I Park My Car in the Weyl Alcove by a Labeled Tree

Outline

{Parking Functions} ⟷ {Labeled trees} Comb.

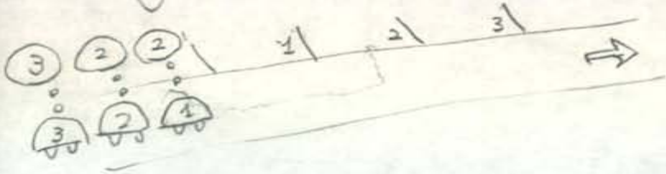
⇕ twilight zone.

{Weyl alcoves}

⇕ Repn

Repn theory

§1 Parking Functions



A parking fcn is a fcn : $\{1,\dots,n\} \to \{1,\dots,n\}$ s.t. any driver has a spot.
driver ↦ fav. spot

⇕

a seq $(a_1,\dots,a_n)$ s.t. its non-decr rearr $b_1 \le \dots \le b_n$

s.t. $b_i \le i \quad \forall\ 1 \le i \le n$

eg. $n=1$: 1 $n=2$: 11 21 12 $n=3$: 111 112 113 121 122 123 131 132 211 212 213 221 231 311 312 321

1 3 16 125

⇒ Thm [Konheim–Weiss, 66] $\# PF_n = P_n = (n+1)^{n-1}$

k-leading $P_{n,k} = \{\vec{a} \mid a_1 = k\}$

Thm [Foata, Riordan, 74] gener. fcn.

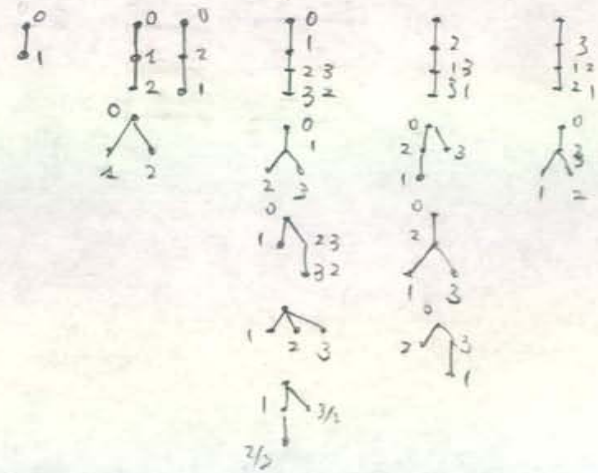
Will give a comb. proof.

$P_{n,k}$ n\k	1	2	3	4	5	
1	1	0				
2	2	1	0			
3	8	5	3	0		
4	50	34	25	16	0	
5	432	307	243	189	125	0

§2 Labeled Trees

$\mathcal{T}_n = \#$ Labeled trees w/ $n+1$ nodes $= (n+1)^{n-1}$

e.g. $n=1$ $n=2$ $n=3$



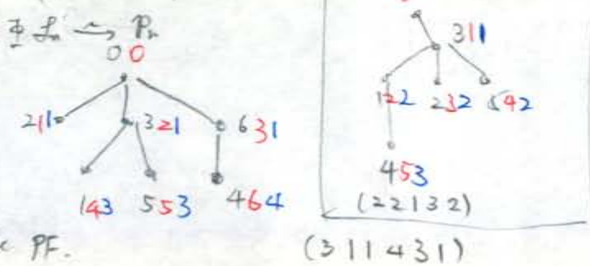
Thm [Eu, Fu, L—, 05] $(x, y(x), z(x))$

(Triple-label algorithm) $\Phi: \mathcal{L}_n \to P_n$

Given a labeled tree

- BFS ⇒ y
- $z(x) = y(\hat{x}) + 1$

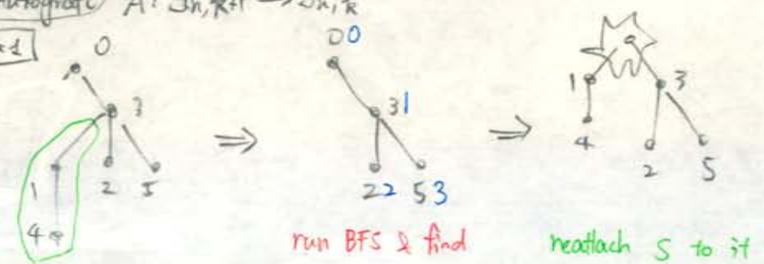
⇒ $(z(1),\dots,z(n))$ is the PF.



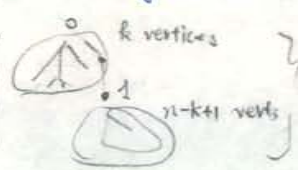
$\mathcal{T}_{n,k} = \Phi^{-1}(P_{n,k})$

Autograft $A: \mathcal{T}_{n,k+1} \to \mathcal{T}_{n,k}$

e.g. $k=1$



Remove S = subtree of 1 run BFS & find where $y(x) = k-1$ reattach S to it

Cor $\mathcal{T}_{n,k} \setminus \mathrm{Im}A = \{$  $\}$ has card $\binom{n-1}{k-1} k^{k-2} (n-k+1)^{n-k-1}$

ways of dividing two subtrees

$$\Rightarrow P_{n,k} = \sum_{i=1}^{k} \binom{n-1}{i-1} i^{i-2} (n-i+1)^{(n-i-1)}$$

§ $\vec{X}$-PF

$\vec{a}$ is a $\vec{1}$-PF $\iff b_i \le i = \underbrace{1+\dots+1}_{i}$ $\forall i$

$\vec{a}$ is a $\vec{x}$-PF $\iff b_i \le x_1+\dots+x_i$ $\forall i$

$\boxed{\text{Thm}}$ [EFL, 05]

Triple-label alg & Autograft. ~~meth~~ work for {rooted labeled forest}

$\Longrightarrow P_{n,k}^{(a,\bar{b})}$ $\updownarrow$ $\{(a,\bar{b})\text{-PF}\}$

$\boxed{\text{Thm}}$ [Pitman, Stanley, 86] $P_n^{(a\bar{b})}$ alg.

[Yan, 01] $P_n^{(a\bar{b})}$ comb

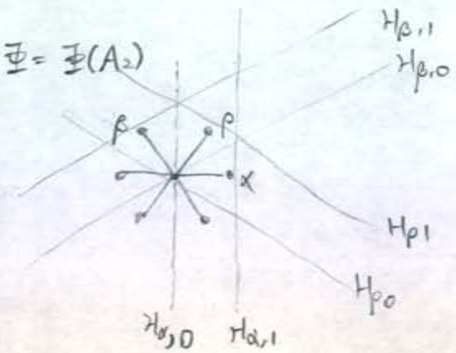
$= \boxed{a(a+bn)^{n-1}}$

§3 Hyperplane Arr.

Given a root sys Φ, for each root α we define a hyperplane

$$H_{\alpha,k} = \{x \mid (x,\alpha) = k\}$$

e.g. $\Phi = \Phi(A_2)$



· Let R^m_Φ = {regions sep. by $H_{\alpha,k}$ | $\alpha \in \Phi^+$, $k = m, m-1, \dots, -(m-1)$}

$\lim_{m\to\infty} R^m_\Phi$ = The Weyl alcove.

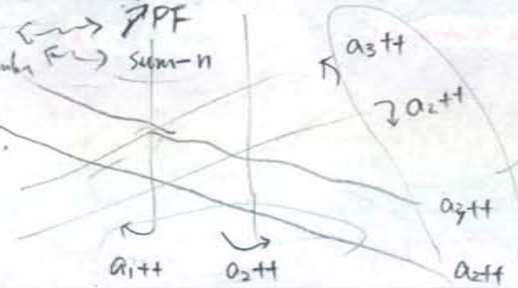
e.g.

$|R^1_{\Phi(A_2)}| = 16 \longleftrightarrow$ PF

dom. reg. $= 5 \longleftrightarrow$ ↗PF

distance w/ fund. chamber $\longleftrightarrow$ sum − n

Draw:



$\boxed{\text{Thm}}$ [Yoshinaga, 04]

$|R^m_\Phi| = (1+mh)^r$ where r = rank of Φ, h = Coxeter # of Φ

Table

Type	h	$\lvert R^m_\Phi \rvert$		
A_r	$n+1$	$(1+m(n+1))^n$	$=$	$\lvert P_{n+1}^{(1,\bar{m})}\rvert$
B_r	$2n$	$(1+2mn)^n$	$=$	$\lvert P_n^{(1,\overline{2m})}\rvert(1+2mn)$
D_r	$2n-2$	$(1+2m(n-1))^n$	$=$	$\lvert P_{n-1}^{(1,\overline{2m})}\rvert(1+2m(n-1))^2$

$\boxed{\text{Thm}}$ [Athanasiadis, 04]

bijections $R^m_{\Phi(A_r)} \longleftrightarrow P_{n+1}^{(1,\bar{m})}$

$\boxed{\text{Obs}}$ crossing hyperplanes = applying autograft. operation.

$\boxed{\text{Conj}}$

$\exists$ bijections $R^m_{\underset{P_r}{\Phi(B_r)}} \longleftrightarrow P_*^{=} \times$ extra label