MATH 381 Section 9.5

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Definition A relation R on a set A is a relation from A to A.

$$xRy \leftrightarrow (x, y) \in R$$

A matrix can be defined which is associated with a relation R

$$a_{ij} = \begin{cases} 1 & (i, j) \in R \\ 0 & (i, j) \notin R \end{cases}$$

The relation R can be interpreted as a directed graph. The vertices of the directed graph consist of the points of A.

1. $\overline{xy}$ is a directed edge iff $(x, y) \in R$
2. $x \circlearrowright$ is a directed loop iff $(x, x) \in R$

Section 9.5 Equivalence Relations

Definition A relation R on a set A is an equivalence relation if it satisfies:

1. reflexivity
$$\forall a \in A(a \overset{R}{\sim} a)$$
2. symmetry
$$\forall a, b \in A(a \overset{R}{\sim} b \to b \overset{R}{\sim} a)$$
3. transitivity
$$\forall a, b, c \in A(a \overset{R}{\sim} b \wedge b \overset{R}{\sim} c \to a \overset{R}{\sim} c)$$

Remark None of the three conditions for an equivalence relation is redundant

Definition Two elements a and b that are related by an equivalence relation are called equivalent (denoted by $a \sim b$ if a and b are equivalent elements with respect to some equivalence.)

Example Let R be a relation.
$a \overset{R}{\sim} b \iff$ either $a = b$ or $a = -b$.
Prove that this is an equivalence relation.

1. if $a \overset{R}{\sim} b$ then also $b \overset{R}{\sim} a$
   i.e. if $a = b$ or $a = -b$ then either $b = a$ or $b = -a$.

2. $\forall a \in A(a \overset{R}{\sim} a)$
   i.e. either $a = a$ or $a = -a$

3. $\forall a, b, c \in A$
$$a \overset{R}{\sim} b \wedge b \overset{R}{\sim} c \to a \overset{R}{\sim} c$$
$$\begin{cases} a = b \vee a = -b \\ b = c \vee b = -c \end{cases} \to a = c \vee a = -c$$

Example $A = \mathbb{R}$

$$a \overset{R}{\sim} b \iff a - b \in \mathbb{Z}$$

Is this an equivalence relation?

Example Is $a \mid b$ an equivalence relation? No

Example Prove that congruence modulo m is an equivalence relation.

$$m \mid a - b \to a \overset{R}{\sim} b$$

Example $A = \mathbb{R} \forall a, b \in \mathbb{R}$
Define $a \overset{R}{\sim} b \iff |a - b| < 1$
Prove that it is not an equivalence relation.

Definition R is an equivalence relation on a set A.
The set of elements that are related to $a \in A$ is called the **equivalence class** of a. The equivalence class of $a \in A$ with respect to relation R can be written as

$$[a]_R = \{s \in A \mid a \overset{R}{\sim} s\}$$

Theorem 0.1 *If R is an equivalence relation on a set A and taking $a, b \in A$, then the following two statements are equivalent*

1. $a \overset{R}{\sim} b$
2. $[a]_R = [b]_R$
3. $[a]_R \cap [b]_R \neq \emptyset$

Corollary 0.2 *This theorem implies*

$$A = \bigsqcup_{a \in A} [a]_R$$

- *i.e. if* $a \overset{R}{\nsim} b$ *then* $[a]_R \cap [b]_R = \emptyset$
- *if* $a \overset{R}{\sim} b$ *then* $[a]_R = [b]_R$

Theorem 0.3 *Let R be an equivalence relation on a set S. The equivalence classes with respect to R form a partition of S.*
Conversely, given any partition $\{A_i \mid i \in I\}$ of subsets of S then there exists an equivalence relation R that has these subsets as an equivalence class.

Example of partition

$$\begin{aligned} S &= \{1, 2, 3, 4, 5, 6\} \\ A_1 &= \{1, 2, 3\} \\ A_2 &= \{4, 5\} \\ A_3 &= \{6\} \\ S &= A_1 \sqcup A_2 \sqcup A_3 \end{aligned}$$