## Reverse mathematics in constructive set theory

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Proof and Computation, Fischbachau, 26 September – 1 October, 2022

### Overview

- Lecture 1
  - Reverse mathematics and set theory
  - Intuitionistic logic
- ► Lecture 2
  - Classical Zermelo-Fraenkel set theory ZF
  - Basic constructive set theory BCST
  - Elementary constructive set theory ECST
  - Constructive Zermelo-Fraenkel set theory CZF
- Lecture 3
  - Set-generated classes and NID principles
  - Equivalents of the nullary NID
  - ► Equivalents of the elementary NID
  - Equivalents of the finitary NID

### Lecture 3

- Set-generated classes and NID principles
- ► Equivalents of the nullary NID
- ► Equivalents of the elementary NID
- ► Equivalents of the finitary NID

Set-generated classes and NID principles

### Set-generated classes

#### Definition 1

Let S be a set, and let X be a subclass of Pow(S). Then X is set-generated if there exists a subset G, called a generating set, of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G(x \in \beta \subseteq \alpha).$$

#### Remark 2

The power class Pow(S) of a set S is set-generated with a generating set

$$\{\{x\}\mid x\in\mathcal{S}\}.$$

### Rules

#### Definition 3

Let S be a set. Then a rule on S is a pair (a, b) of subsets a and b of S. A rule is called

- nullary if a is empty;
- elementary if a is a singleton;
- finitary if a is finitely enumerable.

A subset  $\alpha$  of S is closed under a rule (a, b) if

$$a \subseteq \alpha \rightarrow b \ \ \alpha.$$

For a set R of rules on S, we call a subset  $\alpha$  of S R-closed if it is closed under all rules in R.

#### Remark 4

Note that if a rule is nullary or elementary, then it is finitary.



### NID principles

#### Definition 5

Let NID denote the principles that

▶ for each set *S* and set *R* of rules on *S*, the class of *R*-closed subsets of *S* is set-generated.

The principles obtained by restricting R in NID to a set of nullary, elementary and finitary rules are denoted by  $\mathrm{NID}_0$ ,  $\mathrm{NID}_1$  and  $\mathrm{NID}_{<\omega}$ , respectively.

#### Remark 6

Note that  $\mathrm{NID}_{<\omega}$  implies  $\mathrm{NID}_0$  and  $\mathrm{NID}_1$ .

# Equivalents of the nullary NID

### **Fullness**

The class of total relations between A and B is denoted by mv(A, B):

$$r \in \operatorname{mv}(A, B) \Leftrightarrow r \subseteq A \times B \wedge \forall x \in A \exists y \in B (\langle x, y \rangle \in r);$$

Fullness:

$$\forall a \forall b \exists c (c \subseteq mv(a, b))$$
$$\land \forall r \in mv(a, b) \exists s \in c (s \subseteq r)).$$

### $\mathrm{NID}_0$ and Fullness

### Theorem 7

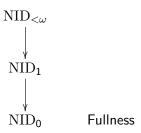
The following are equivalent over ECST.

- 1.  $NID_0$ .
- 2. Fullness.

### Proposition 8

 $\mathrm{NID}_1$  implies  $\mathrm{NID}_0$ .

## $\mathrm{NID}_0$ and Fullness



### NID<sub>0</sub> and Fullness

### Proof of Theorem 7.

(1)  $\Rightarrow$  (2): Suppose NID<sub>0</sub>. For sets A and B, define a set R of nullary rules on  $(A \times B) \cup \{R_{A \times B}\}$  by

$$R = \{(\emptyset, \{x\} \times B) \mid x \in A\} \cup \{(\emptyset, \{R_{A \times B}\})\}.$$

Then there exists a subset G of the class X of R-closed subsets of  $(A \times B) \cup \{R_{A \times B}\}$  such that

$$\forall \alpha \in X \, \forall z \in \alpha \, \exists \beta \in G \, (z \in \beta \subseteq \alpha).$$

Let  $C = \{\beta \cap (A \times B) \mid \beta \in G\}$ . Then  $C \subseteq \operatorname{mv}(A, B)$ . For each  $r \in \operatorname{mv}(A, B)$ , since  $r \cup \{R_{A \times B}\} \in X$ , there exists  $\beta \in G$  such that  $R_{A \times B} \in \beta \subseteq r \cup \{R_{A \times B}\}$ . Therefore

$$\beta \cap (A \times B) \subseteq (r \cup \{R_{A \times B}\}) \cap (A \times B) = r. \qquad \Box$$

## Equivalents of the elementary NID

# The principle $\mathrm{NID}_{\mathrm{bi}}$

### Definition 9

Let S be a set. Then a subset  $\alpha$  of S is biclosed under a rule (a,b) if

$$a \between \alpha \leftrightarrow b \between \alpha$$
.

For a set R of rules on S, we call a subset  $\alpha$  of S R-biclosed if it is biclosed under all rules in R.

#### Definition 10

Let  $\mathrm{NID}_{\mathrm{bi}}$  denote the principle that

▶ for each set *S* and set *R* of rules on *S*, the class of *R*-biclosed subsets of *S* is set-generated.

# The principle $\mathrm{NID}_{\mathrm{bi}}$

### Proposition 11

- 1. NID<sub>1</sub>.
- 2. NID<sub>bi</sub>.

# The principle $\mathrm{NID}_{\mathrm{bi}}$

### Proof of Proposition 11.

Suppose  $NID_1$ , and let R be a set of rules on a set S. Define a set R' of elementary rules on S by

$$R' = \{(\{x\}, b) \mid (a, b) \in R \land x \in a\} \cup \{(\{y\}, a) \mid (a, b) \in R \land y \in b\}.$$

Then a subset  $\alpha\subseteq S$  is R-biclosed if and only if it is R'-closed. Conversely, suppose  $\mathrm{NID_{bi}}$ , and let R be a set of elementary rules on a set S. Define a set R' of rules on S by

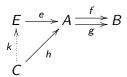
$$R' = \{(a \cup b, b) \mid (a, b) \in R\}.$$

Then a subset  $\alpha \subseteq S$  is R-closed if and only if it is R'-biclosed.



### Definition 12

An equaliser of a parallel pair  $A \stackrel{r}{\underset{g}{\rightleftharpoons}} B$  in a category C is a pair of an object E and a morphism  $E \stackrel{e}{\underset{g}{\rightleftharpoons}} A$  such that  $f \circ e = g \circ e$ , and it satisfies a universal property in the sense that for any morphism  $C \stackrel{h}{\rightarrow} A$  with  $f \circ h = g \circ h$ , there exists a unique morphism  $C \stackrel{k}{\rightarrow} E$  for which the following diagram commutes.



A equaliser without the uniqueness condition is called a weak equaliser.



### Proposition 13

- 1. NID<sub>bi</sub>.
- 2. Rel has weak equalisers.

### Proof.

(1)  $\Rightarrow$  (2): Suppose NID<sub>bi</sub>, and let  $r_1, r_2 \subseteq X \times Y$  be a parallel pair of relations. Consider a subclass

$$\mathcal{E} = \{U \in \mathrm{Pow}(X) \mid r_1(U) = r_2(U)\}$$

of Pow(X), and define a set R of rules on X by

$$R = \{ (r_1^{-1}(\{y\}), r_2^{-1}(\{y\})) \mid y \in Y \}.$$

Then  $\mathcal E$  is the class of R-biclosed subsets of X, and hence has a generating set E by  $\mathrm{NID_{bi}}$ . Define a relation  $e\subseteq E\times X$  by

$$U e x \Leftrightarrow x \in U$$
.

Then e is a weak equaliser of  $r_1$  and  $r_2$  in Rel.



### Proof.

(2)  $\Rightarrow$  (1): Suppose that Rel has weak equaliser, and let R be a set of rules on a set S. Consider a parallel pair  $r_1, r_2 \subseteq S \times R$  of relations given by

$$x r_1(a, b) \Leftrightarrow x \in a,$$
  $x r_2(a, b) \Leftrightarrow x \in b,$ 

and let  $e \subseteq E \times S$  be a weak equaliser of  $r_1$  and  $r_2$  in Rel. Then

$$G = \{e(\{c\}) \mid c \in E\}$$

is a generating set of the class of R-biclosed subsets of S.



# The category of basic pairs

### Definition 14

A basic pair is a triple  $(X, \Vdash, S)$  of sets X and S, and a relation  $\Vdash$  between X and S.

#### Notation 15

For a basic pair  $(X, \Vdash, S)$ , we write

$$\Diamond D = \Vdash (D)$$
 and  $\operatorname{ext} U = \Vdash^{-1} (U)$ 

for  $D \in Pow(X)$  and  $U \in Pow(S)$ .

## The category of basic pairs

#### Definition 16

A relation pair between basic pairs  $\mathcal{X}_1=(X_1,\Vdash_1,S_1)$  and  $\mathcal{X}_2=(X_2,\Vdash_2,S_2)$  is a pair (r,s) of relations  $r\subseteq X_1\times X_2$  and  $s\subseteq S_1\times S_2$  such that

$$\Vdash_2 \circ r = s \circ \Vdash_1$$
,

that is, the following diagram commutes.

$$X_{1} \xrightarrow{\parallel_{1}} S_{1}$$

$$\downarrow r \qquad \qquad \downarrow s$$

$$X_{2} \xrightarrow{\parallel_{2}} S_{2}$$

# The category of basic pairs

#### Definition 17

Two relation pairs  $(r_1, s_1)$  and  $(r_2, s_2)$  between basic pairs  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are equivalent, denoted by  $(r_1, s_1) \sim (r_2, s_2)$ , if

$$\Vdash_2 \circ r_1 = \Vdash_2 \circ r_2$$
,

or equivalently  $s_1 \circ \Vdash_1 = s_2 \circ \Vdash_1$ .

#### Notation 18

We write BP for the category of basic pairs and relation pairs.



### Equalisers in BP

### Proposition 19

- 1. Rel has weak equalisers.
- 2. BP has equalisers.

### Equalisers in BP

#### Remark 20

- ▶ The categories Rel and BP are self dual, that is, Rel  $\simeq$  Rel  $^{\mathrm{op}}$  and BP  $\simeq$  BP $^{\mathrm{op}}$ ;
- ▶ Rel has weak equaliser if and only if Rel has weak coequaliser, and BP has equaliser if and only if BP has coequaliser;
- in ECST, Rel has small products and hence has small coproducts, and BP has small products and coproducts;
- the following are equivalent over ECST.
  - 1. BP has (co)equalisers.
  - 2. BP is (co)complete.

## The elementary NID

### Theorem 21

- 1. NID<sub>1</sub>.
- 2. NID<sub>bi</sub>.
- 3. Rel has weak (co)equalisers.
- 4. BP has (co)equalisers.
- 5. BP is complete and cocomplete.

### The elementary NID



# Equivalents of the finitary NID

# Models of geometric theories

#### Definition 22

Given a set S, a geometric theory (GT) over S is a set T of formulae of the form

$$\wedge \sigma \to \bigvee_{i \in I} \wedge \tau_i,$$

where I is a set, and  $\sigma$  and  $\tau_i$  are finitely enumerable subsets of S.

### Definition 23

A model of T is a subset  $\alpha$  of S such that

$$\sigma \subseteq \alpha \to \exists i \in I(\tau_i \subseteq \alpha)$$

for all formula  $\wedge \sigma \to \bigvee_{i \in I} \wedge \tau_i$  in T.

## Models of geometric theories

### **Definition 24**

Let  $\mathrm{NID}_{\leq 2}$  be the principle obtained from  $\mathrm{NID}$  by restricting the set R to those rules (a,b) where a is a surjective image of  $n\leq 2$ .

### Proposition 25

- 1.  $NID_{\leq 2}$ .
- 2. NID $<\omega$ .
- 3. The class of models of a GT is set-generated.

### *n*-ary NID

### **Definition 26**

A rule (a, b) on a set S is called *n*-ary if there exists a surjection  $n \to a$ .

#### Remark 27

Note that if a rule is n + 1-ary, then it is n + 2-ary.

#### Definition 28

The principle obtained by restricting R in NID to a set of n-ary rules is denoted by  $NID_n$ .

# *n*-ary NID

#### Lemma 29

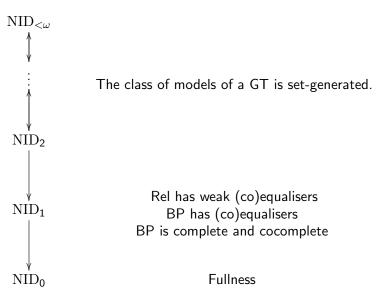
The following are equivalent over ECST.

- 1.  $NID_{\leq 2}$ .
- 2. NID<sub>2</sub>.

### Proposition 30

- 1.  $NID_{<\omega}$ .
- 2. NID<sub>n</sub>  $(n \ge 2)$ .

# *n*-ary NID



### **Definition 31**

A formal topology (FT)  $(S, \leq, \lhd)$  is a preordered set  $(S, \leq)$  equipped with a subclass  $\lhd \subseteq S \times \operatorname{Pow}(S)$  such that

- 1.  $a \in U \Rightarrow a \triangleleft U$ ,
- 2.  $a \triangleleft U$  and  $\forall c \in U(c \triangleleft V) \Rightarrow a \triangleleft V$ ,
- 3.  $a \triangleleft U$  and  $a \triangleleft V \Rightarrow a \triangleleft U \downarrow V$ ,
- 4.  $a \leq b \Rightarrow a \triangleleft \{b\}$ .

### **Definition 32**

A formal topology  $(S, \leq, \lhd)$  is inductively generated (i.g.) by an axiom-set (I, C) if  $\lhd$  is the smallest among the relation  $\lhd'$  such that

- 1.  $a < b \triangleleft' U \Rightarrow a \triangleleft' U$ .
- 2.  $a \triangleleft' C(a, i)$  for all  $i \in I(a)$ ,

and which makes  $(S, \leq, \lhd')$  a formal topology.

### **Definition 33**

A formal point (f.p.) of a formal topology ( $S, \leq, \lhd$ ) is a subset  $\alpha \subseteq S$  such that

- 1.  $\alpha$  is inhabited,
- 2.  $a, b \in \alpha \Rightarrow (a \downarrow b) \Diamond \alpha$
- 3.  $a \in \alpha$  and  $a \triangleleft U \Rightarrow U \lozenge \alpha$ .

#### Remark 34

If  $(S, \leq, \lhd)$  is inductively generated by an axiom-set (I, C), then the condition 3 is equivalent to

- 1.  $a \leq b$  and  $a \in \alpha \Rightarrow b \in \alpha$ ,

### Finite Powers Axiom (FPA):

$$\forall a \forall n \in \omega \, \exists b (b = a^n).$$

### Proposition 35

- 1.  $NID_{<\omega}$ .
- 2. The class of f.p. of an i.g. FT is set-generated + FPA.

## The category of concrete spaces

#### Notation 36

Let  $(S, \leq)$  be a preordered set, and let D and E be subsets of S. Then

- $\blacktriangleright \downarrow D = \{a \in S \mid \exists b \in D(a \leq b)\};$
- $\triangleright$   $D \downarrow E = \downarrow D \cap \downarrow E$ ;
- $ightharpoonup \downarrow a = \downarrow \{a\} \text{ and } a \downarrow b = \{a\} \downarrow \{b\}.$

#### Remark 37

Given a basic pair  $(X, \Vdash, S)$ , we can define a preorder  $\leq$  on S by

$$a \le b \Leftrightarrow \operatorname{ext} a \subseteq \operatorname{ext} b$$
.

## The category of concrete spaces

### **Definition 38**

A concrete space is a basic pair  $(X, \Vdash, S)$  which satisfies

- 1.  $\operatorname{ext} a \cap \operatorname{ext} b = \operatorname{ext}(a \downarrow b)$ ,
- 2. X = ext S

### **Definition 39**

A relation pair (r,s) between basic pairs  $\mathcal{X}_1$  and  $\mathcal{X}_2$  is said to be convergent if

- 1.  $\exp_1(s^{-1}a \downarrow s^{-1}b) = r^{-1} \exp_2(a \downarrow b),$
- 2.  $\operatorname{ext}_1 S_1 = r^{-1} \operatorname{ext}_2 S_2$

for all a and b in  $S_2$ .

### Notation 40

We write CSpa for the category of concrete spaces and convergent relation pairs.



## Equalisers in CSpa

### Proposition 41

- 1.  $NID_{<\omega}$ .
- 2. CSpa has equalisers + FPA.

### Equalisers in CSpa

#### Remark 42

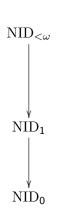
- ► CSpa has small products using  $NID_{<\omega}$ ;
- if CSpa has equalisers, then CSpa is complete;
- coequalisers in CSpa can be constructed exactly as in BP;
- ▶ CSpa is cocomplete under  $NID_1$ , and hence under  $NID_{<\omega}$ ;
- ▶ the following are equivalent over ECST + FPA.
  - 1. CSpa has equalisers.
  - 2. CSpa is complete and cocomplete.

## The finitary NID

### Theorem 43

- 1. NID $<\omega$ .
- 2. NID<sub>n</sub>  $(n \ge 2)$ .
- 3. The class of models of a GT is set-generated.
- 4. The class of f.p. of an i.g. FT is set-generated + FPA.
- 5. CSpa has equalisers + FPA.
- 6. CSpa is complete and cocomplete + FPA.

## The finitary NID



The class of models of a GT is set-generated The class of f.p. of an i.g. FT is set-generated + FPA CSpa has equalisers + FPA CSpa is complete and cocomplete + FPA

Rel has weak (co)equalisers BP has (co)equalisers BP is complete and cocomplete

Fullness

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