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Math3600 Finial Project Report

SIR Model（epidemic model）

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Introduction

SIR model is a kind of disease spread model, it's abstract description for information spreading. The full name of SIR is Susceptible Infected Recovered Model. It divided into three kinds of individual person. Through this model, I will use differential equation to solve it and plot the population of disease trend by the time.

Theory

s(t) means people who are susceptible to the disease after getting in touch to people who already infected. i(t) means people already infected to the diseases and they're easily to spread to s(t) population. r(t) means people that infected to diseases already recovered, so remove them from the disease. This population won't be infected again and won't infect to other people. The total number should be N, then we can get these functions.

$$\frac{S(t)}{N} = s(t)$$
$$\frac{I(t)}{N} = i(t)$$
$$\frac{R(t)}{N} = r(t)$$

set λ be the infection rate, μ be the daily recovery rate. We can know the rate of susceptible population is $-\lambda s(t)i(t)$. the rate of infected population is $\lambda s(t)i(t) - \mu i(t)$. the rate of recovery rate is $\mu i(t)$

Therefore, we can get three differential equations

$$\frac{di}{dt} = \lambda s(t)i(t) - \mu i(t)$$
$$\frac{ds}{dt} = -\lambda s(t)i(t)$$
$$\frac{dr}{dt} = \mu i(t)$$

Depend on the Euler method $\boldsymbol{y_{i+1} = y_i + hf(t_i, y_i)}$**,** h is size step.

Then we can transform these equations to

$$i_{t+1} - i_t = \lambda s(t)i(t) - \mu i(t)$$

$$s_{t+1} - s_t = -\lambda s(t)i(t)$$

$$r_{t+1} - r_t = \mu i(t)$$

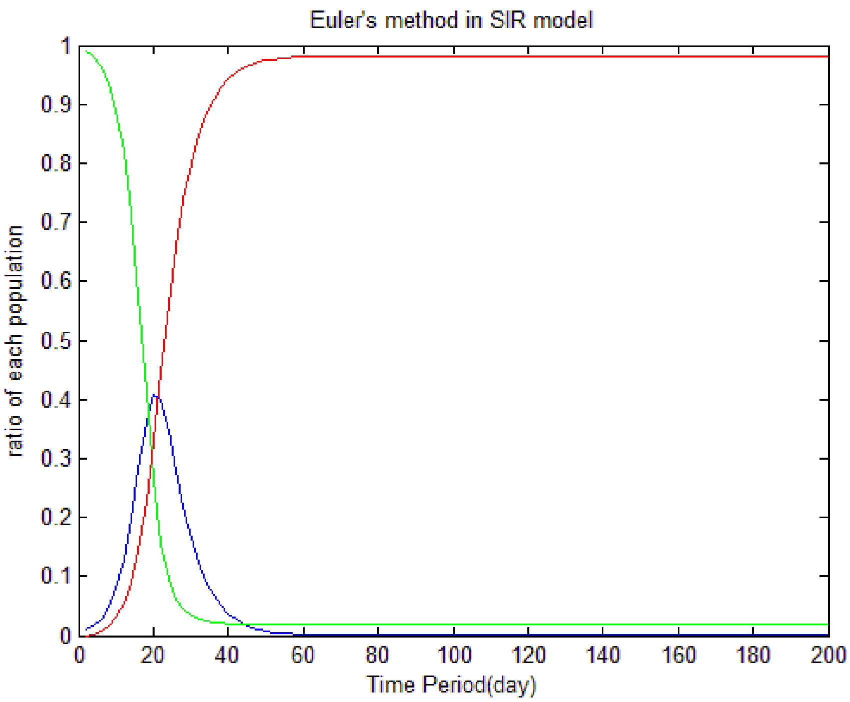
Implementation

Then we can use Euler method to calculate these differential equations and plot the trend about Susceptible, Infected and Recovered.

Here is my code and plot

```
T=200;%set time period
n=100;%size step
 h=T/n;%time interval
E=zeros(3,T/h);%set initial matrix
lamda=0.5;%set infection rate
mu=1/7;%set recovery rate
E(1,1)=0.01;%the initial rate of daily Infected people
E(2,1)=0.99;%the initial rate of daily susceptible people
E(3,1)=0;%the initial rate of daily recover people
for i=1:(T/h)-1
    %follow the equation
    E(1,i+1)=E(1,i)+(lamda*E(1,i)*E(2,i)-mu*E(1,i))*h;
    E(2,i+1)=E(2,i)+(-lamda*E(1,i)*E(2,i))*h;
    E(3,i+1)=E(3,i)+(mu*E(1,i))*h;
end
c=E(3,:);
b=E(2,:);
a=E(1,:);
d=zeros(1,n);
for j=1:n
    d(:,j)=j*h;
end

plot(d,a,'blue');hold all;
plot(d,b,'green');
plot(d,c,'red');
```



The plot shows three different trends. The green line is Susceptible population, the blue line is infectious population, the red line is recovered population.

Another way to implement SIR model is to use modified Euler's method, it changes the

Euler method to take the average of sum of k1 and k2.

According to modified Euler's method, we can know

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_i + h, y_i + k_1)$$

$$y_{i+1} = y_i + 0.5(k_1 + k_2)$$

Then apply it in SIR model,

$k1 = (\lambda s(t)i(t) - \mu i(t))$h

$s1 = (-\lambda s(t)i(t))$h

$r1 = (\mu(i(t))$h

k2=(λ(i(t)+k1)(i(t)+s1)- μ(i(t)+k1))h

s2=(- λ*(i(t)+k1)*(i(t)+s(t)1))h

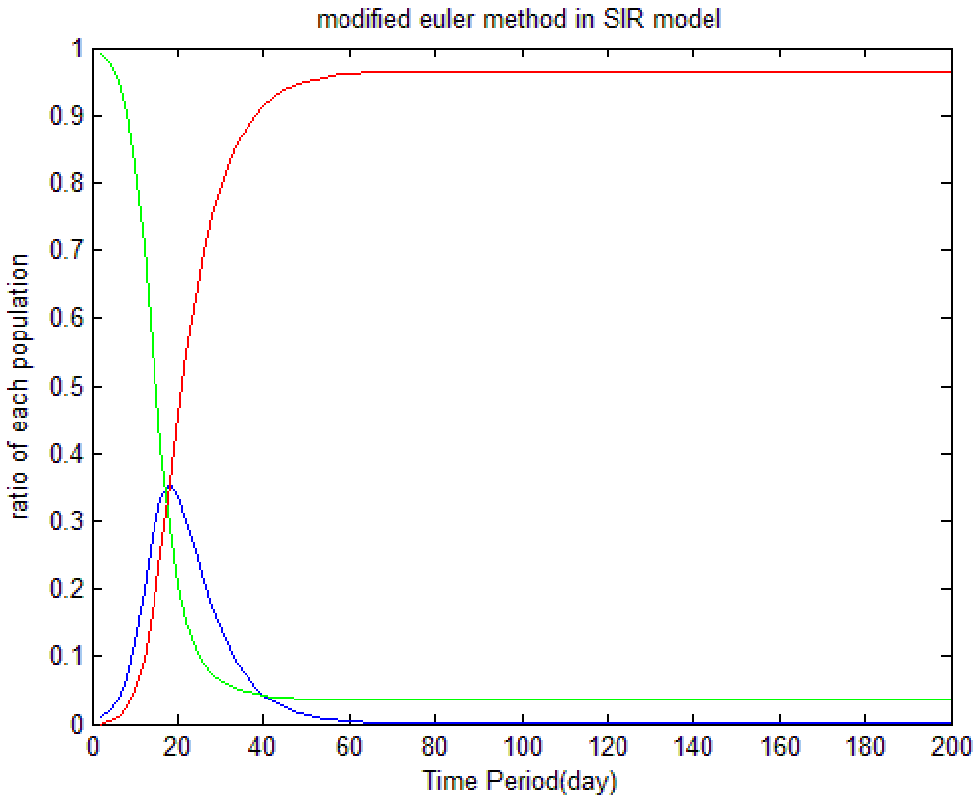
r2= (μ(i(t)+k1))h

here is my code and plot

```
T=200;%set time period
n=100;%size step
 h=T/n;%time interval
E=zeros(3,T/h);%set initial matrix
lamda=0.5;%set infection rate
mu=1/7;%set recovery rate
E(1,1)=0.01;%the initial rate of daily Infected people
E(2,1)=0.99;%the initial rate of daily susceptible people
E(3,1)=0;%the initial rate of daily recover people

for i=1:(T/h)-1
    %follow the equation
    r1=(mu*E(1,i))*h;
    k1=(lamda*E(1,i)*E(2,i)-mu*E(1,i))*h;
    s1=(-lamda*E(1,i)*E(2,i))*h;
    r2=(mu*(E(1,i)+k1))*h;
    k2=(lamda*(E(1,i)+k1)*(E(2,i)+s1)-mu*(E(1,i)+k1))*h;
    s2=(-lamda*(E(1,i)+k1)*(E(2,i)+s1))*h;

    E(1,i+1)=E(1,i)+0.5*(k1+k2);
    E(2,i+1)=E(2,i)+0.5*(s1+s2);
    E(3,i+1)=E(3,i)+0.5*(r1+r2);
end
c=E(3,:);
b=E(2,:);
a=E(1,:);
d=zeros(1,n);
for j=1:n
    d(:,j)=j*h;
end
plot(d,a,'blue');hold all;
plot(d,b,'green');
plot(d,c,'red');
```



Another way to implement SIR model is to use ODE function. It can find the true solution for differential equations, then we can analyze the truncation error by comparing to Euler method solution.

Through these three differential equations,

$$\frac{di}{dt} = \lambda s(t)i(t) - \mu i(t)$$

$$\frac{ds}{dt} = -\lambda s(t)i(t)$$

$$\frac{dr}{dt} = \mu i(t)$$

We set x(1) =s(t),x(2)=i(t), then we get the function

f=@(t,x)[lamta*x(1)*x(2),lamta*x(1)*x(2)-mu*x(2),mu*x(2)]';

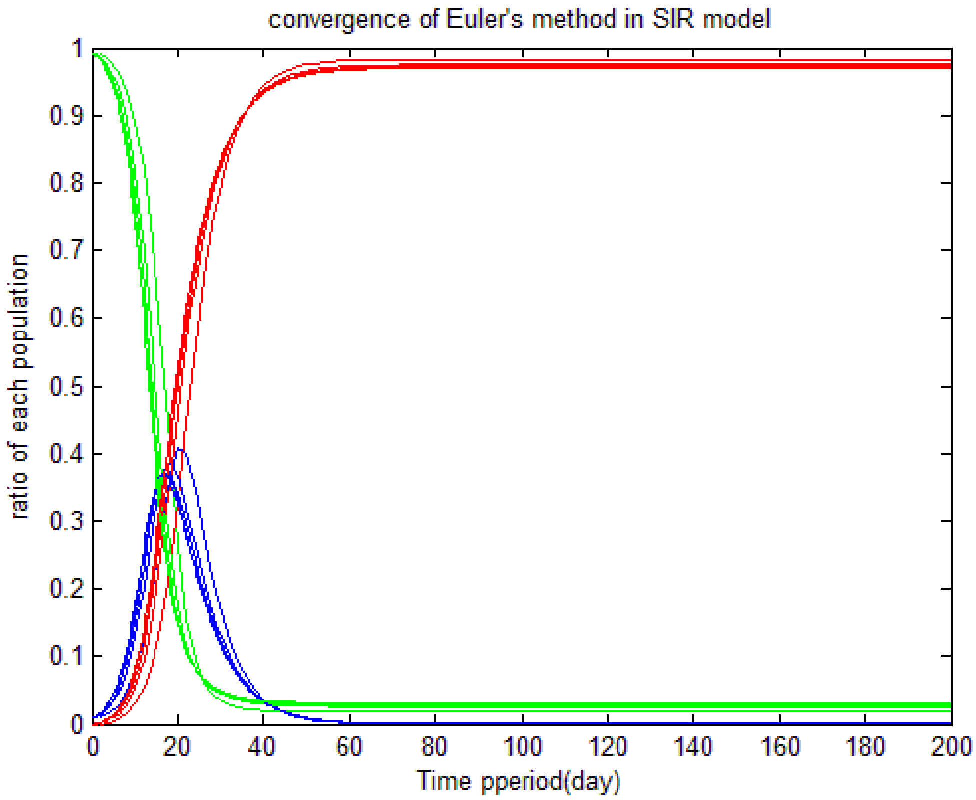
```
lamta=0.5;%set infection rate
mu=1/7;%set recovery rate
%set x(1)=s(t),x(2)=i(t)
f=@(t,x)[-lamta*x(1)*x(2),lamta*x(1)*x(2)-mu*x(2),mu*x(2)]';
[T,Y]=ode45(f,[0:1:200],[0.99 0.01 0]);%ode function
plot(T,Y(:,1),'green');
hold on
plot(T,Y(:,2),'blue');
hold on
plot(T,Y(:,3),'red');
hold on
```

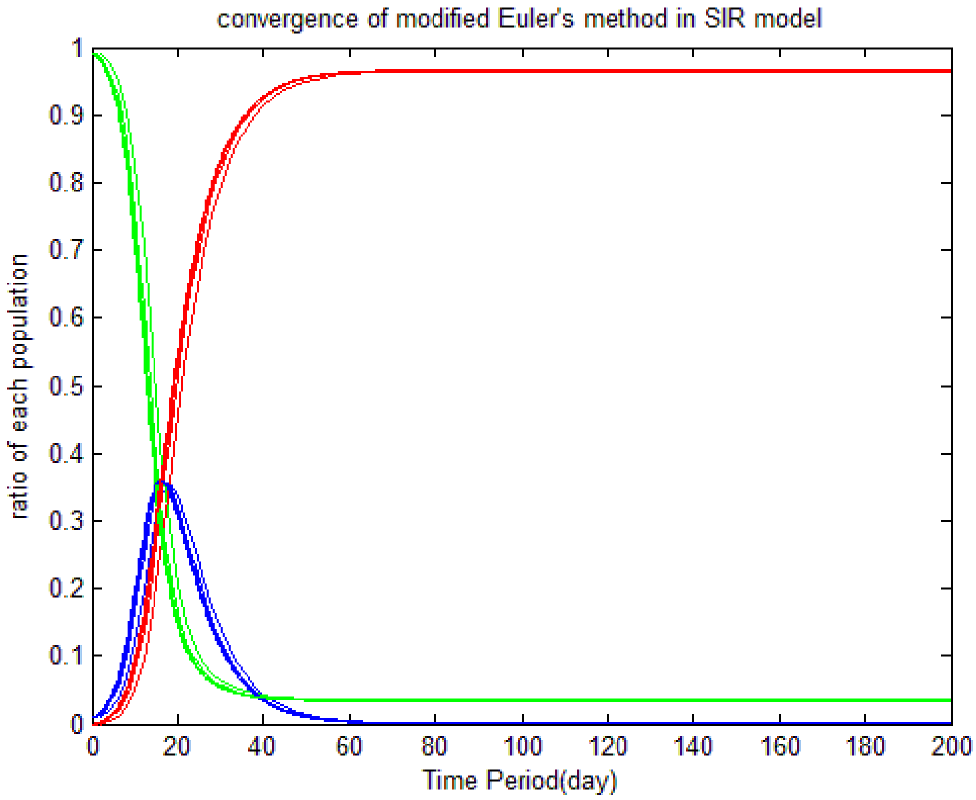
truncation error

Then we output the data about time, susceptible, infected and recovered that comes from ode45, modified Euler method and Euler method. Find the true error that at the same time. The time period is 200 days, so I separate it into 9 parts to find accurate error. According to the following table, we can see the modified Euler method's results are more accurate to the exact solution, it has very small error.

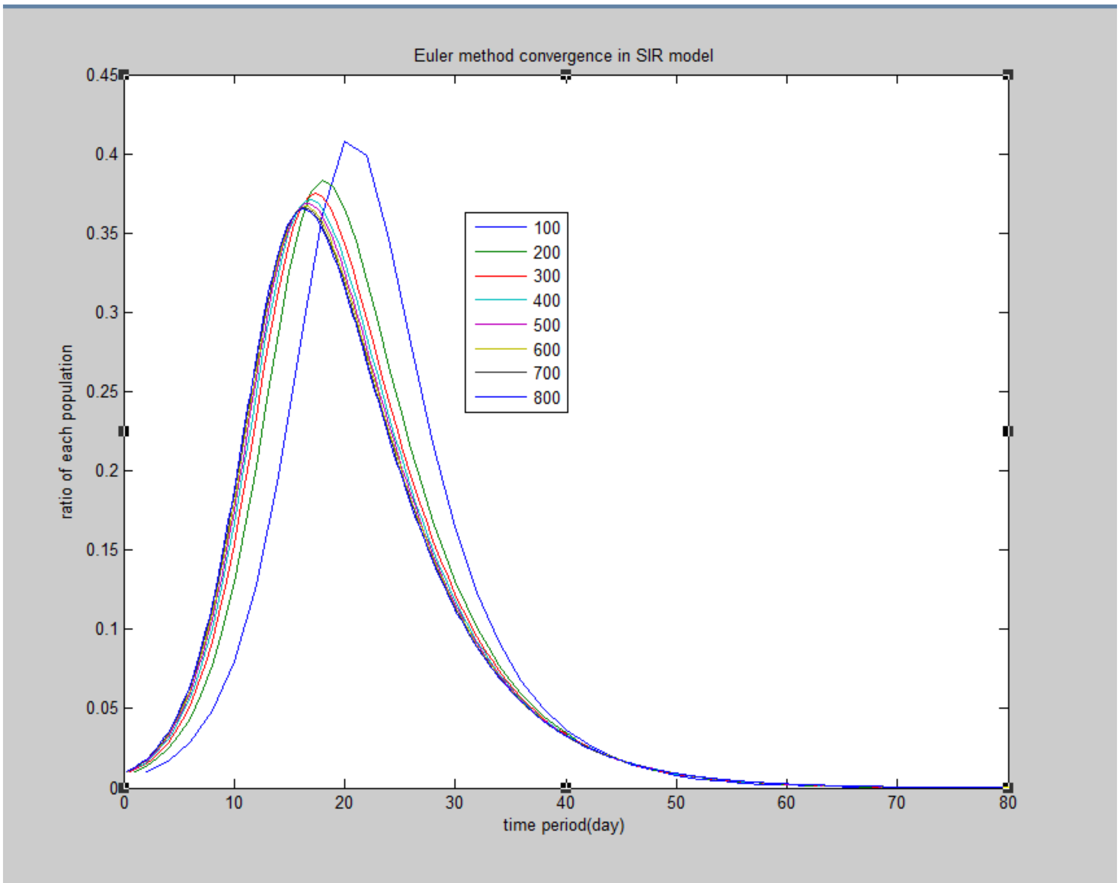
P	Q	R	S	T	U
time	s(ode45)	s(euler)	true error1	s(modi-euler)	ture error2
20	0.138941496	0.148402966	0.0680968	0.145509559	0.047272
40	0.038415575	0.034842258	0.0930174	0.038509295	0.00244
60	0.034136801	0.030869894	0.0957004	0.034039203	0.002859
80	0.033810425	0.030596826	0.0950476	0.033699758	0.003273
100	0.033784267	0.030577006	0.0949336	0.033672565	0.003306
120	0.033782144	0.030575562	0.0949194	0.033670377	0.003308
140	0.033781984	0.030575457	0.0949183	0.033670201	0.003309
160	0.033781969	0.030575449	0.0949181	0.033670187	0.003309
180	0.033781966	0.030575449	0.094918	0.033670186	0.003309

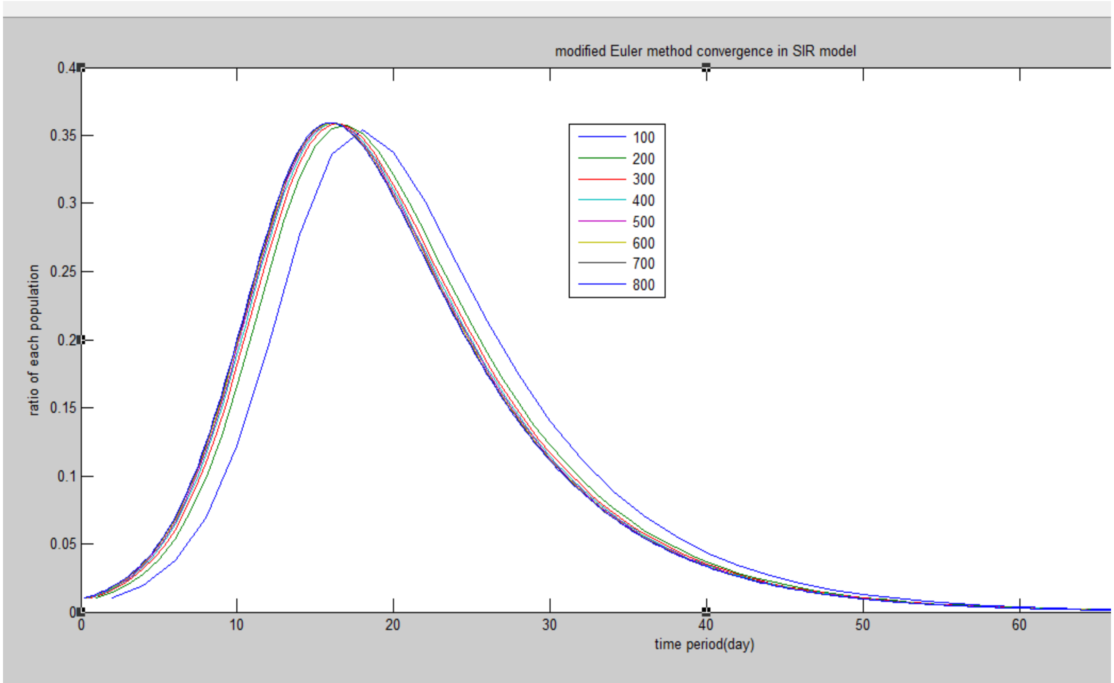
Convergence


convergence of Euler's method in SIR model
ratio of each population
Time pperiod(day)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100
120
140
160
180
200



Then I choose the infection(I) curves to clearly show the process of convergence below. It's obviously to see when n =600, the plot doesn't have too much change, and the larger time step plots look at the same line. So when n=600, Euler method already convergence. For modified Euler plot, we can see when n>=400, the plot almost doesn't change, which means it convergence at n=400. The other two kinds of curves are similar to it.


Euler method convergence in SIR model
ratio of each population
time period(day)
100
200
300
400
500
600
700
800
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
10
20
30
40
50
60
70
80

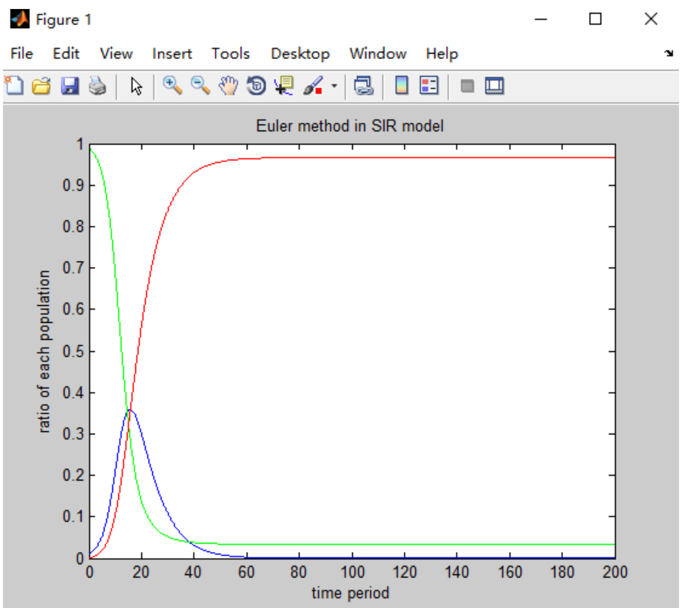


For Euler method plots, it converges at n=600 while the modified Euler plots converge at n=400. Thus the modified Euler method has much faster speed to converge.

From these two plots, we can see the modified Euler's method is more accurate and converge faster than Euler's method. As we can see, the green line is Susceptible population, the blue line is infectious population, the red line is recovered population. Under the shortest time, the recovery population increase rapidly, the infected population will come to a peak number in the short time first, then decrease to a low number, the susceptible population keeps decrease.

Stability

For SIR model, we tried very large time step to see if we can get a large, oscillating outputs. But it doesn't work. This ordinary differential equation doesn't have instability.



Conclusion

Through the results and plot, we can observe the disease trend for human. But this model can only analyze the specific time period, it just shows the rate of different population in a limit period, it can't be a disease model to predict the future trend of disease. In addition, SIR model doesn't consider the incubation period, so it can't be very accurate to simulate disease trend. However, SIR model is still a good disease model to predict a general trend in a limit period.

Appendix: Self-Assessment on Requirements

Up to now, we already come up two ways to set up the SIR model, describe the project content and explain the logic in it. Also, we analyze the convergence between Euler method and modified Euler's method, and compare each result to true solution which I used ode45 to find, so we can analyze truncation error. But SIR model can't be stable, since we tried very large time step, but it still output a satisfy result.

Reference

Smith, D., & Moore, L. (n.d.). The sir model for spread of disease - the differential equation model. Retrieved April 15, 2021, from https://www.maa.org/press/periodicals/loci/joma/the-sir-model-for-spread-of-disease-the-differential-equation-model