod_{\alpha \in \Phi^+} \frac{1}{1 - e(-\alpha)}$ and $P * e(\lambda) = \mathrm{ch} M(\lambda)$

(2) $e(\lambda)$ commutes with any fcn f.

$W \curvearrowright e(\lambda)$ by $w e(\lambda) = e(w(\lambda))$

In particular, $s_\beta \left(e(\tfrac{\beta}{2}) - e(\tfrac{-\beta}{2}) \right) = - \left(e(\tfrac{\beta}{2}) - e(\tfrac{-\beta}{2}) \right)$

- Let Weyl vector $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$
- Let $q = e(\rho) * \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))$

$= \prod_{\alpha \in \Phi^+} \left(e(\tfrac{\alpha}{2}) - e(\tfrac{-\alpha}{2}) \right)$

Fact

(3) $s_\beta(q) = -q \quad \forall \beta \in \Delta$

(4) $q * p = e(\rho) \xRightarrow{(1)} q * \mathrm{ch} M(\lambda) = e(\lambda+\rho)$

(pf)

(3) $s_\beta(q) = s_\beta \left(e(\tfrac{\beta}{2}) - e(\tfrac{-\beta}{2}) \right) \prod_{\alpha \in \Phi^+ \setminus \{\beta\}} \left(e(\tfrac{\alpha}{2}) - e(\tfrac{-\alpha}{2}) \right) = -q$

(4) $q * p = e(\rho) * \left(\prod_{\alpha \in \Phi^+} (1 - e(-\alpha)) \right) * \left(\prod_{\alpha \in \Phi^+} \frac{1}{1 - e(-\alpha)} \right) = e(\rho)$

Thm [Weyl char. formula]

If $\lambda \in \Lambda^+ := \{ \lambda \in \mathfrak{h}^* \mid (\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta \}$ ← dom. int. wt

then $\mathrm{ch} L(\lambda) = \dfrac{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)}$ where $w \cdot \lambda = w(\lambda+\rho) - \rho$ ← dot-action

(pf) Assume we know $\lambda \in \Lambda^+ \Rightarrow \mathrm{ch} L(\lambda) = \sum_{w \in W} c_w \, \mathrm{ch} M(w \cdot \lambda)$ for $\begin{cases} c_w \in \mathbb{Z}_{\geq 0} \\ c_{\mathbb{1}} = 1 \end{cases}$

We show first $q * \mathrm{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e(w(\lambda+\rho))$

- Now $q * \mathrm{ch} L(\lambda) = \Sigma c_w \, q * \mathrm{ch} M(w(\lambda+\rho) - \rho) \overset{(4)}{=} \Sigma c_w \, e(w(\lambda+\rho))$
- OOH, $s_\alpha(q * \mathrm{ch} L(\lambda)) = \Sigma c_w e(s_\alpha w(\lambda+\rho)) = \Sigma c_{s_\alpha w} e(w(\lambda+\rho))$

$\quad \| $

$-q * \mathrm{ch} L(\lambda) = \Sigma (-c_w) e(w(\lambda+\rho)) \longrightarrow c_{s_\alpha w} = -c_w$

$\xRightarrow{\text{ind}} c_w = (-1)^{\ell(w)}$

In particular,

$q = q * e(0) = q * \mathrm{ch} L(0) = \sum_{w \in W} (-1)^{\ell(w)} e(w(\rho)) \rightsquigarrow \divideontimes$

Cor [Kostant dim. formula]

If $\mu \le \lambda \in \Lambda^+$ then

$$\dim L(\lambda)_\mu = \sum_{w\in W} (-1)^{\ell(w)} p(w\cdot\lambda-\mu)$$

(Pf)
$$\dim L(\lambda)_\mu = \sum_{w\in W} (-1)^{\ell(w)} \dim M(w\cdot\lambda)_\mu$$

where $\dim M(w\cdot\lambda)_\mu = [e(\mu)]\ p * e(w\cdot\lambda) = p(w\cdot\lambda-\mu)$ ※

Thm [Weyl's dim. formula]

If $\lambda \in \Lambda^+$ then

$$\dim L(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha^\vee)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha^\vee)}$$

Lemma
- $(w\mu, w\nu) = (\mu,\nu)\ \forall w\in W$
- s_α permutes $\Phi^+\setminus\{\alpha\}$
- $\ell(w) = |\Phi^+ \cap w\Phi^-|$

(Pf) $\forall\ \alpha\in\Phi^+$, define $\partial_\alpha : e(\mu) \mapsto (\mu,\alpha^\vee)\,e(\mu) \rightsquigarrow \partial = \prod_{\alpha\in\Phi^+}\partial_\alpha$,

(i) $\boxed{\partial\, e(\alpha+\beta) = (\partial e(\alpha)) * e(\beta) + e(\alpha) * \partial e(\beta)}$

(ii) Define $\nu : \sum c_\mu e(\mu) \mapsto \sum c_\mu$ so that $\nu(\mathrm{ch}\,L(\lambda)) = \dim L(\lambda)$

Then $\boxed{\nu\Big(\partial \sum_{w\in W}(-1)^{\ell(w)} e(w\mu)\Big) = |W| \prod_{\alpha\in\Phi^+}(\mu,\alpha^\vee)} \leftarrow \nu(\partial\, q * \mathrm{ch}\,L(\mu-\rho)) \rightsquigarrow \boxed{\nu(\partial q) = |W|\prod(\rho,\alpha^\vee)}$

(pf of ii)

$$\mathrm{LHS} = \sum_{w\in W}(-1)^{\ell(w)} \prod_{\alpha\in\Phi^+}(w\mu,\alpha^\vee)$$

Note that $(\cdot,\cdot)$ is W-invariant

$$\rightsquigarrow \prod_{\alpha\in\Phi^+}(w\mu,\alpha^\vee) = \prod_{\alpha\in\Phi^+}(\mu, w^{-1}\alpha^\vee) = \Big(\prod_{\substack{\alpha\in\Phi^+\\ w^{-1}\alpha\in\Phi^+}}(\mu,w^{-1}\alpha^\vee)\Big)\Big(\prod_{\substack{\alpha\in\Phi^+\\ w^{-1}\alpha\in\Phi^-}}(\mu,w^{-1}\alpha^\vee)\Big)$$

$$= \Big(\prod_{\substack{w\beta\in\Phi^+\\ \beta\in\Phi^+}}(\mu,\beta^\vee)\Big)\Big(\prod_{\substack{w\beta\in\Phi^+\\ \beta\in\Phi^-}}(\mu,\beta^\vee)\Big)$$

Note that $\prod_{\substack{w\beta\in\Phi^+\\ \beta\in\Phi^-}}(\mu,\beta^\vee) = \prod_{\substack{w\alpha\in\Phi^-\\ \alpha\in\Phi^+}}(\mu,-\alpha^\vee) = \underset{\boxed{\ell(w)\text{ many!}}}{\prod_{\alpha\in\Phi^+\cap w^{-1}\Phi^-}}(\mu,-\alpha^\vee) = (-1)^{\ell(w)}\prod_{\alpha\in\Phi^+\cap w^{-1}\Phi^-}(\mu,\alpha^\vee)$

$$\rightsquigarrow \mathrm{LHS} = \sum_{w\in W}\prod_{\alpha\in\Phi^+}(\mu,\alpha^\vee).$$

(iii') $\nu(q) = 0$

(pf of iii)

Note that $\nu(e(\alpha)-1) = 0\ \forall\ \alpha\in\Delta$, so

$$\nu(q) = \nu(e(-\rho))\prod \nu(e(\alpha)-1) = 0$$

$$\nu(\partial(q * \mathrm{ch}\,L(\lambda))) \overset{(ii)}{=\!=} |W|\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha^\vee)$$

$\|$ (i)

$$\underbrace{\nu(\partial q)}_{|W|\prod_{\alpha\in\Phi^+}(\rho,\alpha^\vee)}\ \underbrace{\nu(\mathrm{ch}\,L(\lambda))}_{\dim L(\lambda)} + \underbrace{\nu(q)}_{0\text{ by (iii')}}\ \nu(\partial\,\mathrm{ch}\,L(\lambda)) \longrightarrow \dim L(\lambda) = \frac{\prod(\lambda+\rho,\alpha^\vee)}{\prod(\rho,\alpha^\vee)}$$ ※

Remark

1. Weyl char formula only computes $\mathrm{ch}\,L(\lambda)$ for $\lambda\in\Lambda^+$, Weyl's dim formula suggests $\dim L(\lambda) < \infty$ in this case.

2. If $\lambda \notin \Lambda^+$, we have

Conj [KL] [thm of BB, B-K]

$$\mathrm{ch}\,L(w\cdot 0) = \sum_{x\le w}(-1)^{\ell(x)-\ell(w)} P_{w_0w,w_0x}(1)\,\mathrm{ch}\,M(x\cdot 0)$$

or

$[M(w\cdot 0), L(x\cdot 0)] = P_{x,w}(1)$ ← KL polyn

Thm [Soergel]

If $\lambda \in \mathfrak{h}^*\setminus\Lambda$ then

(a) $\exists\ \lambda^\natural \in \Lambda^\natural$ for some other Lie alg $\mathfrak{g}^\natural$

(b) $[M(\lambda) : L(\mu)] = [M(\lambda^\natural) : L(\mu^\natural)]$.

Thm [Jantzen]

If $\lambda\in\Lambda$ then $[M(w\cdot\lambda) : L(x\cdot\lambda)] = [M(w\cdot 0) : L(x\cdot 0)]$

§3 Multiplicity Questions

Recall comp. series for finite grp G :

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = 1$$

s.t. G_i/G_{i+1} is simple

Q: Does $M \in \mathcal{U}(\mathfrak{g})$ has a finite comp. series ?

(i.e. $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$
s.t. M_i/M_{i+1} is simple ($\cong L(\mu_i)$ for some μ_i))

A: No, in general.

We'll see:

- Verma modules have finite comp. series.
- The multiplicity $[M(\lambda):L(\mu)]$ of $L(\mu)$ occurs in $M(\lambda)$ is well-defined
- Finding $\mathrm{ch}\,L(\lambda) = \Sigma\, ?\, \mathrm{ch}\,M(\mu) \Leftrightarrow$ finding $[M(\lambda):L(\mu)]$

§4 Central characters

- Z = center of $\mathcal{U}(\mathfrak{g})$

Fact

(1) $\forall z \in Z$, $m \in M(\lambda)$, $zm = \chi_\lambda(z)\, m$
for some central character $\chi_\lambda : Z \to \mathbb{C}$

(2) If $[M(\lambda):L(\mu)] \neq 0$ then $\chi_\mu = \chi_\lambda$ and $\mu \leq \lambda$

(3) [Harish-Chandra] $\chi_\mu = \chi_\lambda \Leftrightarrow \mu \in W \cdot \lambda$

(pf)

(1) $\forall h \in \mathfrak{h}$, $m \in M(\lambda)_\lambda$

$$hzm = zhm = z\lambda(h)m = \lambda(h)zm \rightsquigarrow zm \in M(\lambda)_\lambda$$

$\because \dim M(\lambda)_\lambda = 1$, $zm \in \mathbb{C}m \rightsquigarrow zm = \chi_\lambda(z)\, m$

$\forall h \in \mathfrak{h}$, $m \in M(\lambda)$

$$m = \underline{f}^{c} v_\lambda^+ \rightsquigarrow zm = z\underline{f}^{c} v_\lambda^+ = \underline{f}^{c} z v_\lambda^+ = \chi_\lambda(z)\, m.$$

(2) ✓

(3) HARD! (study $Z \xrightarrow{\sim} S(\mathfrak{h})^W$)

Example

(A1) The casimir operator $2\Omega = h^2 + 2h + 4fe \in Z$

$$\begin{aligned} e(h^2+2h+4fe) &= (he-2e)h + 2eh + 4(h+fe)e \\ &= h(he-2e) + 4he + 4fe^2 \\ &= (h^2+2h+4fe)e \end{aligned}$$

$$2\Omega v_\lambda^+ = (\lambda^2+2\lambda) v_\lambda^+$$

- $[M(\lambda):L(\mu)] \neq 0 \Leftrightarrow \lambda^2 + 2\lambda = \mu^2 + 2\mu \Leftrightarrow \overset{(\lambda+\mu+2)(\lambda-\mu)}{\overset{\shortparallel}{\lambda^2-\mu^2+2(\lambda-\mu)}} = 0$

$\Leftrightarrow \mu = \lambda$ or $\mu = -\lambda-2$

$\Leftrightarrow \mu \in W \cdot \lambda$

Lemma $M(\lambda)$ has fin. comp. series

(pf) Let $V = \sum_{w \in W} M(\lambda)_{w \cdot \lambda}$ be a f.d. vec sp.
if $M(\lambda) = M_0 \supsetneq M_1 \supsetneq \cdots$ non-stop, so is $V \cap M_i \supsetneq \cdots$
Pick $v_\mu^+ \in M_i/M_{i+1}$, $\chi_\lambda = \chi_\mu \Rightarrow \mu = w \cdot \lambda$ for some $w \in W$
$\Rightarrow M_i \cap V \neq 0$ and $\dim(M_i \cap V) > \dim(M_{i+1} \cap V) \rightarrow \times$

Day 3

Strong linkage principle

Goal: (A) Strong linkage principle
(⇒) Verma thm
(⇐) BGG thm
↑ Jantzen filtration

(B) Category $\mathcal{O}$
& translation functor

Recall:

- Study $L(\lambda)$
  ↑ study $\mathrm{ch}\,L(\lambda)$
  case 1: $\lambda \in \Lambda^+$ ⇝ solved by Weyl char formula.
  case 2: $\lambda \notin \Lambda^+ \xrightarrow[\text{[Soergel]}]{} \lambda \in \Lambda \xrightarrow[\text{[Jantzen]}]{} \lambda \in W\cdot 0$
  ↑ solved by KL theory.
  ↑ based on strong LP
  ↑ based on central char
  ↑ based on HC thm

§1 Verma thm

- We say $\mu \le \lambda$ if $\lambda - \mu = \sum_{\alpha \in \Phi^+} c_\alpha \alpha$ for $c_\alpha \in \mathbb{Z}_{\ge 0}$
- We say $\mu \uparrow \lambda$ if $\mu = s_\alpha \cdot \lambda \le \lambda$ for some $\alpha \in \Phi^+$
- We say μ is strongly linked to λ (write $\mu \uparrow\uparrow \lambda$) if

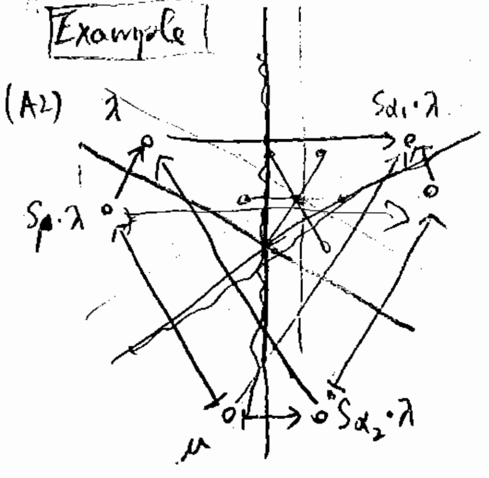
$$\mu = \lambda \quad \text{or} \quad \mu = \mu_0 \uparrow \mu_1 \uparrow \mu_2 \uparrow \cdots \uparrow \mu_n = \lambda$$

Thm SLP

$$\begin{tikzcd}{[M(\lambda) : L(\mu)] \neq 0} \arrow[r, shift left, "\text{BGG}"] & \mu \uparrow\uparrow \lambda \arrow[l, shift left, "\text{Verma}"]\end{tikzcd}$$

Thm (Verma)
If $\lambda, \mu \in \mathfrak{h}^*$ and $\mu \uparrow\uparrow \lambda$ then
(i) $\exists\ M(\mu) \hookrightarrow M(\lambda)$
(ii) $[M(\lambda) : L(\mu)] \neq 0$

Example



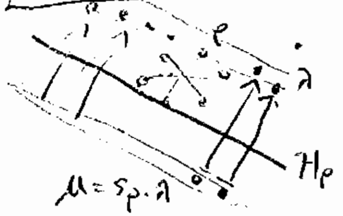
$\lambda \uparrow s_{\alpha_1}\cdot\lambda$
$s_{\alpha_2}\cdot\lambda \uparrow \lambda$
$s_\beta\cdot\lambda \uparrow \lambda$
$\mu \uparrow s_\beta\cdot\lambda$ $(\Rightarrow \mu \uparrow\uparrow \lambda)$
$\mu \uparrow s_{\alpha_2}\cdot\lambda$

Recall that $e f^k v_\lambda^+ = k(\lambda - (k-1)) f^{k-1} v_\lambda^+$.

⇝ If $\lambda \in \Lambda^+$, $\mu = s_\alpha \cdot \lambda = \lambda - k\alpha$ for some $\alpha \in \Delta$, $k \in \mathbb{Z}_{>0}$ then $M(\mu) \hookrightarrow M(\lambda)$ is a nonzero hom.

$$v_\mu^+ \longmapsto f_\alpha^k v_\lambda^+$$

Thm [Shapovalov]
$\exists$ Shapovalov element $S = S(\gamma, k)$, $\gamma \in \Phi^+$, $k \in \mathbb{Z}_{>0}$ s.t.
$M(\mu) \hookrightarrow M(\lambda)$ is a nonzero hom
$v_\mu^+ \mapsto S v_\lambda^+$
for ANY pair (μ, λ) s.t. $\mu = s_\gamma \cdot \lambda = \lambda - k\gamma$



§2 BGG thm

$\boxed{\text{Thm}}$

(1) [Jantzen sum formula]

$M(\lambda)$ has a <u>Jantzen filtration</u>

$$M(\lambda) = M(\lambda)^0 \supsetneq M(\lambda)^1 \supsetneq \cdots \supsetneq M(\lambda)^n = 0 \quad \text{s.t.}$$

(a) $M(\lambda)^0 / M(\lambda)^1 \cong L(\lambda)$

(b) $$\sum_{i>0} \operatorname{ch} M(\lambda)^i = \sum_{\alpha \in \Phi^+} \sum_{s_\alpha \cdot \lambda < \lambda} \operatorname{ch} M(s_\alpha \cdot \lambda)$$

(2) [BGG]

If $[M(\lambda) : L(\mu)] \neq 0$ then $\mu \uparrow\uparrow \lambda$

(pf)

$[M(\lambda) : L(\mu)] \neq 0 \rightsquigarrow \eta := \lambda - \mu = \sum_{\alpha \in \Phi^+} c_\alpha \alpha$

induction on $\mathrm{ht}(\eta) := \sum c_\alpha$

- $\underline{\mathrm{ht}(\eta) = 0}$: $\lambda = \mu$ ✓
- $\underline{\mathrm{ht}(\eta) > 0}$: $\lambda \neq \mu \rightsquigarrow L(\mu) \neq L(\lambda) \overset{\text{"survive first round"}}{\cong} M(\lambda)^0 / M(\lambda)^1$

$\rightsquigarrow [M(\lambda)^i : L(\mu)] \neq 0$ for some $i > 0$

$\rightsquigarrow L(\mu)$ occurs on RHS

$\rightsquigarrow [M(s_\alpha \cdot \lambda) : L(\mu)] \neq 0$ for some $s_\alpha \cdot \lambda < \lambda$

$\rightsquigarrow s_\alpha \cdot \lambda - \mu < \eta$

$\overset{\text{ind}}{\rightsquigarrow} \mu \uparrow\uparrow s_\alpha \cdot \lambda \uparrow \lambda$

※

§3 KL theory

Recall that $\operatorname{ch} L(\lambda)$ is computed only for $\lambda \in \Lambda^+$

$\boxed{\text{Thm}}$ [Soergel]

If $\lambda \in \mathfrak{h}^*$ then

(a) $\exists \lambda^\natural \in \Lambda^\natural$ (for some other Lie alg $\mathfrak{g}^\natural$)

(b) $[M(\lambda) : L(\mu)] = [M(\lambda^\natural) : L(\mu^\natural)]$

pf need $\begin{cases} \text{end alg of proj. gen} \\ \text{Soergel's } \mathbb{V}\text{-functor} \end{cases}$

$\boxed{\text{Thm}}$ [Jantzen]

If $\lambda \in \Lambda^+$ then

$[M(w \cdot \lambda^+) : L(x \cdot \lambda^+)] = [M(w \cdot 0) : L(x \cdot 0)]$

pf need Jantzen's translation functor.

$\boxed{\text{Conj}}$ [KL]

$$[M(w \cdot 0) : L(x \cdot 0)] = \overset{\text{KL-polyn}}{P_{x,w}}(1)$$

equiv,

$$\operatorname{ch} L(w \cdot 0) = \sum_{\substack{x \le w \\ \uparrow \\ \text{Bruhat order}}} (-1)^{l(x) - l(w)} \underset{\substack{\uparrow \\ \text{longest elt in } W}}{P_{w_0 w, w_0 x}}(1) \operatorname{ch} M(x \cdot 0)$$

§ 3 Homological algebra

- A SES of $U(\mathfrak{g})$-mod is

$$0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$$

s.t. $\operatorname{Im} f \overset{=}{=} \ker g$

$\left(\Leftrightarrow L \cong M/N\right)$ $\left(\overset{\text{def}}{\Leftarrow} M \text{ is an } \underline{\text{extension}} \text{ of } N \text{ by } L\right)$

$\Uparrow$ If $\operatorname{Ext}(L,N)=0$ then $M=N\oplus L$

Example

- $0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$ is a SES
- $0 \to \underset{(=L(-2))}{M(-2)} \to M(0) \to L(0) \to 0$ is a SES
- If $M = M^0 \supset \cdots \supset M^n = 0$ s.t. $M^i/M^{i+1} \cong L(\mu_i)$ then $\exists$ SES$_s$ $0 \to M^1 \to M^0 \to L(\mu_1) \to 0$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

- A module $M\in\mathcal{O}$ has a $\underline{\text{Verma flag}}$ if

$$M = M^1 \supset \cdots \supset M^n = 0 \text{ s.t. } M^i/M^{i+1} \cong M(\mu_i)$$

$\rightsquigarrow \exists$ SES$_s$ ——

Note that $M \neq \bigoplus M(\mu_i)$ in general

Fact

(1) $\forall M\in\mathcal{O}$, $M = \bigoplus_{\lambda} M^{\chi_\lambda}$

$\overset{e.g.}{\Rightarrow}$ If M has a Verma flag $M \supseteq M^1 \supseteq M^2 = 0$ s.t. $\mu_2 \notin W\cdot\mu_1$
$$ M(\mu_1) \quad M(\mu_2)$$

then $M = M(\mu_1) \oplus M(\mu_2)$

(2) If L is f.d., $M = L\otimes M(\lambda)$ has a Verma flag s.t. quot. $\exists M(\lambda+\mu)$ and $(M : M(\lambda+\mu)) = \dim M_\mu$

e.g. $L = L(3)$, $\lambda = 0 \rightsquigarrow \operatorname{ch} M = \operatorname{ch} M(3) + \operatorname{ch} M(+1) + \operatorname{ch} M(-1) + \operatorname{ch} M(-3)$

$\rightsquigarrow M = M(3) \oplus M(-1) \oplus$ [illegible]

§ 4 translation functor

For any $\lambda,\mu\in\Lambda$, define translation

$$T_\lambda^\mu(M) = \left(L(\bar{\nu}) \otimes M^{\chi_\lambda}\right)^{\chi_\mu}$$

where $\bar{\nu}$ is the uniq elt in $W\cdot\overset{\lambda-\mu}{\nu} \in \Lambda^+$

e.g.

$$T_0^{3}\left(M(0)\oplus M(3)\right) = M(3)$$

Thm [Jantzen]

(1) Under some good cond.

$T_\lambda^\mu : \mathcal{O}_{\chi_\lambda} \xrightarrow{\sim} \mathcal{O}_{\chi_\mu}$ is an equiv. of cat

Verma $\mapsto$ Verma

Simple $\mapsto$ simple

(2) $\lambda = w\lambda^+$ for $\lambda^+\in\Lambda^+$

$[M(w\cdot\lambda^+) : L(x\cdot\lambda^+)] = [M(w\cdot 0) : L(x\cdot 0)]$

Day 4

Coxeter groups & Hecke algebras

Goal: (A) Coxeter group $W \rightsquigarrow$ (i) length fcn (ii) Bruhat ordering (iii) Hecke alg (iv) KL polyn.

(B) Count $|W|$ via
(i) transitive action on root sys
(ii) deg of fundamental invariants

(C) Hecke alg $\longleftrightarrow$ Cat. $\mathcal{O}_0$ (principal block) $\longleftrightarrow$ Tensor space of quantum alg

std basis T_w $\leftrightarrow$ Verma mod. M_w $\leftrightarrow$ std basis
KL basis C_w $\leftrightarrow$ Simple mod L_w $\leftrightarrow$ dual canonical basis
dual KL basis C'_w $\leftrightarrow$ Tilting mod. T_w $\leftrightarrow$ canonical basis

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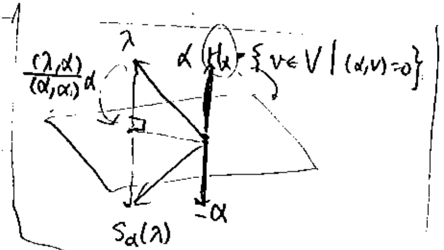
§1 Root sys

- $(V, (\cdot,\cdot))$ is a f.d. Euclidean space
$\left(V = \mathbb{R}\varepsilon_1 \oplus \dots \oplus \mathbb{R}\varepsilon_d \text{ and } (\varepsilon_i, \varepsilon_j) = \delta_{ij} \right)$



- For any $\underset{\text{root}}{\alpha} \in V \setminus \{0\}$, we define reflection $s_\alpha : V \to V$ by
$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \text{where } \underbrace{\alpha^\vee}_{\text{coroot}} = \frac{2\alpha}{(\alpha,\alpha)}$$

Example

(1) $\alpha = \varepsilon_1 - \varepsilon_2 \rightsquigarrow (\alpha,\alpha) = (\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2) = 2 \rightsquigarrow \alpha^\vee = \alpha$

(2) $\alpha = \varepsilon_1 + \varepsilon_2 \rightsquigarrow \alpha^\vee = \alpha$

(3) $\alpha = \varepsilon_1 \rightsquigarrow \alpha^\vee = 2\alpha$

(4) $\alpha = 2\varepsilon_1 \rightsquigarrow \alpha^\vee = \frac{1}{2}\alpha$

- A root sys Φ is a finite subset of $V \setminus \{0\}$ s.t.

(R1) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\} \quad \forall\, \alpha \in \Phi$
(R2) $s_\alpha(\Phi) = \Phi \quad \forall\, \alpha \in \Phi$
(R3) $(\alpha, \beta^\vee) \in \mathbb{Z} \quad \forall\, \alpha, \beta \in \Phi$

Example

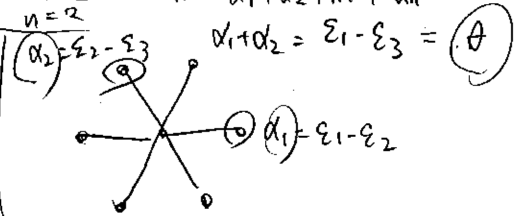
(A_n) $V = \mathbb{R}\varepsilon_1 \oplus \dots \oplus \mathbb{R}\varepsilon_{n+1}$

$\Delta = \{\alpha_1^A, \dots, \alpha_n^A\}$ where $\alpha_i^A = \varepsilon_i - \varepsilon_{i+1}$

$\rightsquigarrow \Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n+1\} \rightsquigarrow |\Phi| = 2\binom{n+1}{2} = n^2+n$

$\theta = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \alpha_2 + \dots + \alpha_n$

e.g. $n=2$



(B_n) $V = \mathbb{R}\varepsilon_1 \oplus \dots \oplus \mathbb{R}\varepsilon_n$

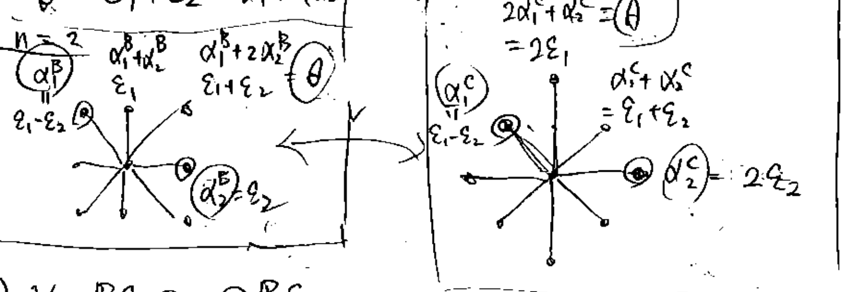
$\Delta = \{\alpha_1^B, \dots, \alpha_n^B\}$ where $\alpha_i^B = \begin{cases} \alpha_i^A & \text{if } i = 1, \dots, n-1 \\ \varepsilon_n & \text{if } i = n \end{cases}$

$\rightsquigarrow \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\pm\varepsilon_i\}_{i=1}^n$

$\rightsquigarrow |\Phi| = 4\binom{n}{2} + 2n = 2n^2$

$\theta = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2(\alpha_2 + \dots + \alpha_n)$

e.g. $n=2$



(C_n) $V = \mathbb{R}\varepsilon_1 \oplus \dots \oplus \mathbb{R}\varepsilon_n$

$\Delta = \{\alpha_1^C, \dots, \alpha_n^C\}$ where $\alpha_i^C := (\alpha_i^B)^\vee = \begin{cases} \alpha_i^A & \text{if } i = 1, \dots, n-1 \\ 2\varepsilon_n & \text{if } i = n \end{cases}$

$\rightarrow \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{\pm 2\varepsilon_i\}_{i=1}^n$

$\theta = 2\varepsilon_1 = 2(\alpha_1 + \dots + \alpha_{n-1}) + \alpha_n$

Facts:

(1) [Finiteness lemma] $(\alpha, \beta^\vee)(\beta, \alpha^\vee) \in \{0, 1, 2, 3\} \quad \forall\, \alpha, \beta \in \Phi$

(2) $\exists$ simple sys $\Delta \subseteq \Phi$ s.t. if $\beta \in \Phi$ then $\beta = \sum c_i \alpha_i$ where c_i all pos or all neg.
$\rightsquigarrow \Phi = \Phi^+ \sqcup \Phi^-$ w.r.t Δ

(3) $\exists!$ highest root $\theta \in \Phi$ (i.e. $\sum c_i$ is the largest)

§2 Coxeter grp

Given $S = \{s_i\} \subseteq W$

- (W,S) is a Coxeter sys (W is a Coxeter grp) if
  $W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where $m_{ii} = 1$, $m_{ij} \geq 2$ if $i \neq j$
- A reduced expression of $w \in W$ is a prod. $w = t_1 \cdots t_N$ where $t_i \in S$ s.t. N is minimal among all

$\rightsquigarrow$ $\ell(w) = N$ is the length of w.

Fact

(1) $\ell(w)$ is well-defined.

(2) $\exists$ root sys $\Phi \subset V := \bigoplus_{s \in S} \mathbb{R}\alpha_s$ with simple sys $\Delta = \{\alpha_s \mid s \in S\}$

(3) $\ell(ws_\alpha) > \ell(w) \iff w\alpha \in \Phi^+$

(4) $\ell(w) = n(w) := |\Phi^+ \cap w^{-1}\Phi^-|$

(5) Del cond. & exchange cond.

(6) $\exists$ Bruhat ordering $w \leq w' \iff (\sim\sim)$

(7) Fix a reduced expression $w = t_1 \cdots t_N$, then
$w' \leq w \iff w'$ is a subexpression of $t_1 \cdots t_N$

§3 Weyl grp

Given a root sys $\Phi \subset V$, its Weyl grp $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$

Facts

(1) $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$

(2) W is a Coxeter grp (m_{ij} given by Dynkin diagram)

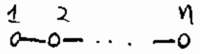
(3) If $|W| < \infty$ then $\exists!$ longest elt $w_0 \in W$ s.t.

(a) $w_0 \Phi^+ = \Phi^-$

(b) $w_0 = w_0^{-1}$

(c) $\forall w \in W$, $\exists w' \in W$ s.t. $ww' = w_0$ and $\ell(w) + \ell(w') = \ell(w_0)$

Example

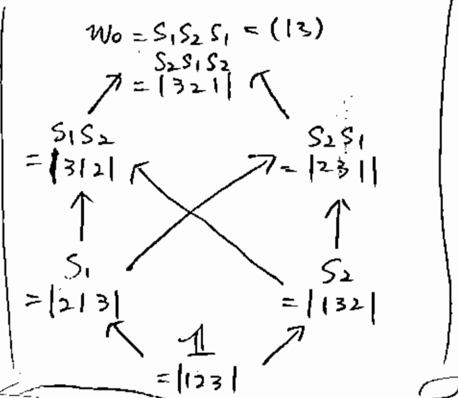


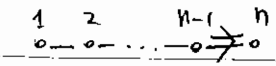
(A_n) $W_n^A = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where $m_{ij} = \begin{cases} 3 & \text{if } |i-j| = 1 \\ 2 & \text{if } |i-j| > 1 \end{cases}$

i.e. $\begin{cases} \cdot\ s_i^2 = \mathbb{1} \\ \cdot\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{if } 1 \leq i \leq n-1 \\ \cdot\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \end{cases}$

- $W_n^A \xrightarrow{\sim} S_{n+1}$
  $s_i \mapsto (i\ i{+}1)$
  $w_0 \mapsto (1\ n{+}1)$
- $\ell_A(w) = \#\left\{(i,j) \;\middle|\; \begin{matrix} i<j \\ w(i) > w(j) \end{matrix}\right\}$

e.g. $n=2$

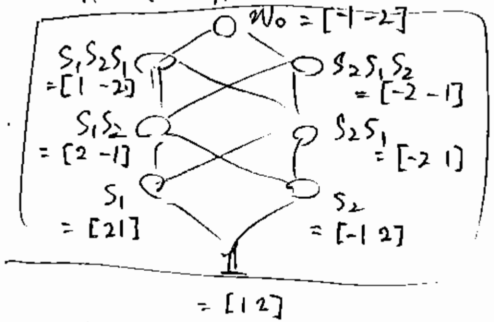




(B_n) $W_n^B = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where $m_{ij} = \begin{cases} 3 & \text{if } |i-j| = 1 \text{ \& not } n \\ 4 & \text{if } |i-j| = 1 \text{ \& is } n \\ 2 & \text{otw} \end{cases}$

i.e. $\begin{cases} \cdot\ s_i^2 = \mathbb{1} \\ \cdot\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{if } 1 \leq i \leq n-2 \\ \cdot\ s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1} \\ \cdot\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \end{cases}$

- $W_n^B \xrightarrow{\sim} \{ w \in \mathrm{Perm}[-n,n] \mid w(-j) = -w(j)\ \forall j = 1, \dots, n \} \cong \mathbb{Z}_2 \ltimes S_n$
  $s_i \mapsto \begin{cases} (-i\ -(i{+}1))(i\ i{+}1) & \text{if } i \neq n \\ (-1\ 1) & \text{if } i = n \end{cases}$
  $w_0 \mapsto (-1\ 1)(-2\ 2) \cdots (-n\ n)$
- $\ell_B(w) = \ell_A(w) - \sum_{\substack{i \in [1,n] \\ w(i) < 0}} w(i)$



$\boxed{\text{Fact 1}}$

(4) $\Phi = \Phi_\ell \sqcup \Phi_s$ and $\theta \in \Phi_\ell$ ← set of long roots.

(5) $W \curvearrowright \Phi_\ell$ transitively (i.e. $\forall\, \alpha, \beta \in \Phi_\ell$, $\exists\, w \in W$ s.t. $w\alpha = \beta$)
$(W \curvearrowright \Phi_s)$

$\rightsquigarrow |W| = |\Phi_\ell|\,|W_\beta|$ for any $\beta \in \Phi_\ell$

where $W_\beta = \{w \in W \mid w(\beta) = \beta\}$ is the <u>isotropy subgrp</u>

$= \langle s_\alpha \in S \mid (\alpha, \beta) = 0 \rangle$

IP. this is true for $\beta = -\theta$, the lowest long root

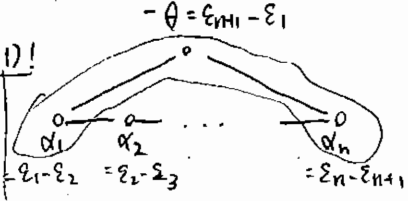
(6) Dynkin (AFF type X) = Dynkin (FIN type X $\sqcup \{-\theta\}$)

<u>Example</u>

(An) <u>Known: $|W| = |S_{n+1}| = (n+1)!$</u>

$|\Phi_\ell| = |\Phi| = n^2 + n = (n+1)n$

$\theta = \varepsilon_1 - \varepsilon_{n+1}$

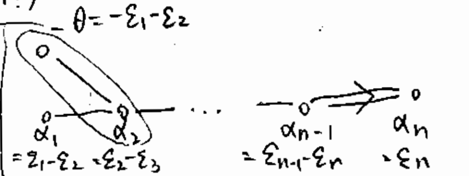


$\rightsquigarrow |W_n^A| = (n+1)n\,|W_{n-2}^A| = \cdots = (n+1)!$

(Bn) <u>Known: $|W| = 2^n|S_n| = 2^n(n!)$</u>

$|\Phi| = 2n^2$, $|\Phi_\ell| = 2n(n-1)$

$\theta = \varepsilon_1 + \varepsilon_2$



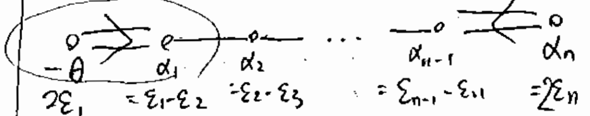
$\rightsquigarrow |W_n^B| = 2n(n-1)\,|W_1^A|\cdot|W_{n-2}^B| = 2^n(n!)$

(Cn) <u>Known: $|W| = 2^n(n!)$</u>

$|\Phi_\ell| = 2n$

$\theta_\ell = 2\varepsilon_1$

$|W_n^C| = 2n\,|W_{n-1}^B| = 2^n(n!)$

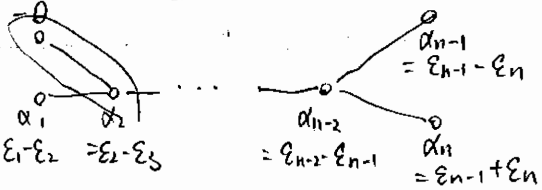


(Dn) $|\Phi_\ell| = |\Phi| = 2n(n-1)$

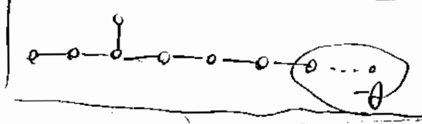
$\theta = \varepsilon_1 + \varepsilon_2$

$|W_n^D| = 2n(n-1)\,|W_1^A|\cdot|W_{n-2}^D|$

$= 2^{n-1}(n!)$



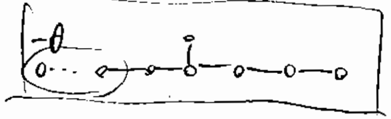
(E8) $|\Phi_8^E| = 240 = 2^4\cdot3\cdot5$



$|W_8^E| = 2^4\cdot3.5\,|W_7^E|$

$= 2^{14}3^5 5^2 7$

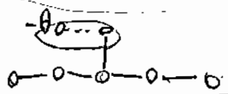
(E7) $|\Phi_7^E| = 126 = 2\cdot3^2\cdot7$



$|W_7^E| = 2\cdot3^2\cdot7\,|W_6^D| = 2\cdot3^2\cdot7\,(2^5\cdot6!)$

$= 2^{10}\cdot3^4\cdot5\cdot7$

(E6)



$|W_6^E| = 72\,|W_5^A| = 72\cdot6!$

§4 Invariant polynomials

- $\mathcal{S} := P(\mathfrak{h})$: polynomials on $\mathfrak{h}$
  $= S(\mathfrak{h}^*)$ sym. alg on $\mathfrak{h}^*$ $\left(= T(\mathfrak{h}^*)/\langle f\otimes g - g\otimes f\rangle_{f,g\in\mathfrak{h}^*}\right)$
- $W \curvearrowright \mathfrak{h}^*$ by reflection (i.e. $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i \quad \forall \lambda \in \mathfrak{h}^*$)
  $\rightsquigarrow W \curvearrowright T(\mathfrak{h}^*)$ by $w(\lambda_1\otimes\cdots\otimes\lambda_r) = w\lambda_1\otimes\cdots\otimes w\lambda_r$
  $\rightsquigarrow W \curvearrowright S(\mathfrak{h}^*) = \mathcal{S}$
- $\mathcal{R} := \mathcal{S}^W$: W-invariant polyn. on $\mathfrak{h}$
  $\mathcal{R}^+ :=$ W-invariant polyn on $\mathfrak{h}$ with const $= 0$

<u>Example</u>

(A_1) $s_{\alpha_1}(h) = -h \quad \forall\, h \in \mathfrak{h}$

$h^2 \in \mathcal{R}$, $h^2 \in \mathcal{R}^+$

$h^2 - 1 \in \mathcal{R}$, $h^2 - 1 \notin \mathcal{R}^+$

Fact

(1) [Chevalley]

$\exists$ homogeneous, alg. ind, elts $\underbrace{f_1, \dots, f_n}_{\text{fund. invariants}} \in \mathcal{R}^+$ s.t.

(a) $\mathcal{R} \cong \mathbb{C}[f_1, \dots, f_n]$

(b) $n = \dim \mathfrak{h}$

(c) $\{\underbrace{\deg f_i}_{=: d_i}\}$ is ind. of choice of f_i's

(2) $|W| = d_1 \cdots d_n$

(3) The Poincaré series $\sum_{w \in W} t^{l(w)}$ = Poincaré polyn. $\left(\sum_{i=0}^{l(w)} \dim H^{2i}(\overline{X_w}) q^i \right)$ // $\prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$

(specialize $t=1 \rightsquigarrow |W| = d_1 \cdots d_n$)

Example

(A_2) $d_1 = 2$, $d_2 = 3$, $|W_2^A| = |S_3| = 6 = 2 \cdot 3$

w	1	s_1	s_2	s_1s_2	s_2s_1	w_0
$l(w)$	0	1	1	2	2	3

$$LHS = 1 + 2t + 2t^2 + t^3 = (1+t)(1+t+t^2) = \left(\frac{1-t^2}{1-t}\right)\left(\frac{1-t^3}{1-t}\right)$$

The <u>Coxeter number</u> h of type X_n is the order of $s_1 \cdots s_n$

Fact

(4) (a) $h \cdot n = |\Phi|$

(b) $h = 1 + \sum c_i$ where $\theta = \sum c_i \alpha_i$

(c) $h = d_n$

Example

(A^2) $h = \operatorname{ord}(s_1 s_2) = 3$

$h \cdot n = 3 \cdot 2 = 6 = |\Phi|$

$\theta = \varepsilon_1 + \varepsilon_3 = \alpha_1 + \alpha_2 \rightsquigarrow 1 + \sum c_i = 1 + 2 = 3$

$3 = d_2$

✓

§5 Hecke alg

Given a Coxeter group (W, S)

Recall: W is gen. by s_i subject to

$(s_i s_j)^{m_{ij}} = 1$

$\Leftrightarrow \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$ $\forall\ i, j$

- $\mathcal{A}$: a $\overset{\text{comm.}}{\text{ring}}$ containing q & q^{-1}

  (e.g. $\mathbb{Z}[q, q^{-1}]$, $\mathbb{Q}(q)$; $\mathbb{Z}[v, v^{-1}]$, $\mathbb{Q}(v)$ where $v^2 = q$)

- The <u>Hecke alg.</u> $\mathcal{H}$ is the unital assoc. $\mathcal{A}$-alg gen by s.t.

  (H1) Braid rel'n $\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$ $\forall\ i, j \in S$

  (H2) Hecke rel'n $(T_i - q)(T_i + 1) = 0$ $\forall\ i \in S$

  (H2′) $T_i^2 = (q-1) T_i + q$ $\forall\ i \in S$

Fact

(1) $T_w = T_{t_1} \cdots T_{t_N}$ if $w = t_1 \cdots t_N$ is a reduced expr. is well-defined.

(2) $s \in S$, $T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w) \\ (q-1) T_w + q T_{sw} & \text{if } l(sw) < l(w) \end{cases}$ — (H1′)

(3) $\mathcal{H} = \langle T_w \mid w \in W \rangle / (H1') \;\&\; (H2)$

(4) $w \in W$, T_w is invertible, and

$$(T_{w^{-1}})^{-1} = q^{-l(w)} \sum_{x \leq w} (-1)^{l_{w,x}} R_{x,w}(q) T_x$$

where $R_{x,w} \in \mathbb{Z}[q]$ of deg $l_{w,x} := l(w) - l(x)$ and

$R_{w,w}(q) \equiv 1$ $\forall\ w \in W$

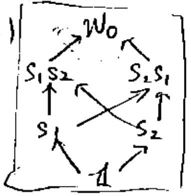
Algorithm computing $R_{x,w}$

1. $R_{x,w} = 0$ if $x \not\leq w$
2. $R_{x,w} = 1$ if $x = w$
3. Now $x < w$: pick any $s \in S$ s.t. $sw < w$ $\left(\begin{array}{l}\text{e.g. } w = t_1 \dots t_N \text{ reduced} \\ \rightsquigarrow s = t_1\end{array}\right)$

$$R_{x,w} = \begin{cases} R_{sx,sw} & \text{if } sx < x \\ (q-1) R_{x,sw} + q R_{sx,sw} & \text{otw} \end{cases}$$

Example

(A2)



- $\underline{R_{\mathbb{1},s_1}}$: $x = \mathbb{1}$, $w = s_1$, pick $s = s_1$

$sx = s_1 > \mathbb{1} = x$

$\rightsquigarrow R_{\mathbb{1},s_1} = (q-1)\overset{1}{\cancel{R_{\mathbb{1},\mathbb{1}}}} + q\,\overset{0}{\cancel{R_{s_1,\mathbb{1}}}} = (q-1)$

§6 KL basis

- $\overline{} : \mathcal{H} \to \mathcal{H}$ by $\bar{q} = q^{-1}$ & $\overline{T_w} = (T_{w^{-1}})^{-1}$ $\left(\overline{R(q)} = R(q^{-1}) \ \forall R \in \mathbb{Z}[q] \right)$

Fact

(1) $\overline{}$ is an alg. involution

(2) $\overline{R_{y,w}} = (-1)^{l_{x,w}} q^{l_{x,w}} R_{x,w} T_x$

$\overline{T_w} = \sum_{x \leq w} q^{-l(x)} \overline{R_{x,w}} T_x$

In particular,

$\overline{T_s} = \overline{R_{\mathbb{1},s}} T_{\mathbb{1}} + q^{-1} \overline{R_{s,s}} T_s = q^{-1} - 1 + q^{-1} T_s$

(3) $C_s = v^{-1}(T_s - q)$ are bar-invariant

$(\dots C'_s = v^{-1}(T_s + 1)$

$\left(\because \overline{C_s} = v(\overline{T_s} - q^{-1}) = v(q^{-1}T_s - 1) = v^{-1}(T_s - q) \ \checkmark \right)$

(4) [KL]

$\exists$ bar-inv. basis $\{C_w\}$ s.t. $C_w = \sum_{x \leq w} (-1)^{l_{x,w}} v^{l(w) - 2l(x)} \overline{P_{x,w}} T_x$,

where $\begin{cases} P_{x,w} \in \mathbb{Z}[q] \\ P_{w,w}(q) = 1 \\ \deg P_{x,w} \leq \frac{l_{w,x} - 1}{2} \text{ if } x < w \end{cases}$

(5) $q^{l_{w,x}} \overline{P_{x,w}} - P_{x,w} = \sum_{x < y \leq w} R_{x,y} P_{y,w}$ ← algorithm

(6) Shortcuts:

(a) $P_{x,w} = 1$ if $0 \leq l_{w,x} \leq 2$

(b) $P_{x,w_0} = 1$ $\forall x \in W$

(c) $P_{x,sw}(0) = 1$ if $x \leq w$

Example

(A3) $x = s_2$; $w = s_2s_1s_3s_2$

(5) $q^3 \overline{P_{x,w}} - P_{x,w} =$

y	$R_{x,y}$	$P_{y,w}$
s_2s_1	1	1
s_{23}	1	
s_{12}	$q-1$	
s_{32}		
s_{212}	$(q-1)^2$	
s_{213}		
s_{232}		
s_{132}		
s_{2132}	$(q-1)^3$	

$\rightsquigarrow q^3 \overline{P_{x,w}} - P_{x,w} = q^3 + q^2 - q - 1$

$\rightsquigarrow P_{x,w} = 1 + q$

§7 KL theory

Thm [KL79]

intercaction cohomology

Schubert var.

$P_{x,w}(q) = \sum_{i=0}^{l(w)} q^i \dim IH_x^{2i}(\overline{X_w})$

IP, $P_{x,w} \in \mathbb{Z}_{\geq 0}[q]$ (No geom. intepretation for $R_{x,w}$)

bdd der. cat of const. sheaves on X

regular holonomic D-mod.

$$\begin{tikzcd}
 & \mathcal{D}^b_c(X) \arrow[rr, "\text{gen. RH corr.}"] & & D^{rh}_X\text{-mod} \arrow[r] & U(\mathfrak{g})\text{-mod} \\
\text{Hecke alg } \mathcal{H} \arrow[r, squiggly, "{[KL]}"] & \text{Perverse sheaves PERV} \arrow[u, phantom, "\cup"] \arrow[rr, leftrightarrow, "{[Kashinara-Mebkhout]}"] & & C'_0 \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow, "{[BB]}", "{[BK]}"'] & \mathcal{O}_0 \arrow[u, phantom, "\cup"] \\
\text{KL basis } \{C_w\} \arrow[r, leftrightarrow] & \text{Perv. sheaves } Perv_w \arrow[rr, leftrightarrow] & & \mathcal{M}_w \arrow[r, leftrightarrow] & \text{Verma } M(w \cdot 0) \\
\text{std basis } \{T_w\} \arrow[r, leftrightarrow] & \text{IC cpx } IC_x \arrow[rr, leftrightarrow] & & \mathcal{L}_x \arrow[r, leftrightarrow] & \text{simple } L(x \cdot 0)
\end{tikzcd}$$

Thm [BB, BK] [KL conj]

(iii) $[M(w\cdot 0) : L(x\cdot 0)] = P_{x,w}(1)$

Pf, $P_{x,w}(1)$

$\| \leftarrow$ [KL] IC sheaf

perv. sheaf $\wr$

$[Perv_w : IC_x]$: in $PERV \subseteq \mathcal{D}^b_c(X)$

abelian subcat

bdd derv. cat of constructible sheaves on X $\leftarrow$ flag var.

$\| \leftarrow$ [Kashiwara - Mebkhout]

$[\mathcal{M}_w : \mathcal{L}_x]$ in regular holonomic $\mathcal{D}$-mod on X

$\| \leftarrow$ [Beilinson - Bernstein] [Brylinski - Kashiwara]

$[M(w\cdot 0) : L(x\cdot 0)]$ in $\mathcal{O}_0$

---

§8 Schur-Jimbo duality

- Classical Schur duality :

$$\mathfrak{gl}_n \overset{\varphi}{\curvearrowright} \underbrace{V\otimes\cdots\otimes V}_{d} \overset{\psi}{\curvearrowleft} S_d \ , \quad \overset{\mathbb{C}^n}{\overset{\shortparallel}{V}}: \text{natural repn of } \mathfrak{gl}_n$$

Thm (a) On $V\otimes\cdots\otimes V$, $\mathfrak{gl}_n$-action commutes w/ S_d-action.

(b) $\mathfrak{gl}_n$ and S_d have double centralizer property

i.e.

$$\varphi(U(\mathfrak{gl}_n)) = \mathrm{End}_{\psi(S_d)}(\underbrace{V\otimes\cdots\otimes V}_{d})$$

$$\psi(S_d) = \mathrm{End}_{\varphi(U(\mathfrak{gl}_n))}(\underbrace{V\otimes\cdots\otimes V}_{d})$$

$$\begin{tikzcd}
\mathfrak{gl}_n \arrow[d, squiggly] & \curvearrowright & \underbrace{V\otimes\cdots\otimes V}_{d} \arrow[d, squiggly] & \curvearrowleft & S_d \arrow[d, squiggly, "q\text{-analog.}"] \\
U_q(\mathfrak{gl}_n) & \curvearrowright & \underbrace{\mathbb{V}\otimes\cdots\otimes\mathbb{V}}_{d} & \curvearrowleft & \mathcal{H}^A_d
\end{tikzcd}$$

$\mathcal{H}^A_d \leftarrow$ Hecke alg of type A_d.

$\mathbb{V} \leftarrow$ natural repn of $U_q(\mathfrak{gl}_n)$ $= \mathbb{Q}(q)^n$

Thm [Jimbo, 90]

$U_q(\mathfrak{gl}_n)$ and $\mathcal{H}^A_{d}$ have double centralizer property.

Thm

(a) $\exists$ Canonical & dual canon. basis on $U_q(\mathfrak{gl}_n)$

(b) 〜〜〜 $\mathbb{V}$ and hence $\mathbb{V}\otimes\cdots\otimes\mathbb{V}$

(c) $\mathcal{H}_d \xrightarrow{\sim} \mathbb{V}\otimes\cdots\otimes\mathbb{V} \xrightarrow{\sim} \mathcal{O}_0$

std basis $\mapsto$ std basis $\mapsto$ Verma

KL basis $\mapsto$ dual can. basis $\mapsto$ simple

dual KL basis $\mapsto$ can. basis. $\mapsto$ Tilting

- $\imath$- Schur duality [Bao-Wang 2013]

$$\begin{array}{c} U_q(\mathfrak{gl}_{2n}) \\ \cup\!\!\shortmid \\ U^\imath \end{array} \curvearrowright \mathbb{V}\otimes\cdots\otimes\mathbb{V} \curvearrowleft \mathcal{H}^B_d$$

$U^\imath \uparrow$ coideal subalg

$\mathbb{V} \uparrow$ natural repn of $U^\imath$

Thm

(a) $U^\imath$ and $\mathcal{H}^B_d$ have double centraliser property

(b) $\exists$ can. & dual can basis on $U^\imath$ and hence $\mathbb{V}\otimes\cdots\otimes\mathbb{V}$

(c) $\mathbb{V}\otimes\cdots\otimes\mathbb{V} \longrightarrow \mathcal{O}_0$

std $\mapsto$ Verma

dual can. $\mapsto$ simple

can. $\mapsto$ Tilting

[new]

(d) generalize to cat $\mathcal{O}$ of Lie superalgebra $osp(2m+1|2n)$.

Day 5

Advanced topics

§1 Kac-Moody algebras

- $A \in M_{I\times I}(\mathbb{Z})$ is a GCM if

(G1) $a_{ii} = 2 \quad \forall \; i \in I$
(G2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$
(G3) $a_{ij} = 0 \iff a_{ji} = 0$

- A is $\begin{cases} \underline{\text{finite}} & \text{if } A \text{ is a CM} \\ \underline{\text{affine}} & \text{if } \exists\, \vec{a} > 0 \text{ s.t. } A\vec{a} = 0 \\ \underline{\text{indefinite}} & \text{otw} \end{cases}$

- A is a GCM $\rightsquigarrow$ realization $(\mathfrak{h}, \Delta^\vee, \Delta)$ s.t.

(i) $\mathfrak{h}$ is a $\mathbb{C}$-vector space
(ii) $\Delta^\vee \subseteq \mathfrak{h}$ has a basis $\{h_i\}_{i\in I}$
(iii) $\Delta \subseteq \mathfrak{h}^*$ has a basis $\{\alpha_i\}_{i\in I}$ s.t. $\alpha_j(h_i) = a_{ij}$

$\boxed{\text{Fact}}$
(1) Each realization $\rightsquigarrow$ Lie algebra $\mathfrak{g}(A) = \langle e_i, f_i, \mathfrak{h} \rangle / \sim$
(2) $\exists!$ minimal realization (i.e. $\dim \mathfrak{h} = 2|I| - \text{rank}(A)$) up to isom.
$\rightsquigarrow$ Kac-Moody alg $\mathfrak{g}(A) = \langle e_i, f_i, \mathfrak{h} \rangle / \sim$

§2 Affine Lie alg.

$\boxed{\text{Fact}}$ Let A: affine, $I = \{0, 1, \dots, n\}$, $I_0 = \{1, \dots, n\}$

(1) rank $A = n \rightsquigarrow \dim \mathfrak{h} = n+2$

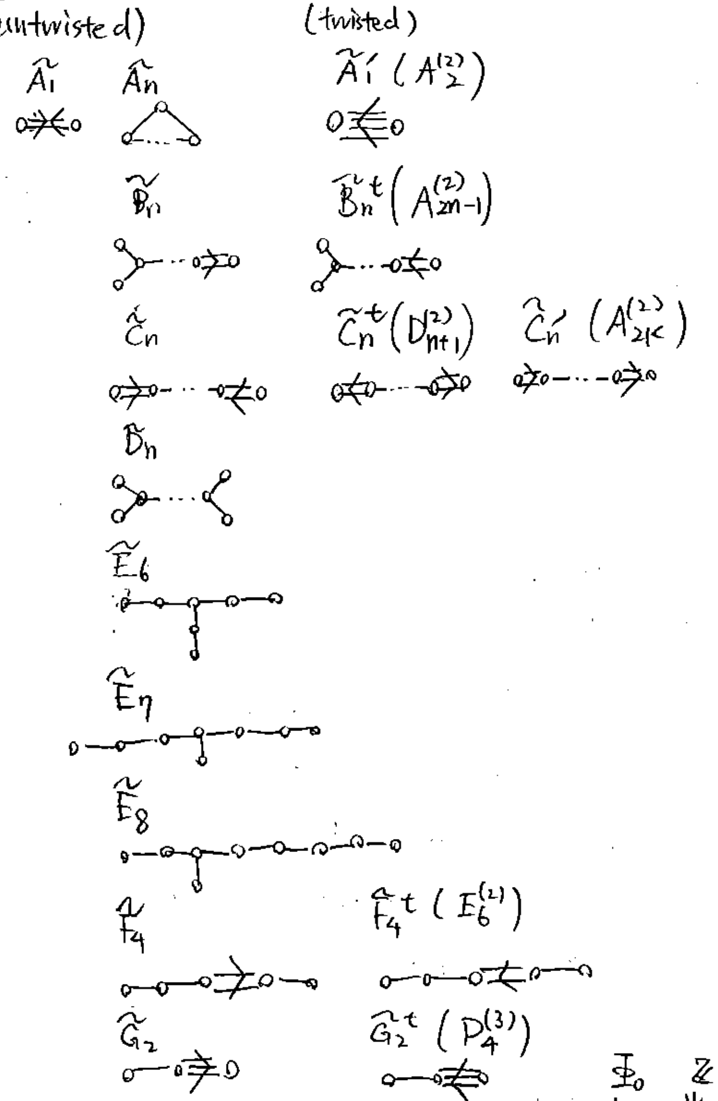
$\mathfrak{h} = \text{Span}_{\mathbb{C}}\{h_0, \dots, h_n, d\}$ where d is the scaling elt s.t. $\alpha_i(d) = \delta_{i,0}$

$\mathfrak{h}^* = \text{Span}_{\mathbb{C}}\{\alpha_0, \dots, \alpha_n, \varpi_0\}$ where $\varpi_i(h_j) = \delta_{ij}$ is the fund wt

$= \text{Span}_{\mathbb{C}}\{\varpi_0, \dots, \varpi_n, \delta\}$ $\quad \delta = \sum_{i\in I} a_i \alpha_i$ is the basic img root

(i.e. $(\delta, \delta) = 0$)

(2) affine Dynkin diagram



(3) $\alpha \in \Phi$ is real iff $(\alpha, \alpha) > 0$ iff $\alpha = \overset{\overset{\Phi_0}{\downarrow}}{\alpha_0} + \overset{\overset{\mathbb{Z}}{\downarrow}}{k}\delta$ i
img iff $(\alpha, \alpha) \leq 0$ iff $\alpha = \underset{\underset{\mathbb{Z}\setminus 0}{\cap}}{k}\delta$

$\Phi = \Phi_{re} \sqcup \Phi_{im} \leftarrow$ has mult n

(4) $W = \langle s_\alpha \mid \alpha \in \Phi_{re} \rangle = \langle s_{\alpha,m} \mid \alpha \in \Phi_0, m \in \mathbb{Z} \rangle$
where $s_{\alpha,m}(\lambda) = s_\alpha(\lambda) + m\lambda$

(5) (W, S) is a Coxeter sys with $S = \{s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\theta,1}\}$

Fact

(1) For $\lambda \in \mathfrak{h}^*$, we can define $M(\lambda) \rightsquigarrow L(\lambda)$

(2) If $\lambda \in \Lambda^+$ then [Weyl-Kac char for]

$$\mathrm{ch}\, L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e(w \cdot \rho)}{\sum_{w \in W} (-1)^{l(w)} e(w \cdot 0)}$$

(3) [SLP] ✓

(4) [KL conj] ✓ by Casian & Kashiwara-Tanisaki

§3 Categorification

Defn 1. A *cat'n* is an assignment

$$\begin{aligned} &\text{object} &&\mapsto &&\text{category} \\ &\text{map} &&\mapsto &&\text{functor} \\ &= &&\mapsto &&\text{isomorphism} \\ &\vdots && &&\vdots \\ &\text{etc} && &&\text{etc} \end{aligned}$$

Example

$\mathbb{N} \longrightarrow \mathrm{FinVect}_{\mathbb{R}}$ $\rightsquigarrow$ " $\mathrm{FinVect}_{\mathbb{R}}$ categorifies $\mathbb{N}$ "

$n \mapsto \mathbb{R}^n$ or, "$\mathrm{FinVect}_{\mathbb{R}}$ is a cat'n of $\mathbb{N}$"

$1+1=2 \mapsto \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^2$ " $\mathrm{FinVect}_{\mathbb{R}}$ = categorified $\mathbb{N}$ "

$2 \times 3 = 6 \mapsto \mathbb{R}^2 \otimes \mathbb{R}^3 \cong \mathbb{R}^6$

$$\begin{tikzcd} & \mathrm{FinVect}_{\mathbb{R}} \arrow[d, bend left, "\text{decat'n} = \dim_{\mathbb{R}}(\cdot)"] \\ \text{cat'n} \quad \begin{matrix} \mathbb{R}^n \\ \uparrow \\ n \end{matrix} & \mathbb{N} \arrow[u, bend left] \end{tikzcd}$$

Recall : Euler formula $\chi(\Box) = F - E + V = 2$

$$\rightsquigarrow \underset{\substack{\uparrow \\ \text{CW cpx}}}{\chi(X)} = \sum_{i=0}^{\infty} (-1)^i \dim H_i(X)$$

[Khovanov, 2000]

For each knot K, define Khovanov homology $Kh(K) = (Kh^i(K))_{i \in \mathbb{Z}}$

$$\rightsquigarrow \chi(Kh(K)) = \sum_{i \in \mathbb{Z}} (-1)^i \,\mathrm{gdim}\, Kh^i(K) = (q+q^{-1}) \underbrace{J(K)}_{\text{Jones polyn.}}$$

Example

$$\begin{tikzcd} \text{Khovanov homology} \arrow[d, bend left, "\text{decat'n} = \text{Euler char.}"] \\ \text{Jones polyn.} \arrow[u, bend left, "\text{cat'n (complicated)}"] \end{tikzcd}$$

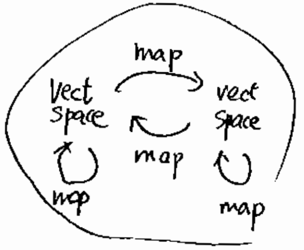
Cor Milnor conj. [Rassmussen, 2010]

Defn 2 Define a new homology to understand structure.

Recall $\mathfrak{sl}_2(\mathbb{C}) = \langle e, f, h \rangle / \sim$

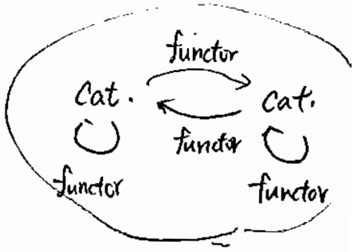
$V: \mathfrak{sl}_2(\mathbb{C})\text{-mod} \rightsquigarrow V = \bigoplus_{n \in \mathbb{Z}} V_n$ (i.e. $v \in V_n \Leftrightarrow h \cdot v = nv$)

$$\begin{tikzcd} \cdots \arrow[r, bend left, "e"] & V_{n-2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] \arrow[loop below, "h"] & V_n \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] \arrow[loop below, "h"] & V_{n+2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] \arrow[loop below, "h"] & \cdots \arrow[l, bend left, "f"] \end{tikzcd}$$



$$\Big\Downarrow \text{ categorification}$$

$$\begin{tikzcd} \cdots & \mathcal{C}_{n-2} \arrow[r, bend left, "\mathcal{E}"] \arrow[loop below, "\mathcal{H}"] & \mathcal{C}_n \arrow[l, bend left, "\mathcal{F}"] \arrow[r, bend left, "\mathcal{E}"] \arrow[loop below, "\mathcal{H}"] & \mathcal{C}_{n+2} \arrow[l, bend left, "\mathcal{F}"] \arrow[loop below, "\mathcal{H}"] & \cdots \end{tikzcd}$$



- $[e,f] = h \rightsquigarrow ef = fe + h$

$$\rightsquigarrow ef|_{V_n} = fe|_{V_n} + n$$

$$\Downarrow$$

$$\mathcal{E}\mathcal{F}|_{C_n} \cong \mathcal{F}\mathcal{E}|_{C_n} \oplus \underbrace{\mathbb{1}_{C_n} \oplus \cdots \oplus \mathbb{1}_{C_n}}_{n}$$

$\boxed{\text{Example}}$

(Naive cat'n)

$$\begin{tikzcd}
\{C_n\}_n \arrow[d, bend left, "\text{decat'n} = \text{Grothendieck group } \mathcal{K}(-)"] \\
\oplus V_n \arrow[u, bend left, "\text{cat'n}"]
\end{tikzcd}$$

(Chuang-Rouquier's cat'n)

$q, a \in \Bbbk^\times$, $x \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$

$$\begin{tikzcd}
(\{C_n\}_n, \text{extra data}) \arrow[d, bend left, "\text{decat'n} = \text{Grot. grp}"] \\
\oplus V_n \arrow[u, bend left, "\text{cath}"]
\end{tikzcd}$$

adjointness
nil affine Hecke rel'n

$\boxed{\text{Conj}}$ [Broue's conj.]

Recall $\mathbb{C}\mathfrak{S}_d$ is semisimple while $\overline{\mathbb{F}_p}\,\mathfrak{S}_d$ is NOT

$\rightarrow$ blocks $\{B_i\}$ $\rightsquigarrow$ defect grp $D(B)$

If defect groups are isom. $\left(\underset{\text{mod}}{\mathcal{D}(B_i)} \cong \underset{\text{mod}}{D(B_j)}\right)$ then $B_i \cong B_j$? No

$\Rightarrow \mathcal{D}^b(\underset{\text{mod}}{B_i}) = \mathcal{D}^b(\underset{\text{mod}}{B_j})$

$\longrightarrow$ Cyclotomic Hecke alg (Hecke alg of cpx refl'n grp $G(d,1,n)$)

Defn 3: construct a "higher" cat. $\mathcal{C}$ s.t. $\mathbb{Q}\otimes\mathcal{K}(\mathcal{C}) \cong (\quad)$

Defn 4:

Given a (n-1)-cat. and a decat'n : from n-cat.
A cat'n is the inverse for this decat'n.

$$\begin{tikzcd}
n\text{-cat.} \arrow[d, bend left, "\text{decat'n} = \begin{matrix}\dim(-)\\ \chi(-)\\ \mathcal{K}(-)\end{matrix}"] \\
(n-1)\text{-cat.} \arrow[u, bend left, "\text{Cat'n}"]
\end{tikzcd}$$