MATH 381 Section 1.4

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Predicates and Quantifiers

Definition **Predicates** are statements involving equations and inequalities.

$$x > 3 \quad x = y + 3 \quad x + y = z$$

Example $P(x)$: the statement ”x ¿ 3” $P(4)$: ”4 ¿ 3” -¿ true $P(2)$: 2 ¿ 3 -¿ false

Example $R(x, y, z)$: ”x + y = z” $R(1, 2, 3)$: ”1 + 2 = 3” -¿ true $R(0, 0, 1)$: 0 + 0 = 1 -¿ false

Definition

1. The notation $\forall x P(x)$ denotes the **universal quantifier** of P(x) i.e. “$P(x)$ holds for all x in the domain.”

2. The notation $\exists x P(x)$ is the **existential quantifier** of P(x) i.e. “there exists an element x in the domain so that P(x) holds.”

Remark The domain must always be specified.

- there is x so that $P(x)$
- there is at least one x so that $P(x)$
- for some x $P(x)$

$\exists!$ means “there exists a unique element.”

Example $P(x)$: statement $x + 1 > x$
What are the truth values for the quantifications? [domain = $\mathbb{R}$]

- $\forall x P(x)$ is true.
- $\exists x P(x)$ is true.

When are the universal and existential quantifiers false?

- $\forall x P(x)$ is false when not every x makes $P(x)$ hold.
- $\exists x P(x)$ is true when every x does not make $P(x)$ hold.

Example $Q(x)$: $x < 2$
What is the truth value for $\forall x Q(x)$ where $x \in \mathbb{R}$?
False for e.g. $x = 3, x = 4, x = 5$

Example $P(x)$ is $x^2 > x$. $[x \in \mathbb{R}]$

- $\forall x P(x)$ is false e.g. $x = 1$.
- $\exists x P(x)$ is true.

Example What are the truth values for $p : \forall x(x^2 \geq x)$?

1. $x \in \mathbb{Z} \implies p \equiv T$
2. $x \in \mathbb{N} \implies p \equiv T$
3. $x \in \mathbb{Q} \implies p \equiv T$

Example $Q(x)$: $x = x + 1$

Quantifiers over finite domains

$$x \in \{x_1, x_2, \ldots, x_n\}$$

$$\begin{aligned}
\forall x P(x) &= P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n) \\
&= \bigwedge i = 1 n P(x_i) \\
\exists x P(x) &= P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n) \\
&= \bigvee i = 1 n P(x_i)
\end{aligned}$$

Example $P(x)$: $x^2 < 10 \quad x \in \{1, 2, 3, 4\}$

- $\forall x P(x)$ is false e.g. $x = 4$.
- $\exists x P(x)$ is true e.g. $x = 1$.

Negation of quantifiers in finite domains

1. Negation of universal quantifier

$$\begin{aligned} \forall x P(x) &= \bigwedge_{i=1}^{n} P(x_i) \\ \neg(\forall x P(x)) &= \neg(\bigwedge_{i=1}^{n} P(x_i)) \\ &= \bigvee_{i=1}^{n} \neg P(x_i) \\ &= \neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n) \end{aligned}$$

2. Negation of existential quantifier

$$\begin{aligned} \exists x P(x) &= \bigvee_{i=1}^{n} P(x_i) \\ \neg(\exists x P(x)) &= \neg(\bigvee_{i=1}^{n} P(x_i)) \\ &= \bigwedge_{i=1}^{n} \neg P(x_i) \\ &= \neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n) \end{aligned}$$

Negation of quantifiers in finite and infinite domains

1. Negation of universal quantifier

$$\neg(\forall x P(x)) \text{ is true if } \exists x \in \text{Domain so that } \neg P(x).$$

2. Negation of existential quantifier

$$\neg(\exists x P(x)) \text{ is true if } \forall x \in \text{Domain so that } \neg P(x).$$

Example Show $\exists x(P(x) \wedge \neg Q(X)) \equiv \neg(\forall x P(x) \to Q(x))$

Example

$$\begin{aligned} \neg(\forall x(x^2 > x)) &\equiv \exists x(x^2 \le x) = false \\ \neg(\exists x(x^2 = 2)) &\equiv \forall x(x^2 \ne 2) = false \end{aligned}$$