MATH 381 Section 1.8

Prof. Olivia Dumitrescu

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Proof methods and strategies

Suppose we want to prove

$$p_1 \vee p_2 \vee p_3 \vee \cdots \vee p_n \to q$$

is equivalent to

$$(p \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)$$

.

Example Prove $(n+1)^3 \geq 3^n$ for positive integers $n \leq 4$.

Example Prove that for any $n \in \mathbb{Z}$, $n^2 \geq n$

Example Use a proof by cases to show that

$$|x \cdot y| = |x| \cdot |y| \quad x, y \in \mathbb{R}$$

Example Show there are no integer solutions x and y to the following

$$x^2 + 3y^2 = 18$$

Fermat's Last Theorem

The equation $x^n + y^n = z^n$ has no integer solutions for $n > 2$.

Example Show that if x and y are even numbers, then both xy and $x + y$ are even integers.

Example Open question: The $3x+1$ or Collatz conjecture

$$\forall x \in \mathbb{Z}, \exists n \ (T^n(x) = 1)$$

Let T be the transformation:

1. if x is an even integer $\rightarrow \frac{x}{2}$
2. if x is an odd interger $\rightarrow T(x) = 3x + 1$

Example Two distinct positive numbers x and y

$$\frac{x+y}{2} > \sqrt{xy}$$

i.e. the arithmetic mean is greater than the geometric mean

Proof Recall that for positive numbers

$$x^2 > y^2 \iff x > y$$

which can be proven through difference of squares.

$$\frac{x+y}{2} \geq \sqrt{xy}$$

It suffices to prove

$$(\frac{x+y}{2})^2 \geq (\sqrt{xy})^2$$

Uniqueness Proofs

$P(x)$: desired property

1. $\exists x(P(x) \land \forall y(y \neq x \rightarrow \neg P(y)))$
2. Assume both x and y satisfy $P(x)$ show that $x = y$.

Example Show that every line that is not horizontal has a unique solution.

Proof

1. existence

2. uniqueness

Assume x_1 and x_2 are 2 solutions for a line. Claim: $x_1 = x_2$

$$y = ax + b$$

is not horizontal if $a \neq 0$

Example Prove that there exists two irrational numbers x, y so that $x^y \in \mathbb{Q}$.

Proof

$$\exists x, y \notin \mathbb{Q} \text{ s.t. } x^y \in \mathbb{Q}$$