Quantum symmetric pairs of affine type: multiplication formulae, canonical bases, and Schur dualities.

joint w/ Fan, Li, Luo and Wang

1. Finite A [Beilinson-Lusztig-MacPherson]
2. Finite B/C [Bao-Kujawa-Li-Wang]
3. Affine C [FL^3W]

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1. Finite A

q-Schur alg $S^{\mathfrak{a}}_{n,d}$ $(d \geq 1)$ $\xrightarrow[d\to\infty]{\text{stab'n}}$ $\dot{U}(\mathfrak{gl}_n)$

(I) Mult. formula — key

$\Downarrow$

(II) monomial basis (MB) — canonical basis (CB)

(III) MB $\xrightarrow{\text{lift}}$ MB

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- $S^{\mathfrak{a}}_{n,d} = \{GL_d\text{-inv't } f: X \times X \to \mathbb{Z}[v^{\pm 1}]\}$

  $X = \{n\text{-step flags in } \mathbb{F}_q^d\} \curvearrowleft GL_d$

- $\{GL_d\text{-orbits on } X \times X\} \ni \mathcal{O}_A$

  $\updownarrow 1:1$

  $\Theta = \{A \in Mat_n(\mathbb{N}) \mid \sum a_{ij} = d\} \ni A$

$\Rightarrow$ basis $\{e_A \mid A \in \Theta\}$ for $S^{\mathfrak{a}}_{n,d}$

char. fcn on $\mathcal{O}_A$

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$S^{\mathfrak{a}}_{n,d}$ has

(i) Std basis $[A] :=$ normalized e_A (s.t. $\overline{[A]} = [A] + \text{lower}$)

WTS (ii) MB m_A s.t.

(M1) $\overline{m_A} = m_A$,

(M2) $m_A = [A] + \text{lower}$

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Prop/defn $\exists$ such $m_A = \prod_k [B^{(k)}]$,

$B^{(k)} = \text{diag} + rE_{i,i+1}$ $\left(\diagdown r\right) \longleftrightarrow e_i^{(r)}$

" $\left(r \diagdown\right) \longleftrightarrow f_i^{(r)}$

(M1) ✓ b/c $\overline{[B^{(k)}]} = [B^{(k)}]$

(M2) $\Leftarrow$ mult. formula.

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Mult. formula

$$e_B * e_A = \sum_T q^{pwr} \begin{bmatrix} A \\ T \end{bmatrix} e_{A(T)}$$

e_B: $e_i^{(r)}$ or $f_i^{(r)}$; e_A: any; $\begin{bmatrix} A \\ T \end{bmatrix}$: q-bino. coeff.

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Cor $[B] * [A] = [M] + \text{lower}$ if ($\exists$ ht term, coeff $= 1$)

(a) $B = \left(\diagdown r\right)$, $A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \geq r & 0 & \cdots & 0 \end{pmatrix}$

(b) $B \leftrightarrow f_i^{(r)}$; similar

Cor $\Rightarrow$ (M2)

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2. Finite B/C [BKLW]

$S^{\jmath}_{n,d}$ $(d \geq 1)$ $\xrightarrow[d\to\infty]{\text{stab'n}}$ $\dot{U}^{\jmath}(\mathfrak{gl}_n)$

- $\dot{U}^{\jmath}$ NOT a Drinfel'd-Jimbo QG

$\Rightarrow$ coideal subalg of $U(\mathfrak{gl}_n)$

$\Delta(U^{\jmath}) \subseteq U(\mathfrak{gl}_n) \otimes U^{\jmath}$

$\Rightarrow$ Quantum sym. pair $(U(\mathfrak{gl}_n), U^{\jmath})$

Q: affinization?

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3. Affine C [FL^3W]

(geom) $\hat{S}^{\mathfrak{c}}_{n,d} = \{Sp_d(K)\text{-inv't } f: X \times X \to \mathbb{Z}[v^{\pm 1}]\}$, $K = \mathbb{F}_q((\varepsilon))$

$X = \{\text{"}n\text{-step" flags of aff } C\} \curvearrowleft Sp_d(K)$

(Hecke) $\hat{S}^{\mathfrak{c}}_{n,d} = End_{\hat{\mathcal{H}}^{\mathfrak{c}}_d}(\mathbb{V}^{\otimes d})$

aff Hecke; $\cong \oplus$ Perm mod

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- $\{Sp_d\text{-orbits on } X \times X\} \ni \mathcal{O}_A$

  $\updownarrow 1:1$

  $\Xi = \{A \in Mat_{\mathbb{Z}}(\mathbb{N}) \mid (X1)\text{-}(X4)\} \ni A$

  (X1) periodic. $a_{ij} = a_{i+2n, j+2n}$

  (X2) centrosym. $a_{ij} = a_{-i,-j}$

  (X3) (X4) size of $A = d$

$\Rightarrow$ basis $\{e_A \mid A \in \Xi\}$, $\{[A] \mid A \in \Xi\}$ for $\hat{S}^{\mathfrak{c}}_{n,d}$

char. fcn. on $\mathcal{O}_A$

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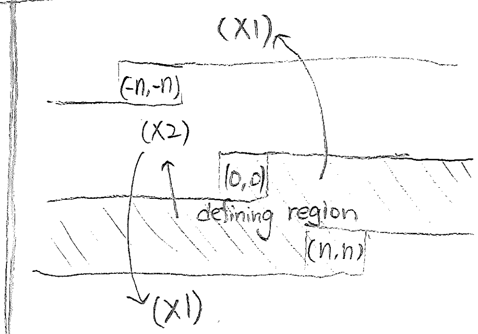

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Goal

$\hat{S}^{\mathfrak{c}}_{n,d}$ $(d \geq 1)$ $\xrightarrow[d\to\infty]{\text{stab'n}}$ $\dot{U}^{\mathfrak{c}}(\hat{\mathfrak{gl}}_N)$, $N = 2n$

modified aff coideal subalg

⚠ $\langle\text{Chevalley}\rangle =: \dot{U}^{\mathfrak{c}}(\hat{\mathfrak{sl}}_N)$

$\not\cap$

$\langle\text{tridiag}\rangle = \dot{U}^{\mathfrak{c}}(\hat{\mathfrak{gl}}_N)$

$\Rightarrow$ need stronger mult. formula. (involving struc. const. on $\hat{\mathcal{H}}^{\mathfrak{c}}_d$)

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Mult. formula

$$e_B * e_A = \sum_{T,S} (q-1)^{pwr} q^{pwr} [A;S;T] \, e_{A(T-S)}$$

e_B: tridiag.; e_A: any; $(q-1)^{pwr}$: Hecke rel'n; $[A;S;T]$: $\prod$ q-bino coeff.

Rmk

- It encodes the mult. formulas for fin A/B/C and affine A [Du-Fu 13]
- It's obtained in {geom. approaches / Hecke

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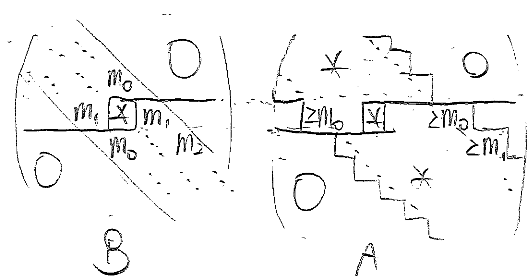
Cor $[B] * [A] = [M] + \text{lower}$ if



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Thm [FL^3W]

(a) $\hat{S}^{\mathfrak{c}}_{n,d}$ admits a CB (w/ positivity)

(b) $\dot{U}^{\mathfrak{c}}(\hat{\mathfrak{gl}}_N)$ admits a CB

Rmk

- CB in (b) doesn't have positivity (observed in [Li-Wang 15] for fin/aff $\mathfrak{gl}_N$)
- CB for $\dot{U}^{\mathfrak{c}}(\hat{\mathfrak{sl}}_N)$ has positivity (wrt mult. and comult.) (see [Fan-Li 15] for fin C, comult.)

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Prop [FL^3W] (Schur-type duality)

$\dot{U}^{\mathfrak{c}}(\hat{\mathfrak{sl}}_N) \subsetneq \dot{U}^{\mathfrak{c}}(\hat{\mathfrak{gl}}_N)$

$\downarrow$

$\hat{S}^{\mathfrak{c}}_{n,d} \curvearrowright \mathbb{V}^{\otimes d} \curvearrowleft \hat{\mathcal{H}}^{\mathfrak{c}}_d$

Prop [FL^3W] We have QSP:

$(U(\hat{\mathfrak{gl}}_N), U^{\mathfrak{c}}(\hat{\mathfrak{gl}}_N))$, $(U(\hat{\mathfrak{sl}}_N), U^{\mathfrak{c}}(\hat{\mathfrak{sl}}_N))$

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