Reverse mathematics in constructive set theory

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Overview

- Lecture 1
  - Reverse mathematics and set theory
  - Intuitionistic logic
- Lecture 2
  - Classical Zermelo-Fraenkel set theory ZF
  - Basic constructive set theory BCST
  - Elementary constructive set theory ECST
  - Constructive Zermelo-Fraenkel set theory CZF
- Lecture 3
  - Set-generated classes and NID principles
  - Equivalents of the nullary NID
  - Equivalents of the elementary NID
  - Equivalents of the finitary NID

Lecture 1

- Reverse mathematics and set theory
- Intuitionistic logic
- Classical Zermelo-Fraenkel set theory ZF
- The axiom of choice

Reverse mathematics and set theory

Classical reverse mathematics

The Friedman-Simpson-program (classical reverse mathematics) is

- a formal mathematics using classical logic (in the language of second order arithmetic);
- assuming a very weak set existence axiom (recursive comprehension axiom RCA_0);
- main question is "What set existence axioms are needed to prove the theorems of ordinary mathematics?";
- many theorems have been classified by set existence axioms of various strengths (the weak König lemma WKL, arithmetical comprehension axiom ACA, ...).

Classical reverse mathematics

Since classical reverse mathematics is formalised with classical logic, we cannot

- classify theorems in intuitionism nor in constructive recursive mathematics which are inconsistent with classical mathematics (Brouwer's continuity principle BCP);
- distinguish theorems from their contrapositions (the fan theorem FAN from WKL);
- classify nonconstructive theorems provable in the base theory RCA_0 (binary expansion theorem BE and the intermediate value theorem IVT).

Bishop's constructive mathematics

Bishop's constructive mathematics is

- an informal mathematics using intuitionistic logic (in the language of set theory?);
- assuming some function existence axioms (countable choice axiom);
- a core of the varieties of mathematics which can be extended to
  - intuitionism (by adding $\mathrm{WC\text{-}N}$ and FAN),
  - constructive recursive mathematics (by adding ECT_0 and MP),
  - classical mathematics (by adding LEM).

Constructive reverse mathematics (CRM)

The purpose of constructive reverse mathematics is

- to classify various theorems in intuitionistic, constructive recursive and classical mathematics;
- by logical principles, function existence axioms and their combinations

in an intuitionistic (second-order or finite-type) arithmetic.

The constructive set theory, CZF

The constructive Zermelo-Fraenkel set theory, CZF (Aczel, 1978)

- is a set theory for Bishop's constructive mathematics;
- has a very natural interpretation in the Martin-Löf type theory;
- is a predicative theory
  - without power set axiom,
  - without full separation axiom.

Predicativity

The following is an example of impredicative definition of a set:

$$S = \{x \in \mathbb{N} \mid \forall a \in \mathrm{Pow}(\mathbb{N})\,(x \in a \to \cdots)\}$$
$$= \{x \in \mathbb{N} \mid \forall a(a \subseteq \mathbb{N} \land x \in a \to \cdots)\}.$$

- The set S is a subset of $\mathbb{N}$, that is, $S \in \mathrm{Pow}(\mathbb{N})$;
- the variable a ranges over $\mathrm{Pow}(\mathbb{N})$; hence we may take the set S being defined as a.

A predicative set theory

- does not allow this kind of circular argument in defining sets;
- does allow only constructions of sets from sets already constructed.

Reverse mathematics in constructive set theory

RM = reverse mathematics
CST = constructive set theory

	language	logic	objects
classical RM	arithmetic	classical	$\mathbb{N}, \mathrm{Pow}(\mathbb{N})$
constructive RM	arithmetic	intuitionistic	$\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \ldots$
RM in CST	set theory	intuitionistic	(predicative) sets

Intuitionistic (constructive) logic

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- primitive logical operators $\wedge, \vee, \to, \perp, \forall, \exists$.

We introduce the abbreviations

- $\neg A \equiv A \to \perp$;
- $A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$.

The BHK interpretation

The Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operators is the following.

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.
- A proof of $A \to B$ is a construction which transforms any proof of A into a proof of B.
- Absurdity $\perp$ has no proof.
- A proof of $\forall x A(x)$ is a construction which transforms any t into a proof of $A(t)$.
- A proof of $\exists x A(x)$ is given by presenting a t and a proof of $A(t)$.

Natural Deduction System

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Deduction (Basis)

For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Deduction (Induction step, $\to$I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

is a deduction with conclusion $A \to B$ and assumptions $\Gamma \setminus \{A\}$.
We write

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

Deduction (Induction step, $\to$E)

If

$$\begin{array}{ccc} \Gamma_1 & \quad & \Gamma_2 \\ \mathcal{D}_1 & & \mathcal{D}_2 \\ A \to B & & A \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A \to B & A \end{array}}{B} \to\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Example

$$
\begin{prooftree}
\AxiomC{$[\neg\neg A]$}
\AxiomC{$[\neg\neg(A \to B)]$}
\AxiomC{$[\neg B]$}
\AxiomC{$[A \to B]$}
\AxiomC{$[A]$}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{B}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$\neg(A \to B)$}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$\neg A$}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$\neg\neg B$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$\neg\neg A \to \neg\neg B$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$\neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$}
\end{prooftree}
$$

Minimal logic

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\;{\to}\mathrm{I} \qquad\qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\;{\to}\mathrm{E}$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A \wedge B}\;\wedge\mathrm{I} \qquad\qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{A}\;\wedge\mathrm{E}_r \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{B}\;\wedge\mathrm{E}_l$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{A \vee B}\;\vee\mathrm{I}_r \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ B\end{array}}{A \vee B}\;\vee\mathrm{I}_l \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]\\ \mathcal{D}_3\\ C\end{array}}{C}\;\vee\mathrm{E}$$

Minimal logic

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{\forall y A[x/y]}\ \forall\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \forall x A\end{matrix}}{A[x/t]}\ \forall\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ A[x/t]\end{matrix}}{\exists x A}\ \exists\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \exists y A[x/y]\end{matrix} \quad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix}}{C}\ \exists\mathrm{E}$$

- In $\forall\mathrm{E}$ and $\exists\mathrm{I}$, t must be free for x in A.
- In $\forall\mathrm{I}$, $\mathcal{D}$ must not contain assumptions containing x free, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.
- In $\exists\mathrm{E}$, $\mathcal{D}_2$ must not contain assumptions containing x free except A, $x \notin \mathrm{FV}(C)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[(A \to B) \land (A \to C)]}{A \to B}\ \land\mathrm{E}_r \qquad [A]}{B}\ \to\mathrm{E} \qquad \cfrac{\cfrac{[(A \to B) \land (A \to C)]}{A \to C}\ \land\mathrm{E}_l \qquad [A]}{C}\ \to\mathrm{E}}{B \land C}\ \land\mathrm{I}}{A \to B \land C}\ \to\mathrm{I}}{(A \to B) \land (A \to C) \to (A \to B \land C)}\ \to\mathrm{I}
$$

Example

$$
\dfrac{\dfrac{[A \vee B] \qquad \dfrac{\dfrac{[(A \to C) \wedge (B \to C)]}{A \to C}\,\wedge\mathrm{E}_r \qquad [A]}{C}\,\to\mathrm{E} \qquad \dfrac{\dfrac{[(A \to C) \wedge (B \to C)]}{B \to C}\,\wedge\mathrm{E}_l \qquad [B]}{C}\,\to\mathrm{E}}{\dfrac{C}{A \vee B \to C}\,\to\mathrm{I}}\,\vee\mathrm{E}}{(A \to C) \wedge (B \to C) \to (A \vee B \to C)}\,\to\mathrm{I}
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \to \forall x B] \quad [A]}{\forall x B}\,\to\mathrm{E}}{B}\,\forall\mathrm{E}}{A \to B}\,\to\mathrm{I}}{\forall x(A \to B)}\,\forall\mathrm{I}}{(A \to \forall x B) \to \forall x(A \to B)}\,\to\mathrm{I}
$$

where $x \notin \mathrm{FV}(A)$.

Example

$$
\dfrac{[\exists x(A \to B)] \qquad \dfrac{\dfrac{[A \to B] \quad [A]}{B}\ {\to}\mathrm{E}}{\exists x B}\ \exists\mathrm{I}}{\dfrac{\dfrac{\exists x B}{A \to \exists x B}\ {\to}\mathrm{I}}{\exists x(A \to B) \to (A \to \exists x B)}\ {\to}\mathrm{I}}\ \exists\mathrm{E}
$$

where $x \notin \mathrm{FV}(A)$.

Intuitionistic logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \perp\end{array}$$

is a deduction, then

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \dfrac{\perp}{A}\ \perp_i\end{array}$$

is a deduction with conclusion A and assumptions Γ.

Example

$$
\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\,{\to}\mathrm{E}}{B}\,{\color{red}\bot_i}}{A \to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{\neg\neg B}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{[B]}{A \to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg B}\,{\to}\mathrm{I}}{\dfrac{\bot}{\neg\neg(A \to B)}\,{\to}\mathrm{I}}}{(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)}\,{\to}\mathrm{I}
$$

Example

$$\dfrac{\dfrac{\dfrac{[\neg A \vee B] \qquad \dfrac{\dfrac{[A] \quad [\neg A]}{\bot}\ \to\mathrm{E}}{B}\ {\color{red}\bot_i} \qquad [B]}{B}\ \vee\mathrm{E}}{A \to B}\ \to\mathrm{I}}{\neg A \vee B \to (A \to B)}\ \to\mathrm{I}$$

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \perp\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \perp\end{array}}{A}\ \perp_c$$

is a deduction with conclusion A and assumption $\Gamma \setminus \{\neg A\}$.

Example (classical logic)

The double negation elimination (DNE):

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\ {\to}\mathrm{E}}{A}\ {\color{red}\bot_c}}{\neg\neg A \to A}\ {\to}\mathrm{I}$$

Example (classical logic)

The law of excluded middle (LEM):

$$
\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{[A]}{A \vee \neg A}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{A \vee \neg A}\,\vee\mathrm{I}_l}{\dfrac{\bot}{A \vee \neg A}\,\bot_c}\,\to\mathrm{E}
$$

Example (classical logic)

De Morgan's law (DML):

$$
\dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \wedge B)] \qquad \dfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathrm{I}}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg B}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_l}{\bot}\,\to\mathrm{E}}{\neg A \vee \neg B}\,\bot_c}{\neg(A \wedge B) \to \neg A \vee \neg B}\,\to\mathrm{I}
$$

$\bot_c$ and DNE

An application of $\bot_c$:

$$\dfrac{\begin{matrix}[\neg A]\\ \mathcal{D}\\ \bot\end{matrix}}{A}\ \bot_c$$

can be simulated using DNE:

$$\dfrac{\neg\neg A \to A \qquad \dfrac{\begin{matrix}[\neg A]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg\neg A}\ {\to}\mathrm{I}}{A}\ {\to}\mathrm{E}$$

$\bot_c$ and LEM

An application of $\bot_c$:

$$\dfrac{\begin{array}{c}[\neg A]\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

can be simulated using LEM:

$$\dfrac{A \lor \neg A \quad [A] \quad \dfrac{\begin{array}{c}[\neg A]\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_i}{A}\ \lor\mathrm{E}$$

$\to\mathrm{I}$ vs $\bot_c$

$\to\mathrm{I}$: deriving $\neg A$ by deducing absurdity ($\bot$) from A.

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg A}\;\to\mathrm{I}$$

$\bot_c$: deriving A by deducing absurdity ($\bot$) from $\neg A$.

$$\dfrac{\begin{matrix}[\neg A]\\ \mathcal{D}\\ \bot\end{matrix}}{A}\;\bot_c$$

$\to$I vs $\perp_c$

There is a (constructive) proof $\mathcal{D}_{\sqrt{2}}$ that "$\sqrt{2}$ is rational" entails a contradiction.

$$\begin{array}{c} \text{“}\sqrt{2}\text{ is rational”} \\ \mathcal{D}_{\sqrt{2}} \\ \perp \end{array}$$

Note that, since a real number is irrational if it is not rational,

$$\text{“}\sqrt{2}\text{ is irrational”} \equiv \neg\text{“}\sqrt{2}\text{ is rational”}.$$

$\to\mathrm{I}$ vs $\bot_c$

$\to\mathrm{I}$:

$$\dfrac{\begin{array}{c}[\text{“}\sqrt{2}\text{ is rational”}]\\ \mathcal{D}_{\sqrt{2}}\\ \bot\end{array}}{\text{“}\sqrt{2}\text{ is irrational”}}\ \to\mathrm{I}$$

$\bot_c$:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\text{“}\sqrt{2}\text{ is irrational”}] \quad [\text{“}\sqrt{2}\text{ is irrational”}]}{\bot}\ \to\mathrm{E}}{\text{“}\sqrt{2}\text{ is rational”}}\ \bot_c\\ \mathcal{D}_{\sqrt{2}}\\ \bot\end{array}}{\text{“}\sqrt{2}\text{ is irrational”}}\ \bot_c$$