MATH 381 Section 6.5 Generalized Permutations and Combinations

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1 April 2024

Section 6.5 Permutations with Repetitions

Theorem 0.1 *The number of r-permutations of a set of n-objects with repetition allowed is n^r.*

Combinations with Repetitions

Theorem 0.2 *There are $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ number of r-combinations from a set with n elements when repetition of elements is allowed.*

Remark Recall that Gauss' formula for the sum of a series up to n gives the number of lattice points in triangle.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

Theorem 0.3 *Say we have a hyperplane in $\mathbb{R}^n$*

$$x_1 + x_2 + x_3 + \cdots + x_n = d$$

Then the number of integer lattice points is

$$\binom{d+n}{n} \qquad \dim_k k[x_1, \ldots, x_n]$$

$$\binom{d+2}{2} = \frac{(d+2)(d+1)}{2} = 1 + 2 + \cdots + (d+1)$$

the space of polynomials in n variables $x_1, \dots, x_n$ of degree $\leq d$ is a vector space of dimension

$$\binom{n+d}{n} = \binom{n+d}{d}$$

Corollary 0.4 *The number of lattice points with non-negative integer coefficients inside the hyperplane is*

$$\binom{n+d-1}{n-1} = \binom{n+d}{n} - \binom{n+d-1}{n}$$

Corollary 0.5 *i.e. homogeneous polynomial in $n+1$ variables $x_0, x_1, \dots, x_n$ of total degree d forms a vector space $\binom{n+d}{n}$.*

$$a + bx + cy + dxy + ex^2 + fy^2 \qquad \dim_k k[x,y]_{\leq d=2} = \binom{2+2}{2} = 6$$

Example How many solutions with non-negative integers are there to $x_1 + x_2 + x_3 = 11$?

1. $$\binom{11+3}{3} - \binom{10+3}{3} = \binom{13}{2}$$

2. Number of ways to select 11 items from a set with 3 elements so that
   x_1 of first element
   x_2 second element
   x_3 third element

Permutations with Indistinguishable Objects

Example How many different words do we have by rearranging the word SUCCESS?

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{7!}{3!2!}$$

Theorem 0.6 *The number of different permutations of n objects*
$n_1 =$ *Indistinguishable objects of type 1*
$n_2 =$ *Indistinguishable objects of type 2*
$\vdots$
$n_k =$ *Indistinguishable objects of type k*

$$\frac{n!}{n_1! n_2! n_3! \dots n_k!}$$

Example What is the number of ways to distribute hands of 5 cards to 4 players from a standard deck of 52 cards?

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}$$

Theorem 0.7 *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are in box i*

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

Definition The Striling Numbers
$S(n,j)$ = number of ways to distribute n distinguishable objects into j indistinguishable boxes. A closed formula is not known.

Theorem 0.8

$$S(n,j) = \frac{1}{j!}\cdot\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$

so the number of ways to distribute n distinguishable objects into k Indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k}\frac{1}{j!}\cdot\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$