MATH 381 Special Topics

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$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges}$$

Let $s \in \mathbb{R}$.

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \begin{cases} \text{converges if } s > 1 \\ \text{diverges if } s \leq 1 \end{cases}$$

Let $s \in \mathbb{C}$. $\zeta : \mathbb{C} \to \mathbb{C}$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The zeta function extends the real function f by analytic continuation i.e. $\zeta(s) = f(s)$ wherever $s \in \mathbb{R} \quad s > 1$.

Theorem 0.1 *There is exactly one way to extend over $\mathbb{C}$ the real function $f(s) \quad s > 1$.*

Basel Problem - Bernoulli

Euler proved

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

with which we can actually compute

$$\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \qquad \begin{matrix} s > 1 \\ s \in \mathbb{Z} \end{matrix}$$

$$\sin : \mathbb{R} \to \mathbb{R}$$

has a Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$R = \infty$$

$$f(x) \simeq \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} \qquad \begin{array}{r} |x - x_0| < R \\ x_0 \in \mathbb{R} \end{array}$$

$$\begin{aligned} i^2 &= -1 \\ e^{i\pi} &= -1 \qquad \text{Euler's formula} \\ e^{i\theta} &= \cos\theta + i \sin\theta \end{aligned}$$

Take $P(x) : \mathbb{R} \to \mathbb{R}$

$$P(a) = 0 \text{ for } a \in \mathbb{R}$$
$$\text{then } P(x) = (x - a)Q(x)$$

$\sin x$ has $x = k\pi \qquad k \in \mathbb{Z}$

$$\sin(k\pi) = 0$$

$$\sin x = (x - \pi)\frac{\sin x}{x - \pi}$$

$$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = \frac{\cos(\pi)}{1} = -1$$

$$h(x) = \begin{cases} \dfrac{\sin x}{x - \pi}, & x \neq \pi \\ -1, & x = \pi \end{cases}$$

$$\frac{\sin x}{x(x - \pi)(x + \pi)}$$

$$\frac{\sin x}{\prod_{k \in \mathbb{Z}} (x - k\pi)} = 1 \qquad k \in \mathbb{Z}$$

$$\frac{1}{-k\pi}(x - k\pi) = 1 - \frac{x}{k\pi} \xrightarrow{k \to \infty} 1 - 0 = 1$$

$$\frac{\sin x}{(1+\frac{x}{2\pi})(1+\frac{x}{\pi})x(1-\frac{x}{\pi})(1-\frac{x}{2\pi})} \to 1$$

$$\begin{aligned}
\sin x &= x \prod_{k\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{x}{k\pi}\right) \\
&= x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) \\
&= x - \frac{x^3}{\pi^2}(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}) + +\frac{x^5}{\pi^4}(\dots)
\end{aligned}$$

Coeffiencts of x^3 in $R_k(x) = x\prod_{n=1}^{k}\left(1-\dfrac{x^2}{n^2\pi^2}\right)$

1.
$$\sum_{n=1}^{k}\frac{1}{n^2\pi^2} = -\frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$$

2.
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$-\frac{1}{3!} = -\frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Bernoulli Numbers

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

$$\begin{aligned}
S_0(n) &= n && k=0 \\
S_1(n) &= \frac{n(n+1)}{2} && k=1 \\
S_2(n) &= \frac{n(n+1)(2n+1)}{6} && k=2 \\
S_3(n) &= \left(\frac{n(n+1)}{2}\right)^2 && k=3 \\
S_4(n) &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{20} && k=4 \\
S_5(n) &= \frac{n^2(n+1)^2(2n^2+2n-1)}{12} && k=5
\end{aligned}$$

Faulhaber calculated all numbers $k \leq 17$.

Bernoulli Formula

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} B_j \cdot n^{k+1-j}$$

B_j = Bernoulli numbers

$$\begin{aligned} B_0 &= 1 \\ B_1 &= \frac{-1}{2} \\ B_2 &= \frac{1}{6} \\ B_3 &= 0 \\ B_4 &= \frac{-1}{30} \\ B_5 &= 0 \end{aligned}$$

$B_{\text{odd}} = 0$ besides $k = 1$
Numerators of $B_{2n} = 1, -1, -1, 1, -1, 5$
Denominators of $B_{2n} = 1, 6, 30, 42, 30, 66$

$$\sum_{m=1}^{n} (m^k - (m-1)^k) = n^k$$

$$m^k - (m-1)^k = \binom{k}{1} m^{k-1} + \binom{k}{2} m^{k-2} + \cdots + (-1)^{k+1} m^0$$

$$\sum_{m=1}^{n} \left[\binom{k}{1} m^{k-1} 0 \binom{k}{2} m^{k-2} + \cdots + (-1)^{k+1} m^0 \right] = n^k$$

$$
\begin{aligned}
&k=0 \qquad S_0 = n \\
&k=1 \qquad S_0 = n \\
&k=2 \qquad \sum_{m=1}^{n}\left[\binom{2}{1}m - \binom{2}{2}1\right] = n^2 \\
&\qquad\qquad \sum_{m=1}^{n}(2m-1) = 2S_1 - S_0 = n^2 \\
&k=3
\end{aligned}
$$

Taylor Series

$$
\begin{aligned}
\sin x &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \\
\cos x &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \\
\tan x &= \sum_{k=1}^{\infty}(-1)^k b_{2k}(1-2^k)\frac{(2x)^{2k-1}}{(2k)!} \\
x\cot x &= 1 + \sum_{k=1}^{\infty}(-1)^k b_{2k}\frac{(2x)^{2k}}{(2k)!}
\end{aligned}
$$

Euler

$$
\begin{aligned}
\sin x &\cong A\prod_{k\in\mathbb{Z}}(x-k\pi) \\
&= x\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{x}{k\pi}\right) \\
&= x\prod_{k\in\mathbb{N}}\left(1-\frac{x^2}{k^2\pi^2}\right) \\
&= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots \\
&= x\left[1-x^2\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\dots\right)\right]
\end{aligned}
$$

Euler

$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$