MS4131 Lecture 19: Complex Eigenvalues and Linear Independence

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27 November 2023

Complex eigenvalues

Complex numbers

A **complex number** $z \in \mathbb{C}$ takes the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.
The **complex conjugate** of z is $\bar{z} = x - iy$.

Modulus

$$z\bar{z} = (x + iy)(x - iy) = x^2 + ixy - ixy + y^2 = x^2 + y^2$$

The **modulus** of z is

$$r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$$

If the **imaginary part** of z is zero; i.e., if $y = 0$:

$$|z| = \sqrt{x^2} = |x|$$

then the modulus is the **absolute value** of x (where x is the **real part** of z).

Complex vectors and matrices

Similarly a complex vector $u \in \mathbb{C}^n$ can be written as $u = v + iw$, where v and w are real valued vectors.
The norm of the complex vector

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

is

$$||u|| = \sqrt{\sum_{i=1}^{n} u_i \bar{u_i}} = \sqrt{u_1\bar{u_1} + u_2\bar{u_2} + \cdots + u_n\bar{u_n}}$$

Remark If λ is a complex eigenvalue of a real matrix A then its conjugate $\bar{\lambda}$ is also an eigenvalue.

Theorem 0.1 *If $A \in \mathbb{R}^{n\times n}$ is symmetric then all its eigenvalues are **real**.*

Linear independence

Definition If $S = \{v_1, v_2, ..., v_r\}$ is a set of vectors, then the equation

$$k_1v_1 + k_2v_2 + ... + k_rv_r = 0$$

has **at least one solution**: $k_i = 0\ \forall i$.
If there is no other solution, then S is a linearly independent set. Otherwise S is linearly dependent.

Remark If S is linearly dependent, then at least one of the scalars $\{k_i\}$ is non-zero. Assume $k_j \neq 0$; then

$$v_j = -\frac{k_1}{k_j}v_1 - \frac{k_2}{k_j}v_2 - \cdots - \frac{k_r}{k_j}v_r$$

i.e. v_j is a linear combination of the other vectors in the set.

Remark 2 vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are lienarly dependent if and only if they lie on the same line through the origin and 3 vectors in $\mathbb{R}^3$ are linearly dependent if and only if they lie on the same plane through the origin.

Matrix with linearly independent columns

Let X be the matrix $(x_1, x_2, \ldots, x_n)$ whose columns $x_1, x_2, \ldots, x_n$ are linearly independent.

Let $k = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$. Then

$$X^T k = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = 0 \iff k = 0$$

Thus X^T is invertible and so X is invertible.