MATH 381 HW 9 part 2

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1. Find all integer solutions to the following. If there are no integer solutions, explain why.

   (a) $3 + 2x \equiv -2 \pmod 7$

$$\begin{aligned}
3 + 2x &\equiv -2 \pmod 7 \\
2x &\equiv -5 \pmod 7 \\
2x + 0 &\equiv -5 + 7 \pmod 7 \\
2x &\equiv 2 \pmod 7 \\
x &\equiv 1 \pmod 7 \\
x \in \hat{1} &= \{z \in \mathbb{Z} \mid z \bmod 7 = 1\} \\
&= \{\dots, -13, -6, 1, 8, 15, \dots\}
\end{aligned}$$

   (b) $2x - 4 \equiv 0 \pmod 6$

$$\begin{aligned}
2x - 4 &\equiv 0 \pmod 6 \\
2x &\equiv 4 \pmod 6 \\
x &\equiv 2 \pmod 6 \\
x \in \hat{2} &= \{z \in \mathbb{Z} \mid z \bmod 6 = 2\} \\
&= \{\dots, -10, -4, 2, 8, 14, \dots\}
\end{aligned}$$

   (c) $x + y \equiv x - y \pmod 5$

$$\begin{aligned}
x + y &\equiv x - y \pmod 5 \\
y &\equiv -y \pmod 5 \\
1 &\equiv -1 \pmod 5 \qquad y \neq 0 \\
1 &\equiv 4 \pmod 5
\end{aligned}$$

Let $y = 0$.

$$\begin{aligned} x + 0 &\equiv x - 0 \pmod 5 \\ x &\equiv x \pmod 5 \\ x &\in \mathbb{Z} \end{aligned}$$

The equation can be simplified to the equivalent equation that 1 and -1 are equivalent modulo 5. This is a contradiction because 1 and -1 are in their own equivalence classes: $\hat{1}$ and $\hat{4}$, respectively. However, this simplification assume $y \neq 0$. Consequently, in the case where $y = 0$, the equation is satisfied for any $x \in \mathbb{Z}$. Therefore, there are inifinite pairs of solutions: $\{(x, 0) \mid x \in \mathbb{Z}\}$.

2. Prove that for all integers $n \geq 0$, $10^n \equiv 1 \pmod 9$. Then, use that result to show that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
Basis step

$$\begin{aligned} & 10^0 \equiv 1 \pmod 9 \\ \iff & 1 \equiv 1 \pmod 9 \\ \therefore & P(0) \end{aligned}$$

Inductive step; assume $P(k)$.

$$\begin{aligned} 10^k &\equiv 1 \pmod 9 \\ 10^k \cdot 10 &\equiv 1 \cdot 1 \pmod 9 \\ 10^{k+1} &\equiv 1 \pmod 9 \end{aligned}$$

$$\therefore P(k) \to P(k+1)$$

$\therefore \forall n \in \{z \in \mathbb{Z} \mid z \geq 0\}(10^n \equiv 1 \pmod 9)$ ■

Every positive integer can be written as a sum of its digits weighted by its place value in base-10.

$$k = k_0 + 10k_1 + 100k_2 + \cdots + 10^n k_n = \sum_{i=0}^{n} 10^i k_i$$

$$\begin{aligned} 10^0 \equiv 1 \pmod 9 &\implies 10^0 k_0 \equiv k_0 \mod 9 \\ 10^1 \equiv 1 \pmod 9 &\implies 10^1 k_1 \equiv k_1 \mod 9 \\ 10^2 \equiv 1 \pmod 9 &\implies 10^2 k_2 \equiv k_2 \mod 9 \\ &\vdots \\ 10^n \equiv 1 \pmod 9 &\implies 10^n k_n \equiv k_n \mod 9 \end{aligned}$$

$\therefore \forall 0 \leq i \leq n(10^i k^i \equiv k_i \pmod 9)$

$$\begin{aligned} 9 \mid k \iff & 0 \equiv k \pmod 9 \\ & 0 \equiv \sum_{i=0}^{n} 10^i k_i \pmod 9 \\ & 0 \equiv \sum_{i=0}^{n} k_i \pmod 9 \end{aligned}$$

$\therefore 9 \mid k \iff 9 \mid (k_0 + k_1 + k_2 + \cdots + k_n)$ ■

3. Show that if n is any integer, then n^2 is congruent modulo 4 to either 0 or 1.

$$n \in \mathbb{Z} \to n^2 \equiv 0 \vee n^2 \equiv 1 \pmod 4$$

If n is odd:

$$\exists k \in \mathbb{Z}(n = 2k + 1)$$

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4(k^2 + k) + 1 \\ \implies n^2 &\equiv 4(k^2 + k) + 1 \pmod 4 \\ &\equiv 1 \pmod 4 \end{aligned}$$

If n is even:

$$\exists k \in \mathbb{Z}(n = 2k)$$

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ \implies n^2 &\equiv 4(k^2) \pmod 4 \\ &\equiv 0 \pmod 4 \end{aligned}$$

Therefore, n^2 is congruent to either 0 or 1 modulo 4 for all integers, since all integers must be either odd or even. $\blacksquare$