MATH 381 Section 6.4

Prof. Olivia Dumitrescu

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Section 6.4 The Binomial Theorem

Theorem 0.1 *Binonmial Theorem*
Let x, y be variables and $n \geq 0$.

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n \\
&= x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots + n x y^{n-1} + y^n
\end{aligned}$$

Corollary 0.2

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

$x = y = 1$
$|\mathcal{P}(S)|$

$n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Taylor Series $\qquad x_0 \in \operatorname{Dom} f$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Maclaurin Series Expansion if $x_0 = 0$.
$f(x) = (1 + x)^n$
$m \notin \mathbb{N}, m \in \mathbb{R}$

$$(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

$$\binom{m}{k} = \begin{cases} \dfrac{m(m-1)(m-2)\dots(m-k+1)}{k!} & k > 0 \\ 1 & k = 0 \end{cases}$$

$$\binom{-2}{3} = \frac{-2 \cdot -3 \cdot -4}{1 \cdot 2 \cdot 3} = \frac{-4}{1} = -4$$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{1 \cdot 2 \cdot 3} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}}{2 \cdot 3} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-1}{2}}{2} = \frac{1}{16}$$

$$\binom{-1}{k} = \frac{-1 \cdot -2 \cdot \dots \cdot (-1 - k + 1)}{1 \cdot 2 \cdot \dots \cdot k} = \frac{-1 \cdot -2 \cdot \dots \cdot -k}{1 \cdot 2 \cdot 3 \cdot \dots \cdot k}$$

$$= (-1)^k = \begin{cases} -1, & k \text{ is odd} \\ +1, & k \text{ is even} \end{cases}$$

Use binomial formula

$$\frac{1}{1-x} = (1 - x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots \qquad |x| < 1$$

$$\begin{aligned} \frac{1}{1-x} &= (1 - x)^{-1} \\ &= \sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k \\ &= \sum_{k=0}^{\infty} (-1)^k (-1)^k \cdot x^k = \sum_{k=0}^{\infty} x^k \end{aligned}$$

Corollary 0.3 $n \in \mathbb{N}$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$$

Proof $x = 1, y = -1$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$$

Corollary 0.4

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Proof $x = 2, y = 1$

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (2+1)^n = 3^n$$

Corollary 0.5

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = \sum_{k=1}^{n} k \binom{n}{k}$$

Proof

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiate (with respect to x) [possible $\because$ we expanded the domain to $\mathbb{R}$]

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

Let $x = 1$.

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

Remark If $n \in \mathbb{N}, k \in \mathbb{N}, k \geq n+1$

$$\binom{n}{k} = 0$$

$$\binom{n}{n+1} = \frac{n(n-1)\dots(n-(n+1)+1)}{(n+1)!} = \frac{n(n-1)\dots(0)}{(n+1)!} = 0$$

Theorem 0.6 *Pascal's Identity*
Let $n, k \in \mathbb{N} \qquad k \leq n$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof

$$\begin{aligned}
\binom{n+1}{k} &= \frac{(n+1)n(n-1)\dots((n+1)-k+1)}{k!} \\
\binom{n}{k-1} &= \frac{n(n-1)\dots(n-(k-1)+1)}{(k-1)!} \\
\binom{n}{k} &= \frac{n(n-1)\dots(n-k+1)}{k!}
\end{aligned}$$

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k-1} + \binom{n}{k} \\
\frac{(n+1)!}{k!(n+1-k)!} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right] \\
&= \frac{n!}{(k-1)!(n-k)!}\left[\frac{k}{k(n-k+1)} + \frac{(n-k+1)}{k(n-k+1)}\right] \\
&= \frac{n!}{(k-1)!(n-k)!}\left[\frac{k+(n-k+1)}{k(n-k+1)}\right] \\
&= \frac{n!}{(k-1)!(n-k)!} \cdot \frac{n+1}{k(n-k+1)} \\
&= \frac{(n+1)!}{k!(n-k+1)!}
\end{aligned}$$

Theorem 0.7 *Vandermonde's Identity*
Let $m, n, r \in \mathbb{N}$. $r \leq m, n$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof

$$\begin{aligned}
(1+x)^{m+n} &= (1+x)^m \cdot (1+x)^n \\
\sum_{r=0}^{m+n} \binom{m+n}{r} x^r &= \left(\sum_{i=0}^{m} \binom{m}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right) \\
&= \sum_{k=0}^{m+n} \left(\sum_{i+j=k} \binom{m}{i}\binom{n}{j}\right) x^k \qquad k = r
\end{aligned}$$

Corollary 0.8 $n \in \mathbb{N}$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof $m = n = r$

$$\begin{aligned}
\binom{2n}{n} &= \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2
\end{aligned}$$

Theorem 0.9 $n, r \in \mathbb{N} \qquad r \leq n$

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof

$$\begin{aligned}
\binom{n+1}{r+1} = \binom{n}{r} + &\binom{n}{r+1} \\
&\binom{n-1}{r} + \binom{n-1}{r+1} \\
&\qquad\qquad \binom{n-2}{r} + \binom{n-2}{r+1} \\
&\qquad\qquad\qquad\qquad \binom{r}{r}
\end{aligned}$$

Example What is the coefficient of $x^{12} \cdot y^{13}$ in the expansion $(2x - 3y)^{35}$?

Proof

$$(2x - 3y)^{25} = \sum_{k=0}^{25} \binom{25}{k} (2x)^{25-k} \cdot (-3y)^k$$

So, $k = 13$.

$$\binom{25}{13} \cdot 2^{12} \cdot (-3)^{13}$$