MATH 381 Section 1.5

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Nested Quantifiers

$$\forall x \exists y Q(x, y)$$

x and y are free variables within $Q(x, y)$

Example Let $Q(x, y) : x + y = 0; x, y \in \mathbb{R}$
What are the truth values for the quantifiers?

$\exists y \forall x Q(x, y) =$ true: There exists a y so that regardless of x, we have $y + x = 0$.
$\forall x \exists y Q(x, y) =$ true: For any x there exists a y so that $x + y = 0$.

Example Let $Q(x, y, z)$ be $x + y = z$; $x, y, z \in \mathbb{R}$.
Find truth values

1. $\forall x \forall y \exists z Q(x, y, z)$ is True
2. $\exists z \forall x \forall y Q(x, y, z)$ is False

The negation of nested qualifiers

1.
$$\neg(\forall x \forall y \exists z Q(x, y, z)) = F$$
$$\equiv \exists x \exists y \forall z (\neg Q(x, y, z))$$
$$\equiv \exists x \exists y \forall z (x + y \neq z)$$

2.
$$\exists z \forall x \forall y Q(x, y, z) = F$$
$$\neg(\exists z \forall x \forall y Q(x, y, z)) \equiv T$$
$$\equiv \forall z \exists x \exists y \neg Q(x, y, z)$$

Take $f : \mathbb{R} \to \mathbb{R}$ and $A, L \in \mathbb{R}$.

$$\lim_{x \to A} f(x) = L$$

Definition For any $\epsilon > 0$ (consider ϵ to be small) there exists a $\delta > 0$ so that if $|x - A| < \delta$ then $|f(x) - L| < \epsilon$. [delta becomes a function of epsilon]

Example Express the definition of a limit of a real-valued function as quantifiers. Then, negate it.

1. The definition of $\lim_{x \to A} f(x) = L$ is

$$\forall \epsilon > 0 \; \exists \delta_\epsilon > 0 \text{ so that } (|x - A| < \delta \to |f(x) - L| < \epsilon)$$

2. Write $\lim_{x \to A} f(x)$ does not exist

   (a) $\lim_{x \to A} f(x)$ exists?

$$\exists L \; \forall \epsilon > 0 \; \exists \delta_\epsilon > 0 \; (|x - A| < \delta \to |f(x) - L| < \epsilon)$$

   (b) $\lim_{x \to A} f(x)$ does not exist if

$$\neg(\exists L \; \forall \epsilon > 0 \; \exists \delta_\epsilon > 0 \; (|x - A| < \delta \to |f(x) - L| < \epsilon))$$

$$\equiv \forall L \; \exists \epsilon > 0 \; \forall \delta_\epsilon > 0 \; \neg(|x - A| < \delta \to |f(x) - L| < \epsilon)$$
$$\equiv \forall L \; \exists \epsilon > 0 \; \forall \delta_\epsilon > 0 \; (|x - A| < \delta \land |f(x) - L| \geq \epsilon)$$

Example Mean Value Theorem

Any Theorem is a conditional statement.

$$p \to q$$

p is the hypothesis, q is the conclusion.
To prove a Theorem is false i.e. find a counterexample i.e. find an f(x) within hypothesis where the conclusion does not hold.

Example Every non-zero real number has a multiplicative inverse.

$$\forall x \; \exists y \; (x \neq 0 \implies xy = 1); \quad x, y \in \mathbb{R}$$

Definition

1. A ring is a set with an addition operation where the additive inverse exists for every member and is itself a member.

2. A field is a set with a multiplication operation where the multiplicative inverse exists for every member and is itself a member..

- $(\mathbb{R}, +, dot)$ is a ring and a field
- $(\mathbb{Q}, +, dot)$ is a ring and a field
- $(\mathbb{C}, +, dot)$ is a ring and a field
- $(\mathbb{Z}, +, dot)$ is a ring