MATH 381 HW 2

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1. Show that $(p \to q) \lor (p \to r)$ is logically equivalent to $p \to (q \lor r)$.

$$
\begin{aligned}
& (p \to q) \lor (p \to r) & \\
\equiv\ & \neg p \lor q \lor \neg p \lor r & \text{[Conditional-disjunction equivalence]} \\
\equiv\ & \neg p \lor \neg p \lor q \lor r & \text{[Commutative law]} \\
\equiv\ & \neg p \lor (q \lor r) & \text{[Idempotent law]} \\
\equiv\ & p \to (q \lor r) & \text{[Conditional-disjunction equivalence]}
\end{aligned}
$$

2. Show that $(q \land (p \to \neg q)) \to \neg p$ is a tautology using propositional equivalence and the laws of logic. Do not use a truth table.

$$
\begin{aligned}
& (q \land (p \to \neg q)) \to \neg p & \\
\equiv\ & \neg(q \land (p \to \neg q)) \lor \neg p & \text{[Conditional-disjunction equivalence]} \\
\equiv\ & (\neg q \lor \neg(p \to \neg q)) \lor \neg p & \text{[De Morgan's law]} \\
\equiv\ & (\neg q \lor \neg(\neg p \lor \neg q)) \lor \neg p & \text{[Conditional-disjunction equivalence]} \\
\equiv\ & (\neg q \lor (p \land q)) \lor \neg p & \text{[De Morgan's law]} \\
\equiv\ & (\neg p \lor \neg q) \lor (\neg p \lor (p \land q)) & \text{[Distributive law]} \\
\equiv\ & \neg p \lor \neg q \lor ((\neg p \lor p) \land (\neg p \lor q)) & \text{[Distributive law]} \\
\equiv\ & \neg p \lor \neg q \lor (T \land (\neg p \lor q)) & \text{[Negation law]} \\
\equiv\ & \neg p \lor \neg q \lor \neg p \lor q & \text{[Identity law]} \\
\equiv\ & \neg p \lor \neg p \lor \neg q \lor q & \text{[Commutative law]} \\
\equiv\ & \neg p \lor (\neg q \lor q) & \text{[Idempotent law]} \\
\equiv\ & \neg p \lor T & \text{[Negation law]} \\
\equiv\ & T & \text{[Domination law]}
\end{aligned}
$$

3. Create a compound logical expression composed of at least two propositions (p, q, e.g.) that is a contradiction. Show that your expression really is a contradiction.

$$
\begin{aligned}
& (\neg q \land \neg p) \land (p \lor q) \\
\equiv\ & (\neg q \land (p \lor q)) \land (\neg p \land (p \lor q)) && \text{[Distributive law]} \\
\equiv\ & ((\neg q \land p) \lor (\neg q \land q)) \land ((\neg p \land p) \lor (\neg p \land q)) && \text{[Distributive law]} \\
\equiv\ & ((\neg q \land p) \lor F) \land (F \lor (\neg p \land q)) && \text{[Negation law]} \\
\equiv\ & (\neg q \land p) \land (\neg p \land q) && \text{[Identity law]} \\
\equiv\ & \neg q \land (p \land \neg p) \land q && \text{[Associative law]} \\
\equiv\ & \neg q \land F \land q && \text{[Negation law]} \\
\equiv\ & F && \text{[Domination law]}
\end{aligned}
$$

4. Determine whether each of the following propositions is true or false, with explanation.

   (a) $\forall x P(x) \to \exists x P(x)$, where $P(x)$ is an arbitrary predicate function.

   This is necessarily **true** because if $P(x)$ holds for every x in the domain, then there certainly exists an x for which $P(x)$ holds. Only a single solution satisfies the conclusion, and the hypothesis states that every single input satisfies the conclusion.

   (b) $\exists x P(x) \to \forall x P(x)$, where $P(x)$ is an arbitrary predicate function.

   This is not necessarily true and therefore **false** in the general case because the hypothesis only supposes that at least one input exists such that $P(x)$ holds, so it is not certain that every element satisfies the conclusion. While it is true that it is *possible* that every input satisfies the conclusion, the hypothesis only gurantees one.

   (c) $\forall x(2x \geq x)$, where the domain consists of all real numbers.

$$
2x \geq x \implies \begin{cases} 2 \geq 1 & x \neq 0 \\ 2(0) \geq 0 & x = 0 \end{cases}
$$

   The predicate $2x \geq x$ is equivalent to $2 \geq 1$ (which is obviously true) by the division property of equality for $x \in \mathbb{R}$, $x \neq 0$. For the case that $x = 0$, $2(0) \geq 0 \implies 0 \geq 0$, which is true. Therefore, this proposition is **true**.

(d) $\exists n(n^2 < n)$, where the domain consists of all natural numbers (positive integers).

$$\begin{gathered} n^2 < n \implies n^2 - n < 0 \\ \text{Let } f(n) = n^2 - n = n(n-1) \\ \implies f(n) = 0 \text{ for } x \in \{0, 1\} \\ f(2) = 2^2 - 2 = 4 - 2 = 2 > 0 \\ \therefore f(n) \geq 0 \text{ for } n \in \mathbb{N} = \mathbb{Z}^+ \end{gathered}$$

We can rearrange $n^2 < n$ to get the equivalent proposition $n^2 - n < 0$ by the subtraction property of equality. If we declare a function $f(n) = n^2 - n$, we can find the roots to get any possible interval for which $n^2 - n < 0$ and therefore $n^2 < n$, where n would fulfill the proposition. There are only two real roots ($x \in \{0, 1\}$), which is the maximum number for a quadratic function by the fundamental theorem of algebra. Since we are only considering $n \geq 1$ since $n \in \mathbb{N} = \mathbb{Z}^+$, we only need to find whether $f(n)$ is positive or negative after the root $n = 1$, as we know the function won't cross the x-axis again for $x > 1$. Accordingly, $f(2) > 0$, so a natural number does not exist such that the proposition will hold, meaning it is **false**.

(e) $\exists! x(x^3 = -1)$, where the domain consists of all real numbers.

$$x^3 = -1 \implies x = \sqrt[3]{-1} = -1$$

The quanitfied proposition can be rearranged algebraically to yield a single solution $x = -1$. We also know that this because the function $f(x) = x^3$ is an injective function i.e. a one-to-one mapping of elements $f : \mathbb{R} \to \mathbb{R}$. If we look at the graph of f, it passes a "horizontal line test," where every single number y is only the output that corresponds to a single input x. Therefore, there exists a unique x that fulfills the predicate, so the proposition is **true**.

5. Translate the following English statements to logical expressions, introducing notation as needed for predicates. Then, give a negation of the logical expression as well as a negation of the English statement.

   (a) Some of the students in the class are not here today.

   $$\exists s(\neg P(s))$$

   The domain of s is the set of students in the class.
   $P(s)$: s is present today.

   $$\neg[\exists s(\neg P(s))] = \forall s P(s)$$

   "Every student in the class is here today."

   (b) The number $\sqrt{x}$ is rational if x is an integer.

   $$\begin{aligned} &\forall x(\sqrt{x} \in \mathbb{Q}) && x \in \mathbb{Z} \\ \neg[&\forall x(\sqrt{x} \in \mathbb{Q})] = \exists x(\sqrt{x} \notin \mathbb{Q}) && x \in \mathbb{Z} \end{aligned}$$

   "There is an integer x such that $\sqrt{x}$ is irrational"