MATH 381 HW 7 part 4

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x - y$. Determine, with proofs, whether f is one-to-one, onto, both, or neither. Be sure to fully justify both your statement of one-to-one and your statement of onto.

   (a) $P : f$ is an injective (i.e. one-to-one) function

   $$\begin{aligned} P &\equiv \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 (f(x_1, y_1) = f(x_2, y_2) \to (x_1, y_1) = (x_2, y_2)) \\ &\equiv \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 ((x_1, y_1) \neq (x_2, y_2) \to f(x_1, y_1) \neq f(x_2, y_2)) \end{aligned}$$

   Suppose $(x_1, y_1) \neq (x_2, y_2)$ and let $x_1 = y_1$ and $x_2 = y_2$.

   $$\left.\begin{aligned} f(x_1, y_1) = x_1 - y_1 = x_1 - x_1 = 0 \\ f(x_2, y_2) = x_2 - y_2 = x_2 - x_2 = 0 \end{aligned}\right. \implies f(x_1, y_1) = f(x_2, y_2)$$

   $\therefore \exists (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 ((x_1, y_1) \neq (x_2, y_2) \to f(x_1, y_1) = f(x_2, y_2))$
   $\implies \neg P$

   (b) $Q : f$ is a surjective (i.e. onto) function

   $$Q \equiv \forall z \in \mathbb{R}\ \exists x, y \in \mathbb{R}^2 (f(x, y) = z)$$

   Let $z \in \mathbb{R}$.

   $$\begin{aligned} &\text{Fix } y = 0 \in \mathbb{R} \\ \implies z = x - y &\iff z = x \\ z \in \mathbb{R} &\implies x \in \mathbb{R} \end{aligned}$$

   $\therefore Q \quad \blacksquare$

Consequently, f **is a surjective (i.e. onto) function**, but **it is not an injective (i.e. one-to-one) function**, and thus it cannot be a bijective function. Intuitively, for every value z on the z-axis of the 3-D graph of f, there must be a one-dimensional (linear) slice through the plane. Consequently, there are an infinite amount of solutions along each line, and therefore it cannot be injective. As for the surjective property, the plane spans all real numbers in its range since it is angled nonparallel to the x-axis, as seen by its coefficients.

2. Suppose $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one. Prove $g \circ f : A \to C$ is also one-to-one.

$$\forall x_A, y_A \in A(f(x_A) = f(y_A) \to x_A = y_A) \\ \wedge \forall x_B, y_B \in B(g(x_B) = g(y_B) \to x_B = y_B)$$

$$P : \forall x_A, y_A \in A((g \circ f)(x_A) = (g \circ f)(y_A) \to x_A = y_A)$$

Let $x_A, y_A \in A$ and suppose $(g \circ f)(x_A) = (g \circ f)(y_A)$.

$$\begin{aligned} &(g \circ f)(x_A) = (g \circ f)(y_A) \\ \implies &g(f(x_A)) = g(f(y_A)) \end{aligned}$$

$$\text{Let } x_B = f(x_A) \in B$$
$$\text{Let } y_B = f(y_A) \in B$$

$$\begin{aligned} &g(f(x_A)) = g(f(y_A)) \\ \implies &g(x_B) = g(y_B) \\ \implies &x_B = y_B \\ \implies &f(x_A) = f(y_A) \\ \implies &x_A = y_A \end{aligned}$$

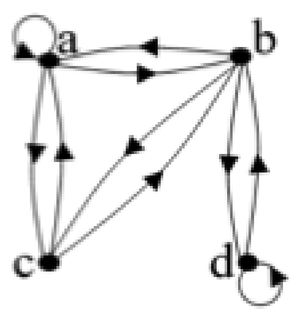
$\therefore P$ $\blacksquare$

3. Let $A = \{1,2,3,4,5\}$ and define relations R and S on A by $R = \{(1,2),(2,2),(4,3)\}$ and $S = \{(2,1),(3,3),(3,5),(5,4)\}$. Determine the matrix for the relation $S \circ R$.

$$S := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad R := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S \circ R := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Given the relation R on $\{a,b,c,d\}$ defined by the digraph below, determine, with justification, whether R is symmetric, transitive, and/or reflexive. Provide explanation for each of the three properties.



(a) The relation is **not reflexive** because only a and d point back to themselves. b and c do not share this characteristic.

(b) The relation is **symmetric** because each element that is related to another shares a relation from that element. In other words, each element is bidirectionally related to its related elements.

(c) The relation is **not transitive** because related elements do not necessarily share relations with the related elements' related elements. For example, $(a,b) \in R$ and $(b,d) \in R$, however $(a,d) \notin R$.

5. Let R be a relation on $\mathbb{N}$ that is symmetric and transitive. Further, suppose that $(x, 7) \in R$ for all $x \in \mathbb{N}$. Show that R induces a partition on $\mathbb{N}$.

   Since R is symmetric:

   $$\forall x \in \mathbb{N}((x, 7) \in R) \rightarrow \forall x \in \mathbb{N}((7, x) \in R)$$

   Let $x, y \in \mathbb{N} \quad x \neq y$. Since R is transitive:

   $$(x, 7) \in \mathbb{R} \wedge (7, y) \in \mathbb{R} \rightarrow (x, y) \in \mathbb{R}$$

   Since x, y are arbitrary positive integers, any two are related. Consequently, every element from $\mathbb{N}$ will be mapped to every element of $\mathbb{N}$. Therefore, every element in $\mathbb{N}$ is in the same subset which forms a partition over itself.

6. Let $S = \mathcal{P}(\{1, 2, 3, 4, 5\})$, the power set of the set of the first five positive integers. Define a relation R on S as follows: set A is related to set B via R if and only if $|A| = |B|$.

   (a) Determine the equivalence class $[\{2, 3\}]$.

   $[\{2, 3\}] = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

   (b) Determine the equivalence class $[\emptyset]$.

   $$[\emptyset] = \{\emptyset\}$$

   (c) How many distinct equivalence classes does R have? Explain your answer.

   R has six distinct equivalence classes because the relation is dependent on the cardinality of subsets. Since a subset can have zero elements at minimum (this would be the empty set) and five elements at maximum (the set itself), the possible cardinalities of subsets are 0,1,2,3,4,5. There are 6 possible cardinalities of subsets, so there should be **6** distinct equivalence classes.