Affine Hecke algebras and Quantum Symmetric Pairs

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The BLM construction

Beilinson, Lusztig and MacPherson ('90) developed a geometric construction for (idempotented) quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ together with canonical basis.

An overview:

1. Convolution algebra $\mathbf{S}_{n,d}$ via partial flag varieties
2. Multiplication formulas on $\mathbf{S}_{n,d}$ with Chevalley generators
3. Monomial basis for $\mathbf{S}_{n,d}$
4. Stabilization algebra $\dot{\mathbf{K}}_n = \underset{\longleftarrow}{\mathrm{Stab}}\, \mathbf{S}_{n,d}$ as $d \to \infty$

$\Rightarrow$ $\dot{\mathbf{K}}_n \simeq \dot{\mathbf{U}}(\mathfrak{gl}_n)$ admits canonical bases

Convolution algebra of pairs of partial flags

- Fix $n, d \in \mathbb{N}$, let
  $\mathcal{X}_{n,d} = \{n\text{-step flags in } \mathbb{F}_q^d\}$: partial flag variety
  $GL_d = GL_d(\mathbb{F}_q)$: general linear group

- $GL_d \curvearrowright \mathcal{X}_{n,d}$

$\Rightarrow$ $GL_d \curvearrowright \mathcal{X}_{n,d} \times \mathcal{X}_{n,d}$ diagonally

$\Rightarrow$ Convolution algebra

$$\mathbf{S}_{n,d} := \{GL_d\text{-invariant functions on } \mathcal{X}_{n,d} \times \mathcal{X}_{n,d}\}$$

Chevalley generators

- The bases for $\mathbf{S}_{n,d}$ are parametrized by

$$\begin{array}{ccc} \{GL_d\text{-orbits on } \mathcal{X}_{n,d} \times \mathcal{X}_{n,d}\} & \xleftrightarrow{1:1} & \Theta^{\text{fin}}_{n,d} = \left\{A \in \text{Mat}_{n\times n}(\mathbb{N}) \,\middle|\, \sum a_{ij} = d\right\} \\ \mathcal{O}_A & \leftrightarrow & A \end{array}$$

- $\{[A] \mid A \in \Theta^{\text{fin}}_{n,d}\}$: standard basis, i.e.
  $[A]$ = normalized characteristic function on orbit $\mathcal{O}_A$
- A (divided power of) Chevalley generator $\mathbf{e}_i^{(R)} \leftrightarrow$

$$B = \begin{bmatrix} * & & & & \\ & \ddots & & & \\ & & * & R & \\ & & & \ddots & \\ & & & & * \end{bmatrix}$$

Multiplication formulas (finite A)

Lemma (Beilinson-Lusztig-MacPherson 90)

Assume that $B \leftrightarrow$ Chevalley generator $\mathbf{e}_i^{(R)}$. Then

$$[B] * [A] = \sum_{\substack{T \leq A \\ (+\text{ cond.})}} (\text{coeff}) \; [A - T + \widehat{T}],$$

where $A - T + \widehat{T} \leftarrow\!\!\sim$ shifting T up by 1 row in A.

- (There is also a counterpart for $\mathbf{f}_i^{(R)}$)
- The Chevalley generators form a generating set of $\mathbf{S}_{n,d}$

Multiplication formulas (finite A)

Example

$$B = \begin{bmatrix} * & & \\ & * & 3 \\ & & * \end{bmatrix} \leftrightarrow \mathbf{e}_2^{(3)}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

We have $[B] * [A] = (\text{coeff})[A_1] + (\text{coeff})[A_2]$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 4+3 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2+1 & 1 & 4+2 \\ 0 & 0 & 1 \end{bmatrix}$$

Monomial basis (finite A)

Facts

- [BLM 90] Multiplying Chevalley generators in a suitable order $\Rightarrow$ monomial basis $\{m_A\}_A$ for $\mathbf{S}_{n,d}$.
- monomial basis $\Rightarrow$ canonical basis

$$
\begin{tikzcd}
\underset{\text{convolution algebras}}{\mathbf{S}_{n,d}\ (d \in \mathbb{N})} \arrow[r, Rightarrow] & \underset{\text{stabilization algebra}}{\dot{\mathbf{K}}_n} \arrow[r, phantom, "\simeq"] & \underset{\text{idempotented quantum group}}{\dot{\mathbf{U}}(\mathfrak{gl}_n)} \\
\text{monomial basis} \arrow[r, mapsto] & \text{monomial basis} \arrow[d, Rightarrow] & \\
 & \text{canonical basis} &
\end{tikzcd}
$$

Another approach

- $\mathbf{S}_{n,d}$ admits a Schur-type duality

$$\begin{array}{ccccc} \mathbf{U}(\mathfrak{gl}_n) & & & & \\ \downarrow & & & & \\ \mathbf{S}_{n,d} & \curvearrowright & \underset{\text{tensor space}}{\mathbb{V}^{\otimes d}} & \curvearrowleft & \underset{\text{Hecke algebra}}{\mathbf{H} = \mathbf{H}(\mathfrak{S}_d)} \end{array}$$

- In other words, $\mathbf{S}_{n,d}$ = *q-Schur algebra* as an endomorphism algebra

$$\mathbf{S}_{n,d} = \mathrm{End}_{\mathbf{H}}(\mathbb{V}^{\otimes d})$$

where $\mathbb{V}^{\otimes d} \simeq \bigoplus_{\lambda} x_\lambda \mathbf{H}$: permutation module

$\Rightarrow$ The BLM construction can be done in two approaches:

1. (Geometric) partial flags and counting over finite fields
2. (Hecke algebraic) permutation modules and combinatorics

BLM-type constructions

- There are generalizations to other types

Realization of Schur algebras	Partial flag varieties (dimension counting)	Hecke algebras (combinatorics)
Affine A	Ginzburg-Vasserot[1] ('93) Lusztig[1] ('99)	Du-Fu ('14)
Finite B/C	Bao-Kujawa-Li-Wang ('14)	
Finite D	Fan-Li ('14)	

[1] partially generalized

BLM-type constructions

- [Fan-L.-Li-Luo-Wang '16] We provide a BLM-type construction for affine type C in both approaches

Realization of Schur algebras	Partial flag varieties (dimension counting)	Hecke algebras (combinatorics)
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Finite B/C	Bao-Kujawa-Li-Wang ('14)	
Finite D	Fan-Li ('14)	
Affine C	FLLLW1 ('16)	FLLLW2 ('16)

[1] partially generalized

Difficulties in affine type A

- An overview for affine type A (Du and Fu):
  1. Hecke algebraic realization of affine Schur algebra $\mathbf{S}_{n,d}^{\mathrm{affA}}$
  2. Multiplication formulas on $\mathbf{S}_{n,d}^{\mathrm{affA}}$
  3. Monomial basis for $\mathbf{S}_{n,d}^{\mathrm{affA}}$
  4. Stabilization algebra $\dot{\mathbf{K}}_n^{\mathrm{affA}} = \underset{\longleftarrow}{\mathrm{Stab}}\, \mathbf{S}_{n,d}^{\mathrm{affA}}$ as $d \to \infty$

⚠ Chevalley generators do not generate affine Schur algebra

$\Rightarrow$ Du and Fu establish 2 with bidiagonal generators instead.

⚠ 3 is non-trivial

$\Rightarrow$ Du and Fu use monomial bases from Hall algebras of the cyclic quiver due to Deng-Du-Xiao

Multiplication formulas / monomial basis (affine A)

Lemma (Du-Fu 15)

Let B be upper bidiagonal. Then

$$[B] * [A] = \sum_{\substack{T \leq A \\ (+\text{cond.})}} (\text{coeff}) \ [A - T + \widehat{T}].$$

- (There is also a counterpart for lower bidiagonal generators)
- The bidiagonal generators form a generating set of $\mathbf{S}_{n,d}^{\text{affA}}$

Theorem A (L.-Luo 15)

Multiplying bidiagonal generators in a suitable order
$\Rightarrow$ monomial basis for $\mathbf{S}_{n,d}^{\text{affA}}$.

Difficulties in affine type C

- An overview for affine type C:
  1. Hecke algebraic realization of affine Schur algebra $\mathbf{S}_{n,d}^{\mathrm{affC}}$
  2. Multiplication formulas on $\mathbf{S}_{n,d}^{\mathrm{affC}}$
  3. Monomial basis for $\mathbf{S}_{n,d}^{\mathrm{affC}}$
  4. Stabilization algebra $\dot{\mathbf{K}}_n^{\mathrm{affC}} = \underset{\longleftarrow}{\mathrm{Stab}}\, \mathbf{S}_{n,d}^{\mathrm{affC}}$ as $d \to \infty$

⚠ Chevalley/bidiagonal generators do not generate the entire Schur algebra

$\Rightarrow$ We establish 2 with tridiagonal generators instead

⚠ Du-Fu's monomial bases do not generalize to 3 since no Hall algebra approach is known in affine type C

$\Rightarrow$ We adapt Theorem A

Finite type B/C

- We review first finite type B/C
- Fix $n = 2r + 1$. The bases for $\mathbf{S}^{\text{finBC}}_{n,d}$ are parametrized by

$$\Xi^{\text{fin}}_{n,d} = \left\{ A \in \text{Mat}_{[-r,r]\times[-r,r]}(\mathbb{N}) \;\middle|\; (1),\ (2) \right\}$$

(1) (centro-symmetry) $a_{ij} = a_{-i,-j} \quad (i, j \in \mathbb{Z})$
(2) (size) $\sum a_{ij} = d$ over "half index set"

- A (divided power of) Chevalley generator $\mathbf{e}_i^{(R)} \leftrightarrow$

$$B = \left[\begin{array}{ccc|c|ccc} * & & & & & & \\ & * & & & & & \\ & R & \ddots & & & & \\ \hline & & & b_{00} & & & \\ \hline & & & & \ddots & R & \\ & & & & & * & \\ & & & & & & * \end{array}\right]$$

Multiplication formula (finite B/C)

Let $T^\theta = (t^\theta_{ij})$ is the centro-symmetrization of T, i.e., $t^\theta_{ij} = t_{ij} + t_{-i,-j}$

Lemma (Bao-Kujawa-Li-Wang 14)

Assume that $B \leftrightarrow$ Chevalley generator. Then

$$[B] * [A] = \sum_{\substack{T \text{ s.t.} \\ T^\theta \leq A \\ (+\text{cond.})}} (\text{coeff}) \; [A - T^\theta + (\widehat{T})^\theta],$$

where $A - T^\theta + (\widehat{T})^\theta \rightsquigarrow$ shifting T up by 1 row, and its counterpart down by 1 row in A.

Multiplication formula (finite B/C)

Example

$$B = \begin{bmatrix} * & & \\ 3 & * & 3 \\ & & * \end{bmatrix} \leftrightarrow \mathbf{e}_1^{(3)}, \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

We have $[B] * [A] = (\text{coeff})[A_1] + (\text{coeff})[A_2]$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 2{\color{red}+3} & 1 & 2{\color{red}+3} \\ 1 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ {\color{red}{}_{+2}} & & {\color{red}{}_{+1}} \\ 2 & 1 & 2 \\ {\color{red}{}_{+1}} & & {\color{red}{}_{+2}} \\ 0 & 0 & 1 \end{bmatrix}$$

Affine Schur algebra of type C

- Let's switch back to affine type C.
  The affine Schur algebra $\mathbf{S}^{\mathrm{affC}}_{n,d}$ can be defined:
- (geometric)
  $Sp_d = Sp_d(K)$ = symplectic group over K: formal Laurent series $/\mathbb{F}_q$
  $\mathcal{X}_{n,d} = \{n\text{-step affine partial flags of type C}\} \curvearrowleft Sp_d$

$$\mathbf{S}^{\mathrm{affC}}_{n,d} = \{Sp_d\text{-invariant functions on } \mathcal{X}_{n,d} \times \mathcal{X}_{n,d}\}$$

- (Hecke algebraic)
  $\mathbf{H}^{\mathrm{affC}}_d$ = non-extended affine Hecke algebra of type C_d
  tensor space $\mathbb{V}^{\otimes d} \simeq \bigoplus_\lambda x_\lambda \mathbf{H}^{\mathrm{affC}}_d$: permutation module

$$\mathbf{S}^{\mathrm{affC}}_{n,d} = \mathrm{End}_{\mathbf{H}^{\mathrm{affC}}_d}(\mathbb{V}^{\otimes d})$$

Tridiagonal generators

- Fix $n = 2r + 1$. The bases for $\mathbf{S}^{\mathrm{affC}}_{n,d}$ are parametrized by

$$\Xi_{n,d} := \{A \in \mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{N}) \mid (1) - (3)\},$$

  (1) (periodicity) $a_{ij} = a_{i+n,j+n}$
  (2) (centro-symmetry) $a_{-i,-j} = a_{ij}$
  (3) (size) $\sum a_{ij} = d$ over any "half period"

- A tridiagonal generator $\leftrightarrow$

$$B = \left[\begin{array}{c|c|cc|c|c}
\ddots & R_n & & & & \\
\hline
R_1 & b_{00} & R_1 & & & \\
\hline
 & R_n & * & \ddots & & \\
 & & \ddots & * & R_r & \\
\hline
 & & & R_{r+1} & b_{rr} & R_{r+1} \\
\hline
 & & & & R_r & \ddots
\end{array}\right]$$

Multiplication formula (affine C)

Theorem B

Let B be tridiagonal. Then

$$[B] * [A] = \sum_{\substack{T^\theta \leq A \\ (+\text{cond.})}} \sum_{\substack{S \leq T \\ (+\text{cond.})}} (\text{coeff}) \ [A - (T-S)^\theta + (\widehat{T-S})^\theta],$$

where $A - (T-S)^\theta + (\widehat{T-S})^\theta \leadsto$ shifting $T - S$ up by 1 row, and its counterpart down by 1 row in A.

- The tridiagonal generators form a generating set of $\mathbf{S}^{\text{affC}}_{n,d}$

Monomial basis (affine C)

Theorem C

The Schur algebra $\mathbf{S}^{\mathrm{affC}}_{n,d}$ admits both monomial and canonical bases.

Example

Monomial basis element m_A is obtained in the following sense

$$A = \left[\begin{array}{c|c|cccc} \ddots & 4 & 5 & & & \\ \hline 1 & 0 & 1 & 2 & & \\ \hline 5 & 4 & 3 & 4 & 5 & \\ & 2 & 1 & 0 & 1 & 2 \\ & & 5 & 4 & \ddots & \end{array}\right] \leftsquigarrow \left[\begin{array}{c|c|cccc} \ddots & 6 & & & & \\ \hline 6 & 0 & 6 & & & \\ \hline & 6 & 3 & 6 & & \\ & & 6 & 0 & 6 & \\ & & & 6 & \ddots & \end{array}\right] \leftsquigarrow \left[\begin{array}{c|c|ccc} \ddots & & & & \\ \hline & 12 & & & \\ \hline & & 15 & & \\ & & & 12 & \\ & & & & \ddots \end{array}\right].$$

Stabilization algebra (affine C)

$$\begin{array}{ccc} \mathbf{S}^{\mathrm{affC}}_{n,d}\ (d \in \mathbb{N}) & \Longrightarrow & \dot{\mathbf{K}}^{\mathrm{affC}}_{n} \\ \text{Affine } q\text{-Schur algebras} & & \text{Stabilization algebra} \\ \text{monomial basis} & \longmapsto & \text{monomial basis} \\ & & \Big\Downarrow \\ & & \text{canonical basis} \end{array}$$

Theorem D

We have an algebra $\dot{\mathbf{K}}^{\mathrm{affC}}_{n}$ from stabilization on $\mathbf{S}^{\mathrm{affC}}_{n,d}$. Moreover, $\dot{\mathbf{K}}^{\mathrm{affC}}_{n}$ admits both monomial and canonical bases.

Quantum symmetric pairs

- A quantum symmetric pair $(\mathbf{U}, \mathbf{B})$ (cf. Letzter('02), Kolb('14)) is a q-deformation of the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ where

$$\begin{aligned} \mathfrak{g} &: \text{ Lie algebra} \\ \theta &: \text{ involution on } \mathfrak{g} \\ \mathfrak{g}^\theta &: \text{ fixed-point subalgebra} \end{aligned}$$

- One essential property of quantum symmetric pairs is that $\mathbf{B}$ is a coideal subalgebra of $\mathbf{U}$, i.e., the comultiplication $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ satisfies that

$$\Delta(\mathbf{B}) \subset \mathbf{B} \otimes \mathbf{U}$$

Non-idempotented quantum algebras

By taking certain infinite sum, one can obtain the non-idempotented stabilization algebras:

$$\dot{\mathbf{K}}_n \Rightarrow \mathbf{K}_n \simeq \mathbf{U}(\mathfrak{gl}_n) \quad \text{(finA)}$$

$$\dot{\mathbf{K}}_n^{\text{affA}} \Rightarrow \mathbf{K}_n^{\text{affA}} \simeq \mathbf{U}(\widehat{\mathfrak{gl}}_n) \quad \text{(affA)}$$

$$\dot{\mathbf{K}}_n^{\text{finBC}} \Rightarrow \mathbf{K}_n^{\text{finBC}} \simeq \mathbf{iU}(\mathfrak{gl}_n) \quad \text{(finBC)}$$

$(\mathbf{U}(\mathfrak{gl}_n), \mathbf{iU}(\mathfrak{gl}_n))$ forms a quantum symmetric pair

$$\dot{\mathbf{K}}_n^{\text{affC}} \Rightarrow \mathbf{K}_n^{\text{affC}} \quad \text{(affC)}$$

Proposition

The pair $(\mathbf{U}(\widehat{\mathfrak{gl}}_n), \mathbf{K}_n^{\text{affC}})$ forms a quantum symmetric pair.

Thank you for your attention