MATH 381 Section 2.5

Olivia Dumitrescu

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Countable Sets

Remark Two finite sets $|A| = |B|$.
There are 2 ways of proving the equiality of 2 sets.

1. prove $A \mapsto B \wedge B \mapsto A \implies A = B$
2. Assume A, B are finite. $A \mapsto B \implies A = B$.

Definition

1. A set is countable if either it is finite or it has a bijective correspondence with $\mathbb{N}$.
$$\exists f : \mathbb{N} \to A$$
where f is a bijection. i.e. The set has the same cardinality as $\mathbb{N}$ which is countable.
2. A set is uncountable if it is not countable. Ex:
$$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, (a, b)$$

Example

$$E = \{2, 4, 6, 8, \dots\}$$
$$O = \{1, 3, 5, 7, \dots\}$$

Prove that E and O are countable sets.

1.

$$f : \mathbb{N} \rightleftarrows E$$
$$f(n) = 2n \forall n \in \mathbb{N}$$

$$g : E \to \mathbb{N}$$
$$g(m) = \frac{m}{2} \in \mathbb{N}$$

Check

$$g \circ f = Id_{\mathbb{N}}$$
$$f \circ g = Id_E$$

2.

$$h : \mathbb{N} \rightleftarrows O$$
$$h(n) = 2n - 1 \forall n \in \mathbb{N}$$

$$k : O \to \mathbb{N}$$
$$k(m) = \frac{k+1}{2} \in \mathbb{N}$$

Check

$$k \circ h = Id_{\mathbb{N}}$$
$$h \circ k = Id_O$$

Example Prove that $\mathbb{Z}$ is countable.

Proof Need $f : \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even} \end{cases}$$

Example

1. The function $f(x) = \dfrac{x}{x^2 - 1} : (-1, 1) \to \mathbb{R}$ is a bijection i.e. $(-1, 1)$ $\mathbb{R}$.

2. $f(x) = \arctan(x) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

Definition

1. f is increasing if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$.
2. f is decreasing if $x_1 \leq x_2$, then $f(x_1) \geq f(x_2)$.
3. f is monotone if it is either increasing or decreasing over its domain.

Remark If f is monotone, then f is injective.

Proof Assume f is increasing.

$$\forall x_1, x_2 in \mathbb{D}(x_1 \leq x_2 \to f(x_1) \leq f(x_2))$$

Theorem 0.1 *We claim*

1. $\mathbb{Q}$ *is countable*

   Proof
   $$\mathbb{Q} = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \cup \ldots$$

   $$\begin{aligned} A_1 &= \left\{ \frac{0}{1} \right\} \\ A_2 &= \left\{ \frac{1}{1}, -\frac{1}{1} \right\} \\ A_3 &= \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1} \right\} \\ A_4 &= \left\{ \frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1} \right\} \\ A_5 &= \left\{ \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1} \right\} \end{aligned}$$

   For a $z \in \mathbb{Q} \implies z = \pm\frac{p}{q} \quad (p, q) = 1 \implies z \in A_n \quad |p| + |q| = n.$

2. $\mathbb{R}$ *is uncountable*

Theorem 0.2 *Nested Interval Property*
For each $n \in \mathbb{N}$ assume we have

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$$

$$I_n \supseteq I_{n+1} \text{ i.e. } I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

$$\bigcap_{n=1}^{\infty} I_i \neq \emptyset \quad \mathbb{R} = \{x_1, x_2, x_3, \dots\} \qquad \mathbb{R}\{x_1, x_2, x_3, \dots\}$$

Proof *Assume by contradiction $\mathbb{R}$ are countable and a bijection.*

$$f : \mathbb{N} \to \mathbb{R}$$
$$f(1) = x_1$$
$$f(2) = x_2$$
$$f(3) = x_3$$

Choose I_1 which is a closed interval so that $x_i \notin I_i$.

(a) $I_{n+1} \subseteq I_n$

(b) $x_{n+1} \notin I_{n+1}$

For any real number x_{n_0}

$$x_{n_0} \notin I_{n_0} \iff \bigcap_{i=1}^{\infty} I_i = \emptyset$$

Thus, we have contradiction with the previous theorem.

3. $(0, 1)$ *is uncountable (Cantor)*

Proof *We assume by contradiction that $(0, 1)$ is countable i.e. $\exists f : \mathbb{N} \to (0, 1)$ that is a bijection i.e.*

$$f(n) \in (0, 1) \forall n \in \mathbb{N}$$

$$f(n) = .a_{n_1} a_{n_2} a_{n_3} a_{n_4} \dots$$
$$a_{n_i} \in \{0, 1, 2, 3, \dots, 9\}$$

$$
\begin{aligned}
1 &\leftrightarrow .a_{11}a_{12}a_{13}a_{14}a_{15}\dots\\
2 &\leftrightarrow .a_{21}a_{22}a_{23}a_{24}a_{25}\dots\\
3 &\leftrightarrow .a_{31}a_{32}a_{33}a_{34}a_{35}\dots\\
4 &\leftrightarrow .a_{41}a_{42}a_{43}a_{44}a_{45}\dots
\end{aligned}
$$

Define $x \in (0,1)$ *such that* $x = .b_1b_2b_3b_4$.

$$
b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}
$$

$$
\begin{aligned}
&x \neq f(1) \implies x \notin f(\mathbb{N}) = Range f = (0,1)\\
&x \neq f(n) \quad \forall n \in \mathbb{N}
\end{aligned}
$$

C

Corollary 0.3 $\mathbb{R}$ *are uncountable* $\because (0,1) \subset \mathbb{R}$.

$$
\begin{aligned}
\exists f : \mathbb{R} \to (0,1) &a \text{ bijection}\\
f(x) = \frac{2}{\pi} \cdot &\frac{\arctan(x) + 1}{2}
\end{aligned}
$$

Corollary 0.4 *Every open interval is uncountable.*

$$
\begin{aligned}
\exists f(0,1) &\xrightarrow{f} (a,b) \ \text{a bijection}\\
f(x) &= mx + n\\
f(0) &= n = a\\
f(1) &= m + n = m + a = b \implies m = b - a\\
f(x) &= (b-a)x + a
\end{aligned}
$$

Theorem 0.5 *Schröder-Bernstein*
If A, B *are two infinite sets where* $|A| \leq |B| \leq |A| \implies |A| = |B|$ *(A and B are in bijective correspondence). Assume* $\exists f : A \to B$ *is an injective map.*

$$
\begin{aligned}
&(\exists g : B \to A \text{ which is an injective map})\\
&\qquad\qquad\to (\exists h : A \to B \text{ which is a bijection})
\end{aligned}
$$

Corollary 0.6 *Show that* $(0,1]$ *is uncountable or* $|(0,1)| = |(0,1]|$.

Proof

$$
\begin{aligned}
(0,1) &\xrightarrow{f} (0,1] \quad \textit{one-to-one} \\
(0,1] &\xrightarrow{g} (0,1) \quad \textit{one-to-one} \\
g(x) &= \frac{x}{2} \\
\therefore \exists h : (0,1) &\to (0,1]
\end{aligned}
$$

Proposition 0.7 *You can construct a surjective function from a finite set* A *to* $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Proof *Assume* A *is a finite set and* $\exists f : A \to \mathcal{P}(A)$ *for any* $a \in A$ *then*

$$f(a) \in \mathcal{P}(A)$$

i.e. $\forall a \in A,\ f(a) \subseteq A$ *in such a way that it is an injection.*

Theorem 0.8 *Let* A, B *be finite sets and* $f : A \to B$ *be a function between these finite sets.*

1. *If* f *is injective, then* $|A| \leq |B|$
2. *If* f *is surjective, then* $|B| \leq |A|$
3. *If* f *is bijective, then* $|A| = |B|$

Therefore, this proposition would imply $|A| \geq |\mathcal{P}(A)|$.

Theorem 0.9 *If* A *is a finite set,*

$$|\mathcal{P}(A)| = 2^{|A|}$$

Recalling the above theorem, the proposition must be ***false***.

Theorem 0.10 *Cantor's*
If A *is an infinite set, then there is no surjective function from*

$$A \to \mathcal{P}(A).$$

Proof *Assume there exists a surjection*

$$f : A \to \mathcal{P}(A)$$

A is infinite.

$$\forall a \in A \implies \begin{matrix} f(a) \in \mathcal{P}(A) \\ f(a) \subseteq A \end{matrix}$$

We claim that there exists a subset of A that is not in the range of f.

$$B \subseteq a$$
$$B = \{a \in A \mid a \notin f(a)\}$$

$$a \notin f(a) \subseteq A \qquad f : A \to \mathcal{P}(A)$$

Assume that f is surjective. Then $\exists a' \in A$ *so that* $f(a') = B$.

1. *Assume* $a' \in B$
$$\implies a' \notin f(a') = B \equiv F$$
2. *Assume* $a' \notin B$
$$\implies a' \in f(a') = B \equiv F$$

Therefore, we cannot create a surjective function between A and $\mathcal{P}(A)$ *by proof of contradiction.*

Example Let S be the set consisting of all sequences of 0 and 1.

$$S = \{(a_1, a_2, a_3, a_4, \dots) \mid a_i \in \{0, 1\}\}$$

Prove that S is not countable.

Proof Assume S is countable i.e. $\exists f : \mathbb{N} \to S$.

$$f(1) = (1000\dots)$$
$$f(2) = (0100\dots)$$
$$f(n) = (a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4})$$

$$b_n = \begin{cases} 1 & a_{nn} = 0 \\ 0 & a_{nn} = 1 \end{cases}$$
$$b_{nn} = b_1 b_2 b_3 b_4 \dots$$
$$b_{nn} \notin Image f = Range f$$

$$|S| = 2^{|\mathbb{N}|}$$

This is because for every natural number i representing the ith digit, there are 2 possibilities for i, 0 and 1.

$$|\mathbb{N}| = \chi_0$$

$$|S| = 2^{|\mathbb{N}|} = |\mathbb{R}| \implies |\mathbb{R}| = 2^{\chi_0}$$

Proposition 0.11 *Continuum Hypothesis*
Cantor presented in 1877, and Hilbert added it to his famous list of Open Problems at the ICM in 1900.

$$\nexists S : \aleph_0 < |S| < 2^{\aleph_0}$$

The Cantor Set (Topology of $\mathbb{R}$)

Define

$$\begin{aligned}
C_0 &:= [0,1] \\
C_1 &:= [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
C_2 &:= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \sqcup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]
\end{aligned}$$

For every $n = 0, 1, 2, \ldots$ then C_n consists of 2^n closed intervals of length $\frac{1}{3^n}$.

Define the Cantor Set to be

$$C := \bigcap_{n=1}^{\infty} C_n.$$

$$C = [0,1] \setminus [\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots]$$

$$0, 1 \in C$$

If $y \in C_n$ is an endpoint for one of these subintervals then $y \in C$.

Example 1. Is C a countable set?

Let S consist of sequeneces of $\{0\}$ and $\{1\}$.

$$S = \{(a_1, a_2, a_3, a_4, \ldots) \mid a_i \in \{0, 1\}\}$$

The S is not countable.

$$2^{|\mathbb{N}|} \sim |\mathbb{R}|$$

For each $c \in C$, set $a_1 = 0$ if it falls in the left component of C_1 or $a_1 = 1$ if it falls in the right component of C. Once a_1 is chosen, we have 2 possibilities for a_2.

$$a_2 = \begin{cases} 0 & \text{if on left of } C_2 \\ 1 & \text{if on right of } C_2 \end{cases}$$

Therefore, we can associate every element in C with a unique sequence a_n where $n \in \mathbb{N}$, so C is not countable.

2. Does it contain any subinterval? Length of C:

$$\begin{aligned} & 1 - \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{2^{n-1}}{3^n} \\ = \; & 1 - \frac{1}{3}(1 + \frac{2}{9} + \frac{4}{9} + \cdots + \frac{2^{n-1}}{3^{n-1}}) \\ = \; & 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) \\ = \; & 1 - \frac{1}{3} \cdot 3 \\ = \; & 1 - 1 \\ = \; & 0 \end{aligned}$$

Therefore, there are no elements in C which contain an interval, since each element is a single point (zero-dimensional). [Known as Cantor Dust.]

3. Does it contain any rational or irrational numbers?

Definition A is a **closed** set if it is the complement of an open set.

Definition O is an **open** set if $\forall x \in O \implies \exists N(x) \subseteq O$.

Definition A is **compact** if and only if it is closed and bounded.

Definition A is **perfect** if it is closed and contains no isolated points.

The Cantor Set is a **closed** and **compact** set.
The Cantor Set is also a **perfect** set.

$$\forall c \in C \quad \exists (x_n)_n \subset C \implies \lim_{n\to\infty} x_n = c$$

$$C = \left\{ \sum_{n=1}^{\infty} \frac{C_n}{3^n}, \quad C_n \in \{0,2\} \right\}$$

An irrational number is in the Cantor set if and only if in a base 3 expansion it consists only of 0 and 2.

Remark

$$C \rightsquigarrow \dim \frac{\log 2}{\log 3} \approx 0.631$$