1 Overview

We solve for the dynamics of a fluid/fluid interface in two dimensional low Reynolds flow. The fluid interface is modeled using the level set method and surface tension effects are included as body forces in the momentum equation governing the flow.

1.1 Model Equations and Boundary Conditions

The fluid flow is governed by Stokes equations (for $i = 1, 2$):

$$\mu^{(i)}\nabla^2\mathbf{u} = \boldsymbol{\nabla} p + \mathbf{f}_b \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \tag{1b}$$

Here $\mu^{(i)}$ is the fluid viscosity in fluid i, $\mathbf{u} = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y$ is the velocity field, p is the pressure, and $\mathbf{f}_b$ is a body force (we will use this term to include surface tension).

This problem is solved on a rectangular domain $[x_{min}, x_{max}] \times [y_{min}, y_{max}] \subset \mathbb{R}^2$. Along the boundary of the domain we enforce a no-slip/no-penetration condition at $y = y_{min}$ and $y = y_{max}$ (with the option to prescribe wall velocities), and enforce periodicity in the x direction.

The fluid/fluid interface is modeled using the level set method [1]. In the level set framework the fluid interface Γ is defined as the zero-level set of a signed distance function ϕ. i.e.

$$\Gamma := \left\{\mathbf{x}\in\mathbb{R}^2 : \phi^{-1}(\mathbf{x}) = 0\right\},\ \|\boldsymbol{\nabla}\phi\| = 1$$

Along the fluid-fluid interface we have continuity of tangential stress and a jump in the normal stress due to surface tension. Following [2, 3], we include the effect of surface tension through a body force term in the momentum equation. Thus, in (1a) we put

$$\mathbf{f}_b = \sigma\kappa(\phi)\boldsymbol{\nabla} H(\phi) \tag{2}$$

where σ is surface tension, κ is curvature, and $H(\phi)$ is a Heaviside function. This gradient of the Heaviside function will in practice be replaced by a smoothed delta function (see below).

Finally, the fluid interface moves according to the normal component of the local fluid velocity. This can be used to derive the evolution equation for ϕ:

$$\phi_t + \mathbf{u}\cdot\boldsymbol{\nabla}\phi = 0 \tag{3}$$

2 Numerical Details

2.1 Timestep overview

We begin by presenting an overview of a timestep of the algorithm. Details for each sub-step can be found in the following sections.

Algorithm 1: Temporal Update

for *each timestep* **do**
- -Compute the surface tension body force term (2)
- -Solve (1) for $\mathbf{u}, p$
- -Solve (7) for the speed F
- -Update interface location by advancing (6)

2.2 Solving the Stokes Equations

We discretize (1) on the domain $[x_{min}, x_{max}] \times [y_{min}, y_{max}] \subset \mathbb{R}^2$ using a staggered grid of size $N_x \times N_y$. The fluid velocity nodes are located on cell walls while pressure nodes are located on cell centers. i.e. for the fluid variables we have

$$u_{i,j} \approx u\left(x_{min} + i\Delta x, y_{min} + \left(j + \frac{1}{2}\right)\Delta y\right)$$

$$v_{i,j} \approx v\left(x_{min} + \left(i + \frac{1}{2}\right)\Delta x, y_{min} + j\Delta y\right)$$

$$p_{i,j} \approx p\left(x_{min} + \left(i + \frac{1}{2}\right)\Delta x, y_{min} + \left(j + \frac{1}{2}\right)\Delta y\right)$$

where $\Delta x = \dfrac{x_{max} - x_{min}}{N_x - 1}$ and similarly for Δy. (1) is then discretized using a standard 5 point stencil for the Laplacian and centered differences for the gradient and divergence terms. Details for computing the surface tension body forces are in the next section. We proceed now assuming a generic body force term $\mathbf{f} = f_x\hat{\mathbf{e}}_x + f_y\hat{\mathbf{e}}_y$.

Putting this together leads to our discrete version of (1).

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} - \frac{p_{i,j} - p_{i-1,j}}{\Delta x} = (f_x)_{i,j} \tag{4a}$$

$$\frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{\Delta x^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j+1}}{\Delta y^2} - \frac{p_{i,j} - p_{i,j-1}}{\Delta y} = (f_y)_{i,j} \tag{4b}$$

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y} = 0 \tag{4c}$$

(4) is solved using a modified version of SOR [4]. In this modified version, a single update of the momentum equations (4a) and (4b) are performed assuming fixed pressure data. Then the pressure is updated according to the discrete divergence of the velocity. The detailed algorithm is below.

Algorithm 2: Stokes SOR Solver

while $maxU > tol$ *OR* $maxV > tol$ *OR* $maxP > tol$ **do**

  for *each* u *gridpoint* **do**

    $r_u = \frac{\Delta y}{\Delta x}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + \frac{\Delta x}{\Delta y}\left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right) - \Delta y\left(p_{i,j} - p_{i-1,j}\right) - \Delta x\Delta y(f_x)_{i,j}$

    Update $u_{i,j} = u_{i,j} + \eta r_u$

    if $|r_u| > maxU$ **then**

      $maxU = |r_u|$

  for *each* v *grid point* **do**

    $r_v = \frac{\Delta y}{\Delta x}\left(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}\right) + \frac{\Delta x}{\Delta y}\left(v_{i,j-1} - 2v_{i,j} + v_{i,j+1}\right) - \Delta x\left(p_{i,j} - p_{i,j-1}\right) - \Delta x\Delta y(f_y)_{i,j}$

    Update $v_{i,j} = v_{i,j} + \eta r_v$

    if $|r_v| > maxV$ **then**

      $maxV = |r_v|$

  for *each* p *grid point* **do**

    $r_p = -\left(u_{i+1,j} - u_{i,j}\right) - \frac{\Delta x}{\Delta y}\left(v_{i,j+1} - v_{i,j}\right)$

    Update $p_{i,j} = p_{i,j} + \eta_p r_p$

    if $|r_p| > maxP$ **then**

      $maxP = |r_p|$

Here η and η_p are relaxation parameters. Notes: For periodic boundary conditions in x, all operations on the i indices are done under modular arithmetic. For Dirichlet BCs the grid points are either set directly (when the grid point lies along the boundary) or enforced using an interpolation (by introducing a ghost point and averaging to set the value at the boundary). Also, the system as described is singular due to the constant pressure mode. We can eliminate this by enforcing

$$\int_\Omega p\,\mathrm{d}A = p_{constant} \tag{5}$$

where Ω is the computational domain. In practice this is enforced by subtracting the mean of the pressure over the domain from the value at each node. This constant mode can also be eliminated by prescribing the pressure at a given grid point but this leads to poor conditioning of the system.

2.3 The Level Set Method

Full details of the level set method would require a fairly lengthy discussion. For these notes we restrict our discussion to the application of the level set method in a fluid problem. Further details can be found in [1].

Relative to our discretization of (1), the discrete level set data $\phi_{i,j}$ that defines the interface are located on cell corners. i.e.

$$\phi_{i,j} \approx \phi(x_{min} + i\Delta x, y_{min} + j\Delta y)$$

2.3.1 Computing Surface Tension

Surface tension is computed from the discrete level set data. The curvature term is computed as

$$\kappa(\phi) = \frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2}{\left(\phi_x^2 + \phi_y^2\right)^{3/2}}$$

and all derivatives are computed using centered differences. For the Heaviside term, we follow [5] and introduce a regularized delta function δ_ε to write

$$\boldsymbol{\nabla} H(\phi) \approx \boldsymbol{\nabla}\phi\delta_\varepsilon(\phi) = \boldsymbol{\nabla}\phi \begin{cases} \frac{1}{2}\left[1 + \cos(\pi\phi/\varepsilon)\right]/\varepsilon, & |\phi| < \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

Note: The code currently assumes $\varepsilon = 1.5\Delta x$ but this is easily adjustable. Because the level set data are offset from the fluid velocity data, we use linear interpolation to populate body force terms needed for solving (4).

2.3.2 Velocity Extensions

Using (3) to update ϕ will not in general preserve ϕ as a signed distance function. This difficulty can be overcome by advancing ϕ according to

$$\phi_t + F\|\boldsymbol{\nabla}\phi\| = 0 \tag{6}$$

where $F\Big|_\Gamma = \dfrac{\mathbf{u}\cdot\boldsymbol{\nabla}\phi}{\|\boldsymbol{\nabla}\phi\|}$ and $\boldsymbol{\nabla}F \cdot \boldsymbol{\nabla}\phi = 0$ away from Γ. It can be easily shown that solving (6) preserves the signed distance property.

Constructing F requires us to solve the Eikonal equation

$$\boldsymbol{\nabla}F \cdot \boldsymbol{\nabla}\phi = 0 \tag{7}$$

We use the fast marching method to solve (7) and prescribe initial data by using a bicubic fit to locate the fluid interface. Details of this procedure are presented in [6].

2.3.3 Temporal Update

We solve (6) using upwind differences on the $\|\boldsymbol{\nabla}\phi\|$ term giving us the following discretization:

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} + F_{i,j}\sqrt{A_{i,j}^2 + B_{i,j}^2} = 0 \tag{8}$$

where

$$A = \max\left(\text{sgn}\left(F_{i,j}\right) D^{-x}\phi_{i,j}^n, -\text{sgn}\left(F_{i,j}\right) D^{+x}\phi_{i,j}^n, 0\right)$$
$$B = \max\left(\text{sgn}\left(F_{i,j}\right) D^{-y}\phi_{i,j}^n, -\text{sgn}\left(F_{i,j}\right) D^{+y}\phi_{i,j}^n, 0\right)$$

References

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