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2. (Linear Independence, 9pt). For each of the following sets of (column) vectors determine whether they are linearly independent. If the vectors are linearly dependent, exhibit a non-trivial linear combination of the vectors that sums up to the zero vector

if they are linearly independent briefly argue how you can tell. Use Matlab or any other reasoning.

a.)

```
>> U=horzcat(u1,u2,u3,u4)

U =

    -4     5   -13     1
    -2    11   -15     1
    -6    18     8    15
    11    19   -17    -1

>> rref(U)

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

The rref has full column rank. There are 4 columns and the rank is 4. That means the four vectors:

u1 =[-4;-2;-6;11]

u2 = [5;11;18;19]

u3 = [-13;-15;8;-17]

u4 = [1;1;15;-1]

are linearly independent.

There is no way you can make them add up to 0 without making each of the coefficients = to 0.

The matrix spans a 4 dimensional subspace.

b.)

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```
>> U=horzcat(u1,u2,u3)

U =

    -4    -7    41
     6     5   -23
    -1    -1     5
   -10    -7    29

>> R=rref(U)

R =

     1     0     2
     0     1    -7
     0     0     0
     0     0     0
```

```
>> R*[2;-7;-1]

ans =

     0
     0
     0
     0
```

Since the rref does not exhibit full column rank, the vectors in the linearly dependent.

The rref also tells us that the matrix has a 2 dimensional subspace. If we take the first column twice, subtract the second column we get the 3rd column which is represented by [2;-7;-1]. This results in the non-trivial linear combination of the vectors that sums to the zero vector

c.)

```
>> U=horzcat(u1,u2,u3,u4,u5,u6)

U =

     3    -3     3    -1    12    24
    -3     9     2    -6   -67   -47
     7     8    -6     7    26    28
     1     1    -4    -6   -41   -23

>> R=rref(U)

R =

     1     0     0     0     2     5
     0     1     0     0    -3    -2
     0     0     1     0     1     2
     0     0     0     1     6     3
```

Since the rref does not exhibit full column rank, the vectors are linearly dependent.

The two non trivial linear combinations of vectors are shown to the left and below:

[2;-3;1;6;-1;0]

[5;-2;2;3;0-1]

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```
>> R*[2;-3;1;6;-1;0]

ans =

     0
     0
     0
     0
```

```
>>  R*[5;-2;2;3;0;-1]

ans =

     0
     0
     0
     0
```

3.) How do we test whether a vector b lies in the span of U = {u1, ... , un }? Easily, by testing whether Ux = b has a solution (where U is the matrix whose columns are the ui). But how do we do that in general, if U is not invertible, or not even a square matrix? Decide whether b lies in span(U) in the following examples, and if it does write down the linear combination of the ui that yields b.

a) [3pt] U =[5 7 13 -14 11 -3;17 3 5 0 3 9;-7 11 -6 2 3 -23;5 31 -29 28 -1 -51]; b =[101;37;63;125] .

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```
>> U =[5 7 13 -14 11 -3;17 3 5 0 3 9;-7 11 -6 2 3 -23;5 31 -29 28 -1 -51]

U =

     5     7    13   -14    11    -3
    17     3     5     0     3     9
    -7    11    -6     2     3   -23
     5    31   -29    28    -1   -51

>> b =[101;37;63;125]

b =

   101
    37
    63
   125

>> rref(horzcat(U,b))

ans =

    1.0000         0         0    0.3611   -0.0926    0.7037   -0.2778
         0    1.0000         0   -0.1944    0.5370   -1.4815    7.6111
         0         0    1.0000   -1.1111    0.5926    0.2963    3.7778
         0         0         0         0         0         0         0
```

```
>> U*[-0.2778;7.611;3.778;0;0;0]

ans =

  101.0020
   37.0004
   62.9976
  124.9900
```

```
>> b

b =

   101
    37
    63
   125
```

Under the assumption of approximation, we can say that b lies in the span of U

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b) [3pt] U =[5 7 13 -14 11 -3;17 3 5 0 3 9;-7 11 -6 2 3 -23;5 31 -29 28 -1 -51]; b =[49;9;-59;9] .

```
>> U =[5 7 13 -14 11 -3;17 3 5 0 3 9;-7 11 -6 2 3 -23;5 31 -29 28 -1 -51]

U =

     5     7    13   -14    11    -3
    17     3     5     0     3     9
    -7    11    -6     2     3   -23
     5    31   -29    28    -1   -51

>> b =[49;9;-59;9]

b =

    49
     9
   -59
     9

>> R=rref(horzcat(U,b))

R =

    1.0000         0         0    0.3611   -0.0926    0.7037         0
         0    1.0000         0   -0.1944    0.5370   -1.4815         0
         0         0    1.0000   -1.1111    0.5926    0.2963         0
         0         0         0         0         0         0    1.0000
```

The system is not solvable because the rank[A] < rank[M]. In other words there is a pivot in the last column…. b does not lie in the span of U.

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Hint: use rref. If you use linsolve, make sure you know what you're doing.

c) [2pt] For the matrix U in a), span(U) is a subspace of 4-dimensional space R4 (since each column has four entries). What is the dimension of span(U)? Hint: how many vectors in a basis?

6 vectors in other words, 6 dimensions in the vector space

d) [2pt] Consider the matrix U = [-30 6 -2 -19 -48 73;-24

6 0 -15 -42 63;24 -30 -32 11 114 -155;0 6 8 1 -18 23;24 -12 -8 14 60 -86;-18 -30 -46 -17 72 -85] .

```
>>  U = [-30 6 -2 -19 -48 73;-24 6 0 -15 -42 63;24 -30 -32 11 114 -155;0 6 8 1 -18 23;24 -

U =

   -30     6    -2   -19   -48    73
   -24     6     0   -15   -42    63
    24   -30   -32    11   114  -155
     0     6     8     1   -18    23
    24   -12    -8    14    60   -86
   -18   -30   -46   -17    72   -85
```

```
>> rref(U)

ans =

    1.0000         0    0.3333    0.6667    1.0000   -1.6667
         0    1.0000    1.3333    0.1667   -3.0000    3.8333
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
```

What is the dimension of the (ambient) space that span(U) is contained in, that is, for what d would we say that span(U) is contained in Rd ? Hint: how many entries does each vector have.

6

What is the dimension of span(U) as a vector space? Hint: how many vectors in a basis?

Spans a 2-dimensional subspace, since there are two columns that have a pivot. In other words full column rank would be 6 and we only have a rank of 2, giving us a 2 dimensional subspace.

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4. (Finding a Basis, 8pt). You are given a set of (column) vectors U = {[9;-3;-6;-5;2;-1], [-3;7;2;1;9;-4] , [39;-37;-26;-19;-30;13], [-16;-21;0;5;26;28], [1;6;9;-8;1;-1] } which is a set of vectors in R6.

a) [3pt] Use the column-space algorithm to find a basis of span(U) which consists of vectors of U. Hint: First build the matrix U.

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```
>> u1=[9;-3;-6;-5;2;-1]

u1 =

     9
    -3
    -6
    -5
     2
    -1

>> u2=[-3;7;2;1;9;-4]

u2 =

    -3
     7
     2
     1
     9
    -4

>> u3=[39;-37;-26;-19;-30;13]

u3 =

    39
   -37
   -26
   -19
   -30
    13

>> u4=[-16;-21;0;5;26;28]

u4 =

   -16
   -21
     0
     5
    26
    28

>> u5=[1;6;9;-8;1;-1]

u5 =

     1
     6
     9
    -8
     1
    -1
```

```
>> U=horzcat(u1,u2,u3,u4,u5)

U =

     9    -3    39   -16     1
    -3     7   -37   -21     6
    -6     2   -26     0     9
    -5     1   -19     5    -8
     2     9   -30    26     1
    -1    -4    13    28    -1

>> rref(U)

ans =

     1     0     3     0     0
     0     1    -4     0     0
     0     0     0     1     0
     0     0     0     0     1
     0     0     0     0     0
     0     0     0     0     0
```

```
>> B = U(:, [1 2 4 5]);
>> B

B =

     9    -3   -16     1
    -3     7   -21     6
    -6     2     0     9
    -5     1     5    -8
     2     9    26     1
    -1    -4    28    -1
```

Spans same as U

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b) [3pt] Use the row-space algorithm to find a basis of span(U).

```
U =

     9    -3    39   -16     1
    -3     7   -37   -21     6
    -6     2   -26     0     9
    -5     1   -19     5    -8
     2     9   -30    26     1
    -1    -4    13    28    -1

>> U

U =

     9    -3    39   -16     1
    -3     7   -37   -21     6
    -6     2   -26     0     9
    -5     1   -19     5    -8
     2     9   -30    26     1
    -1    -4    13    28    -1

>> U'

ans =

     9    -3    -6    -5     2    -1
    -3     7     2     1     9    -4
    39   -37   -26   -19   -30    13
   -16   -21     0     5    26    28
     1     6     9    -8     1    -1
```

```
>> R=rref(U')

R =

    1.0000         0         0         0   -4.4597   -0.9339
         0    1.0000         0         0    1.1310   -0.8101
         0         0    1.0000         0   -3.9879   -0.1702
         0         0         0    1.0000   -4.3206   -0.7907
         0         0         0         0         0         0

>> R=R(1:4,:)

R =

    1.0000         0         0         0   -4.4597   -0.9339
         0    1.0000         0         0    1.1310   -0.8101
         0         0    1.0000         0   -3.9879   -0.1702
         0         0         0    1.0000   -4.3206   -0.7907
```

```
>> B=R'

B =

    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000
   -4.4597    1.1310   -3.9879   -4.3206
   -0.9339   -0.8101   -0.1702   -0.7907
```

Spans the same as U

Can test using

```
rref(horzcat(U,B)
rref(horzcat(B,U))
```

c) [2pt] What is the dimension of span(U) as a subspace of R6 ?

4 Dimensional subspace

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5. (Solving Ax = b, 5pt) In class we saw that the set of solutions of a non-homogenous system Ax = b can be written as x0 + u, where x0 is a (fixed, but arbitrary) solution of Ax = b, and u is any (varying) solution of the homogenous system Au = 0. Since the solution set of a homogenous system is a vector space, we can write the general solution of Ax = b as x0 + λ1 u1 + ... + λk uk , where u1, ..., uk is a basis of that vector space (the nullspace of A), and the λi are arbitrary real numbers. We want to find a general solution of the system Ax = b for

A = [-1 -4 -13 1 -4;-3 1 13 -3 0;-1 -3 -9 -2 2;0 -1 -4 1 -2]; b = [-4;-1;7;-3].

a) [1pt] Find a specific solution x0 of Ax = b.

horzcat(A,b)

```
>> M

M =

    -1    -4   -13     1    -4    -4
    -3     1    13    -3     0    -1
    -1    -3    -9    -2     2     7
     0    -1    -4     1    -2    -3
```

```
>> rref(M)

ans =

     1     0    -3     0     2     4
     0     1     4     0     0    -1
     0     0     0     1    -2    -4
     0     0     0     0     0     0
```

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```
>> x=[4;-1;0;-4;0]

x =

     4
    -1
     0
    -4
     0

>> A*x

ans =

    -4
    -1
     7
    -3

>> b

b =

    -4
    -1
     7
    -3

>> x0=[4;-1;0;-4;0]

x0 =

     4
    -1
     0
    -4
     0
```

x0=[4;-1;0;0;4;0]

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b) [2pt] What is the dimension k of the nullspace of A? (Show calculation based on rank, and #of columns, don't use null except to verify)

```
>> rref(A)

ans =

     1     0    -3     0     2
     0     1     4     0     0
     0     0     0     1    -2
     0     0     0     0     0
```

Dimension of null space A is 2 based on rank of 3 in a 5 column matrix

c) [2pt] Find a basis u1,..., uk of the nullspace of A.

```
>> null(A)

ans =

    0.6844   -0.3198
   -0.6820   -0.5120
    0.1705    0.1280
   -0.1730    0.7038
   -0.0865    0.3519
```

```
>> n1=[2;0;0;-2;-1]

n1 =

     2
     0
     0
    -2
    -1
```

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x=x0+lamda*n1

```
>> x=x0+.6844*n1

x =

    5.3688
   -1.0000
         0
   -5.3688
   -0.6844

>> A*x

ans =

    -4
    -1
     7
    -3

>> rref(A)*n1

ans =

     0
     0
     0
     0
```

```
>> b

b =

    -4
    -1
     7
    -3
```

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Hint: You can use linsolve() and null() (except for 2b), or any other Matlab function you may find useful.

6. (Orthogonality and Projections, 8pt) Do the following problems:

a) [1pt] You are given the vector [10;-2;3;-5]. Find its corresponding unit vector (in the Euclidean norm).

```
u =

    10
    -2
     3
    -5

>> norm(u)

ans =

   11.7473
```

```
>> v=u/norm(u)

v =

    0.8513
   -0.1703
    0.2554
   -0.4256
```

```
>> norm(v)

ans =

     1
```

"v" is the corresponding unit vector

b) [1pt] Show that [3; -3; 4; 2] and [5; 3; -2; 1] are orthogonal.

```
>> u=[3; -3; 4; 2]

u =

     3
    -3
     4
     2

>> v=[5; 3; -2; 1]

v =

     5
     3
    -2
     1

>> abs(dot(u,v))

ans =

     0
```

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c) [2pt] Find a vector n that is orthogonal to all the column vectors in the matrix O = [-3 -1 -4;2 -5 4;2 -3 0;3 5 5] and then verify that n is orthogonal to all the columns of O. Hint: both parts (finding n, and verifying n) can be done by a single Matlab operation each (not the same one).

Create the below transposed null vector:

```
>> n=null(O')

n =

    0.8015
    0.1753
    0.2755
    0.5009
```

Prove orthogonal – close to zero can assume using approximation

```
>> abs(dot(n,O(:,1)))

ans =

   8.8818e-16

>> abs(dot(n,O(:,2)))

ans =

   4.4409e-16

>>
>> abs(dot(n,O(:,3)))

ans =

   8.8818e-16
```

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d) [2pt] Given u = [5; -6; 2; 5] and v = [1;1;1;1] find projv(u), the projection of u onto v (the book calls this the parallel projection of u along v).

```
>> lamdba = dot(u,v)/dot(v,v)

lamdba =

    1.5000
```

e) [2pt] For u, v as in d), find the orthogonal projection orthv(u). Verify in Matlab that orthv(u) is orthogonal to v.

```
>> p=lambda*v

p =

    1.5000
    1.5000
    1.5000
    1.5000
```

P and v are on the same line

```
>> abs(dot(u-p,v))

ans =

     0
```