MATH 381 Section 2.3

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Functions

Example Prove that there are ∞ many pairs of integer solutions on the cone

$$x^2 + y^2 - z^2 = 0$$

Definition Let A, B be non-empty sets.
A function $f : A \to B$ to be an assignment of exactly one element of B to each element of A.

$$f(a) = b$$

$$A = \text{domain}, B = \text{codomain}, b = \text{image of } a, a = \text{preimage of } b$$

$Imf \subseteq B$ is called the range of f.

$$Imf = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$$

Example The circle $x^2+y^2 = 1$ is not a function but the union of 2 functions.

$$y = \begin{cases} \sqrt{1-x^2} \\ -\sqrt{1-x^2} \end{cases}$$

Example Now take

$$\begin{aligned} f_1 &: A \to B \\ f_2 &: A \to B \quad B \subseteq \mathbb{R} \end{aligned}$$

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \quad \forall x \in A \\ (f_1 \cdot f_2)(x) &= f_1(x) \cdot f_2(x) \end{aligned}$$

Definition A function $f : A \to B$ is **injective** if and only if $f(a) = f(b) \implies a = b$.
This means that distinct points in the domain have different heights i.e. if $a \neq b$, then $f(a) \neq f(b)$.

Remark To disprove that a function is injunctive, it is enough to find two points of the domain $a \neq b$ so that $f(a) = f(b)$.
If you know its graph, how can you test if a function is injective?

1. A, B are finite sets

2. Continuous functions
   Horizontal Line Test: A function is injective if any horizontal line intersects the graph at at most 1 point.

Example $y = \sqrt{1 - x^2}$ is not injective because $f(x) = f(-x)$.

Definition We say a function $f : A \to B$ is **surjective** if the range of f is the codomain B.

$$Imf = \{b \in B \mid \exists a \in A,\ f(a) = b\} = B$$

To prove a function is surjective:

$$\forall b \in B\ \exists a \in A \text{ so that } f(a) = b$$

Remark We can always make a function surjective by reducing the codomain.

$$f : A \to B \qquad f : A \to Imf \subseteq B$$

Example To make $f : x \mapsto x^2$ surjective,

$$f : \mathbb{R} \to \mathbb{R}_+$$

To make f injective,

$$f : \mathbb{R}_+ \to \mathbb{R}$$

Definition The function $f : A \to B$ is **bijective** if f is both injective and surjective.

Remark Any continuous function can be transformed into a bijective function, but not in a unique way necessarily.

Remark Let f be a bijection

$$A \xrightarrow{f} B$$

between finite sets.

$$\implies |A| = |B|$$

Example Bijection between open and bounded interval and $\mathbb{R}$.

$$\tan(x) : \mathbb{R} - \{\frac{2k+1}{2}\pi \mid k \in \mathbb{Z}\} \to \mathbb{R}$$

So, tan is sujective but not injective.

$$\arctan(x) : \mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$$

arctan is injective but not surjective.

Definition If f is bijective between A and B, then there exists an inverse function f^{-1}

$$\begin{aligned} &f : A \to B \\ &f : A \to \text{Range} f = \text{Im} f \end{aligned}$$

$$\begin{gathered} \exists f^{-1} : \text{Im} f \to A = A \xleftarrow{f^{-1}} \text{Im} f \\ \text{if } f(a) = b \implies f^{-1}(b) = a \qquad a \in A \wedge b \in B \end{gathered}$$

$$\begin{gathered} f : A \to \text{Im} f = C \\ \forall b \in \text{Im} f \; \exists! a \in A \, (b = f(a)) \end{gathered}$$

Proposition 0.1

$$\begin{cases} f : A \to B \\ k : Imf \Rightarrow C \end{cases} \implies k \circ f : A \to C$$

Definition We say that $f : A \to B$ is invertible if $\exists g = f^{-1} : B \to A$ so that

1. $f \circ g = Id_B (\forall x \in B \implies f(g(x)) = x)$
2. $g \circ f = Id_A (\forall x \in A \implies g(f(x)) = x)$

$$A \xrightarrow{f} B \xrightarrow{g} A \implies g \circ f : A \to A$$

Remark Start from a bijective function $f : A \to B$ then $g : B \to A$.
Can we find g?

1. $f(x) = y$
2. Solve it for x $\quad x = g(y)$
3. Interchange x and y $\quad y = g(x) = f^{-1}$

Proposition 0.2 *$f : A \to B$ is bijective and continuous $\implies$ the inverse $g : B \to A$ exists.*
The graph of g is obtained from the graph of f by reflecting along $y = x$.