MS4131 Lecture 18: Eigenvalues and eigenvectors

Kevin Moroney

23 November 2023

If A is an $n \times n$ matrix and x is a vector in $\mathbb{R}^n$ (i.e. x is a $n \times 1$ column vector), then the matrix product Ax is also a vector in $\mathbb{R}^n$ that we can denote by y i.e.

$$Ax = y$$

Therefore A is a **mapping** from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$A : \mathbb{R}^n \to \mathbb{R}^n.$$

$$x \mapsto Ax$$

Remark It makes then sense to consider the following problem:
For which $x \in \mathbb{R}^n$ does there exist a scalar $\lambda \in \mathbb{R}$ such that

$$Ax = \lambda x?$$

What are the possible values of λ?

Eigenvalues and eigenvectors

Definition Let A be an $n \times n$ matrix. We say that a vector $x \in \mathbb{R}^n$ is an **eigenvector** of A if $x \neq 0$ and there exists $\lambda \in \mathbb{R}$ such that

$$Ax = \lambda x.$$

Such a λ is called an **eigenvalue** of A and x is said to be an **eigenvector** of A corresponding to λ.

Finding the eigenvalues

Let A be an $n \times n$ matrix. We want to find λ such that, for some $x \in \mathbb{R}^n, x \neq 0$,

$$Ax = \lambda x$$

or equivalently

$$(A - \lambda I)x = 0$$

This is a homogeneous linear system; so we know that it either has a unique solution $x = 0$ (which is not of interest here) or an **infinite number of solutions** $x \neq 0$.
We can have non-zero solutions only if $\det(A - \lambda I) = 0$.
So we have the following:

Definition λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0 \quad (\text{or} \det(\lambda I - A) = 0)$$

(Characteristic equation of A)
The **characteristic polynomial** of A is the polynomial

$$P(\lambda) = \det(\lambda I - A).$$

Remark $P(\lambda)$ has the form

$$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n,$$

where $c_1, \ldots, c_n$ are constants.
$P(\lambda)$ has therefore at most n distinct solutions (by the Fundamental Theorem of Algebra).
Therefore, if A is an $n \times n$ matrix, then A has at most n distinct eigenvalues.

Finding the eigenvectors

Suppose λ is an eigenvalue of the matrix A and we want to find the eigenvectors of A corresponding to λ.
x is an eigenvector corresponding to λ if $x \neq 0$ and

$$Ax = \lambda x$$

i.e.

$$(\lambda I - A)x = 0.$$

The set of eigenvectors of A corresponding to λ is therefore given by

$$\{x \in \mathbb{R}^n \mid (\lambda I - A)x = 0, \quad x \neq 0\}.$$