MATH 381 HW 8

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1. Prove that $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all positive integers n.

   (a) Basis step:

$$\begin{aligned}
P(1):&\quad 1^3 = \frac{1^2(1+1)^2}{4} &&\iff 1 = 1\\
P(2):&\quad 1^3 + 2^3 = \frac{2^2(2+1)^2}{4} &&\iff 9 = 9\\
P(3):&\quad 1^3 + 2^3 + 3^3 = \frac{3^2(3+1)^2}{4} &&\iff 36 = 36
\end{aligned}$$

   (b) Induction step; Assume $P(k)$.

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 &= \frac{k^2(k+1)^2}{4}\\
\implies 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3\\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}\\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\
&= \frac{(k+1)^2(4(k+1) + k^2)}{4}\\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4}\\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

$\therefore P(k) \to P(k+1) \quad \blacksquare$

2. Prove that for every positive integer n, $3^{2n} - 1$ is divisible by 8.

(a) Basis step:

$$\begin{aligned} P(1): &\quad 8 \mid 3^{2\cdot 1} - 1 \iff 8 \mid 3^2 - 1 \iff 8 \mid 8 \\ P(2): &\quad 8 \mid 3^{2\cdot 2} - 1 \iff 8 \mid 3^4 - 1 \iff 8 \mid 80 \end{aligned}$$

$\therefore P(1), P(2)$

(b) Induction step:
Assume $P(k)$.

$$8 \mid 3^{2k} - 1 \iff \exists q \in \mathbb{Z} \left(\frac{3^{2k} - 1}{8} = q \right)$$

$$\begin{aligned} q &= \frac{3^{2k} - 1}{8} \\ 8q &= 3^{2k} - 1 \\ 8 \cdot 3^{2k} + 8q &= 3^{2k} - 1 + 8 \cdot 3^{2k} \\ 8(q + 3^{2k}) &= 9(3^{2k}) - 1 \\ 8(q + 3^{2k}) &= 3^2 \cdot 3^{2k} - 1 \\ 8(q + 3^{2k}) &= 3^{2k+2} - 1 \\ 8(q + 3^{2k}) &= 3^{2(k+1)} - 1 \\ q + 3^{2k} &= \frac{3^{2(k+1)} - 1}{8} \end{aligned}$$

$$\begin{aligned} k \in \mathbb{N} \wedge q \in \mathbb{Z} &\implies q + 3^{2k} \in \mathbb{Z} \\ &\implies 8 \mid 3^{2(k+1)} - 1 \end{aligned}$$

$\therefore P(k) \to P(k+1) \quad \blacksquare$

3. Prove that $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer for every positive integer n.

(a) Basis step:

$$\begin{aligned} P(1) :& \frac{1^3}{3} + \frac{1^5}{5} + \frac{7}{15} \in \mathbb{Z} \\ \iff & \frac{1}{3} + \frac{1}{5} + \frac{7}{15} \in \mathbb{Z} \\ \iff & \frac{5}{15} + \frac{3}{15} + \frac{7}{15} \in \mathbb{Z} \\ \iff & \frac{15}{15} \in \mathbb{Z} \\ \iff & 1 \in \mathbb{Z} \end{aligned}$$

$\therefore P(1)$

(b) Induction step:
Assume $P(k)$.

$$\begin{aligned} & \frac{k^3}{3} + \frac{k^5}{5} + \frac{7k}{15} \in \mathbb{Z} \\ \iff & \frac{5k^3}{15} + \frac{3k^5}{15} + \frac{7k}{15} \in \mathbb{Z} \\ \iff & \frac{3k^5 + 5k^3 + 7k}{15} \in \mathbb{Z} \\ \iff & 15 \mid 3k^5 + 5k^3 + 7k \end{aligned}$$

We want $P(k+1)$.

$$\begin{aligned} P(k+1) :& \frac{(k+1)^3}{3} + \frac{(k+1)^5}{5} + \frac{7(k+1)}{15} \in \mathbb{Z} \\ \iff & 15 \mid 3(k+1)^5 + 5(k+1)^3 + 7(k+1) \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} & 3(k+1)^5 + 5(k+1)^3 + 7(k+1) \\ & = (3k^5 + 15k^4 + 30k^3 + 30k^2 + 15k + 3) + (5k^3 + 15k^2 + 15k + 5) + (7k + 7) \\ & = 3k^5 + 15k^4 + 35k^3 + 45k^2 + 37k + 15 \\ & = (3k^5 + 5k^3 + 7k) + 15k^4 + 30k^3 + 45k^2 + 30k + 15 \\ & = (3k^5 + 5k^3 + 7k) + 15(k^4 + 2k^3 + 3k^2 + 2k + 1) \end{aligned}$$

$$k^4 + 2k^3 + 3k^2 + 2k + 1 \in \mathbb{Z} \implies 15 \mid 15(k^4 + 2k^3 + 3k^2 + 2k + 1)$$
$$P(k) : 15 \mid 3k^5 + 5k^3 + 7k \equiv T$$
$$\implies 15 \mid (3k^5 + 5k^3 + 7k) + 15(k^4 + 2k^3 + 3k^2 + 2k + 1)$$
$$\therefore 15 \mid 3(k+1)^5 + 5(k+1)^3 + 7(k+1)$$

$\therefore P(k) \to P(k+1)$ ■

4. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a horizontal or vertical line separating the squares. Assuming that only one piece can be broken at a time, prove that $n-1$ successive breaks are needed to separate the bar into its n separate squares, using strong induction.

   (a) Basis step

   A whole chocolate bar (1 piece) when divided (1 break) leaves 2 pieces.

   $$m = 2 \text{ pieces from } m - 1 = 2 - 1 = 1 \text{ pieces}$$

   $\therefore P(2)$

   (b) Induction step

   Assume $P(j) \quad 1 \leq j \leq n$. Upon the first break of the bar, two rectangles are left, each composed of a and b squares, respectively.

   $$a + b = n \qquad a < n, \quad b < n$$

   Since we assumed $P(j)$, the breaks needed to separate the first part into its a squares is $a-1$ and similarly $b-1$ breaks for the second's b squares. But we also must take into account the break we did in order to separate a and b.

   $$1 + (a-1) + (b-1) = (a+b) - 1 = n - 1 \quad ■$$

   Therefore, $n-1$ breaks are necessary to separate a bar into its n squares.

Let f_n be the nth Fibonacci number, defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$.

5. Show that $f_{n+6} = 4f_{n+3} + f_n$ for all positive integers n.

$$\begin{aligned}
f_{n+6} &= f_{n+5} + f_{n+4} \\
&= f_{n+4} + f_{n+3} + f_{n+4} \\
&= f_{n+3} + f_{n+2} + f_{n+3} + f_{n+3} + f_{n+2} \\
&= 3f_{n+3} + f_{n+2} + f_{n+2} \\
&= 3f_{n+3} + f_{n+2} + f_{n+1} + f_n \\
&= 4f_{n+3} + f_n
\end{aligned}$$

$\therefore f_{n+6} = 4f_{n+3} + f_n \quad \blacksquare$

6. Make and prove a conjecture about sums of the form $f_2 + f_4 + f_6 + \cdots + f_{2n}$.

$$f = (1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$$

$$\begin{aligned}
f_2 + f_4 &= 1 + 3 &&= 4 = 5 - 1 &&= f_5 - 1 \\
f_2 + f_4 + f_6 &= 1 + 3 + 8 &&= 12 = 13 - 1 &&= f_7 - 1 \\
f_2 + f_4 + f_6 + f_8 &= 1 + 3 + 8 + 21 &&= 33 = 34 - 1 &&= f_9 - 1
\end{aligned}$$

$$\sum_{i=1}^{n} f_{2i} = f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$$

Assume $P(k)$

$$\begin{aligned}
\sum_{i=1}^{k} f_{2i} &= f_{2k+1} - 1 \\
f_{2k+2} + \sum_{i=1}^{k} f_{2i} &= f_{2k+1} - 1 + f_{2k+2} \\
f_2 + f_4 + f_6 + \cdots + f_{2k} + f_{2k+2} &= (f_{2k+2} + f_{2k+1}) - 1 \\
\sum_{i=1}^{k+1} f_{2i} &= f_{2k+3} - 1 \\
\sum_{i=1}^{k+1} f_{2i} &= f_{2(k+1)} - 1
\end{aligned}$$

$\therefore P(k) \to P(k+1) \quad \blacksquare$