MATH 381 Section 1.7

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Introduction to Proofs

Definition

1. The integer $n =$ even number if $\exists$ an integer k so that $n = 2k$.
2. The integer $n =$ odd number if $\exists$ an integer k so that $n = 2k + 1$.

If $x \in \mathbb{Z}$, either $x =$ odd or $x =$ even.

Example Give a proof for the statement
"If $n =$ odd integer, then n^2 is an odd integer".

$$\begin{aligned} \text{proof } P(n) : n = &\text{ odd integer} \\ Q(n) : n^2 = &\text{ odd integer} \end{aligned}$$

claim $\forall n(P(n) \to Q(n))$

Example Give a direct proof to
"If m and n are both perfect squares, then $m \times n$ is also a perfect square."

Proof Since m and n are both perfect squares $\implies \exists a, b \in \mathbb{Z}^+$ so that

1. $m \times n = a^2 \times b^2 = (ab)^2$ $\blacksquare$

Section 1.7.6 proof by negation

Assume we want to prove a conditional statement $p \to q \equiv \neg q \to \neg p$.

1. prove it by **contraposition** i.e. prove

$$\neg q \to \neg p.$$

2. prove it by **contradiction** i.e. assume
   $p \to q$ is False i.e. $p = T$ and $q = F$.

Example Prove that if $n = a \cdot b$ where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

$$\begin{aligned} q: \quad & a \leq \sqrt{n} \vee b \leq \sqrt{n} \\ \neg q: \quad & \neg(a \leq \sqrt{n}) \wedge \neg(b \leq \sqrt{n}) \\ & (a > \sqrt{n}) \wedge (b > \sqrt{n}) \end{aligned}$$

Example Prove that $\sqrt{2}$ is an irrational number.

Proof Assume by contradiction that $\sqrt{2} \in \mathbb{Q}$.

$$\mathbb{Q} = \{\frac{a}{b},\ (a,b) = 1,\ b \neq 0;\ a, b \in \mathbb{Z}\}$$

$$\sqrt{2} \in \mathbb{Q} \implies \exists a, b \in \mathbb{Z},\ b \neq 0,\ (a,b) = 1$$

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ \implies 2 &= (\frac{a}{b})^2 = \frac{a^2}{b^2} \\ . \implies a^2 &= 2b^2 \end{aligned}$$

Proof of Equivalences

To prove a theorem that is a biconditional statement of the form $p \leftrightarrow q$, you must prove two conditional statements: $p \to q$ and $q \to p$.

$$p \leftrightarrow q = (p \to q) \wedge (q \to p)$$

Sometimes a theorem stating that many propositions are equivalent

$$p_1, p_2, p_3, \ldots, p_n$$

Show that TFAE (that the following are equivalent)

1. p_1
2. p_2
3. p_3

n. p_n

bu definition it means prove

$$p_1 \leftrightarrow p_2$$
$$p_2 \leftrightarrow p_3$$
$$\vdots$$
$$p_{n-1} \leftrightarrow p_n$$

i.e. $2(n-1)$ statements to prove

Theorem 0.1 *If we can prove a loop in the form*

$$p_1 \rightarrow p_2$$
$$p_2 \rightarrow p_3$$
$$\vdots$$
$$p_{n-1} \rightarrow p_n$$
$$p_n \rightarrow p_1$$

i.e. n statements then it suffices to show that $p_1, \ldots, p_n$ are equivalent statements.

Example Let $n \in \mathbb{Z}$
Prove that n is odd if and only if n^2 is odd.

Proof

$$p : n \text{ is odd}$$
$$q : n^2 \text{ is odd}$$

1. Show that $p \rightarrow q$ i.e. if n is odd then n^2 is odd.
2. Show that $q \rightarrow p$ i.e. if n^2 is odd then n is odd. We prove $q \rightarrow p$ by showing the contrapositive $\neg p \rightarrow \neg q$ i.e. if n is even then n^2 is even.

Example Let $n \in \mathbb{Z}$.
Show that the following are equivalent:

$$p_1 : n \text{ is even}$$
$$p_2 : n-1 \text{ is odd}$$
$$p_3 : n^2 \text{ is even}$$

$p_1 \rightarrow p_2$ is obvious.
$p_2 \rightarrow p_3$ can be proven with another integer k.
$p_3 \rightarrow p_1$ is proven by its equivalence with $\neg p_1 \rightarrow \neg p_3$.

Example Show that the statement “every positive integer is the sum of two squares of integers” is false.
Counterexample is $n = 3$.