Quantum Schur-type dualities of affine type B

Chun-Ju Lai

University of Virginia

cl8ah@virginia.edu

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(joint work with L. Luo and W. Wang)

Outline

- (old) Schur-type dualities (of finite type A)

$$\vdots$$

- (new) Schur-type dualities of affine type B [?]

Schur-type dualities of type A

$V := \mathbb{C}^n$ natural representation of general linear Lie algebra $\mathfrak{gl}_n$

[Classical Schur duality]	$U(\mathfrak{gl}_n)$ universal enveloping algebra	$\curvearrowright$	$V^{\otimes d}$	$\curvearrowleft$	$\mathbb{C}[S_d]$ group algebra of symmetric group
	$\downarrow$ q-deformation				$\downarrow$ q-deformation
[Schur-Jimbo duality]	$U_q(\mathfrak{gl}_n)$ quantum group	$\curvearrowright$	$V_q^{\otimes d}$	$\curvearrowleft$	$\mathcal{H}_d^A$ Hecke algebra

$V_q = \mathbb{Q}(q)^n$ is the natural representation of $U_q(\mathfrak{gl}_n)$.

Hecke algebra (of finite type A)

The Hecke algebra $\mathcal{H}_d^A$ is a $\mathbb{Q}(q)$-algebra generated by

$$\{T_i \mid i = 1, \ldots, d-1\}$$

subject to

- Braid relations (among T_i's)
- Hecke relations $(T_i - q^{-1})(T_i + q) = 0$.

Hecke algebra action (of finite type A)

- $V_q = \sum_i \mathbb{Q}(q) v_i$
- e.g. $d = 2$
  $V_q^{\otimes 2} \curvearrowleft \mathcal{H}_2^A$ by

$$(v_a \otimes v_b)T_1 = \begin{cases} v_b \otimes v_a & \text{if } b > a \\ q^{-1} v_a \otimes v_b & \text{if } b = a \\ v_b \otimes v_a + (q^{-1} - q) v_a \otimes v_b & \text{if } b < a \end{cases}$$

Specializing $q = 1 \Rightarrow (v_a \otimes v_b)T_1 = v_b \otimes v_a = (v_a \otimes v_b)(12)$

Quantum group (of finite type A)

The quantum group $U_q(\mathfrak{gl}_n)$ is a $\mathbb{Q}(q)$-algebra generated by

$$\{E_i, F_i, D_j, D_j^{-1}\}$$

subject to

- q-Chevalley relations
- q-Serre relations

Remark: The quantum group $U_q(\mathfrak{sl}_n) = \{E_i, F_i, K_i, K_i^{-1}\}$ is related via the embedding

$$\begin{array}{ccc} U_q(\mathfrak{sl}_n) & \hookrightarrow & U_q(\mathfrak{gl}_n) \\ K_i & \mapsto & D_i D_{i+1}^{-1} \end{array}$$

Schur-Jimbo duality

$$\underset{\text{quantum group}}{U_q(\mathfrak{gl}_n)} \quad \overset{\psi}{\curvearrowright} \quad V_q^{\otimes d} \quad \overset{\varphi}{\curvearrowleft} \quad \underset{\text{Hecke algebra}}{\mathcal{H}_d^A}$$

Schur-Jimbo duality (Jimbo1986)

The algebras $U_q(\mathfrak{gl}_n)$ and $\mathcal{H}_d^A$ satisfy double centralizer property. That is,

$$\begin{aligned} \mathrm{End}_{\psi(U_q(\mathfrak{gl}_n))}(V_q^{\otimes d}) &= \varphi(\mathcal{H}_d^A) \\ \psi(U_q(\mathfrak{gl}_n)) &= \mathrm{End}_{\varphi(\mathcal{H}_d^A)}(V_q^{\otimes d}) \end{aligned}$$

Quantum Schur duality of finite type A

q-Schur algebra $S_q^A(n,d) := \mathrm{End}_{\mathcal{H}_d^A}\left(V_q^{\otimes d}\right)$.

$$\begin{array}{ccccc} U_q(\mathfrak{gl}_n) & & & & \\ \text{quantum group} & & & & \\ \downarrow & & & & \\ S_q^A(n,d) & \curvearrowright & V_q^{\otimes d} & \curvearrowleft & \mathcal{H}_d^A \\ q\text{-Schur algebra} & & & & \text{Hecke algebra} \end{array}$$

Proposition (Dipper-James1991)

The algebras $S_q^A(n,d)$ and $\mathcal{H}_d^A$ satisfy double centralizer property.

Affine Hecke algebra of type B

$$\underset{0}{\bigcirc} \Longleftarrow \underset{1}{\bigcirc} \text{———} \cdots \text{———} \underset{d-1}{\bigcirc}$$

The affine Hecke algebra $\widehat{\mathcal{H}}_d^B$ is a $\mathbb{Q}(q)$-algebra generated by

$$\{T_i \mid i = 0, 1 \ldots, d-1\} \cup \{X_j, X_j^{-1} \mid j = 1, \ldots, d\}$$

such that

- $\langle T_i \rangle \simeq \mathcal{H}_d^B$
- $\langle X_j, X_j^{-1} \rangle$ is a Laurent polynomial ring
- T_i, X_j satisfy some mixed relations

Affine Hecke algebra action of type B

- $V_q = \sum_{i \in I} \mathbb{Q}(q) v_i$ where $I = \left\{-n + \frac{1}{2}, \ldots, n - \frac{1}{2}\right\}$
- $V_q^{\otimes d} \curvearrowleft T_0 \approx q$-sign change on the 1st copy.
- We have an explicit but complicated description [?] of $\widehat{\mathcal{H}}_d^B$-action on the infinite-dimensional tensor space

$$\mathbb{V}_q = V_q \otimes \mathbb{Q}(q)[z, z^{-1}]$$

- We define the q-Schur algebra $\widehat{S}_q^\imath(n, d)$ of affine type B similarly, i.e.

$$\widehat{S}_q^\imath(n, d) := \mathrm{End}_{\widehat{\mathcal{H}}_d^B}\left(\mathbb{V}_q^{\otimes d}\right)$$

Quantum Schur duality of affine type B

$$\widehat{S}^{\imath}_q(n,d) \curvearrowright \mathbb{V}_q^{\otimes d} \curvearrowleft \widehat{\mathcal{H}}_d^B$$

affine q-Schur algebra | affine Hecke algebra

Proposition (L-Luo-Wang, 2014)

The algebras $\widehat{S}^{\imath}_q(n,d)$ and $\widehat{\mathcal{H}}_d^B$ satisfy double centralizer property.

Quantum Schur-type dualities

finite type A	affine type A
$U_q(\mathfrak{gl}_n)$ $\downarrow$ $S_q^A(n,d) \curvearrowright V_q^{\otimes d} \curvearrowleft \mathcal{H}_d^A$	$U_q(\widehat{\mathfrak{gl}}_n)$ $\downarrow$ $\widehat{S}_q^A(n,d) \curvearrowright \mathbb{V}_q^{\otimes d} \curvearrowleft \widehat{\mathcal{H}}_d^A$
finite type B	**affine type B**
$U^\imath$ $\downarrow$ $S_q^\imath(n,d) \curvearrowright V_q^{\otimes d} \curvearrowleft \mathcal{H}_d^B$	$\widehat{U}^\imath$ $\downarrow$ $\widehat{S}_q^\imath(n,d) \curvearrowright \mathbb{V}_q^{\otimes d} \curvearrowleft \widehat{\mathcal{H}}_d^B$

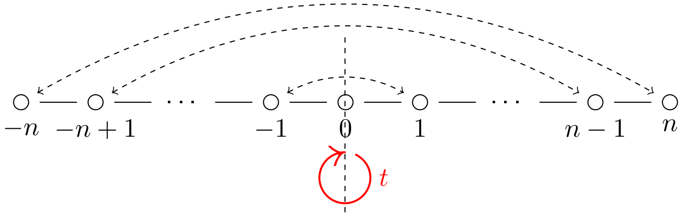
Remark: There is other type B duality replacing $U_q(\mathfrak{gl}_n)$ by $U_q(\mathfrak{so}_{2n+1})$ and $\mathcal{H}_d^B$ by q-Brauer algebra.

Coideal subalgebra (of finite type B)

- The algebra $U^\imath$ is generated by

$$\{e_i, f_i, k_i, k_i^{-1}\} \cup \{t\}$$

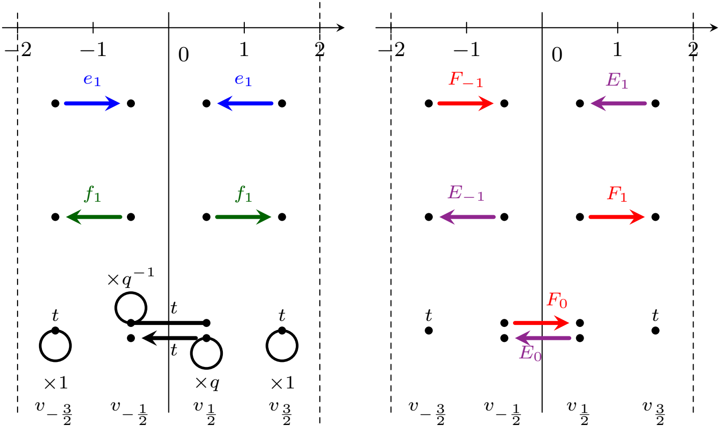
  subject to some Serre-type relations
- t arises from the Dynkin diagram involution



- $U^\imath$ is a subalgebra of $U_q(\mathfrak{sl}_\bullet) = \langle E_i, F_i, K_i, K_i^{-1} \rangle$
- $(U_q(\mathfrak{sl}_\bullet), U^\imath)$ form a quantum symmetric pair

Coideal subalgebra action on V_q

- $U^\imath$-action v.s. U-action



$\imath$-Schur duality of finite type B

$$\begin{array}{ccccccc} U_q(\mathfrak{sl}_\bullet) \supseteq & \underset{\text{coideal subalgebra}}{U^\imath} & \curvearrowright & V_q^{\otimes d} & \curvearrowleft & \underset{\text{Hecke algebra}}{\mathcal{H}_d^B} \\ & \downarrow & & & & \\ & \underset{q\text{-Schur algebra}}{S_q^B(n,d)} & \curvearrowright & V_q^{\otimes d} & \curvearrowleft & \underset{\text{Hecke algebra}}{\mathcal{H}_d^B} \end{array}$$

Theorem (Bao-Wang, 2013)

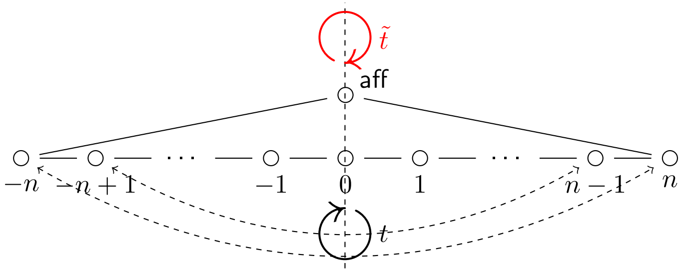
1. The algebras $U^\imath$ and $\mathcal{H}_d^B$ satisfy double centralizer property.
2. $(U, U^\imath)$ gives a new formulation of the KL theory for Lie algebras of type B

Constructing $\widetilde{U}^\imath$ in affine type B

- The algebra $\widetilde{U}^\imath$ is a $\mathbb{Q}(q)$-algebra generated by

$$\{e_i, f_i, k_i, k_i^{-1}\} \cup \{t, \widetilde{t}\}$$

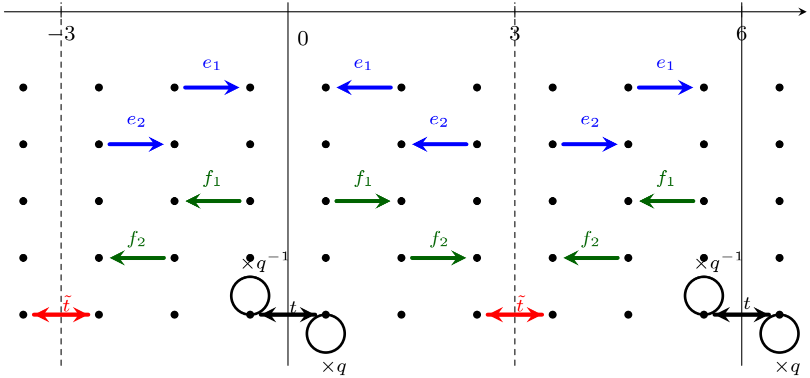
subject to similar Serre-type relations
- $\tilde{t}$ arises from the Dynkin diagram involution



- There is an embedding $\widetilde{U}^\imath \hookrightarrow U_q(\widehat{\mathfrak{sl}}_\bullet)$ and $(U_q(\widehat{\mathfrak{sl}}_\bullet), \widetilde{U}^\imath)$ form a quantum symmetric pair

$\widetilde{U}^\imath$-action on $\mathbb{V}_q$

- $\widetilde{U}^\imath$-action $\approx$ periodic $U^\imath$-action, while $\tilde{t}$ is slightly different.



$\imath$-Schur duality of affine type B

$$U_q(\widehat{\mathfrak{sl}}_\bullet) \supseteq \widetilde{U}^\imath \quad \curvearrowright \quad \mathbb{V}_q^{\otimes d} \quad \curvearrowleft \quad \widehat{\mathcal{H}}_d^B$$

affine Hecke algebra

Theorem (L-Luo-Wang2014)

The actions of $\widetilde{U}^\imath$ and $\widehat{\mathcal{H}}_d^B$ on $\mathbb{V}_q^{\otimes d}$ commute

Question: How do we achieve double centralizer property?

$\imath$-Schur duality of affine type B

We plan to "extend" $\widetilde{U}^\imath$ to an affine coideal subalgebra $\widehat{U}^\imath \subseteq U_q(\widehat{\mathfrak{gl}}_\bullet)$ by adding some central elements

$$\begin{tikzcd}
U_q(\widehat{\mathfrak{sl}}_\bullet) \arrow[d, hook] & \supseteq & \widetilde{U}^\imath \arrow[d, hook] & & & & \\
U_q(\widehat{\mathfrak{gl}}_\bullet) & \supseteq & \underset{\text{affine coideal subalgebra}}{\widehat{U}^\imath} \arrow[d, two heads] & \curvearrowright & \mathbb{V}_q^{\otimes d} & \curvearrowleft & \underset{\text{affine Hecke algebra}}{\widehat{\mathcal{H}}_d^B} \\
& & \underset{\text{affine } q\text{-Schur algebra}}{\widehat{S}_q^\imath(n,d)} & & & &
\end{tikzcd}$$

We expect (work in progress)

1. $\widehat{U}^\imath$ and $\widehat{\mathcal{H}}_d^B$ satisfy double centralizer property.
2. $\widetilde{U}^\imath$ and $\widehat{\mathcal{H}}_d^B$ satisfy double centralizer property if $n > d$.

Remark: Similar result holds for affine type A [Ginzburg-Vasserot, Lusztig, Green].

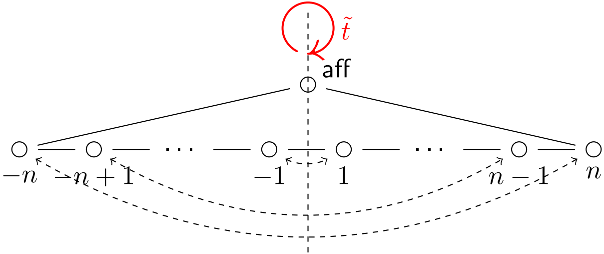
$\imath$-Schur duality of affine type B

Remark: There are two Schur-type dualities depending on the parity of $\dim V_q$

- $\widetilde{U}^\imath$ has a counterpart $\widetilde{U}^\jmath$ generated by

$$\{e_i, f_i, k_i, k_i^{-1}\} \cup \{\widetilde{t}\}$$

- $\widetilde{t}$ arises from the Dynkin diagram involution.



- The above results hold for $\widetilde{U}^\jmath$

Thank you for your attention