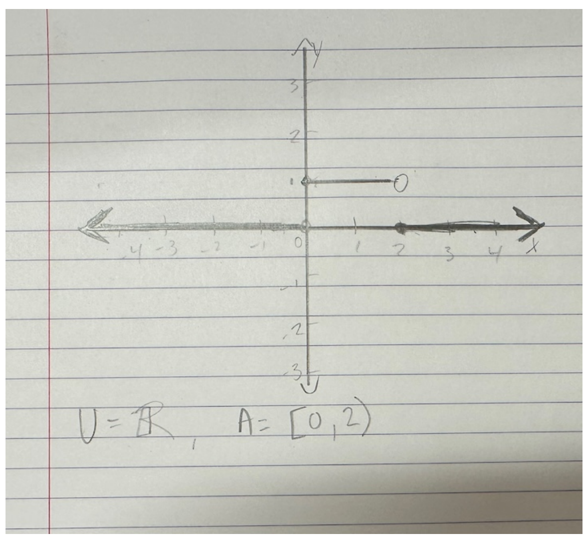
MATH 381 HW 7 part 3

Christian Jahnel

28 February 2024

1. Let U be a universe and $A \subseteq U$. Define the characteristic function of A, $\chi_A : U \to \{0,1\}$, by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

   (a) Consider $A = [0,2) \subseteq \mathbb{R}$. Sketch the graph of χ_A.



   (b) Again consider $A = [0,2) \subseteq \mathbb{R}$. Determine the sets $(\chi_A)^{-1}(1)$ and $(\chi_A)^{-1}(2)$.

$$(\chi_A)^{-1} : \{0,1\} \to U$$
$$(\chi_A)^{-1}(1) = \{(\chi_A)^{-1}(\chi_A(x)) \mid x \in A\} = \{x \mid x \in A\} = [0,2)$$
$$(\chi_A)^{-1}(2) = \emptyset \because 2 \notin \{0,1\} = \text{domain of } (\chi_A)^{-1}$$

2. Give examples of sets $S, T \subseteq \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ for which $f(S \cap T) \neq f(S) \cap f(T)$. Clearly justify why your sets and function satisfy the property.

$$\begin{aligned} f(S) &= \{f(x) \mid x \in S\} \\ f(T) &= \{f(x) \mid x \in T\} \\ f(S \cap T) &= \{f(x) \mid x \in S \cap T\} = \{f(x) \mid x \in S \land s \in T\} \\ f(S) \cap f(T) &= \{f(x) \mid x \in S\} \cap \{f(x) \mid x \in T\} \end{aligned}$$

$$\begin{aligned} S &= \{0\} \\ T &= \{1\} \\ S \cap T &= \emptyset \\ f &: x \mapsto 1 \\ f(S) &= \{1\} \\ f(T) &= \{1\} \\ f(S \cap T) &= \emptyset \\ f(S) \cap f(T) &= \{1\} \\ \therefore f(S \cap T) &\neq f(S) \cap f(T) \quad \blacksquare \end{aligned}$$

Essentially, this phenomenon arises from the fact that the function is not injective since it maps both $x = 0$ and $x = 1$ to $f(x) = 1$. Since the two singleton sets are disjoint, their intersection is the empty set, and thus so is the image. However, the images of both sets are not disjoint, and therefore their intersection is a singleton set.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x - y$. Determine, with proofs, whether f is one-to-one, onto, both, or neither. Be sure to fully justify both your statement of one-to-one and your statement of onto.

   (a) $P : f$ is an injective (i.e. one-to-one) function

   $$\begin{aligned} P &\equiv \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 (f(x_1, y_1) = f(x_2, y_2) \to (x_1, y_1) = (x_2, y_2)) \\ &\equiv \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 ((x_1, y_1) \neq (x_2, y_2) \to f(x_1, y_1) \neq f(x_2, y_2)) \end{aligned}$$

   Suppose $(x_1, y_1) \neq (x_2, y_2)$ and let $x_1 = y_1$ and $x_2 = y_2$.

   $$\left.\begin{aligned} f(x_1, y_1) = x_1 - y_1 = x_1 - x_1 = 0 \\ f(x_2, y_2) = x_2 - y_2 = x_2 - x_2 = 0 \end{aligned}\right. \implies f(x_1, y_1) = f(x_2, y_2)$$

   $$\therefore \exists (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 ((x_1, y_1) \neq (x_2, y_2) \to f(x_1, y_1) = f(x_2, y_2))$$
   $$\implies \neg P$$

   (b) $Q : f$ is a surjective (i.e. onto) function

   $$Q \equiv \forall z \in \mathbb{R}\ \exists x, y \in \mathbb{R}^2 (f(x, y) = z)$$

   Let $z \in \mathbb{R}$.

   $$\begin{aligned} &\text{Fix } y = 0 \in \mathbb{R} \\ \implies z = x - y &\iff z = x \\ z \in \mathbb{R} &\implies x \in \mathbb{R} \end{aligned}$$

   $\therefore Q \quad \blacksquare$

Consequently, f **is a surjective (i.e. onto) function**, but **it is not an injective (i.e. one-to-one) function**, and thus it cannot be a bijective function. Intuitively, for every value z on the z-axis of the 3-D graph of f, there must be a one-dimensional (linear) slice through the plane. Consequently, there are an infinite amount of solutions along each line, and therefore it cannot be injective. As for the surjective property, the plane spans all real numbers in its range since it is angled nonparallel to the x-axis, as seen by its coefficients.

4. Suppose $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one. Prove $g \circ f : A \to C$ is also one-to-one.

$$\forall x_A, y_A \in A(f(x_A) = f(y_A) \to x_A = y_A) \\ \wedge \forall x_B, y_B \in B(g(x_B) = g(y_B) \to x_B = y_B)$$

$$P : \forall x_A, y_A \in A((g \circ f)(x_A) = (g \circ f)(y_A) \to x_A = y_A)$$

Let $x_A, y_A \in A$ and suppose $(g \circ f)(x_A) = (g \circ f)(y_A)$.

$$\begin{aligned} &(g \circ f)(x_A) = (g \circ f)(y_A) \\ \implies &g(f(x_A)) = g(f(y_A)) \end{aligned}$$

$$\text{Let } x_B = f(x_A) \in B$$
$$\text{Let } y_B = f(y_A) \in B$$

$$\begin{aligned} &g(f(x_A)) = g(f(y_A)) \\ \implies &g(x_B) = g(y_B) \\ \implies &x_B = y_B \\ \implies &f(x_A) = f(y_A) \\ \implies &x_A = y_A \end{aligned}$$

$\therefore P \quad \blacksquare$