MATH 381 HW 7 part 1

Christian Jahnel

28 February 2024

1. Prove that the following are equivalent for any subsets A and B of the same universal set U:

   (a) $A \subseteq B$;

   (b) $A \cap \bar{B} = \emptyset$;

   (c) $\bar{A} \cup B = U$.

   i. $A \subseteq B \implies A \cup \bar{B} = \emptyset$

$$\begin{aligned}
A \subseteq B &\equiv \forall x \in U(x \in A \to x \in B) \\
&\equiv \forall x \in U(\neg(x \in A) \vee x \in B) \\
&\equiv \forall x \in U(x \notin A \vee x \in B) \\
&\equiv \forall x \in U(x \in \bar{A} \vee x \in B) \\
&\equiv \forall x \in U(x \in \bar{A} \cup B) \\
&\implies \bar{A} \cup B = U
\end{aligned}$$

   $\therefore a \to c$

   ii. $\bar{A} \cup B = U \implies A \cap \bar{B} = \emptyset$

$$\begin{aligned}
&\bar{A} \cup B = U \\
\implies &\overline{\bar{A} \cup B} = \bar{U} \\
\implies &\bar{\bar{A}} \cap \bar{B} = \bar{U} \\
\implies &A \cap \bar{B} = \bar{U} \\
&\qquad\quad = \{x \in U \mid x \notin U\} \\
&\qquad\quad = \emptyset
\end{aligned}$$

   $\therefore c \to b$

iii. $A \cap \bar{B} = \emptyset \implies A \subseteq B$

$$\begin{aligned} A \cap \bar{B} &= \emptyset \\ \implies A - B &= \emptyset \\ \implies \{x \mid x \in A \land x \notin B\} &= \emptyset \end{aligned}$$

$$\begin{aligned} \{x \mid x \in A \land x \notin B\} = \emptyset &\implies \nexists x \in U(x \in A \land x \notin B) \\ &\equiv \neg(\exists x \in U(x \in A \land x \notin B)) \\ &\equiv \forall x \in U(\neg(x \in A) \lor x \in B) \\ &\equiv \forall x \in U(x \in A \to x \in B) \\ &\implies A \subseteq B \end{aligned}$$

$\therefore b \to a \quad \blacksquare$

2. Prove or disprove: for any sets A, B, and C, if $A \cup B = B \cup C$, then $A = C$. Let $A = \{1\}, B = \{1, 2\}, C = \{2\}$.

$$\left.\begin{aligned} A \cup B = \{1, 2\} = B \\ B \cup C = \{1, 2\} = B \end{aligned}\right. \implies A \cup B = B \cup C$$

$\therefore A \cup B = B \cup C \land A \neq C \quad \blacksquare$

A	B	C	$A \cup B$	$B \cup C$
1	1	1	1	1
1	1	*0*	*1*	*1*
1	0	1	1	1
1	0	0	1	0
0	1	*1*	*1*	*1*
0	1	0	1	1
0	0	1	0	1
0	0	0	0	0

As seen in the above table, there are two cases in which $A \cup B = B \cup C$ does not imply that $A = C$. The starred rows show such cases. Since B already contains the element that happens to be in A and not C or vice-versa depending on the case, the duplicate is not counted, since its membership in B makes it qualify anyway. Therefore, **the proposition has been disproven generally**.

3. Determine and prove a relationship among the sets
$X = (A \cap B) \cup (A \cap C)$, $Y = A \cup (B \cap C)$, and $Z = A \cap (B \cup C)$,
where A, B, and C are any subsets of the same universal set U.

$$
\begin{aligned}
& Z = A \cap (B \cup C) && \\
& Z = (A \cap B) \cup (A \cap C) && \text{(Distributive law)} \\
\therefore\ & Z = X && \text{(Transitive POE)}
\end{aligned}
$$

$$
\begin{aligned}
& X \cup Y \cup Z && \\
&= X \cup Z \cup Y && \text{(Commutative law)} \\
&= X \cup Y && \text{(Idempotent law)} \\
&= (A \cap B) \cup (A \cap C) \cup A \cup (B \cap C) && \text{(Substitution)} \\
&= A \cup (A \cap B) \cup (A \cap C) \cup (B \cap C) && \text{(Commutative law)} \\
&= A \cup (A \cap C) \cup (B \cap C) && \text{(Absorption law)} \\
&= A \cup (B \cap C) && \text{(Absorption law)} \\
&= Y && \text{(Substitution)}
\end{aligned}
$$

$\therefore X \cup Y \cup Z = Y \quad \blacksquare$

4. For each $n \in \mathbb{Z}^+$, let $A_n = [\frac{1}{n}, 2 - \frac{n}{n+1}] \subseteq \mathbb{R}$. Find, with justification, the sets

(a) $\displaystyle\bigcup_{n=1}^{\infty} A_n$;

$$\begin{aligned}
\bigcup_{n=1}^{\infty} A_n &= \left[\frac{1}{1}, 2 - \frac{1}{1+1}\right] \cup \left[\frac{1}{2}, 2 - \frac{2}{2+1}\right] \cup \cdots \cup \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right] \\
&= \left[1, 2 - \frac{1}{2}\right] \cup \left[\frac{1}{2}, 2 - \frac{2}{3}\right] \cup \cdots \cup \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right] \\
&= \left[1, \frac{3}{2}\right] \cup \left[\frac{1}{2}, \frac{4}{3}\right] \cup \cdots \cup \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right]
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} 2 - \frac{n}{n+1} = 2 - \lim_{n\to\infty} \frac{n}{n+1} = 2 - 1 = 1 \\
\lim_{n\to\infty} \frac{1}{n} = 0
\end{aligned}$$

$$\begin{aligned}
1 &= A_{11} > A_{21} > \cdots > A_{n1} = 0 \\
\frac{3}{2} &= A_{12} > A_{22} > \cdots > A_{n2} = 1
\end{aligned}$$

$$\therefore \bigcup_{n=1}^{\infty} A_n = \left[0, \frac{3}{2}\right] \quad \blacksquare$$

The minimum bound of each interval in the series decreases from 1 to approach the limit at 0. The maximum bound of each interval in the series decreases from $\frac{3}{2}$ to approach the limit at 1. Consequently, the union will expand the set to push the minimum to the limit at 0 and keep the maximum at $\frac{3}{2}$. Therefore, the series will resolve to $(0, \frac{3}{2}]$.

(b) $\bigcap_{n=1}^{\infty} A_n$.

$$\begin{aligned}
\bigcap_{n=1}^{\infty} A_n &= \left[\frac{1}{1}, 2 - \frac{1}{1+1}\right] \cap \left[\frac{1}{2}, 2 - \frac{2}{2+1}\right] \cap \cdots \cap \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right] \\
&= \left[1, 2 - \frac{1}{2}\right] \cap \left[\frac{1}{2}, 2 - \frac{2}{3}\right] \cap \cdots \cap \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right] \\
&= \left[1, \frac{3}{2}\right] \cap \left[\frac{1}{2}, \frac{4}{3}\right] \cap \cdots \cap \left[\frac{1}{n}, 2 - \frac{n}{n+1}\right]
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} 2 - \frac{n}{n+1} = 2 - \lim_{n\to\infty} \frac{n}{n+1} = 2 - 1 = 1 \\
\lim_{n\to\infty} \frac{1}{n} = 0
\end{aligned}$$

$$\begin{aligned}
1 &= A_{11} > A_{21} > \cdots > A_{n1} = 0 \\
\frac{3}{2} &= A_{12} > A_{22} > \cdots > A_{n2} = 1
\end{aligned}$$

$$\therefore \bigcap_{n=1}^{\infty} A_n = [1, 1] = \{1\} \quad \blacksquare$$

The minimum bound of each interval in the series decreases from 1 to approach the limit at 0. The maximum bound of each interval in the series decreases from $\frac{3}{2}$ to approach the limit at 1. Consequently, the union will keep the minimum at 1 and contract to pull the maximum to the limit at 1. Therefore, the series will resolve to the singleton set $\{1\}$.