Affine Hecke algebras & quantum symmetric pairs

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joint w/ Fan, Li, Luo & Wang

- §1 finite type A
- §2 beyond type A
- §3 QSP

- [BLM'90] fin A

  (geom.) const'n for $\mathbf{U}_q(\mathfrak{gl}_n)$ via $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ w/ CB

<u>Idea</u>: fix n

$\{\text{Conv. alg } S_{nd} \mid d \geq 1\} \overset{(\mathrm{III})}{\Longrightarrow}$ Stabilization alg.

(I) Mult. formula $\qquad \underset{d\in\mathbb{N}}{\underleftarrow{\mathrm{Stab}}}\, S_{n,d} \cong \dot{\mathbf{U}}_q(\mathfrak{gl}_n)$

$\Downarrow$

(II) Monomial basis (MB) $\underset{\text{lift}}{\rightsquigarrow}$ MB $\Rightarrow$ CB

WTS $\exists$ MB, i.e., $\exists\, m_A$ s.t.

(M1) $\overline{m_A} = m_A$ $\qquad$ (M2) $m_A = [A] + \text{lower}$

$\boxed{\text{Prop/defn}}$ $\exists$ such $m_A = \prod_k [A^{(k)}]$

each $A^{(k)} = \begin{pmatrix} * & & \\ & \ddots\, r & \\ & & * \end{pmatrix} \leftrightsquigarrow$ Chev. $e_i^{(r)}$

or $\begin{pmatrix} * & & \\ r & \ddots & \\ & & * \end{pmatrix}$ $\qquad f_i^{(r)}$

(M1) ✓ b/c $\overline{[\mathrm{Chev}]} = [\mathrm{Chev}]$

(M2) requires mult. formula

Schur duality

$\mathbf{U}_q(\mathfrak{gl}_n)$

$\downarrow$

	tensor sp	Hecke alg
S_{nd} $\curvearrowright$	$V^{\otimes d}$	$\curvearrowleft \mathcal{H}(\mathfrak{S}_d)$
$= \mathcal{A}_G(\mathcal{X}\times\mathcal{X})$	$= \mathcal{A}_G(\mathcal{X}\times\mathcal{Y})$	$= \mathcal{A}_G(\mathcal{Y}\times\mathcal{Y})$
$= \mathrm{End}_{\mathcal{H}}(\bigoplus_\lambda x_\lambda H)$	$= \bigoplus_\lambda x_\lambda H$	$= \langle T_w \mid w\in\mathfrak{S}_d\rangle/\sim$

where $x_\lambda = \sum_{w\in W_\lambda} T_w$, W_λ : parabolic

§1

- [Drinfeld, Jimbo '85]

  $\mathbf{U}_q(\mathfrak{g}) = \langle \text{Chev. gen.}\rangle/\sim$: assoc. alg.

  $=$ q-deform of UEA of Lie alg $\mathfrak{g}$

- [Lusztig '90]

  canonical basis (CB) for $\mathbf{U}^+$ (hence $\dot{\mathbf{U}}$)

  = Kashiwara's global crystal basis

  $\Leftrightarrow$ cat. $\mathcal{O}$, quiver var., categorification

- $S_{nd} = \{GL_d\text{-inv't } f: \mathcal{X}\times\mathcal{X} \to \mathcal{A}\}$

  where $\mathcal{X} = \{n\text{-step flags in } \mathbb{F}_q^d\} \curvearrowleft GL_d$

  $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$

- $\{GL_d\text{-orbits on } \mathcal{X}\times\mathcal{X}\} \ni \mathcal{O}_A$

  $\updownarrow$ 1:1 $\qquad\qquad \uparrow$

  $\Theta := \{A \in \mathrm{Mat}_n(\mathbb{N}) \mid \sum a_{ij} = d\} \ni A$

$\Rightarrow$ basis $\{e_A\}_{A\in\Theta}$, e_A : char fcn on $\mathcal{O}_A$

$\Downarrow$ normalize

$\Rightarrow$ basis $\{[A]\}_{A\in\Theta}$ s.t. $\overline{[A]} = [A] + \text{lower}$

$\boxed{\text{Mult formula}}$ If $B \leftrightarrow$ Chev.

Then $[B][A] = \sum_T q^{\mathrm{pwr}}(q\text{-bino})\,[A^{(T)}]$

$\boxed{\text{Cor}}$

$[B][A] = [M] + \text{lower}$ ($\exists$ ht term & coeff $= 1$)

if (B, A) is admissible

$\prod_k [A^{(k)}]$ is admissible $\Rightarrow$ (M2) ✓

$\Rightarrow \dots \Rightarrow$ $\boxed{\text{Thm}}$ $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ admits CB

§2 BLM-type const'n can be done in

(Geom) flag var. & counting / fin. field

(Alg) Hecke alg & combinatorics

	geom	alg
aff A	GV('93) L('99)	DF ('14)
fin B/C	BKLW ('14)	
aff C	FLLLW 1 ('16)	FLLLW 2 ('16)
fin D	FL ('14)	

Unexpected affine phenomenon

- [Du-Fu '13] aff A

- Schur alg $S^{\hat{A}}_{n,d} := \mathrm{End}_{\mathcal{H}^{\hat{A}}_d}(\bigoplus_\lambda x_\lambda \mathcal{H}^{\hat{A}}_d)$

  ⚠ $\qquad \supsetneq \qquad\qquad =$

  $\qquad \langle\text{Chev.}\rangle \qquad \langle\text{bidiag.}\rangle$

⚠ Const'n of MB is non-trivial

[DF'14] via Hall alg on cyclic quiver

[LL'15] direct const'n

$\boxed{\text{Thm I}}$ If $B \leftrightarrow$ tridiag

Then $[B][A] = \sum_T (q-1)^{\mathrm{pwr}} q^{\mathrm{pwr}}(q\text{-bino})\,[A^{(T)}]$

$\boxed{\text{Thm II}}$ $\exists\, m_A \approx \prod[\text{tridiag}]$

$\boxed{\text{Thm III}}$ $\exists$ Stab'n alg $\dot{\mathbf{U}}^{\tilde{C}}_n := \underleftarrow{\mathrm{Stab}}\, S^{\tilde{C}}_{n,d}$

admitting CB

- [BKLW'14] finite B/C

  $\underleftarrow{\mathrm{Stab}}\, S^{B/C}_{nd} =: \dot{\mathbf{U}}^{B/C}_n \Rightarrow \mathbf{U}^{B/C}_n$

⚠ $\mathbf{U}^{B/C}_n$ is NOT Drinfeld-Jimbo QG

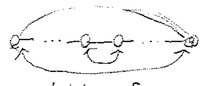
$\boxed{\text{Thm}}$

$(\mathbf{U}_q(\mathfrak{gl}_n), \mathbf{U}^{B/C}_n) = \text{QSP}$

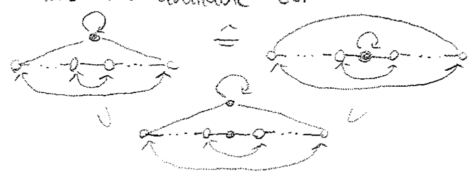
$\boxed{\text{Thm IV}}$

$(\mathbf{U}_q(\widehat{\mathfrak{gl}}_n), \mathbf{U}^{\tilde{C}}_n) = \text{QSP}$

$\boxed{\text{Rmk}}$ The above const'n is associated to



Variants are available for :



- [FLLLW2 '16] aff C

  Schur alg $S^{\tilde{C}}_{nd} := \mathrm{End}_{\mathcal{H}^{\hat{C}}_d}(\bigoplus_\lambda x_\lambda \mathcal{H}^{\hat{C}}_d)$

  ⚠ $\qquad \supsetneq \qquad\qquad =$

  $\qquad \langle\text{bidiag}\rangle \qquad \langle\text{tridiag.}\rangle$

⚠ [tridiag] not bar-inv't

⚠ const'n of MB :

$\not\longrightarrow$ no ~~Hall~~ Hall alg approach in aff C

$\longrightarrow$ can adapt here ✓

§3

- Symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$

  θ : involution on $\mathfrak{g}$: Lie alg

  $\Downarrow$ q-deform.

- [Letzter '02, Kolb '14]

  QSP $(\mathbf{U}, \mathbf{B}) = (\mathbf{U}_q(\mathfrak{g}), \mathbf{B}(\theta))$ s.t.

  $\mathbf{B}$ is a coideal subalg of $\mathbf{U}$ (& more)

  (i.e. $\Delta(\mathbf{B}) \subset \mathbf{B}\otimes\mathbf{U}$)

$\boxed{\text{Rmk}}$

- [BW'13, B'15] X: classical

  CB for $\underleftarrow{\mathrm{Stab}}\, S^X_{nd}$ gives new formulation for KL theory (of cat $\mathcal{O}$ of Lie alg)

  $\Rightarrow$ solves irred char prob for Lie superalg osp for the first time

Q: How about CB for $\underleftarrow{\mathrm{Stab}}\, S^{\hat{X}}_{nd}$ / QSP ?

Examples of QSP

(1) Reflection eqns ($\mathfrak{g}$: classical)

(2) Onsager alg from Ising model ($\mathfrak{g} = \widehat{\mathfrak{sl}}_2$)

(3) (twisted) Yangians ($\mathfrak{g} = \widehat{\mathfrak{gl}}_n$)

(4) GIM Lie alg. ($\mathfrak{g}$ = sym'able KM)

(5) BLM const'n $\begin{pmatrix} \mathfrak{g} = \mathfrak{sl}_n, \mathfrak{gl}_n \\ \widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{gl}}_n \end{pmatrix}$