An elementary approach to monomial bases of quantum affine $\mathfrak{gl}_n$

(joint w/ Li LUO @ ECNU)

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1. Finite A [Beilinson-Lusztig-MacPherson 90]

geom. real'n of q-Schur alg $S(n,d)$ $\xrightarrow[d\to\infty]{\text{stab'n}}$ $\dot{U}(\mathfrak{gl}_n)$

key

monomial basis (MB) $\xrightarrow{\text{lift}}$ MB

$\Downarrow$

canonical basis (CB)

Constructed directly via a multiplication formula

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• q-Schur algebra

$S = \{GL_d\text{-invariant } f: X\times X \to \mathbb{Z}[v^{\pm 1}]\}$

$X = \{n\text{-step flags in } \mathbb{F}_q^d\}$

$GL_d \curvearrowright X$, $GL_d \curvearrowright X\times X$ diagonally

3.

• $\{GL_d\text{-orbits on } X\times X\} \ni \mathcal{O}_A$

$\updownarrow$ 1:1

$\Theta_d = \{A \in \mathrm{Mat}_n(\mathbb{N}) \mid \sum a_{ij} = d\} \ni A$

$\Rightarrow$ basis $\{e_A \mid A \in \Theta_d\}$

char. fcn on $\mathcal{O}_A$

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• S has

(i) Std basis $[A] :=$ normalized e_A (chosen s.t. $\overline{[A]} = [A]$ + lower terms)

Claim

(ii) MB m_A s.t.
- $\overline{m_A} = m_A$
- $m_A = [A]$ + lower terms

(iii) CB

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Prop/defn $\exists$ such $m_A = \prod_i [B_i]$, where

$B_i = \mathrm{diag} + rE_{j,j+1}$ $\begin{pmatrix} * & & \\ & \cdot\, r & \\ & & * \end{pmatrix} \longleftrightarrow e_j^{(r)}$

or

$\mathrm{diag} + rE_{j+1,j}$ $\begin{pmatrix} * & & \\ r\,\cdot & & \\ & & * \end{pmatrix} \longleftrightarrow f_j^{(r)}$

Rmk

$\overline{[B_i]} = [B_i] \Rightarrow \overline{m_A} = m_A$

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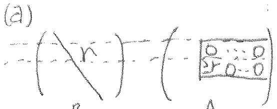
Key lemma (leading coeff = 1)

$[B] * [A] = [M]$ + lower terms if

(a)



(b)

$\Rightarrow m_A := \prod [B_i] = [A]$ + lower terms

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Mult. formula $[B] * [A] = \sum_T (\text{coeff})\, [T]$

divided pwr, general, good enough to afford stab'n

$S \xrightarrow[d\to\infty]{\text{stab'n}} \dot{U}(\mathfrak{sl}_n) \cong \dot{U}(\mathfrak{gl}_n)$

MB $\rightsquigarrow$ MB

$\Downarrow$

CB

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2. Affine A

• [Ginzburg-Vasserot 93]

• [Lusztig 99] geom. real'n $\dot{U}(\widehat{\mathfrak{sl}}_n)$

$\cap$

• [Du-Fu 13-14] Hecke-alg real'n $\dot{U}(\widehat{\mathfrak{gl}}_n)$

missing "semisimple" elts

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Aff. q-Schur alg $\widehat{S}(n,d)$ $\xrightarrow[d\to\infty]{\text{stab'n}}$ $\dot{U}(\widehat{\mathfrak{gl}}_n)$

MB $\xrightarrow{\text{key}}$ MB

$\Downarrow$

CB

① Du-Fu's construction via double Hall alg

② [L-Luo 15] direct construction w/o Hall alg.

Rmk: rel'n btw MB are unclear

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• Affine q-Schur alg

$\widehat{S} \overset{\text{[Green 99]}}{=\!=\!=} \mathrm{End}_{\mathcal{H}_A^{\text{ext}}}(\text{tensor space})$ ⊕ perm. mod of $\mathcal{H}_A^{\text{ext}}$

$\Rightarrow$ basis $\{e_A \mid A \in \widehat{\Theta}\}$

• $\widehat{\Theta} = \{A \in \mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{N}) \mid (T1), (T2)\}$

(T1) (periodic) $a_{ij} = a_{i+n,j+n}$

(T2) (size $= d$) $\sum a_{ij} = d$

Rmk

(T1) $\Rightarrow$ A is uniq. det by any $n\times\mathbb{Z}$ submat.

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Example $n=2$, $d=10$

$$A = \begin{array}{c|cccccc|} -1 & & 4 & & & & \\ 0 & & & 1 & 2 & & \\ \hline 1 & & 3 & & 4 & & \\ 2 & & & & & 1 & 2 \\ \hline 3 & & & & 3 & & 4 \\ 4 & & & & & & \\ \hline & -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

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• $\widehat{S}$ has

(i) Std basis $[A] :=$ normalized e_A (chosen s.t. $\overline{[A]} = [A]$ + lower terms)

Claim

(ii) MB m_A s.t.
- $\overline{m_A} = m_A$
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(iii) CB

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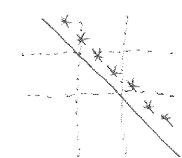
Thm/defn $\exists$ such $m_A = \prod_i [B_i]$, where

B_i or ${}^{+}B_i$: upper bidiag.



$\updownarrow$

semisimple rep'n of Hall alg of cyclic quiver

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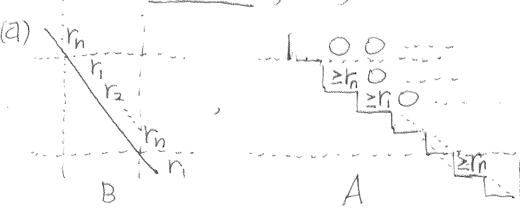
Key lemma

$[B] * [A] = [M]$ + lower terms if

(B, A) is admissible, i.e.,

(a)



(b) similar

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Multiplication formula [Du-Fu 13]

$[B] * [A] = \sum_T (\text{coeff})\, [T]$

bidiag, general, good enough to afford stab'n

$\widehat{S} \xrightarrow[d\to\infty]{\text{stab'n}} \dot{U}(\widehat{\mathfrak{gl}}_n) \supsetneq \dot{U}(\widehat{\mathfrak{sl}}_n)$

MB $\rightsquigarrow$ MB

$\Downarrow$

CB

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