Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria

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Biographie – I am a third-year Phd student at École polytechnique. Prior to my Phd, I got an engineering degree from École polytechnique, a second one from Mines Paristech and the master degree *Optimization* of Université Paris-Saclay. In my thesis, I am interested in the complexity of problems arising in optimal control and zero-sum games. My thesis is funded by the AMX fellowship of École polytechnique.

Resumé :

A *tropical hyperplane* is a set of the form

$$\mathcal{H}_a := \{x \in \mathbb{R}^n \mid \max_{i\in[n]}(a_i + x_i) \text{ achieved at least twice}\},$$

where $a = (a_i) \in \mathbb{R}^n$ is fixed. Tropical hyperplanes are basic objects in tropical geometry [3]. They also arise in economics, in the analysis of auction mechanisms and price response [2]. We will give an application of our results in the next section.

A tropical hyperplane is invariant by the addition of constant vectors, so, we shall think of $\mathcal{H}_a$ as a subset of the tropical projective space $\mathbb{P}(\mathbb{R}^n)$, defined as the quotient of $\mathbb{R}^n$ by the equivalence relation $\sim$ such that $x \sim y$ if and only if $x - y$ is a constant vector. A canonical metric on $\mathbb{P}(\mathbb{R}^n)$ is induced by Hilbert's seminorm, $\|z\|_H := (\max_{i\in[n]} z_i) - (\min_{i\in[n]} z_i)$. Given a finite collection of points $\mathcal{V}$ of $\mathbb{R}^n$, we consider the following *tropical analogue of the linear regression problem:* find a vector $a \in \mathbb{R}^n$ minimizing the one-sided Hausdorff distance

$$\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a) := \max_{v\in\mathcal{V}} \inf_{x\in\mathcal{H}_a} \|v - x\|_H \ .$$

This is a non-convex problem of a unusual disjunctive nature since a tropical hyperplane is the union of n convex sectors. However, we next show that a strong duality theorem holds, leading to an effective algorithmic approach.

The dual problem will involve the tropical convex cone $\mathrm{Sp}(\mathcal{V})$ generated by $\mathcal{V}$, which is the set of tropical linear combinations of elements of $\mathcal{V}$, i.e., of vectors of the form $\sup_{v\in\mathcal{V}}(\lambda_v + v)$, where λ_v are real parameters, and the notation $\lambda + v$ for a scalar λ and a vector v stands for the vector obtained by adding λ to every entry of v. We consider the following *Shapley operator* $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$T_i(x) = \min_{v\in\mathcal{V}}[-v_i + \max_{j\in[n], j\neq i}(v_j + x_j)] \ .$$

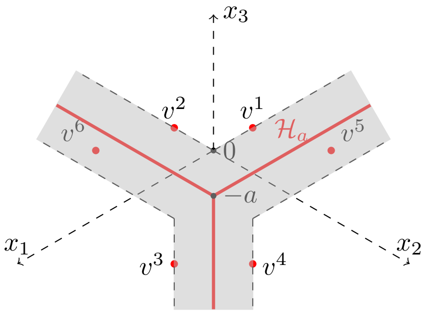
This is the dynamic programming operator of a zero-sum two-player deterministic game, in which i is the current state, $v \in \mathcal{V}$ is the action of player MIN, $j \neq i$ is the next state, chosen by player MAX, and $-v_i + v_j$ is the instantaneous payment made by MIN to MAX. One associates to such an operator a mean payoff game, in which players alternate moves indefinitely, and Player MAX (resp. MIN) maximizes (resp. minimizes) the average per time unit of the instantaneous payments. We denote by $\rho(T)$ the value of this game (which here is independent of the initial state). Computing the value of a mean payoff game is one of the fundamental problems of algorithmic game theory. It

belongs to NP $\cap$ coNP. It is not known whether it belongs to P. Experimentally efficient algorithms, including pseudopolynomial value iteration algorithms, are known. See [1] for background.

Theorem 1 (Strong duality) *The maximal radius of a Hilbert's ball included in* $\mathrm{Sp}(\mathcal{V})$ *is equal to* $-\rho(T)$ *and it coincides with the minimal distance from* $\mathcal{V}$ *to a tropical hyperplane. Moreover, any vector* u *such that* $T(u) \geq \rho(T) + u$ *provides an optimal hyperplane* $\mathcal{H}_u$*, whereas any vector* v *such that* $T(v) \leq \rho(T) + v$ *provides an optimal ball centered at* $-v$*.*

Corollary 2 *The tropical linear regression problem is polynomial-time equivalent to the problem of solving a mean payoff game.*

Here is an optimal regression hyperplane for points $v^1, \dots, v^6$ in $\mathbb{P}(\mathbb{R}^3)$:



As an application of the tropical linear regression, we consider a market with n companies answering repeatedly to invitations to tender. Denote by p_{ij} the price offered by company $i \in [n]$ for the invitation number $j \in [q]$. Assume that the decision maker has a secret evaluation $f_i > 0$ of the technical quality of each company i and that she will select the company which minimizes the expression: $\min_{i\in[n]} p_{ij} f_i^{-1}$. A factor $f_i^{-1} \geq 1$ may be interpreted as a proportional penalty depending on the technical quality f_i of the company (the larger f_i, the better its quality). A factor $f_i^{-1} = 1 - \alpha_i \beta \leq 1$ for some $0 \leq \alpha_i \leq 1$ and $0 \leq \beta < 1$ may represent a proportional bribe: company i promises to secretely give back $\alpha_i p_{ij}$ to the decision maker if its offer is accepted, and the parameter β measures how sensitive is the decision maker to bribery ($\beta = 0$ corresponds to a totally honnest decision maker).

If the same companies answer in a recurrent manner to invitations from the same decision maker, the secret factors f_i being kept constant, we expect the prices to constitute an *equilibrium*, meaning that for each invitation $j \in [q]$, the minimum $\min_{i\in[q]} p_{ij} f_i^{-1}$ is achieved twice at least. Indeed, if company i which wins the invitation offers a price p_{ij} such that $p_{ij} f_i^{-1}$ is strictly smaller than $p_{kj} f_k^{-1}$ for all $k \in [n] \setminus \{i\}$, it may rise its price and still win the offer. We associate to each call for tender j a vector $v^j = (-\log p_{ij})_{i\in[n]}$, so that the decision maker selects the company of index i achieving the maximum in $\max_{i\in[n]}(v_i^j + a_i)$, where $a = (\log(f_i))_{i\in[n]}$. At equilibrium, this can be modeled in terms of membership of the points $(v^j)_{j\in[q]}$ to the tropical hyperplane $\mathcal{H}_a$. More generally, measuring the "distance to equilibrium" in this market and inferring the secret factors f_i are equivalent to finding a tropical linear regression of the points $(v^j)_{j\in[q]}$.

Références

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