From Schur-Weyl duality to quantum symmetric pairs

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Outline

Background

- $\mathbf{GL}_n := \mathbf{GL}_n(\mathbb{C})$ = general linear group of $\mathbb{C}^n$
- Schur's 1901 dissertation on the polynomial representations of $\mathbf{GL}_n$:
  - Each polynomial representation V is semi-simple (i.e. $V = \bigoplus$ irreducible modules)
  - Each polynomial representation is homogeneous for some degree $d \in \mathbb{N}$
  - There is a correspondence

$$\left\{\begin{array}{c}\text{polynomial representation}\\ \text{of } \mathbf{GL}_n \text{ of degree } d\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{representation of}\\ \text{symmetric group } \mathfrak{S}_d\end{array}\right\}$$

$$\{\text{irreducibles}\} \leftrightarrow \{\text{irreducibles}\}$$

$$\updownarrow$$

$$\{\text{partitions } \lambda = (\lambda_1, \ldots, \lambda_n) \vdash d\}$$

Background

- Weyl's 1926 research on representation of semi-simple Lie group
  - based on repn theory of Lie algebra, integration over compact form
  - has no counterpart to $\mathfrak{S}_d$ in Schur's method
- In 1927, Schur re-derived his 1901 dissertation by showing the double centralizer property for $(\mathbf{GL}_n, \mathfrak{S}_d)$, which was publicized by Weyl in his 1939 book "The Classic Groups".
- The methods used in the classical Schur-Weyl are still important in representation theory today

Dual actions

- $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$: general linear Lie algebra
  $V := \mathbb{C}^n$ natural representation of $\mathfrak{gl}_n$
  $\Rightarrow$ action on $V^{\otimes d}$ by the Leibniz rule, e.g.,

$$g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes v_2 + v_1 \otimes (g \cdot v_2)$$

  so that its exponential e^{tg} acts group-like –

$$e^{tg} \cdot (v_1 \otimes v_2) = e^{tg} \cdot v_1 \otimes e^{tg} \cdot v_2$$

- $\mathfrak{S}_d$ has a (right) action on $V^{\otimes d}$ by permuting tensor factors, e.g.,

$$(v_1 \otimes v_2) \cdot (12) = v_2 \otimes v_1$$

$$\begin{array}{ccccc} \mathfrak{gl}_n & \overset{\psi}{\curvearrowright} & V^{\otimes d} & \overset{\varphi}{\curvearrowleft} & \mathfrak{S}_d \\ \text{general linear Lie algebra} & & \text{tensor space} & & \text{symmetric group} \end{array}$$

Example: commutativity

Here's an example showing $\Big(g(v_1 \otimes v_2)\Big)(12) = g\Big((v_1 \otimes v_2)(12)\Big)$

$$\begin{tikzcd}
V \otimes V \arrow[rrr, "(12)"] \arrow[ddd, "g"'] & & & V \otimes V \arrow[ddd, "g"] \\
& v_1 \otimes v_2 \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & v_2 \otimes v_1 \arrow[d, mapsto] & \\
& g v_1 \otimes v_2 + v_1 \otimes g v_2 \arrow[r, mapsto] & g v_2 \otimes v_1 + v_2 \otimes g v_1 & \\
V \otimes V \arrow[rrr, "(12)"'] & & & V \otimes V
\end{tikzcd}$$

Double Centralizer Property

- We have $\psi : \mathfrak{gl}_n \to \mathrm{End}(V^{\otimes d})$, $\varphi : \mathfrak{S}_d \to \mathrm{End}(V^{\otimes d})$ and

$$\underset{\text{general linear Lie algebra}}{\mathfrak{gl}_n} \overset{\psi}{\curvearrowright} \underset{\text{tensor space}}{V^{\otimes d}} \overset{\varphi}{\curvearrowleft} \underset{\text{symmetric group}}{\mathfrak{S}_d}$$

Schur-Weyl duality (1927)

1. The actions of $\mathfrak{gl}_n$ and $\mathfrak{S}_d$ on the tensor space $V^{\otimes d}$ commute
2. The algebras generated by the actions of $\mathfrak{gl}_n$ and $\mathfrak{S}_d$ in $\mathrm{End}(V^{\otimes d})$ are centralizing algebras of each other, i.e.,

$$\mathrm{End}_{\varphi(A)}(V^{\otimes d}) = \psi(B), \quad \mathrm{End}_{\psi(B)}(V^{\otimes d}) = \varphi(A),$$

where $A = \mathbb{C}[\mathfrak{S}_d]$ = group algebra, $B = U(\mathfrak{gl}_n)$ = univ env algebra

Double centralizer property

- The double centralizer property leads to

Corollary

There is a decomposition

$$V^{\otimes d} = \bigoplus_{\lambda \vdash d} V_\lambda \otimes L_\lambda,$$

where $\{V_\lambda\} = \mathrm{Irrep}(\mathfrak{S}_d)$;
$\{L_\lambda\}$ are distinct irreducibles of $\mathfrak{gl}_n$ or 0.

Quantization

$$\begin{array}{ccccc} \underset{\text{univ env algebra}}{U(\mathfrak{gl}_n)} & \curvearrowright & (\mathbb{C}^n)^{\otimes d} & \curvearrowleft & \underset{\text{group algebra of sym group}}{\mathbb{C}[\mathfrak{S}_d]} \\ \text{quantization}\ \Big\downarrow & & & & \Big\downarrow\ \text{quantization} \\ \underset{\text{quantum group}}{\mathbf{U}_q(\mathfrak{gl}_n)} & \curvearrowright & (\mathbb{Q}(v)^n)^{\otimes d} & \curvearrowleft & \underset{\text{Hecke algebra}}{\mathbf{H}(\mathfrak{S}_d)} \end{array}$$

Here the quantized algebras are over $\mathbb{Q}(v)$ with $v = q^{1/2}$: indeterminate

Hecke algebra

- Recall that the symmetric group $\mathfrak{S}_d$ is generated by simple reflections

$$s_1, s_2, \dots, s_{d-1}$$

  subject to the braid relations and $s_i^2 = 1$.
- Hecke algebra $\mathbf{H}(\mathfrak{S}_d)$ is a $\mathbb{Q}(v)$-algebra generated by

$$T_1, T_2, \dots, T_{d-1}$$

  subject to braid relations and the Hecke relations

$$(T_i + 1)(T_i - q) = 0. \tag{1}$$

- Specializing $q \to 1$, (1) recovers $s_i^2 = 1$

Hecke algebra

- $\mathbf{H}(\mathfrak{S}_d)$ has a linear basis (called standard basis)

$$\{T_w \mid w \in \mathfrak{S}_d\}$$

- $\mathbf{H}(\mathfrak{S}_d)$ has a involution $^- : \mathbf{H}(\mathfrak{S}_d) \to \mathbf{H}(\mathfrak{S}_d)$ sending

$$v \mapsto v^{-1}, \quad T_w \mapsto T_{w^{-1}}^{-1}$$

- $\mathbf{H}(\mathfrak{S}_d)$ has a bar-invariant basis $\{C_w\}_{w \in \mathfrak{S}_d}$ called the Kazhdan-Lusztig(KL) basis
- The famous Kazhdan-Lusztig theory ('79) offered a solution the the difficult problem of determining irreducible character problem for category $\mathcal{O}$ of semisimple Lie algebras using the KL basis

Quantum groups

- Recall that the universal enveloping algebra $U(\mathfrak{gl}_n)$ is an associative algebra generated by
$$\{e_i, f_i, d_j, d_j^{-1}\}, \quad \text{subject to}$$
Chevalley/Serre relations (e.g. $e_1^2 e_2 + e_2 e_1^2 = 2e_1 e_2 e_1$)
- Around 1985, Drinfeld and Jimbo introduced the quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{gl}_n)$ as a $\mathbb{Q}(v)$-algebra generated by
$$\{E_i, F_i, D_j, D_j^{-1}\}, \quad \text{subject to}$$
q-Chevellay/Serre relations (e.g. $E_1^2 E_2 + E_2 E_1^2 = (v + v^{-1}) E_1 E_2 E_1$)

Quantum groups

- $\mathbf{U}$ provides solutions to the Yang-Baxter equation
- $\mathbf{U}$ is a Hopf algebra. Particularly it has a comultiplication $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$
- There is a triangular decomposition $\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$
- There is a modified (i.e. idempotented) quantum group $\dot{\mathbf{U}}$:

$$\underset{\text{quantum group}}{\mathbf{U}} \quad \underset{\text{take certain infinite sum}}{\overset{\text{replacing } \mathbb{1} \text{ with idempotents}}{\rightleftharpoons}} \quad \underset{\text{modified quantum group}}{\dot{\mathbf{U}}}$$

$$\{\dot{\mathbf{U}}\text{-modules}\} \overset{1:1}{\leftrightarrow} \{\text{weight modules of } \mathbf{U}\}$$

- $\dot{\mathbf{U}}$ is viewed as a preadditive category in categorification

Canonical basis

- $\mathbf{U}$ has a sheaf theoretic interpretation

$\Rightarrow$ bar involution $^-: \mathbf{U} \to \mathbf{U}$ from the Verdier duality

- $\mathbf{U}^-$ admits canonical basis, i.e., $\mathbf{U}^-$ has a standard basis $\{a_i\}_{i\in I}$ that is "unitriangular" and "integral":

$$\overline{a}_i \in a_i + \sum_{j<i} \mathbb{Z}[v, v^{-1}] a_j,$$

and a (unique) canonical basis $\{b_i\}_{i\in I}$ that is "bar-invariant" and "positive":

$$\overline{b}_i = b_i \in a_i + \sum_{j<i} v^{-1}\mathbb{N}[v^{-1}] a_j$$

- This is analogous to (dual) Kazhdan-Lusztig basis for Hecke algebra

Canonical basis

- Canonical basis = Kashiwara's global crystal basis, i.e., for any weight λ,

  $\{b_i v_\lambda\}$ is a basis of irreducible integrable $\mathbf{U}$-module $L(\lambda) := \mathbf{U}^- v_\lambda$

- Canonical basis $\{b_i\}$ has positivity, i.e.

$$b_i b_j \in \sum_k \mathbb{N}[v, v^{-1}] b_k$$

- $\mathbf{U}$ does not admit canonical basis, while $\dot{\mathbf{U}}$ does
- Existence of canonical basis has connection/application in
  - Categorification
  - Algebraic combinatorics
  - Category $\mathcal{O}$
  - Cluster algebra
  - Geometric representation theory

Outline

q-Schur duality of type A

Our goal here is to describe the quantized Schur-Weyl duality below in examples:

$$\underset{\text{quantum group}}{\mathbf{U}_q(\mathfrak{gl}_n)} \overset{\psi}{\curvearrowright} \mathbb{V}^{\otimes d} \overset{\varphi}{\curvearrowleft} \underset{\text{Hecke algebra}}{\mathbf{H}(\mathfrak{S}_d)}$$

We start with describing first a tensor space $\mathbb{V}^{\otimes d}$ admitting natural actions from both sides

Natural representation $\mathbb{V}$ of $\mathbf{U}_q(\mathfrak{gl}_n)$

- $\mathbb{V} := \sum_{i=1}^{n} \mathbb{Q}(v) v_i$: natural representation of $\mathbf{U}_q(\mathfrak{gl}_n)$, e.g. $n = 3$

$$\begin{array}{cccccc}
 & v_1 & \underset{E_1}{\overset{F_1}{\rightleftarrows}} & v_2 & \underset{E_2}{\overset{F_2}{\rightleftarrows}} & v_3 \\
D_3 & \underset{\times 1}{\circlearrowright} & & \underset{\times 1}{\circlearrowright} & & \underset{\times q}{\circlearrowright} \\
D_2^{-1} & \underset{\times 1}{\circlearrowright} & & \underset{\times q^{-1}}{\circlearrowright} & & \underset{\times 1}{\circlearrowright}
\end{array}$$

- $\mathbf{U}_q(\mathfrak{gl}_n)$ acts on $\mathbb{V}^{\otimes d}$ via comultiplication, e.g.

$$\Delta(F_2) = D_2^{-1} D_3 \otimes F_2 + F_2 \otimes 1$$

Hence

$$F_2(v_3 \otimes v_2) = D_2^{-1} D_3 v_3 \otimes F_2 v_2 + \cancel{F_2 v_3} \otimes v_2 = q v_3 \otimes v_3$$
$$F_2(v_2 \otimes v_3) = D_2^{-1} D_3 v_2 \otimes \cancel{F_2 v_3} + F_2 v_2 \otimes v_3 = v_3 \otimes v_3$$

Hecke algebra action

- For $\mathbf{H}(\mathfrak{S}_2)$, we have the following equalities in one-line notation $(\mathbb{1} = [12], s_1 = [21])$

$$\begin{aligned} T_{[12]}T_1 &= T_{[21]} \\ T_{[21]}T_1 &= qT_{[12]} + (q-1)T_{[21]} \end{aligned}$$

- This leads to a natural Hecke algebra action on $\mathbb{V}^{\otimes 2}$ respecting the multiplication rule, e.g.

$$(v_a \otimes v_b)T_1 = \begin{cases} v_b \otimes v_a & \text{if } b > a \\ qv_b \otimes v_a + (q-1)v_a \otimes v_b & \text{if } b < a \\ qv_a \otimes v_a & \text{if } b = a \end{cases}$$

Example: commutivity

Here's an example showing $\Big(F_2(v_3 \otimes v_2)\Big)T_1 = F_2\Big((v_3 \otimes v_2)T_1\Big)$

$$
\begin{tikzcd}
\mathbb{V} \otimes \mathbb{V} \arrow[rrr, "T_1"] \arrow[ddd, "F_2"'] & & & \mathbb{V} \otimes \mathbb{V} \arrow[ddd, "F_2"] \\
& v_3 \otimes v_2 \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & (q-1)v_3 \otimes v_2 + qv_2 \otimes v_3 \arrow[d, mapsto] & \\
& qv_3 \otimes v_3 \arrow[r, mapsto] & q^2 v_3 \otimes v_3 = (q^2 - q)v_3 \otimes v_3 + qv_3 \otimes v_3 & \\
\mathbb{V} \otimes \mathbb{V} \arrow[rrr, "T_1"'] & & & \mathbb{V} \otimes \mathbb{V}
\end{tikzcd}
$$

q-Schur duality

$$\underset{\text{quantum group}}{\mathbf{U}_q(\mathfrak{gl}_n)} \quad \overset{\psi}{\curvearrowright} \quad \mathbb{V}^{\otimes d} \quad \overset{\varphi}{\curvearrowleft} \quad \underset{\text{Hecke algebra}}{\mathbf{H}(\mathfrak{S}_d)}$$

q-Schur duality of type A (Jimbo'86)

The algebras $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{H}(\mathfrak{S}_d)$ satisfy the double centralizer property:

$$\mathrm{End}_{\varphi(A)}(\mathbb{V}^{\otimes d}) = \psi(B), \quad \mathrm{End}_{\psi(B)}(\mathbb{V}^{\otimes d}) = \varphi(A),$$

where $A = \mathbf{H}(\mathfrak{S}_d)$, $B = \mathbf{U}_q(\mathfrak{gl}_n)$

q-Schur algebra

- In other words, there is a surjection, for each d, from $\mathbf{U}_q(\mathfrak{gl}_n)$ to the q-Schur algebra

$$\mathbf{S}_{n,d} = \mathrm{End}_{\mathbf{H}(\mathfrak{S}_d)}(\mathbb{V}^{\otimes d}),$$

and hence we have

$$\begin{array}{ccccc} \mathbf{U}(\mathfrak{gl}_n) & & & & \\ \text{quantum group} & & & & \\ \downarrow & & & & \\ \mathbf{S}_{n,d} & \curvearrowright & \mathbb{V}^{\otimes d} & \curvearrowleft & \mathbf{H}(\mathfrak{S}_d) \\ q\text{-Schur algebra} & & \text{tensor space} & & \text{Hecke algebra} \end{array}$$

q-Schur duality for other types

- It is known that $\mathbf{U}_q(\mathfrak{gl}_n)$ is a type A quantum group. There are (q-)Schur duality for other types, e.g.,

$$\begin{array}{ccccc}
\begin{array}{c}\mathbf{O}_n\\ \text{orthogonal group}\end{array} & & & & \\
\downarrow & & & & \\
\begin{array}{c}S^{\mathfrak{o}}_{n,d}\\ \text{orthogonal Schur algebra}\end{array} & \curvearrowright & V^{\otimes d} & \curvearrowleft & \begin{array}{c}\mathfrak{B}_d\\ \text{Brauer algebra}\end{array}
\end{array}$$

$$\begin{array}{ccccc}
\begin{array}{c}\mathbf{U}_q(\mathfrak{so}_n)\\ \text{quantum group}\end{array} & & & & \\
\downarrow & & & & \\
\begin{array}{c}\mathbf{S}^{\mathfrak{o}}_{n,d}\\ \text{orthogonal } q\text{-Schur algebra}\end{array} & \curvearrowright & \mathbb{V}^{\otimes d} & \curvearrowleft & \begin{array}{c}\mathbf{BMW}_d\\ \text{Birman-Murakami-Wenzl algebra}\end{array}
\end{array}$$

- In these pictures the "types" are associated to the classical/quantum groups

q-Schur duality for other types

- If we associate the "types" to the Hecke algebra instead, we obtain another family of (q-)Schur duality:

$$\underbrace{\mathbf{S}^{\mathsf{X}}_{n,d} := \mathrm{End}_{\mathbf{H}^{\mathsf{X}}_d}(\mathbb{V}^{\otimes d})}_{q\text{-Schur algebra of type X}} \quad \curvearrowright \quad \mathbb{V}^{\otimes d} \quad \curvearrowleft \quad \underbrace{\mathbf{H}^{\mathsf{X}}_d}_{\text{type X Hecke algebra}}$$

- Question: Is there a bottom-top construction that recovers the quantum group from the Schur algebras?

BLM construction

- The answer is Yes, the construction for type A is provided in 1990 by Beilinson-Lusztig-MacPherson (BLM) geometrically using partial flags and a so-called stabilization procedure. We can paraphrase it algebraically as below:

$$
\begin{array}{ccccc}
\underset{\text{stabilization algebra of type A}}{\dot{\mathbf{K}}_n^{\mathsf{A}}} & & & & \\
\Uparrow & & & & \\
\underset{q\text{-Schur algebra of type A}}{\mathbf{S}_{n,d}^{\mathsf{A}} := \mathrm{End}_{\mathbf{H}_d^{\mathsf{A}}}(\mathbb{V}^{\otimes d})} & \curvearrowright & \mathbb{V}^{\otimes d} & \curvearrowleft & \underset{\text{type A Hecke algebra}}{\mathbf{H}_d^{\mathsf{A}} := \mathbf{H}(\mathfrak{S}_d)}
\end{array}
$$

- Stabilization algebra $\dot{\mathbf{K}}_n^{\mathsf{A}} \approx$ inverse limit $\varprojlim_{d \in \mathbb{N}} \mathbf{S}_{n,d}^{\mathsf{A}}$

BLM construction

$$\dot{\mathbf{K}}_n^{\mathsf{A}}$$
stabilization algebra of type A
$$\Uparrow$$

$$\mathbf{S}_{n,d}^{\mathsf{A}} := \mathrm{End}_{\mathbf{H}_d^{\mathsf{A}}}(\mathbb{V}^{\otimes d}) \quad \curvearrowright \quad \mathbb{V}^{\otimes d} \quad \curvearrowleft \quad \mathbf{H}_d^{\mathsf{A}} := \mathbf{H}(\mathfrak{S}_d)$$

q-Schur algebra of type A — type A Hecke algebra

- Canonical bases for $\mathbf{S}_{n,d}^{\mathsf{A}}(d \in \mathbb{N})$ lift "compatibly" to canonical basis of $\dot{\mathbf{K}}_n^{\mathsf{A}}$
- $\dot{\mathbf{K}}_n^{\mathsf{A}} \simeq \dot{\mathbf{U}}(\mathfrak{gl}_n)$

$\Rightarrow$ A concrete realization of canonical basis of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$

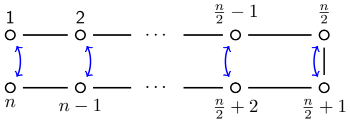
BLM construction

- A BLM construction (for type X) produces from a family of q-Schur algebra $\{\mathbf{S}^{\mathsf{X}}_{n,d} \mid d \in \mathbb{N}\}$ an $\mathbb{Q}(v)$-algebra $\dot{\mathbf{K}}^{\mathsf{X}}_n$ that enjoys similar properties

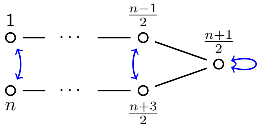
$$\begin{array}{ccccc}
\begin{array}{c}\dot{\mathbf{K}}^{\mathsf{X}}_n \\ \text{stabilization algebra of type X} \\ \Uparrow \\ \mathbf{S}^{\mathsf{X}}_{n,d} := \mathrm{End}_{\mathbf{H}^{\mathsf{X}}_d}(\mathbb{V}^{\otimes d}) \\ q\text{-Schur algebra of type X}\end{array} & \curvearrowright & \mathbb{V}^{\otimes d} & \curvearrowleft & \begin{array}{c}\mathbf{H}^{\mathsf{X}}_d \\ \text{type X Hecke algebra}\end{array}
\end{array}$$

New "quantum groups"

- In type B/C, there are two types of BLM constructions due to Bao-Kujawa-Li-Wang ('14) associated to involutions (type $\imath, \jmath$) of the Dynkin diagram of type A



type $\jmath$: n even



type $\imath$: n odd

⚠ $\dot{\mathbf{K}}_n^{\mathrm{BC}}$ is NOT the modified Drinfeld quantum group of type B/C; it is a modified quantization of certain product $\mathfrak{gl}_\bullet \times \mathfrak{sl}_\bullet$

Applications of new "quantum groups"

- Depending on parity of n, $\mathbf{K}_n^{\mathsf{BC}}$ is isomorphic to involutive quantum groups $\mathbf{U}_n^{\imath}$ or $\mathbf{U}_n^{\jmath}$ introduced by Bao-Wang ('13). The canonical basis theory for $\mathbf{U}_n^{\imath}, \mathbf{U}_n^{\jmath}$
  - gives a new formulation of KL theory for Lie algebra of type B/C
  - establishes for the first time KL theory for Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ (i.e. super type B/C)

$\Rightarrow$ reformulation of KL theory for Lie algebra of type D and for Li superalgebra $\mathfrak{osp}(2m|2n)$ (i.e. super type D) by Bao ('16)

- We expect similar applications for other types (e.g. affine classical)

BLM-type constructions

- Here's a summary on what are known

BLM construction (method)	Geometric (dimension counting on flags)	Algebraic (combinatorics)
Affine A	Ginzburg-Vasserot ('93) Lusztig ('99)	Du-Fu ('14)
Finite B/C	BKLW ('14)	
Finite D	Fan-Li ('14)	
Affine C	Fan-Lai-Li-Luo-Wang1 ('16)	FLLLW2 ('16)

Coideal subalgebras

- It is worth mentioning that $\mathbf{U}_n^\imath, \mathbf{U}_n^\jmath$ are coideal subalgebras of $\mathbf{U}(\mathfrak{gl}_n)$, i.e., the comultiplication $\Delta : \mathbf{U}(\mathfrak{gl}_n) \to \mathbf{U}(\mathfrak{gl}_n) \otimes \mathbf{U}(\mathfrak{gl}_n)$ satisfies that

$$\Delta(\mathbf{U}_n^\imath) \subset \mathbf{U}_n^\imath \otimes \mathbf{U}(\mathfrak{gl}_n)$$

$$\Delta(\mathbf{U}_n^\jmath) \subset \mathbf{U}_n^\jmath \otimes \mathbf{U}(\mathfrak{gl}_n)$$

⚠ A coideal subalgebra is different from a Hopf subalgebra $\mathbf{B}$ of $\mathbf{U}$ s.t. $\Delta(\mathbf{B}) \subset \mathbf{B} \otimes \mathbf{B}$

Outline

Symmetric pairs

- A symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ consists of a Lie algebra $\mathfrak{g}$ and its fixed-point subalgebra $\mathfrak{g}^\theta$ with respect to an involution

$$\theta : \mathfrak{g} \to \mathfrak{g}$$

- It plays an important role in the study of real reductive groups
- Classification of symmetric pairs of finite type
  $=$ Classification of real simple Lie algebras
  $\leftrightarrow$ Satake diagrams (of finite type)

Quantum symmetric pairs

- The quantum symmetric pair(QSP) is introduced by Letzter ('02) for finite type; and by Kolb ('14) for symmetrizable Kac-Moody as a quantization of the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$
- The pair $(\mathbf{U}, \mathbf{B})$ is a QSP if $\mathbf{U} = \mathbf{U}(\mathfrak{g})$ and $\mathbf{B} = \mathbf{B}(\theta)$ is certain coideal subalgebra of $\mathbf{U}$, i.e., the comultiplication $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ satisfies that

$$\Delta(\mathbf{B}) \subset \mathbf{B} \otimes \mathbf{U}$$

Examples

The following are examples of QSP:

- $(\mathbf{U}_q(\widehat{\mathfrak{gl}}_n),$ twisted Yangian) from Yang-Baxter equation
- $(\mathbf{U}_q(\mathfrak{g}), \mathbf{B})$ where $\mathfrak{g}$ is of classical type, and $\mathbf{B}$ is constructed based on solutions of the reflection equation

$$RKRK = KRKR$$

- $(\mathbf{U}_q(\widehat{\mathfrak{sl}}_2), q$-Onsager algebra) from Ising model
- Quantum algebras from BLM construction. In other words, quantum algebras arising from Schur-Weyl duality

QSP from BLM construction

Recall that

$$\{\mathbf{S}^{\mathsf{X}}_{n,d} \mid d \in \mathbb{N}\} \xrightarrow{\text{stabilization}} \dot{\mathbf{K}}^{\mathsf{X}}_n \xrightarrow{\text{take inf sum}} \mathbf{K}^{\mathsf{X}}_n$$

The resulting quantum algebras are:

type	quantum algebra $\mathbf{K}^{\mathsf{X}}_n$
Finite A	$\mathbf{U}_q(\mathfrak{gl}_n)$
Finite B/C	$\mathbf{K}^{\mathsf{BC}}_n \simeq \mathbf{U}^{\imath}_n, \mathbf{U}^{\jmath}_n$
Affine A	$\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$
Affine C	$\mathbf{K}^{\widehat{\mathsf{C}}}_n$ (four variants)

Proposition

The pairs $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{U}^{\imath}_n), (\mathbf{U}(\mathfrak{gl}_n), \mathbf{U}^{\jmath}_n)$ and all four variants of $(\mathbf{U}(\widehat{\mathfrak{gl}}_n), \mathbf{K}^{\widehat{\mathsf{C}}}_n)$ are quantum symmetric pairs.

Applications and future work

- A canonical basis theory for QSP of Satake diagrams finite type is developed recently by Bao-Wang ('16). It may lead to similar application as in $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{K}_n^{\mathsf{X}})$ with further study
- ? canonical basis theory for affine QSP
- ? BLM construction from $\{\mathbf{S}_{n,d}^{\mathsf{X}}\}$ when X is of affine B/D, finite and affine exceptional type
- ? BLM-type construction for other Schur-type dualities (e.g., for Brauer algebras, partition algebras)
- ? BLM-type construction for Howe duality

Thank you for your attention