MATH 381 Homework 5 part 2

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1. Show that an integer is divisible by 4 if and only if it can be written as the sum of two consecutive odd numbers.

$$\forall n \in \mathbb{Z}((4 \mid n) \leftrightarrow (\exists k \in \mathbb{Z}(n = (2k+1) + (2(k+1)+1))))$$

Suppose an integer can be written as the sum of two consecutive odd numbers.

$$\begin{aligned}
&\exists k \in \mathbb{Z}(n = (2k+1) + (2(k+1)+1)) \\
&\implies n = (2k+1) + 2(k+1) + 1 \\
&\implies n = 2k + 1 + 2k + 2 + 1 \\
&\implies n = (2k + 2k) + (1 + 2 + 1) \\
&\implies n = 4k + 4 \\
&\implies n = 4(k+1) \\
&\implies \frac{n}{4} = k + 1 \in \mathbb{Z} \\
&\implies 4 \mid n
\end{aligned}$$

Therefore, the integer is divisible by 4.

Suppose an integer is dibisible by 4.

$$
\begin{gathered}
4 \mid n \iff \frac{n}{4} \in \mathbb{Z} \\
\implies \left(\frac{n}{2}\right)/2 \in \mathbb{Z} \iff 2 \mid \left(\frac{n}{2}\right) \iff \frac{n}{2} \text{ is even} \\
\implies 2\left(\frac{n}{4}\right) \in \mathbb{Z} \implies \frac{n}{2} \in \mathbb{Z} \iff 2 \mid n \iff n \text{ is even} \\
\frac{n}{2} \text{ is even} \implies \frac{n}{2} \pm 1 \text{ is odd}
\end{gathered}
$$

$$
\begin{aligned}
n &= \left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) \\
n &= \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} + 1\right) \\
\exists k, m \in \mathbb{Z}(n &= (2k+1) + (2m+1))
\end{aligned}
$$

$$
\begin{aligned}
\frac{n}{2} - 1 &= 2k + 1 && \implies \frac{n}{2} = 2k + 2 && \implies n = 4k + 4 \\
\frac{n}{2} + 1 &= 2m + 1 && \implies \frac{n}{2} = 2m && \implies n = 4m
\end{aligned}
$$

$$4k{+}4 = n = 4m \implies 4k{+}4 = 4m \implies 4(k{+}1) = 4m \implies k{+}1 = m$$

$$
\begin{aligned}
&\frac{n}{2} - 1 = 2k + 1 \\
&\frac{n}{2} + 1 = 2m + 1 = 2(k+1) + 1
\end{aligned}
$$

$$\therefore n = (2k+1) + 2(k+1) + 1 \quad \blacksquare$$

Therefore, the integer can be written as the sum of two consecutive odd numbers.

2. Prove that if $a^2 + b^2 = c^2$, then abc is even.

Suppose abc is not even, meaning abc is odd.

$$abc = 2k+1 \iff ad = 2k+1 \quad d = bc$$

$$\implies \begin{cases} \quad a \text{ is odd and } d \text{ is even} \implies \begin{cases} & b \text{ is odd and } c \text{ is odd} \\ \text{OR} & b \text{ is even and } c \text{ is even} \end{cases} \\ \text{OR} \quad a \text{ is even and } d \text{ is odd} \implies \begin{cases} & b \text{ is odd and } c \text{ is even} \\ \text{OR} & b \text{ is even and } c \text{ is odd} \end{cases} \end{cases}$$

(a) a is odd, b is odd and c is odd

$$a^2 \text{ is odd} \land b^2 \text{ is odd} \implies a^2 + b^2 \text{ is even}$$
$$a^2 + b^2 \text{ is even} \land c^2 \text{ is odd} \implies a^2 + b^2 \neq c^2$$

(b) a is odd, b is even and c is even

$$a^2 \text{ is odd} \land b^2 \text{ is even} \implies a^2 + b^2 \text{ is odd}$$
$$a^2 + b^2 \text{ is odd} \land c^2 \text{ is even} \implies a^2 + b^2 \neq c^2$$

(c) a is even, b is odd and c is even

$$a^2 \text{ is even} \land b^2 \text{ is odd} \implies a^2 + b^2 \text{ is odd}$$
$$a^2 + b^2 \text{ is odd} \land c^2 \text{ is even} \implies a^2 + b^2 \neq c^2$$

(d) a is even, b is even and c is odd

$$a^2 \text{ is even} \land b^2 \text{ is even} \implies a^2 + b^2 \text{ is even}$$
$$a^2 + b^2 \text{ is even} \land c^2 \text{ is odd} \implies a^2 + b^2 \neq c^2$$

$\therefore (abc \text{ is odd}) \to (a^2 + b^2 \neq c^2) \equiv (a^2 + b^2 = c^2) \to (abc \text{ is even})$ $\blacksquare$

3. Suppose that x and y are real numbers. Prove that if $x+y$ is irrational then x is irrational or y is irrational.

$$\forall x, y \in \mathbb{R}(P(x,y) \to Q(x,y))$$
$$\forall x, y \in \mathbb{R}((x+y \notin \mathbb{Q}) \to (x \notin \mathbb{Q} \lor y \notin \mathbb{Q}))$$

Suppose $\neg Q(x,y) \equiv (x \in \mathbb{Q} \land y \in \mathbb{Q})$.

$$x = \frac{a}{b} \quad y = \frac{c}{d} \qquad a, b, c, d \in \mathbb{Z} \quad b, d \neq 0$$
$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd} = \frac{e}{f} \qquad e, f \in \mathbb{Z} \quad f \neq 0$$
$$\therefore \frac{e}{f} = x + y \in \mathbb{Q} \quad \blacksquare$$

Therefore, $\neg Q \to \neg P \equiv P \to Q$.

4. Prove that there are no positive integer solutions to $x^2 + x + 1 = y^2$.

$$\neg(\exists x \in \mathbb{Z}_+ \exists y \in \mathbb{Z}_+ (x^2 + x + 1 = y^2))$$
$$\equiv \forall x \in \mathbb{Z}_+ \forall y \in \mathbb{Z}_+ (x^2 + x + 1 \neq y^2)$$

Assume $x, y \in \mathbb{Z}_+$.

$$y^2 = x^2 + x + 1$$
$$y^2 - x^2 = x + 1$$
$$\implies (y+x)(y-x) = x + 1$$

There are three possible cases for x and y.

(a) $y = x$

$$(y+x)(y-x) = x + 1$$
$$\implies 2y(0) = y + 1$$
$$0 = y + 1 \equiv F \because y > 0$$
$$\therefore (y+x)(y-x) = x + 1 \equiv F$$

(b) $y < x$

$$\begin{aligned}(y+x)(y-x) &= x+1\\(y+x)(y-x) &< 0 \because (y+x) > 0 \land (y-x<0 \iff y<x)\\x+1 &> 0 \because x>0\end{aligned}$$

$$\therefore (y+x)(y-x) = x+1 \equiv F$$

(c) $y > x$

$$\begin{aligned}(x+y) &\geq (x+1) \because y \geq 1\\\implies (y-x)(x+y) &\geq x+1 \because y-x>0 \iff y>x\\y-x=1 \implies (x+y) &\geq (x+1)\\(x+y) = (x+1) &\implies y=1\\(y-x=1)\land(y=1) &\equiv (1-x=1 \iff -x=0 \iff x=0)\\x=0 \equiv F &\because x \in \mathbb{Z}_+\end{aligned}$$

$$\therefore (y+x)(y-x) = x+1 \equiv F \quad \blacksquare$$

Since in all three cases for positive integer solutions or x and y the proposition we are trying to prove implies a contradiction, the assumption that there are positive integer solutions must be false.

5. Show that if you choose 92 different dates from a calendar, at least 14 of the chosen dates must occur on the same day of the week.

   r: "If you choose 92 different dates from a calendar."
   p: "At least 14 of the 92 chosen dates must occur on the same day of the week."
   $\neg p$: "At most 13 of the 92 chosen days fall on the same day of the week."
   Suppose $\neg p$. Since there are 7 days in a week, 13 on each day would give us 91 total chosen days. Any more days chosen necessarily means that there would be a fourteenth date on a given day of the week. But we have to choose 92 dates. This suggests $\neg p \to (r \land \neg r)$. In other words, if p was false, it would imply a contradiction. Therefore, p must be true, and p is dependent on r for the number of chosen dates that was agreed upon (92). So, $r \to p$ is true. $\blacksquare$