Polynomial Interpolation

Problem outline

Suppose we have census data for a town as follows

Years since 2000	Population (000's)
0	9
5	11
10	10
15	15

- We can represent this as a set of points $\{(0,9),(5,11),(10,10),(15,15)\}$.
- Now suppose we want to estimate the population in 2013 (year 13 here). We need to ***interpolate*** the data.
- One way to do this is with a polynomial of degree 1 less than the number of points:

$$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

where x represents the time and $P_3(x)$ the population at that time We want to find the coefficients of the polynomial so that the polynomial:

- matches the census data at each ***interpolation node*** x
- approximates the true population (hopefully) between interpolation nodes.

To find the coefficients we subsitute the times into the equation for the x values and set equal to the population at that time. So for the first data point we have

$$a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 = 9$$

Repeating for each data point we have

$$a_0 + (0)a_1 + (0)^2a_2 + (0)^3a_3 = 9$$
$$a_0 + (5)a_1 + (5)^2a_2 + (5)^3a_3 = 11$$
$$a_0 + (10)a_1 + (10)^2a_2 + (10)^3a_3 = 10$$
$$a_0 + (15)a_1 + (15)^2a_2 + (15)^3a_3 = 15$$

This is simply a linear system of equations to solve for the polynomial coefficients as unknowns. We solve below in matlab but you could use Gaussian elimination

Solving system

Define coefficient matrix B

```
B = [1 0 0 0;...
    1 5 5^2 5^3;...
    1 10 10^2 10^3;...
    1 15 15^2 15^3]
```

```
B = 4×4
           1           0           0           0
           1           5          25         125
           1          10         100        1000
           1          15         225        3375
```

Define constant vector b

```
b = [9;11;10;15]
```

```
b = 4×1
     9
    11
    10
    15
```

Solve system to find coefficient vector a

Here we could use Gaussian elimination and back substitution or Gauss Jordan elimination. Matlab uses a function called ***mldivide*** which analyses the system and chooses the most efficient method of solution. In the form presented we are solving $B\,a = b$

```
a = B\b
```

```
a = 4×1
    9.0000
    1.3000
   -0.2400
    0.0120
```

Define polynomial function with the calculated coefficients

```
P3 = @(x) polyval([a(4),a(3),a(2),a(1)],x)
```

```
P3 = function_handle with value:
    @(x)polyval([a(4),a(3),a(2),a(1)],x)
```

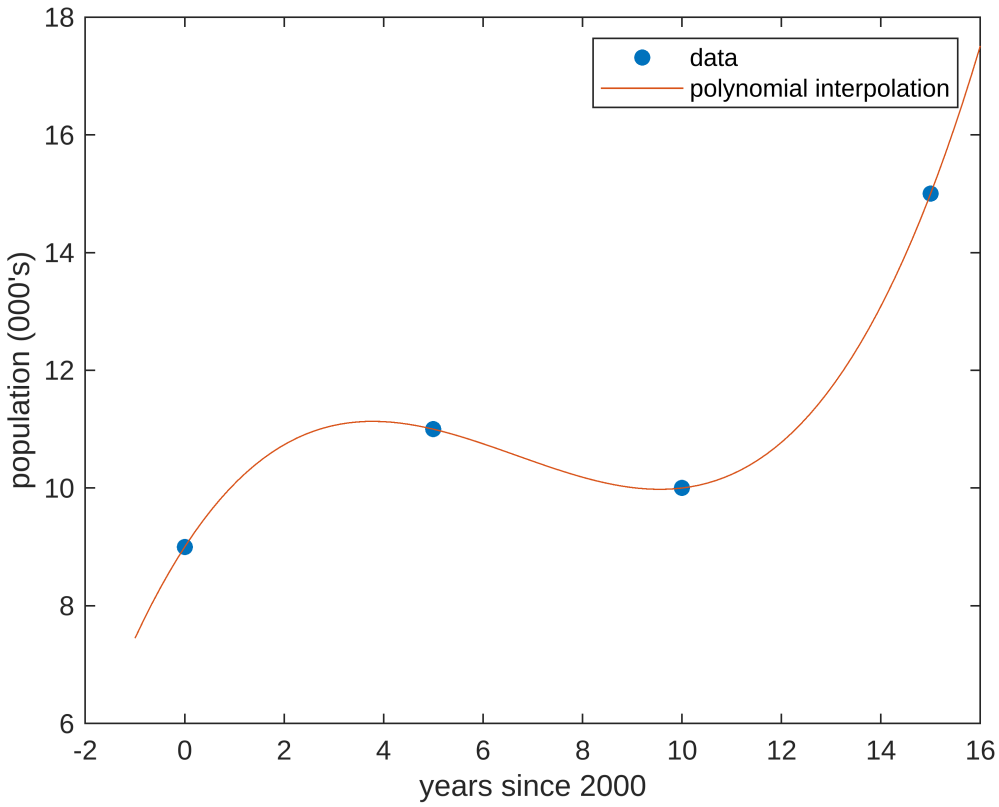
Plot data and interpolation polynomial

```
xdata = [0,5,10,15];
ydata = b;
xval = linspace(-1,16,100);
yval = P3(xval);

close("all")%Reset figure

figure(1)
plot(xdata,ydata,'.',MarkerSize=20)
hold on
```

```
plot(xval,yval,'-')
xlabel("years since 2000")
ylabel("population (000's)")
legend("data","polynomial interpolation")
```



Population estimate in specific year (e.g. 2003)

```
P3(3)
```

```
ans = 11.0640
```

Be careful not to extrapolate from data (e.g. population in 1996 gives negative result)

```
P3(-4)
```

```
ans = -0.8080
```