Affine Schur-Weyl duality & q-Schur algebras

§1: Review aff Weyl grps

§2: Affine q-Schur alg ($n \ge r \ge 3$) — q-tensor sp. — (ext) AHA

$$\hat{S}_q(n,r) \curvearrowright E(n,r) \curvearrowleft \hat{H}$$

std basis: $\{\phi^d_{\lambda\mu} \mid \lambda,\mu,d\}$ — $\{\phi^d_{\lambda\omega} \mid \lambda,d\}$ — $\{T_d = \phi^d_{\omega\omega} \mid d\}$

$\Downarrow$ KL polyn, Double cosets — $\Downarrow$ KL polyn

canonical basis: $\{\theta^d_{\lambda\mu}\}$ — KL basis: $\{C_d\}$

1

§3: Quantum groups

$U = U_q(\hat{\mathfrak{gl}}_n)$ as Hopf alg $\curvearrowright$ V: nat. repn

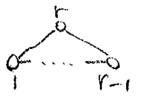
$\Downarrow$ — $\Downarrow$ tensor sp.

$\hat{S}_q(n,r)$ — $V^{\otimes r} \leftarrow \hat{H}$ via U-action

§4: $V^{\otimes r} \updownarrow E(n,r)$ — $\hat{H}$: Bernstein-Lusztig presentation

2

Quick review

Aff Weyl grp $W := \langle s_1, \dots, s_r\rangle$ = Coxeter grp w/ 

Ext ~~~ $\hat{W} := \langle W, \rho\rangle$ admits length fcn ($\ell(\rho) = 0$)

Ext AHA (algebra / $\mathcal{A} := \mathbb{Z}[v^{\pm 1}]$, $q = v^2$)

$$\hat{H} = H(\hat{W}) := \langle T_1, \dots, T_r, T_\rho^{\pm 1}\rangle_{\mathcal{A}} \,/$$

Braid rel'n, Hecke rel'n, $T_\rho T_{i+1} = T_i T_\rho$

$\Rightarrow \hat{H} = \bigoplus_{t\in\mathbb{Z}} T_\rho^t H$ as right mod / $H = H(W)$

3

Compositions $\Lambda = \Lambda(n,r) := \{\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{N}^n \mid \sum \lambda_i = r\}$

For $\lambda \in \Lambda$, define Young subgrp of $\mathfrak{S}_r = \langle s_1, \dots, s_{r-1}\rangle$ by

$$W_\lambda := \mathrm{Stab}_W[1,\lambda_1] \cap \mathrm{Stab}_W[\lambda_1+1, \lambda_1+\lambda_2] \cap \dots$$
$$= \langle s_i \mid i \ne \lambda_1 + \dots + \lambda_j \text{ for some } j\rangle$$

Any finite parabolic subgrp of W, denoted by $W_\pi := \langle s_i \mid i \in \pi\rangle$ for $\pi \subsetneq \{s_1, \dots, s_r\}$, can be written as $W_\pi = W_{\lambda+t} := \rho^{-t} W_\lambda \rho^t$ for some $t \in \mathbb{Z}$, $\lambda \in \Lambda$

4

e.g. $n = r = 3$

(i) $\lambda = (1,2,0) \in \Lambda(3,3) \Rightarrow \lambda_1 = 1$ i.e. $s_1 \notin W_\lambda$,

$\Rightarrow W_\lambda = \langle s_2\rangle$, $W_{\lambda+1} = \langle s_3\rangle$, $W_{\lambda+2} = \langle s_1\rangle, \dots$

(ii) $\pi = \{s_3\}$

$W_\pi \not\subseteq \mathfrak{S}_3$ and hence $W_\pi \ne W_\lambda$ for some $\lambda \in \Lambda$

5

For finite $X \subset \hat{W}$, set

$$T_X := \sum_{w\in X} T_w, \quad x_\lambda := T_{W_\lambda}, \quad x_{\lambda+t} := T_{W_{\lambda+t}} = T_\rho^{-t} x_\lambda T_\rho^t$$

$\Rightarrow$ Right $\hat{H}$-modules $x_\lambda \hat{H}$ and $\bigoplus_{\lambda\in\Lambda} x_\lambda \hat{H}$

$\Rightarrow$ Affine q-Schur algebra (over $\mathcal{A}$)

$$\hat{S} = \hat{S}_q(n,r) := \mathrm{End}_{\hat{H}}\Big(\bigoplus_{\lambda\in\Lambda} x_\lambda \hat{H}\Big)$$
$$= \Big\{f : \bigoplus_{\mu\in\Lambda} x_\mu \hat{H} \to \bigoplus_{\lambda\in\Lambda} x_\lambda \hat{H} \;\Big|\; f \text{ is right } \hat{H}\text{-linear}\Big\}$$
$$= \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{\hat{H}}(x_\mu \hat{H}, x_\lambda \hat{H})$$

Next: describe $\mathcal{A}$-bases for $\hat{S}$.

6

Distinguished coset representatives ($\pi \subsetneq \{s_1, \dots, s_r\}$)

$$\hat{\mathcal{D}}_\pi := \{d \in \hat{W} \mid \ell(wd) = \ell(w) + \ell(d)\ \forall w \in W_\pi\}$$
$$= \{d \in \hat{W} \mid \ell(d) \text{ is min in } (W_\pi)d\}$$
$$\hat{\mathcal{D}}_\pi^{-1} = \{d \in \hat{W} \mid \ell(d) \text{ is min in } dW_\pi\}$$
$$\hat{\mathcal{D}}_{\lambda\mu} := \hat{\mathcal{D}}_\lambda \cap \hat{\mathcal{D}}_\mu^{-1} = \{d \in \hat{W} \mid \ell(d) \text{ is min in } (W_\lambda)d(W_\mu)\}$$
$$= \{\rho^t w \mid t \in \mathbb{Z},\ w \in \mathcal{D}_{\lambda+t,\mu}\}$$

where $\mathcal{D}_\pi, \mathcal{D}_\pi^{-1}, \mathcal{D}_{\lambda\mu}$ are analogs in W.

7

e.g. $n = r = 3$

$\lambda = (1,2,0) \Rightarrow W_\lambda = \langle s_2\rangle$, $W_{\lambda+1} = \langle s_3\rangle, \dots$

$\mu = (3,0,0) \Rightarrow W_\mu = \langle s_1, s_2\rangle$

$\mathcal{D}_{\lambda\mu} = \{\mathbb{1}, s_3, s_1s_2s_3, \dots\}$

$\mathcal{D}_{\lambda+1,\mu} = \{\mathbb{1}, s_2s_3, \dots\}$

$\vdots$

$\Rightarrow \hat{\mathcal{D}}_{\lambda\mu} = \left\{\begin{matrix}\mathbb{1}, s_3, s_1s_2s_3, \dots \\ \rho, \rho s_2 s_3, \dots\end{matrix}\right\}$ ← hard to describe using reduced expr.

Rmk [Du-Fu]

$d \in \hat{\mathcal{D}}_\lambda \iff d^{-1}$ is order-preserving on $[\lambda_1 + \dots + \lambda_i + 1, \lambda_1 + \dots + \lambda_{i+1}]\ \forall i$

8

$\boxed{\text{Thm}}$

$\hat{S}$ has an $\mathcal{A}$-basis $\{\phi^d_{\lambda\mu} \mid \lambda,\mu\in\Lambda,\ d\in\hat{\mathcal{D}}_{\lambda\mu}\}$,

where $\phi^d_{\lambda\mu}\in \mathrm{Hom}_{\hat H}(x_\mu\hat H, x_\lambda \hat H)$, $x_\mu \mapsto T_{W_\lambda d W_\mu}$

(Pf) It suffices to find $\mathcal{A}$-basis for $\mathrm{Hom}_{\hat H}(x_\mu\hat H, x_\lambda\hat H)$

$\Rightarrow$ 〃 $\mathrm{Hom}_H(x_\mu H, x_\lambda \hat H)$

$\Rightarrow$ 〃 $\bigoplus_{t\in\mathbb Z}\mathrm{Hom}_H(x_\mu H, T_\rho^t x_{\lambda+t}H)$

$\Rightarrow$ 〃 $\mathrm{Hom}_H(x_\mu H, x_{\lambda+t}H)$ for each $t\in\mathbb Z$

[Dipper-James] $\mathrm{Hom}_H(x_\mu H, x_{\lambda+t}H)$ has $\mathcal A$-basis

$\psi^w_{\lambda+t,\mu}: x_\mu H \longrightarrow x_{\lambda+t}H$

$(w\in \mathcal D_{\lambda+t,\mu})$ $\quad x_\mu \mapsto T_{(W_{\lambda+t})w(W_\mu)} \overset{[\text{Howlett}]}{=\!=} x_{\lambda+t}T_w T_{\mathcal D_\nu\cap W_\mu}$ for some $\nu\in\Lambda(n',r)$

$\Rightarrow \phi^w_{\lambda+t,\mu}: x_\mu H \longrightarrow T_\rho^t x_{\lambda+t}H$ forms an $\mathcal A$-basis.

$(w\in\mathcal D_{\lambda+t,\mu})$ $\quad x_\mu \longmapsto T_\rho^t T_{(W_{\lambda+t})w(W_\mu)} = T_{W_\lambda d W_\mu}$ $\quad (d=\rho^t w)$

$\Rightarrow$ done.

Canonical basis for $\hat S$

Recall: [Kazhdan-Lusztig]

$\mathcal H$ has $\mathcal A$-basis $\{C'_w\mid w\in W\}$ where

$C'_w := v^{-\ell(w)}\sum_{z\le w} P_{z,w}T_z$

$\le$: Bruhat order on W $\xrightarrow{\text{extend}}$

$P_{z,w}\in\mathbb Z[v^2]$: KL polyn

$$\left\{\begin{array}{l} C'_d = T^t_\rho C'_w \quad (d=\rho^t w) \\ \rho^a z \le \rho^b w \iff \begin{cases} a=b \\ z\le w\end{cases} \\ P_{\rho^a z,\rho^b w} = \delta_{ab}P_{z,w}\end{array}\right.$$

Moreover,

$C'_w\cdot C'_x \in \sum_z \mathbb N[v,v^{-1}]C'_z$

$\boxed{\text{Lemma}}$

$W_\lambda d W_\mu = \{g \mid d\le g\le d^+_{\lambda\mu}\}$ for some $d^+_{\lambda\mu}\in\hat W$

(Pf) from [Curtis],

$(W_{\lambda+t})w(W_\mu) = \{g\mid w\le g\le w^+\}$ for some $w^+\in W$

In particular, if $\lambda=\mu$, $d=1\in\hat{\mathcal D}_{\lambda\mu}$, we have

$W_\mu 1 W_\mu = W_\mu$ and $1^+_{\mu\mu} = w_0^\mu$: longest elt in W_μ.

Rmk

$\phi^d_{\lambda\mu}: T_{W_\mu 1 W_\mu} \longmapsto T_{W_\lambda d W_\mu}$

$\boxed{\text{Thm}}$

$\hat S$ has an $\mathcal A$-basis $\{\theta^d_{\lambda\mu}\}$ where $\theta^d_{\lambda\mu}: x_\mu \hat H \to x_\lambda\hat H$
$\quad C'_{1^+_{\mu\mu}} \mapsto C'_{d^+_{\lambda\mu}}$

In particular,

$$\theta^d_{\lambda\mu} = v^{\ell(w_0^\mu)}\sum_{\substack{z\le d \\ \text{in } \hat{\mathcal D}_{\lambda\mu}}} v^{-\ell(d^+_{\lambda\mu})}P_{z^+_{\lambda\mu},d^+_{\lambda\mu}}\ \phi^d_{\lambda\mu}$$

(Pf) [Curtis] $C'_{1^+_{\mu\mu}} = v^{-\ell(w_0^\mu)}x_\mu$

[Du] $$C'_{w^+_{\lambda+t,\mu}} = \sum_{\substack{z\le w \\ \text{in } \mathcal D_{\lambda+t,\mu}}} v^{-\ell(w^+_{\lambda+t,\mu})}P_{z^+_{\lambda+t,\mu},w^+_{\lambda+t,\mu}}\ T_{(W_{\lambda+t})z(W_\mu)}$$

Rmk: positivity is shown in [Lusztig]

q-tensor space

- Recall that $n\ge r \Rightarrow \exists\ \omega = (\underbrace{1,\dots,1}_{r},\underbrace{0,\dots,0}_{n-r})\in\Lambda$
- We have:

$W_\omega=\{1\}$, $\qquad x_\omega = 1\in\hat H$,

$\hat{\mathcal D}_\omega = \hat W$, $\qquad \hat{\mathcal D}_{\lambda\omega}=\hat{\mathcal D}_\lambda$,

$\phi^d_{\lambda\omega}: x_\omega\hat H\to x_\lambda\hat H$

$\quad 1\mapsto T_{W_\lambda d W_\omega} = x_\lambda T_d$

q-tensor sp. $E(n,r) := \bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\hat H}(\hat H, x_\lambda\hat H) \cong \bigoplus_{\lambda\in\Lambda} x_\lambda\hat H$

$\qquad \phi^d_{\lambda\omega} \qquad\mapsto\qquad x_\lambda T_d$

$E(n,r)$ is a $\hat S$-$\hat H$-bimodule via composition

e.g. $\hat S\times E(n,r)\longrightarrow E(n,r)$

$(\phi^d_{\lambda\mu}, \phi^g_{\nu\omega}) \mapsto \phi^d_{\lambda\mu}\cdot\phi^g_{\nu\omega}: x_\omega \mapsto \phi^d_{\lambda\mu}(x_\nu T_g)$
$\qquad = \delta_{\mu\nu}T_{W_\lambda d W_\mu}T_g$

In particular,

$\phi^d_{\lambda\omega}\cdot\phi^1_{\omega\omega}(x_\omega) = T_{W_\lambda d} = x_\lambda T_d \Rightarrow \phi^d_{\lambda\omega}\cdot\phi^1_{\omega\omega}=\phi^d_{\lambda\omega}$

$\boxed{\text{Lem}}$ $E(n,r) = \langle\phi^1_{\omega\omega}\rangle$ as a left $\hat S$-mod

$\boxed{\text{Thm}}$ (Schur-Weyl duality)

If $n\ge r$, then $\hat S_q(n,r)$ and $\hat{\mathcal H}$ have double centralizer property, i.e.

(i) $\mathrm{End}_{\hat S}(E(n,r)) = \hat H$, $\quad$ (ii) $\mathrm{End}_{\hat H}(E(n,r)) \overset{\text{defn}}{=} \hat S$

(Pf)(i)($\supseteq$) follows from that $\hat H$-action commutes w/ $\hat S$-action.

($\subseteq$): $f\in \mathrm{End}_{\hat S}(E(n,r)) \overset{\text{Lem}}{=\!=} \mathrm{End}_{\hat S}(\langle\phi^1_{\omega\omega}\rangle_{\hat S})$

$\iff$ f is uniquely det. by $f(\phi^1_{\omega\omega}) = \sum_{\mu\in\Lambda, d\in\hat{\mathcal D}_\mu}\phi^d_{\mu\omega}c_{\mu d}$

$\forall\lambda\ne\omega$, $\quad \phi^1_{\lambda\lambda}f(\phi^1_{\omega\omega}) = f(\phi^1_{\lambda\lambda}\phi^1_{\omega\omega}) = 0 \quad\Rightarrow c_{\lambda d}=0$

$\qquad = \phi^1_{\lambda\lambda}\sum\phi^d_{\mu\omega}c_{\mu d} = \sum_d \phi^d_{\lambda\omega}c_{\lambda d}$

$\Rightarrow f(\phi^1_{\omega\omega}) = \phi^1_{\omega\omega}h$ for some $h\in\hat H$ ※