MATH 381 Section 5.1

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Gauss' Formula

$$1+2+3+4+\cdots+n=\frac{n(n+1)}{2}=\binom{n+1}{2}$$
$$1^2+2^2+3^2+4^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$
$$1^3+2^3+3^3+4^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$$

There is no forumla for an arbitrary power k.

5.1.2 Mathematical Induction

Mathematical Induction can be used to prove statements such as $P(n)$ is true for all positive integers.

Proof of Mathematical Induction

1. Basis step
$$P(1) \equiv T$$
2. Inductive step
$$\forall k \in \mathbb{N}(P(k) \to P(k+1))$$

Example Prove Gauss' Formula

Proof 1.
$$P(1)=1=\frac{1\cdot 2}{2}\equiv T$$
$\therefore P(1) \equiv T$

2.

$$P(k) = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$
$$P(k+1) = 1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right) \\
&= (k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}$$

$\therefore P(k) \to P(k+1) \quad \blacksquare$

Francesco Maurolico (1494-1575)
Let $S \subseteq \mathbb{Z}^+, S \neq \emptyset$.
By the well-ordering property, the subset S must have a minimal element.
(i.e. $m \in S$ but $m - 1 \notin S$)
Mathematical Induction is equivalent to the well-ordering axiom.

Example Prove

$$P(n) : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof Mathematical Induction

1. $P(1)$

$$1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

2. Mathematical Induction
   Let $k \in \mathbb{Z}$
   Assume $P(k)$ holds.

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 &= \frac{k(k+1)(2k+1)}{6} \\
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

   $\therefore P(k) \to P(k+1) \quad \blacksquare$

Remark Recall

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \left[\sum_{i=1}^{n} f(x_i^*) \right]$$

Example Compute

$$\begin{aligned}
\int_{a=0}^{b=1} x^2 dx &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*) \\
&= \lim_{n\to\infty} \frac{1}{n}(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^2} \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\frac{n(n+1)(2n+1)}{6}}{n^2} \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&\sim \frac{2n^3}{6n^3} = \frac{1}{3}
\end{aligned}$$

Example Use Mathematical Induction to prove $7^{n+2} + 8^{2n+1}$ is divisible by 57 for any non-negative integer n.

$$P(k) : 57 \mid (7^{n+2} + 8^{2n+1})$$

Proof

1. Basis step

$$\begin{aligned}
&P(1) : 57 \mid (7^{1+2} + 8^{2\cdot 1+1}) \\
&P(1) : 57 \mid (343 + 512) \\
&P(1) : 57 \mid 855 \equiv T
\end{aligned}$$

$\therefore P(1)$

2. Inductive step
   Let $k \in \mathbb{N}$.

$$\begin{aligned}
&7^{n+3} + 8^{2n+3} \\
&= 7 \cdot 7^{n+2} + 8^2 \cdot 8^{2n+1} \\
&= 7 \cdot 7^{n+2} + (57 + 7) \cdot 8^{2n+1} \\
&= 7 \cdot 7^{n+2} + 7 \cdot 8^{2n+1} + 57 \cdot 8^{2n+1}
\end{aligned}$$

$57 \mid 7(7^{n+2}+8^{2n+1}) \wedge 57 \mid 57{\cdot}8^{2n+1} \implies 57 \mid (7{\cdot}7^{n+2}+7{\cdot}8^{2n+1}+57{\cdot}8^{2n+1})$

$\therefore P(k) \to P(k+1) \quad \blacksquare$

Example What is the sum of all positive odd integers?

$$P(k) : 1 + 3 + 5 + \cdots + (2k-1) = k^2$$

Proof 1. Basis step

$$1 = 1$$

$\therefore P(1)$

2. Inductive step
Assume $P(k)$, $k \in \mathbb{N}$.

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k-1) &= k^2 \\ 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$

$\therefore P(k) \to P(k+1) \quad \blacksquare$

Example Use mathematical induction to prove

$$P(n) : 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1} \qquad r \neq 1$$

Proof 1. $P(1) : 1 + r = \dfrac{r^2-1}{r-1} \equiv T$

$\therefore P(1)$

2.

$$\begin{aligned} P(n) + r^{n+1} = 1 + r + r^2 + \cdots + r^n &= \frac{r^{n+1}-1}{r-1} + r^{n+1} \\ &= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+1}(r-1)}{r-1} \\ &= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1} \\ &= \frac{r^{n+1}-1+r^{n+2}-r^{n+1}}{r-1} \\ &= \frac{r^{n+2}-1}{r-1} \\ &= P(n+1) \end{aligned}$$

$\therefore P(n) \to P(n+1) \quad \blacksquare$

Theorem 0.1 *If S is a finite set, $|\mathcal{P}(S)| = 2^{|S|}$.*

Proof 1. Let $S = \{y\}$

$$\text{Subsets of S} = \begin{cases} \emptyset \\ \{y\} \end{cases} \implies |\mathcal{P}(S)| = 2 = 2^{|S|}$$

$\therefore P(1)$

2.

$$S = \{x_1, \ldots, x_n, A\} \qquad |S| = n+1$$

α: Subsets of S that do not include A are subsets of $\{x_1, \ldots, x_n, A\}$. We have 2^n.

β: Subsets of S that include A are subsets of $\{x_1, \ldots, x_n, A\}$ with an A appended to it. Therefore, we have another 2^n.

$$\alpha + \beta = 2^n + 2^n = 2^{n+1}$$

$\therefore P(n) \to P(n+1)$ ■

Example Denote $H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$.
Use mathematical induction to prove

$$H_{2^n} \geq 1 + \frac{n}{2}$$

for any non-negative integer n.

Proof Induction on n

1.

$$P(1) : 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$$

$\therefore P(1)$

2.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$
$$(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}) + (\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}}) \geq 1 + \frac{n+1}{2}$$

$$
\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + (\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}}) \\
\geq 1 + \frac{n}{2} + (\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}})
\end{aligned}
$$

$$
\begin{aligned}
1 + \frac{n}{2} + (\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}}) \geq 1 + \frac{n+1}{2} \\
\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} \geq \frac{1}{2} \\
\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \geq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} = \frac{2^n}{2^{n+1}} = \frac{1}{2}
\end{aligned}
$$

$\therefore P(n) \to P(n+1) \quad \blacksquare$