MATH 381 HW 4

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1. Let the universe for x be students and let the universe for y be courses. Translate the following English statements to logical expressions, using the following notation for predicates:

   - $T(x, y)$: x is taking y
   - $M(y)$: y is a math course
   - $P(x)$: x is a part-time student

   Then, give a negation of your logical expression, simplifying as much as possible.

   (a) Every student is taking at least one course.

   $$\begin{aligned} p: &\quad \forall x \exists y T(x, y) \\ \neg p: &\quad \neg(\forall x \exists y T(x, y)) \equiv \exists x \forall y(\neg T(x, y)) \end{aligned}$$

   (b) Some part-time student is not taking any math courses.

   $$\begin{aligned} q: &\quad \exists x \forall y((P(x) \wedge M(y)) \rightarrow \neg T(x, y)) \\ \neg q: &\quad \neg(\exists x \forall y((P(x) \wedge M(y)) \rightarrow \neg T(x, y))) \\ &\equiv \forall x \exists y \neg((P(x) \wedge M(y)) \rightarrow \neg T(x, y)) \\ &\equiv \forall x \exists y \neg(\neg(P(x) \wedge M(y)) \vee \neg T(x, y)) \\ &\equiv \forall x \exists y(P(x) \wedge M(y) \wedge T(x, y)) \\ &\equiv \forall x \exists y((P(x) \wedge M(y)) \rightarrow T(x, y)) \end{aligned}$$

2. Let the universe for x be people and let the universe for y be movies. Translate the following statements to English without using any variables, given the following notation for predicates:

- $S(x, y)$: x saw y
- $L(x, y)$: x liked y
- $A(y)$: y won an award
- $C(y)$: y is a comedy

(a) $\forall y(C(y) \to L(Max, y))$

Max likes all comedy movies.

(b) $\forall y \exists x(S(x, y) \land A(y))$

Every movie has been seen by someone and won an award.

3. Let $P(x, y)$ be the statement $x + 2y = xy$, where x is an integer and y is a real number. Determine the truth value of each statement, with explanation.

(a) $\exists y P(4, y)$

$$\begin{aligned} P(4, y) \equiv 4 + 2y &= 4y \\ 4 &= 2y \\ 2 &= y \end{aligned}$$

This statement is **true** because the statement which the predicate forms can be solved algebraically to find that $y = 2$ is a solution to the predicate. Therefore, there does exist a real number such that P holds, fulfilling the quantified predicate.

(b) $\forall x \exists y P(x, y)$

$$\begin{aligned} x + 2y &= xy \\ 2y - xy &= -x \\ y(2 - x) &= -x \\ \implies &\begin{cases} y = \dfrac{-x}{2 - x} = \dfrac{x}{x - 2} & \text{for } x \neq 2 \\ y(2 - 2) = -2 \implies 0 = 2 & \text{for } x = 2 \end{cases} \end{aligned}$$

The statement which the predicate forms can be rearranged algebraically to find that y is a function with a domain of

$\{x \mid x \neq 0, x \in \mathbb{Z}\}$ and a range of $\{y \mid y \in \mathbb{R}\}$. Since $x = 0$ is not in the domain, not every x is a solution to the predicate and therefore the statement is **false**.

(c) $\exists x \forall y P(x, y)$

$$\begin{aligned} x + 2y &= xy \\ 2y &= xy - x \\ 2y &= x(y-1) \\ \implies & \begin{cases} x = \dfrac{2y}{y-1} & \text{for } y \neq 1 \\ 2(1) = x(1-1) \implies 2 = 0 & \text{for } y = 1 \end{cases} \end{aligned}$$

The statement which the predicate forms can be rearranged algebraically to find that x is a function of y. Since $y = 1$ produces a statement that implies $2 = 0$ which is obviously false, there is no integer value for x in which $P(x, 1)$ holds. Therefore, the quantified predicate is **false**. Additionally, for certain values of y, x will be a non-integer values (e.g. $y = 4 \implies x = \frac{8}{3} \notin \mathbb{Z}$).

4. Let a and b be real numers with $a < b$, and let the universe for x be $\mathbb{R}$. Write the negation of the following statement without the symbol "$\neg$" and determine, with explanation, whether the original statement or its negation is true.

$$\exists x(a < x < b)$$

$$\neg(\exists x(a < x < b)) \equiv \forall x \neg(a < x \land x < b) \equiv \forall x(a \geq x \lor x \geq b)$$

The original statement is true for $a, b, x \in \mathbb{R}; a < b$ if we declare x to be the arithmetic mean of a and b i.e. $x = \frac{a+b}{2}$.

$$\begin{gathered} x = \frac{a+b}{2} \\ a < b \implies a + a < a + b \implies 2a < a + b \\ a < b \implies a + b < b + b \implies a + b < 2b \\ 2a < a + b < 2b \\ \frac{2a}{2} < \frac{a+b}{2} < \frac{2b}{2} \\ a < \frac{a+b}{2} < b \end{gathered}$$

Since it is demonstrated that the arithmetic mean is a solution to the statement, and additionally that $\frac{a+b}{2} \in \mathbb{R}$ because the real numbers is closed under addition and division (where the denominator is not zero, but it is two), the **original statement is true**.

5. Let the universe for m and n be the set of positive integers and let the universe for z be the set of all integers. Write the negation of the following statement without the symbol "¬" and determine, with explanation, whether the original statement or its negation is true.

$$\forall z \exists n \forall m (n \leq z^2 \vee z < n + m)$$

$$\begin{cases} n \leq z^2 \\ \quad \text{OR} \\ z < n + m \end{cases}$$

$$\begin{aligned} z^2 &\geq 0 \qquad \{z \mid z \in \mathbb{Z}\} \\ z^2 &= 0 \implies z = 0 \\ z^2 &\geq 1 \qquad \{z \mid z \neq 0, z \in \mathbb{Z}\} \end{aligned}$$

$$\begin{gathered} z = 0 \\ z < n + m \\ \implies 0 < n + m \\ \implies n > -m \end{gathered}$$

$$\begin{gathered} m > 0 \\ \implies -m < 0 \end{gathered}$$

$$\begin{gathered} n > 0 \\ \implies n > 0 > -m \\ \implies n > -m \end{gathered}$$

To fulfill the disjunction in the predicate, one of two conditions has to be true. The first condition offers many solutions. For all z, $z^2 \geq 0$

because no real number (and therefore) integer squared is negative (as evidenced by the parabola it makes when graphed, which doesn't dip below the x-axis). Since the only value for which $z^2 = 0$ is $z = 0$, we can conclude that for any integer z, $z^2 \geq 1$, as -1 and/or 1 is the next integer which increases to only 1 for z^2. So far, for all $z \in \mathbb{Z}, z \neq 1$, there exists an $n \in \mathbb{Z}_+$ such that the first condition in the disjunction holds. For the condition where $z = 0$, the second condition holds when $z < n + m$. This can be algebraically rearranged to get $n > -m$. Since m must be a positive integer, its additive inverse must be negative (less than zero) and since n must be greater than zero as a positive integer, we can choose an n which is always greater than any $-m$. So, the second condition in the disjunction holds for the only case in which the first does not. Therefore, the statement is **true**.