MATH 381 Homework 5

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1. Suppose that x and y are real numbers. Prove that if $x+y$ is irrational then x is irrational or y is irrational.

$$
\begin{aligned}
&x, y \in \mathbb{R} \\
&p: \quad x+y \notin \mathbb{Q} \\
&q: \quad x \notin \mathbb{Q} \vee y \notin \mathbb{Q}
\end{aligned}
$$

$$p \to q \equiv \neg q \to \neg p$$

$$
\begin{aligned}
\neg p: &\quad x+y \in \mathbb{Q} \\
\neg q: &\quad x \in \mathbb{Q} \wedge y \in \mathbb{Q}
\end{aligned}
$$

Suppose $\neg q$.

$$
\begin{aligned}
x = \frac{a}{b} \quad y = \frac{c}{d} & \qquad a, b, c, d \in \mathbb{Z} \quad b, d \neq 0 \\
x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd} = \frac{e}{f} & \qquad e, f \in \mathbb{Z} \quad f \neq 0 \\
\therefore \frac{e}{f} = x + y \in \mathbb{Q} \quad \blacksquare &
\end{aligned}
$$

Therefore, $\neg q \to \neg p \equiv p \to q$.

2. Show that any integer is a multiple of 3 if and only if its square is a multiple of 3.

   $P(n)$: "n is a multiple of 3" $\equiv \exists k \in \mathbb{Z}(n = 3k)$
   $Q(n)$: "n^2 is a multiple of 3" $\equiv \exists m \in \mathbb{Z}(n^2 = 3m)$

   $$\begin{aligned} &\forall n \in \mathbb{Z}(P(n) \leftrightarrow Q(n)) \\ &\qquad \equiv \forall n \in \mathbb{Z}(P(n) \to Q(n)) \land \forall n \in \mathbb{Z}(Q(n) \to P(n)) \end{aligned}$$

   Suppose $P(n)$.

   $$n = 3k \quad k \in \mathbb{Z}$$
   $$n^2 = (3k)^2 = (3^2k^2) = 3(3k^2) = 3m \quad m = 3k^2 \in \mathbb{Z}$$

   $\therefore Q(n)$
   Suppose $Q(n)$.

   $$n^2 = 3m \quad m \in \mathbb{Z}$$
   $$\sqrt{n^2} = \sqrt{3m} = \sqrt{3}\sqrt{m}$$

   Since n^2 is a square of an integer:

   $$\begin{aligned} &\sqrt{n^2} \in \mathbb{Z}_+ \\ &\implies \sqrt{3}\sqrt{m} \in \mathbb{Z}_+ \\ &\implies \sqrt{3}\sqrt{3}\sqrt{m/3} \in \mathbb{Z}_+ \\ &\implies 3\sqrt{m/3} \in \mathbb{Z}_+ \\ &\implies \sqrt{m/3} \in \mathbb{Z}_+ \because 3 \in \mathbb{Z}_+ \\ &\therefore \sqrt{n^2} = n = 3k \quad \sqrt{m/3} = k \in \mathbb{Z}_+ \end{aligned}$$

   $\therefore P(n) \quad \blacksquare$

3. Using the definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, prove that for all real numbers x and y, $|x| \leq y$ if and only if $-y \leq x \leq y$.

$P(x, y)$: $|x| \leq y$
$Q(x, y)$: $-y \leq x \leq y$

$$\forall x, y \in \mathbb{R}(P(x, y) \leftrightarrow Q(x, y))$$

Suppose $P(x, y)$.

$$|x| \leq y = \begin{cases} x \leq y & \text{if } x \geq 0 \\ -x \leq y & \text{if } x < 0 \end{cases} = \begin{cases} x \leq y & \text{if } x \geq 0 \\ x \geq -y & \text{if } x < 0 \end{cases}$$

$$\begin{aligned} &\iff (x \leq y \wedge x \geq 0) \vee (x \geq -y \wedge x < 0) \\ &\iff (0 \leq x \leq y) \vee (-y \leq x < 0) \\ &\iff x \in [0, y] \vee x \in [-y, 0) \\ &\iff x \in [0, y] \cup [-y, 0) \\ &\iff x \in [-y, y] \\ &\iff -y \leq x \leq y \end{aligned}$$

$\therefore Q(x, y)$ $\blacksquare$

4. Prove that if $ab > 0$ and $bc < 0$, then $ax^2 + bx + c = 0$ has two real solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic equation has two real solutions when $b^2 - 4ac > 0 \iff b^2 > 4ac$.

$$(ab > 0) \rightarrow (a > 0 \wedge b > 0) \vee (a < 0 \wedge b < 0)$$
$$(bc < 0) \rightarrow (b < 0 \wedge c > 0) \vee (b > 0 \wedge c < 0)$$

There are two possibilities, since a and c are dependent on the value of b to fulfill their inequalities.
If b is negative, then a must be negative as well, and c must be positive.
If b is positive, then a must be positive as well, and c must be negative.

$$(b > 0) \rightarrow (a > 0 \wedge c < 0)$$
$$(b < 0) \rightarrow (a < 0 \wedge c > 0)$$

Assume $b < 0$.

$$b^2 > 0$$
$$a < 0 \implies 4ac < 0$$
$$\therefore b^2 > 4ac$$

Assume $b > 0$.

$$b^2 > 0$$
$$c < 0 \implies 4ac < 0$$
$$\therefore b^2 > 4ac$$

$$(ab > 0 \wedge bc < 0) \rightarrow (b^2 - 4ac > 0) \quad \blacksquare$$

5. Prove that there are no positive integer solutions to $x^2 + x + 1 = y^2$.

$$\neg(\exists x \in \mathbb{Z}_+ \exists y \in \mathbb{Z}_+ (x^2 + x + 1 = y^2))$$
$$\forall x \in \mathbb{Z}_+ \forall y \in \mathbb{Z}_+ (x^2 + x + 1 \neq y^2)$$

Assume $x, y \in \mathbb{Z}_+$.

$$\begin{aligned} y^2 - x^2 &= x + 1 \\ \implies (y+x)(y-x) &= x + 1 \end{aligned}$$

There are three possible cases for x and y

(a) $y = x$

$$\begin{aligned} (y+x)(y-x) &= x + 1 \\ \implies 2y(0) &= y + 1 \\ 0 &= y + 1 \equiv F \because y > 0 \end{aligned}$$

(b) $y < x$

$$\begin{aligned} &(y+x)(y-x) = x + 1 \\ &(y+x)(y-x) < 0 \because (y+x) > 0 \land (y - x < 0 \iff y < x) \\ &x + 1 > 0 \because x > 0 \end{aligned}$$

$\therefore (y+x)(y-x) = x + 1 \equiv F$

(c) $y > x$

$$\begin{aligned} x^2 + x + 1 &= y^2 \\ x^2 + x + \frac{1}{4} + 1 - \frac{1}{4} &= y^2 \\ (x + \frac{1}{2})^2 + \frac{3}{4} &= y^2 \\ (x + \frac{1}{2})^2 + \frac{3}{4} &\notin \mathbb{Z} \\ \implies (x + \frac{1}{2})^2 + \frac{3}{4} &= y^2 \equiv F \quad \blacksquare \end{aligned}$$

Since in all three cases for positive integer solutions or x and y the proposition we are trying to prove implies a contradiction, the assumption that there are positive integer solutions must be false.

6. Show that if you choose 92 different dates from a calendar, at least 14 of the chosen dates must occur on the same day of the week.

   r: "If you choose 92 different dates from a calendar."
   p: "At least 14 of the 92 chosen dates must occur on the same day of the week."
   $\neg p$: "At most 13 of the 92 chosen days fall on the same day of the week."
   Suppose $\neg p$. Since there are 7 days in a week, 13 on each day would give us 91 total chosen days. Any more days chosen necessarily means that there would be a fourteenth date on a given day of the week. But we have to choose 92 dates. This suggests $\neg p \to (r \wedge \neg r)$ is true. In other words, if p was false, it would imply a contradiction. Therefore, p must be true, and p is dependent on r for the number of chosen dates that was agreed upon (92). So, $r \to p$ is true. ■

7. Show that there is a three-digit number less than 400 with distinct digits such that the sum of the digits is 17 and the product of the digits is 108. Is there a unique such number? Explain.

$$n = 100n_2 + 10n_1 + 1n_0 < 400$$
$$n_2 + n_1 + n_0 = 17$$
$$n_2 \cdot n_1 \cdot n_0 = 108$$
$$108 = 12 \cdot 9 = 3 \cdot 4 \cdot 3^2 = 2^2 \cdot 3^3$$
$$n_1, n_2, n_3 \neq 0 \because n_2 \cdot n_1 \cdot n_0 = 108 \neq 0$$

i	j	2^i	3^j	$2^i \cdot 3^j$
0	0	1	1	*1
0	1	1	3	*3
0	2	1	9	*9
0	3	1	27	27
1	0	2	1	*2
1	1	2	3	*6
1	2	2	9	18
1	3	2	27	54
2	0	4	1	*4
2	1	4	3	12
2	2	4	9	36
2	3	4	27	108

$$n_1, n_2, n_3 \in \{1, 2, 3, 4, 6, 9\}$$
$$6 + 3 + 4 = 13 < 17 \implies (n_1 = 9) \vee (n_2 = 9) \vee (n_3 = 9)$$
$$17 - 9 = 8 = 6 + 2$$
$$n_1, n_2, n_3 \in \{2, 6, 9\}$$
$$n_2 \neq 9 \because n \geq 900 > 400$$
$$n_2 \neq 6 \because n \geq 600 > 400$$
$$\implies n_2 = 2 (n_2 = 2 \wedge n_1 = 6 \wedge n_0 = 9) \rightarrow (n = 269)$$
$$(n = 269) \rightarrow (n < 400)$$
$$\therefore (n_2 = 2 \wedge n_1 = 6 \wedge n_0 = 9) \rightarrow (n < 400)(n_2 = 2 \wedge n_1 = 9 \wedge n_0 = 6) \rightarrow (n = 296)$$
$$(n = 296) \rightarrow (n < 400)$$
$$\therefore (n_2 = 2 \wedge n_1 = 9 \wedge n_0 = 6) \rightarrow (n < 400) \quad \blacksquare$$

Yes, there is such a number. One such number is 269. Its digits were found by finding combinations of factors in the prime factorization in 108 that could possibly add to 17. 9 must be a digit to reach 17 as a sum, and the only other factors that add to 8 are 2 and 6. Since 6 and 9 as the first digit would make the number greater than 400, 2 must be the first digit, and a possible number was revealed to be 269 which is indeed less than 400. However, another possible number that fits the restrictions is 296, which is formed by interchanging the last two digits. Therefore, there does exist a number, **but it is not unique**, as there are two possible numbers that fit the requirements.

8. Without trying to evaluate these numbers, show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{2121} - 9^{2399} + 2^{2001}$, and $24^{4493} + 5^{8192} + 7^{1777}$ is non-negative.

$$n_1 = 65^{1000} - 8^{2001} + 3^{177}$$
$$n_2 = 24^{4493} + 5^{8192} + 7^{1777}$$

$$(n_1 * n_2 > 0) \leftrightarrow (n_1 > 0 \land n_2 > 0) \lor (n_1 < 0 \land n_2 < 0) \tag{1}$$
$$n_2 > 0 = T \because (24^{4493} > 0) \land (5^{8192} > 0) \land (7^{1777} > 0) \tag{2}$$
$$\therefore (n_1 < 0 \land n_2 < 0) = F,\ (n_1 > 0 \land n_2 > 0) \equiv (n_1 > 0) \tag{3}$$
$$\implies (n_1 * n_2 > 0) \leftrightarrow (n_1 > 0) \tag{4}$$

$$65^{1000} > 8^{2001} \implies 65^{1000} + 3^{177} > 8^{2001} \tag{5}$$
$$\iff 65^{1000} - 8^{2001} + 3^{177} > 0 \tag{6}$$

p: $65^{1000} - 8^{2001} + 3^{177} > 0$
q: $65^{1000} + 3^{177} > 8^{2001}$
r: $65^{1000} > 8^{2001}$
s: $(n_1 * n_2 > 0)$

$$\begin{array}{ll} r \to q & \text{(From equation 5)} \\ q \leftrightarrow p & \text{(From equation 6)} \\ p \leftrightarrow s & \text{(From equation 4)} \\ & \therefore r \to s \end{array}$$

$$\begin{aligned} & 65^{1000} > 8^{2001} \\ & = (64+1)^{1000} > 8^{2(1000.5)} \\ & = (64+1)^{1000} > (8^2)^{1000.5} \\ & = (64+1)^{1000} > (64)^{1000.5} \\ & = (64+1)^{1000} > 64^{0.5}(64)^{1000} \\ & = (64+1)^{1000} > 8(64)^{1000} \\ & = \sqrt{65}(64+1)^{999.5} > 8(64)^{1000} \\ & = \sqrt{65}(64+1)^{999.5} > 8(64)^{1000} \end{aligned}$$

$$\sqrt{65} > \sqrt{64} = 8$$

$$\therefore (8(64+1)^{999.5} > 8(64)^{1000}) \to (\sqrt{65}(64+1)^{999.5} > 8(64)^{1000})$$

$$\begin{aligned}
&8(64+1)^{999.5} > 8(64)^{1000} \\
&\iff (64+1)^{999.5} > (64)^{1000} \\
&\iff 999.5 \ln(65) > 1000 \ln(64) \\
&\iff \frac{\ln(65)}{\ln(64)} > \frac{1000}{999.5} \quad [1.003727969 > 1.00050025] \quad \blacksquare
\end{aligned}$$

Since we have found a demonstrably true statement ($\frac{\ln(65)}{\ln(64)} > \frac{1000}{999.5}$) and it is logically equivalent to $8(64+1)^{999.5} > 8(64)^{1000}$, this statement must be true, which implies that $\sqrt{65}(64+1)^{999.5} > 8(64)^{1000}$ is true. Since this statement is ultimately equivalent to our original r, r must be true. Finally, we showed that the biconditional statement $r \leftrightarrow s$ is true, so s our original statement that the product of the two numbers we chose is positive is also true.