Reverse mathematics in constructive set theory

Hajime Ishihara

School of Information Science
Japan Advanced Institute of Science and Technology
(JAIST)
Nomi, Ishikawa 923-1292, Japan

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Overview

- Lecture 1
  - Reverse mathematics and set theory
  - Intuitionistic logic
- Lecture 2
  - Classical Zermelo-Fraenkel set theory ZF
  - Basic constructive set theory BCST
  - Elementary constructive set theory ECST
  - Constructive Zermelo-Fraenkel set theory CZF
- Lecture 3
  - Set-generated classes and NID principles
  - Equivalents of the nullary NID
  - Equivalents of the elementary NID
  - Equivalents of the finitary NID

Lecture 3

- Set-generated classes and NID principles
- Equivalents of the nullary NID
- Equivalents of the elementary NID
- Equivalents of the finitary NID

Set-generated classes and NID principles

Set-generated classes

Definition 1

Let S be a set, and let X be a subclass of $\mathrm{Pow}(S)$. Then X is set-generated if there exists a subset G, called a generating set, of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G (x \in \beta \subseteq \alpha).$$

Remark 2

The power class $\mathrm{Pow}(S)$ of a set S is set-generated with a generating set

$$\{\{x\} \mid x \in S\}.$$

Rules

Definition 3

Let S be a set. Then a rule on S is a pair (a, b) of subsets a and b of S. A rule is called

- nullary if a is empty;
- elementary if a is a singleton;
- finitary if a is finitely enumerable.

A subset α of S is closed under a rule (a, b) if

$$a \subseteq \alpha \to b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-closed if it is closed under all rules in R.

Remark 4

Note that if a rule is nullary or elementary, then it is finitary.

NID principles

Definition 5

Let NID denote the principles that

- for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated.

The principles obtained by restricting R in NID to a set of nullary, elementary and finitary rules are denoted by NID_0, NID_1 and $\mathrm{NID}_{<\omega}$, respectively.

Remark 6

Note that $\mathrm{NID}_{<\omega}$ implies NID_0 and NID_1.

Equivalents of the nullary NID

Fullness

The class of total relations between A and B is denoted by $\mathrm{mv}(A, B)$:

$$r \in \mathrm{mv}(A, B) \Leftrightarrow r \subseteq A \times B \wedge \forall x \in A \, \exists y \in B \, (\langle x, y \rangle \in r);$$

Fullness:

$$\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a, b) \\ \wedge \forall r \in \mathrm{mv}(a, b) \, \exists s \in c \, (s \subseteq r)).$$

NID_0 and Fullness

Theorem 7

The following are equivalent over ECST.

1. NID_0.
2. *Fullness.*

Proposition 8

NID_1 *implies* NID_0.

NID_0 and Fullness

$$
\begin{tikzcd}
\mathrm{NID}_{<\omega} \arrow[d] & \\
\mathrm{NID}_1 \arrow[d] & \\
\mathrm{NID}_0 & \text{Fullness}
\end{tikzcd}
$$

NID_0 and Fullness

Proof of Theorem 7.

(1) $\Rightarrow$ (2): Suppose NID_0. For sets A and B, define a set R of nullary rules on $(A \times B) \cup \{\mathsf{R}_{A\times B}\}$ by

$$R = \{(\emptyset, \{x\} \times B) \mid x \in A\} \cup \{(\emptyset, \{\mathsf{R}_{A\times B}\})\}.$$

Then there exists a subset G of the class X of R-closed subsets of $(A \times B) \cup \{\mathsf{R}_{A\times B}\}$ such that

$$\forall \alpha \in X \, \forall z \in \alpha \, \exists \beta \in G \, (z \in \beta \subseteq \alpha).$$

Let $C = \{\beta \cap (A \times B) \mid \beta \in G\}$. Then $C \subseteq \mathrm{mv}(A, B)$. For each $r \in \mathrm{mv}(A, B)$, since $r \cup \{\mathsf{R}_{A\times B}\} \in X$, there exists $\beta \in G$ such that $\mathsf{R}_{A\times B} \in \beta \subseteq r \cup \{\mathsf{R}_{A\times B}\}$. Therefore

$$\beta \cap (A \times B) \subseteq (r \cup \{\mathsf{R}_{A\times B}\}) \cap (A \times B) = r. \qquad \square$$

Equivalents of the elementary NID

The principle $\mathrm{NID}_{\mathrm{bi}}$

Definition 9

Let S be a set. Then a subset α of S is biclosed under a rule (a, b) if

$$a \between \alpha \leftrightarrow b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-biclosed if it is biclosed under all rules in R.

Definition 10

Let $\mathrm{NID}_{\mathrm{bi}}$ denote the principle that

- for each set S and set R of rules on S, the class of R-biclosed subsets of S is set-generated.

The principle $\mathrm{NID}_{\mathrm{bi}}$

Proposition 11

The following are equivalent over ECST*.*

1. NID_1.
2. $\mathrm{NID}_{\mathrm{bi}}$.

The principle $\mathrm{NID}_{\mathrm{bi}}$

Proof of Proposition 11.

Suppose NID_1, and let R be a set of rules on a set S. Define a set R' of elementary rules on S by

$$R' = \{(\{x\}, b) \mid (a, b) \in R \wedge x \in a\} \cup \{(\{y\}, a) \mid (a, b) \in R \wedge y \in b\}.$$

Then a subset $\alpha \subseteq S$ is R-biclosed if and only if it is R'-closed. Conversely, suppose $\mathrm{NID}_{\mathrm{bi}}$, and let R be a set of elementary rules on a set S. Define a set R' of rules on S by

$$R' = \{(a \cup b, b) \mid (a, b) \in R\}.$$

Then a subset $\alpha \subseteq S$ is R-closed if and only if it is R'-biclosed. $\square$

Weak equalisers in Rel

Definition 12

An equaliser of a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a category C is a pair of an object E and a morphism $E \xrightarrow{e} A$ such that $f \circ e = g \circ e$, and it satisfies a universal property in the sense that for any morphism $C \xrightarrow{h} A$ with $f \circ h = g \circ h$, there exists a unique morphism $C \xrightarrow{k} E$ for which the following diagram commutes.

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dotted, "k"] \arrow[ur, "h"'] & &
\end{tikzcd}
$$

A equaliser without the uniqueness condition is called a weak equaliser.

Weak equalisers in Rel

Proposition 13

The following are equivalent over ECST*.*

1. $\mathrm{NID}_{\mathrm{bi}}$.
2. Rel *has weak equalisers.*

Weak equalisers in Rel

Proof.

(1) $\Rightarrow$ (2): Suppose $\mathrm{NID}_{\mathrm{bi}}$, and let $r_1, r_2 \subseteq X \times Y$ be a parallel pair of relations. Consider a subclass

$$\mathcal{E} = \{U \in \mathrm{Pow}(X) \mid r_1(U) = r_2(U)\}$$

of $\mathrm{Pow}(X)$, and define a set R of rules on X by

$$R = \{(r_1^{-1}(\{y\}), r_2^{-1}(\{y\})) \mid y \in Y\}.$$

Then $\mathcal{E}$ is the class of R-biclosed subsets of X, and hence has a generating set E by $\mathrm{NID}_{\mathrm{bi}}$. Define a relation $e \subseteq E \times X$ by

$$U \mathrel{e} x \Leftrightarrow x \in U.$$

Then e is a weak equaliser of r_1 and r_2 in Rel. $\square$

Weak equalisers in Rel

Proof.

(2) $\Rightarrow$ (1): Suppose that Rel has weak equaliser, and let R be a set of rules on a set S. Consider a parallel pair $r_1, r_2 \subseteq S \times R$ of relations given by

$$x \; r_1 \; (a, b) \Leftrightarrow x \in a, \qquad\qquad x \; r_2 \; (a, b) \Leftrightarrow x \in b,$$

and let $e \subseteq E \times S$ be a weak equaliser of r_1 and r_2 in Rel. Then

$$G = \{e(\{c\}) \mid c \in E\}$$

is a generating set of the class of R-biclosed subsets of S. $\square$

The category of basic pairs

Definition 14

A basic pair is a triple $(X, \Vdash, S)$ of sets X and S, and a relation $\Vdash$ between X and S.

Notation 15

For a basic pair $(X, \Vdash, S)$, we write

$$\Diamond D = \Vdash (D) \quad \text{and} \quad \mathrm{ext}\, U = \Vdash^{-1} (U)$$

for $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$.

The category of basic pairs

Definition 16

A relation pair between basic pairs $\mathcal{X}_1 = (X_1, \Vdash_1, S_1)$ and $\mathcal{X}_2 = (X_2, \Vdash_2, S_2)$ is a pair (r, s) of relations $r \subseteq X_1 \times X_2$ and $s \subseteq S_1 \times S_2$ such that

$$\Vdash_2 \circ\, r = s \circ \Vdash_1,$$

that is, the following diagram commutes.

$$\begin{tikzcd} X_1 \arrow[r, "\Vdash_1"] \arrow[d, "r"'] & S_1 \arrow[d, "s"] \\ X_2 \arrow[r, "\Vdash_2"'] & S_2 \end{tikzcd}$$

The category of basic pairs

Definition 17

Two relation pairs (r_1, s_1) and (r_2, s_2) between basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent, denoted by $(r_1, s_1) \sim (r_2, s_2)$, if

$$\Vdash_2 \circ\, r_1 = \Vdash_2 \circ\, r_2,$$

or equivalently $s_1 \circ \Vdash_1 = s_2 \circ \Vdash_1$.

Notation 18

We write BP for the category of basic pairs and relation pairs.

Equalisers in BP

Proposition 19

The following are equivalent over ECST.

1. Rel *has weak equalisers.*
2. BP *has equalisers.*

Equalisers in BP

Remark 20

- The categories Rel and BP are self dual, that is, $\mathrm{Rel} \simeq \mathrm{Rel}^{\mathrm{op}}$ and $\mathrm{BP} \simeq \mathrm{BP}^{\mathrm{op}}$;
- Rel has weak equaliser if and only if Rel has weak coequaliser, and BP has equaliser if and only if BP has coequaliser;
- in ECST, Rel has small products and hence has small coproducts, and BP has small products and coproducts;
- the following are equivalent over ECST.
  1. BP has (co)equalisers.
  2. BP is (co)complete.

The elementary NID

Theorem 21
The following are equivalent over ECST.

1. NID_1.
2. $\mathrm{NID}_{\mathrm{bi}}$.
3. Rel *has weak (co)equalisers.*
4. BP *has (co)equalisers.*
5. BP *is complete and cocomplete.*

The elementary NID

$$\begin{tikzcd}
\mathrm{NID}_{<\omega} \arrow[d] \\
\mathrm{NID}_1 \arrow[d] \\
\mathrm{NID}_0
\end{tikzcd}$$

Rel has weak (co)equalisers
BP has (co)equalisers
BP is complete and cocomplete

Fullness

Equivalents of the finitary NID

Models of geometric theories

Definition 22

Given a set S, a geometric theory (GT) over S is a set T of formulae of the form

$$\wedge\sigma \to \bigvee_{i \in I} \wedge\tau_i,$$

where I is a set, and σ and τ_i are finitely enumerable subsets of S.

Definition 23

A model of T is a subset α of S such that

$$\sigma \subseteq \alpha \to \exists i \in I(\tau_i \subseteq \alpha)$$

for all formula $\wedge\sigma \to \bigvee_{i \in I} \wedge\tau_i$ in T.

Models of geometric theories

Definition 24

Let $\mathrm{NID}_{\leq 2}$ be the principle obtained from NID by restricting the set R to those rules (a, b) where a is a surjective image of $n \leq 2$.

Proposition 25

The following are equivalent over ECST.

1. $\mathrm{NID}_{\leq 2}$.
2. $\mathrm{NID}_{<\omega}$.
3. *The class of models of a GT is set-generated.*

n-ary NID

Definition 26

A rule (a, b) on a set S is called *n-ary* if there exists a surjection $n \to a$.

Remark 27

Note that if a rule is $n + 1$-ary, then it is $n + 2$-ary.

Definition 28

The principle obtained by restricting R in NID to a set of n-ary rules is denoted by NID_n.

n-ary NID

Lemma 29

The following are equivalent over ECST*.*

1. $\mathrm{NID}_{\leq 2}$.
2. NID_2.

Proposition 30

The following are equivalent over ECST*.*

1. $\mathrm{NID}_{<\omega}$.
2. NID_n $(n \geq 2)$.

n-ary NID

$$
\begin{tikzcd}
\mathrm{NID}_{<\omega} \arrow[d, leftrightarrow] \\
\vdots \arrow[d, leftrightarrow] \\
\mathrm{NID}_2 \arrow[d] \\
\mathrm{NID}_1 \arrow[d] \\
\mathrm{NID}_0
\end{tikzcd}
$$

The class of models of a GT is set-generated.

Rel has weak (co)equalisers
BP has (co)equalisers
BP is complete and cocomplete

Fullness

Formal points of formal topologies

Definition 31

A formal topology (FT) $(S, \leq, \lhd)$ is a preordered set $(S, \leq)$ equipped with a subclass $\lhd \subseteq S \times \mathrm{Pow}(S)$ such that

1. $a \in U \Rightarrow a \lhd U$,
2. $a \lhd U$ and $\forall c \in U (c \lhd V) \Rightarrow a \lhd V$,
3. $a \lhd U$ and $a \lhd V \Rightarrow a \lhd U \downarrow V$,
4. $a \leq b \Rightarrow a \lhd \{b\}$.

Formal points of formal topologies

Definition 32

A formal topology $(S, \leq, \lhd)$ is inductively generated (i.g.) by an axiom-set (I, C) if $\lhd$ is the smallest among the relation $\lhd'$ such that

1. $a \leq b \lhd' U \Rightarrow a \lhd' U$,
2. $a \lhd' C(a, i)$ for all $i \in I(a)$,

and which makes $(S, \leq, \lhd')$ a formal topology.

Formal points of formal topologies

Definition 33

A formal point (f.p.) of a formal topology $(S, \leq, \lhd)$ is a subset $\alpha \subseteq S$ such that

1. α is inhabited,
2. $a, b \in \alpha \Rightarrow (a \downarrow b) \between \alpha$
3. $a \in \alpha$ and $a \lhd U \Rightarrow U \between \alpha$.

Remark 34

If $(S, \leq, \lhd)$ is inductively generated by an axiom-set (I, C), then the condition 3 is equivalent to

1. $a \leq b$ and $a \in \alpha \Rightarrow b \in \alpha$,
2. $a \in \alpha \Rightarrow \alpha \between C(a, i)$ for all $i \in I(a)$.

Formal points of formal topologies

Finite Powers Axiom (FPA):

$$\forall a \forall n \in \omega \, \exists b (b = a^n).$$

Proposition 35

The following are equivalent over ECST*.*

1. $\mathrm{NID}_{<\omega}$.
2. *The class of f.p. of an i.g. FT is set-generated* $+ \, \mathrm{FPA}$.

The category of concrete spaces

Notation 36

Let $(S, \leq)$ be a preordered set, and let D and E be subsets of S. Then

- $\downarrow D = \{a \in S \mid \exists b \in D(a \leq b)\}$;
- $D \downarrow E = \downarrow D \cap \downarrow E$;
- $\downarrow a = \downarrow \{a\}$ and $a \downarrow b = \{a\} \downarrow \{b\}$.

Remark 37

Given a basic pair $(X, \Vdash, S)$, we can define a preorder $\leq$ on S by

$$a \leq b \Leftrightarrow \mathrm{ext}\, a \subseteq \mathrm{ext}\, b.$$

The category of concrete spaces

Definition 38

A concrete space is a basic pair $(X, \Vdash, S)$ which satisfies

1. $\mathrm{ext}\, a \cap \mathrm{ext}\, b = \mathrm{ext}(a \downarrow b)$,
2. $X = \mathrm{ext}\, S$

Definition 39

A relation pair (r, s) between basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ is said to be convergent if

1. $\mathrm{ext}_1(s^{-1}a \downarrow s^{-1}b) = r^{-1}\,\mathrm{ext}_2(a \downarrow b)$,
2. $\mathrm{ext}_1\, S_1 = r^{-1}\,\mathrm{ext}_2\, S_2$

for all a and b in S_2.

Notation 40

We write CSpa for the category of concrete spaces and convergent relation pairs.

Equalisers in CSpa

Proposition 41

The following are equivalent over ECST.

1. $\mathrm{NID}_{<\omega}$.
2. CSpa *has equalisers* $+ \mathrm{FPA}$.

Equalisers in CSpa

Remark 42

- CSpa has small products using $\mathrm{NID}_{<\omega}$;
- if CSpa has equalisers, then CSpa is complete;
- coequalisers in CSpa can be constructed exactly as in BP;
- CSpa is cocomplete under NID_1, and hence under $\mathrm{NID}_{<\omega}$;
- the following are equivalent over $\mathsf{ECST} + \mathrm{FPA}$.
  1. CSpa has equalisers.
  2. CSpa is complete and cocomplete.

The finitary NID

Theorem 43

The following are equivalent over ECST.

1. $\mathrm{NID}_{<\omega}$.
2. NID_n $(n \geq 2)$.
3. *The class of models of a GT is set-generated.*
4. *The class of f.p. of an i.g. FT is set-generated* $+ \mathrm{FPA}$.
5. CSpa *has equalisers* $+ \mathrm{FPA}$.
6. CSpa *is complete and cocomplete* $+ \mathrm{FPA}$.

The finitary NID

$$\begin{array}{c}
\mathrm{NID}_{<\omega} \\
\downarrow \\
\mathrm{NID}_1 \\
\downarrow \\
\mathrm{NID}_0
\end{array}$$

$\mathrm{NID}_{<\omega}$:

The class of models of a GT is set-generated
The class of f.p. of an i.g. FT is set-generated $+ \mathrm{FPA}$
CSpa has equalisers $+ \mathrm{FPA}$
CSpa is complete and cocomplete $+ \mathrm{FPA}$

NID_1:

Rel has weak (co)equalisers
BP has (co)equalisers
BP is complete and cocomplete

NID_0:

Fullness

References

- Peter Aczel, *Aspects of general topology in constructive set theory*, Ann. Pure Appl. Logic **137** (2006), 3–29.
- Peter Aczel, Hajime Ishihara, Takako Nemoto and Yasushi Sangu, *Generalized geometric theories and set-generated classes*, Math. Structures Comput. Sci. **25** (2015), 1466–1483.
- Ayana Hirata, Hajime Ishihara, Tatsuji Kawai and Takako Nemoto, *Equivalents of the finitary non-deterministic inductive definitions*, Ann. Pure Appl. Logic **170** (2019), 1256–1272.
- Hajime Ishihara and Erik Palmgren, *Quotient topologies in constructive set theory and type theory*, Ann. Pure Appl. Logic **141** (2006), 257–265.

References

- Hajime Ishihara and Tatsuji Kawai, *Completeness and cocompleteness of the categories of basic pairs and concrete spaces*, Math. Structures Comput. Sci. **25** (2015), 1626–1648.
- Hajime Ishihara and Takako Nemoto, *Non-deterministic inductive definitions and fullness*, Concepts of proof in mathematics, philosophy, and computer science, 163–170, Ontos Math. Log., 6, De Gruyter, Berlin, 2016.
- Giovanni Sambin, *Some points in formal topology*, Topology in computer science (Schloß Dagstuhl, 2000), Theoret. Comput. Sci. **305** (2003), 347–408.
- Benno van den Berg, *Non-deterministic inductive definitions*, Arch. Math. Logic **52** (2013), 113–135.