MS4131 Lecture 20: Diagonalization

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1 Diagonalization

Definition A square matrix A is said to be **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

Theorem 1.1 *If A is an $n \times n$ matrix, then the following are equivalent:*

1. *A is diagonalizable*
2. *A has n linearly independent eigenvectors*

Remark If A does not have n linearly independent eigenvectors, it is not diagonalizable.

Application

$A^n = (PDP^{-1})^n \because P^{-1}AP = D \implies A = PDP^{-1} \therefore A^n = PD^nP^{-1}$

Orthogonal diagonalization

Definition A square matrix A with the property $A^{-1} = A^T$ is said to be an **orthogonal matrix**.

Definition A square matrix A is **orthogonally diagonalizable** if there is an orthogonal matrix P such that $P^{-1}AP = P^TAP$ is diagonal. The matrix P is said to orthogonally diagonalize A.

Remark If A is orthogonally diagonalizable then $P^{-1}AP$ is diagonal where P is orthogonal;
that is $P^TP = I$ or if p_i for $i = 1, 2, \ldots, n$ are the columns of P then

$$p_i^T p_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

that is, the columns of P (the eigenvectors of A) are **orthonormal** and so A has n orthonormal eigenvectors.
Similarly if A has n orthonormal eigenvectors $p_1, p_2, \ldots, p_n$ then the matrix P, whose columns are these eigenvectors, diagonalizes A orthogonally.
Thus A is orthogonally diagonalizable if and only if A has n orthonormal eigenvectors.

Theorem 1.2 *If A is an $n \times n$ matrix, then A is orthogonally diagonalizable if and only if A is symmetric.*

Theorem 1.3 *If A is symmetric, then eigenvectors corresponding to different eigenvalues are orthogonal.*