Estimation of multivariate generalized gamma convolutions through Laguerre expansions

O. Laverny, E. Masiello, V. Maume-Deschamps, D. Rullière
Université Claude Bernard Lyon 1 & SCOR SE, Université Claude Bernard Lyon 1 & École des Mines de Saint-Étienne

Email : laverny@math.univ-lyon1.fr

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Biographie – Oskar Laverny is an actuary by formation, fond of statistics and code. Oskar's PhD is funded through a CIFRE grant between the UCBL and SCOR SE, a reinsurance company. Our research is articulated around the mathematical and statistical study of dependence structures in high-dimensions, and the development of statistical theory and associated computational tools.

Resumé :

The class $\mathcal{G}_1$ of univariate generalized gamma convolution, defined as the class of weak limits of independent convolutions of gamma distributions, was first introduced by Thorin [7, 8] as a tool to show the infinite divisibility of log-normal and Pareto distributions. Although originating from this very practical question, its study led to many improvements of the theory, well summarized by Bondesson [1] and is still an active field nowadays (see, e.g., [4, 6]). By definition, $\mathcal{G}_1$ is closed under independent convolutions, but as it appears recently [2], it is also closed by independent products of random variables. Pareto, log-normal, α-stable, Weibull, and many other distributions are in this class, which makes it a nice framework for many applications fields such as climate events modeling, insurance, etc.

An analogue multivariate class $\mathcal{G}_d$ was constructed by Bondesson [2], following an old idea of Cherian [3], by convoluing comonotonous multivariate gamma distributions. Finally, we also consider $\mathcal{G}_{d,n}$, the subclass containing convolutions of at most n d-variates gamma. Although not much is known about $\mathcal{G}_d$, a random vector $\boldsymbol{X} \in \mathcal{G}_{d,n}$ follows an additive risk-factor structure: there exists gamma random variables $Y_{i,j}$ such that $\forall\, i \in \{1, ..., d\}$,

$$X_i = Y_{i,1} + ... + Y_{i,n}, \tag{1}$$

where each vector $Y_{i,.}$ has independent marginals, and each vector $Y_{.,j}$ has comonotonous marginals. Since on one hand some $Y_{i,j}$ might be identically zero, as $0 \in \mathcal{G}_{1,1}$, and on the other hand every $Y_{i,j}$ is infinite divisible, by increasing n the model can achieve a wide variety of dependence structures and approach any marginal in $\mathcal{G}_1$.

A useful tool for the analysis of these models is the Thorin measure. It can be shown that the cumulant generating function of a random vector in $\mathcal{G}_d$ writes:

$$K(\boldsymbol{t}) = \ln \mathbb{E}\left(e^{\langle \boldsymbol{t}, \boldsymbol{X} \rangle}\right) = -\int\limits_{R_+^d} \ln\left(1 - \langle \boldsymbol{t}, \boldsymbol{s} \rangle\right) \nu\left(\partial \boldsymbol{s}\right),$$

where the Thorin measure ν is discrete with n atoms if $\boldsymbol{X} \in \mathcal{G}_{d,n}$. In this case, atoms and weights of ν are respectively scales and shapes of gamma vectors $Y_{.,j}$. Therefore, the model parameters can be fully summarized by ν, which we would like to estimate.

The deconvolution problem of estimating distributions in $\mathcal{G}_{d,n}$, which is equivalent to the problem of retrieving the measure ν, is an inverse problem known to be numerically challenging, even when

$d = 1, n = 2$. Thus, the current literature contains no estimation procedure for distributions in $\mathcal{G}_d$ or $\mathcal{G}_{d,n}$.

In this work, we investigate the univariate and multivariate estimation of atomic Thorin measures. Through a projection in a Laguerre basis, we provide a deconvolution procedure that produces estimators in $\mathcal{G}_{d,n}$, for any finite d, n, be the distribution given through an exact density or an empirical dataset. To make the estimation viable, we designed fast ad-hoc algorithms to compute Laguerre coefficients of multivariate gamma convolutions, leveraging the clever Faa di bruno trick from Miatto [5].

Through an analytic combinatorics approach, we construct a regularity condition on the random vector that is necessary and sufficient for the (uniform) exponential decay of the Laguerre coefficients, and derive a consistency result for our estimator. Moreover, under the same regularity condition, the approach provides a series expansion for densities in $\mathcal{G}_{d,n}$, where the current literature provides density expansions only when $d = 1$, and which are known to be unstable, or even dramatically failing for certain parameters ranges, noteworthy those which correspond to projection of log-normal, Pareto or Weibull distributions.

We propose some numerical illustrations of the method, and highlight the flexibility of the obtained model.

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