MS4131 Lecture 17: Planes and Lines (in $\mathbb{R}^2$ and $\mathbb{R}^3$)

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1 Lines in $\mathbb{R}^2$

The general equation of a straight line in $\mathbb{R}^2$ is

$$ax + by + c = 0. \tag{1}$$

Theorem 1.1 *The vector $n = (a, b)$ is orthogonal to the line given by equation (1).*

Distance between a point and a line in $\mathbb{R}^2$

Theorem 1.2 *The* ***distance*** *d between a point $P_0 = (x_0, y_0)$ and the line $l : ax + by + c = 0$ is*

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

2 Planes in $\mathbb{R}^3$

A **plane** in $\mathbb{R}^3$ is determined by one of its points and its slope or inclination, which is given by a **normal** to the plane.

Definition A plane π through the point $P_0 = (x_0, y_0, z_0)$, with normal $n = (a, b, c)$, is the set of points $P = (x, y, z)$ such that

$$n \cdot \overrightarrow{P_0P} = 0.$$

Therefore the equation of π is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation is called the **point-normal form** of the equation of the plane.
This can be rewritten as

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

which is called the **vector form** of the equation of the plane.
And in turn can also be rewritten as

$$ax + by + cz + d = 0,$$

called the **general form** of the equation of the plane.

Theorem 2.1 *If $a, b, c, d \in \mathbb{R}$, with a, b, c not all zero, then*

$$ax + by + cz + d = 0$$

is the equation of a plane with $n = (a, b, c)$ as a normal.

Remark Solving the system

$$ax + by = k_1$$

$$cx + dy = k_2$$

means finding the point(s) $P = (x, y)$ of intersection of two lines in $\mathbb{R}^2$.
Solving the system

$$a_{11}x + a_{12}y + a_{13}z = k_1$$

$$a_{21}x + a_{22}y + a_{23}z = k_1$$

$$a_{31}x + a_{32}y + a_{33}z = k_1$$

means finding the point(s) $P = (x, y, z)$ of intersection of three planes in $\mathbb{R}^3$.

3 Lines in $\mathbb{R}^3$

In $\mathbb{R}^3$, a line is determined by a point and a direction: the direction is simply given by a **nonzero vector**.
Let l denote the line passing through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the nonzero vector $v = (a, b, c)$ (or equivalently with direction $v = (a, b, c)$).
Then l is the set of points $P = (x, y, z)$ such that

$$\overrightarrow{P_0P} \text{ is parallel to } v.$$

i.e. l is the set of points $P = (x, y, z)$ such that

$$\overrightarrow{P_0P} = tv, \text{ for some } t.$$

In terms of components, we have

$$(x - x_0, y - y_0, z - z_0) = (ta, tb, tc), \quad \text{for some } t.$$

Therefore, the equation of the line l is

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad \text{for all } -\infty < t < +\infty.$$

If t varies between $-\infty$ and $+\infty$, then the point $P = (x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc)$ describes the entire line l.

Definition Equations

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad \text{for all } -\infty < t < +\infty.$$

are called the **parametric equations** for the line l.

Distance between a point and a plane in $\mathbb{R}^3$

Theorem 3.1 *The* ***distance*** *D between a point $P_0 = (x_0, y_0, z_0)$ and a plane π: $ax + by_cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Distance between two parallel planes

If π and π' are two parallel planes, then if we choose a point $P_0 \in \pi$ for example, the distance between π and π' is equal to the distance between P_0 and π'.