BME 590L: HWK 1

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Problem 1

$$u = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 4 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

(a) $$u^T v = \begin{bmatrix} 1 & 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \\ 1 \end{bmatrix} = 0 + 5 + 5 + 4 = \boxed{14}$$

(b) $$u v^T = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 5 & 1 \\ 0 & 5 & 25 & 5 \\ 0 & 1 & 5 & 1 \\ 0 & 4 & 20 & 4 \end{bmatrix}}$$

(c) $$u \circ v = \begin{bmatrix} 1(0) \\ 5(1) \\ 1(5) \\ 4(1) \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 5 \\ 5 \\ 4 \end{bmatrix}}$$

$$a = \begin{bmatrix} 1 \\ e^{i\pi/4} \\ e^{i\pi/2} \\ e^{i\pi/4} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ e^{i\pi} \\ 5e^{i\pi} \\ e^{-i\pi} \end{bmatrix}$$ recall: $e^{ix} = \cos x + i \sin x$

(d) $$a^H b = \begin{bmatrix} 1 & e^{-i\pi/4} & e^{-i\pi/2} & e^{-i\pi/4} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\pi} \\ 5e^{i\pi} \\ e^{-i\pi} \end{bmatrix} = e^{3i\pi/4} + 5e^{i\pi/2} + e^{-5i\pi/4}$$

$$= \left(\cos\left(3\pi/4\right) + 5\cos\left(\pi/2\right) + \cos\left(-5\pi/4\right)\right) + i\left(\sin\left(3\pi/4\right) + 5\sin\left(\pi/2\right) + \sin\left(-5\pi/4\right)\right)$$

$$\boxed{= -\sqrt{2} + (5 + \sqrt{2})i \approx -1.41 + 6.41i}$$

(e) $$a^H a = \begin{bmatrix} 1 & e^{-i\pi/4} & e^{-i\pi/2} & e^{-i\pi/4} \end{bmatrix} \begin{bmatrix} 1 \\ e^{i\pi/4} \\ e^{i\pi/2} \\ e^{i\pi/4} \end{bmatrix} = 1 + \cancel{e^{0}}^{1} + \cancel{e^{0}}^{1} + \cancel{e^{0}}^{1} = \boxed{4}$$

$$b^H b = \begin{bmatrix} 0 & e^{-i\pi} & 5e^{-i\pi} & e^{i\pi} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\pi} \\ 5e^{i\pi} \\ e^{-i\pi} \end{bmatrix} = 0 + \cancel{e^{0}}^{1} + 25\cancel{e^{0}}^{1} + \cancel{e^{0}}^{1} = \boxed{27}$$

(f) $$a \circ a^* = \begin{bmatrix} 1 \\ e^{i\pi/4} \\ e^{i\pi/2} \\ e^{i\pi/4} \end{bmatrix} \circ \begin{bmatrix} 1 \\ e^{-i\pi/4} \\ e^{-i\pi/2} \\ e^{-i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 \\ \cancel{e^{0}}^{1} \\ \cancel{e^{0}}^{1} \\ \cancel{e^{0}}^{1} \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

(g) $$b \circ b^* = \begin{bmatrix} 0 \\ e^{i\pi} \\ 5e^{i\pi} \\ e^{-i\pi} \end{bmatrix} \circ \begin{bmatrix} 0 \\ e^{-i\pi} \\ 5e^{-i\pi} \\ e^{i\pi} \end{bmatrix} = \begin{bmatrix} 0 \\ \cancel{e^{0}}^{1} \\ 25\cancel{e^{0}}^{1} \\ \cancel{e^{0}}^{1} \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 25 \\ 1 \end{bmatrix}}$$

(h) $$a^H b = -\sqrt{2} + (5+\sqrt{2})i \Rightarrow |a^H b| = \sqrt{2 + (25 + 10\sqrt{2} + 2)} = 6.57$$

$$\|a\|_2 = \sqrt{\sum_{i=1}^{4} |a[i]|^2} = \sqrt{(1)^2 + |e^{i\pi/4}|^2 + |e^{i\pi/2}|^2 + |e^{i\pi/4}|^2}$$

$$= \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$\|b\|_2 = \sqrt{\sum_{i=1}^{4} |b[i]|^2} = \sqrt{0^2 + |e^{i\pi}|^2 + |5e^{i\pi}|^2 + |e^{-i\pi}|^2}$$

$$= \sqrt{1 + 25 + 1} = \sqrt{27}$$

$$\|a\|_2 \|b\|_2 = 2\sqrt{27} \approx 10.39$$

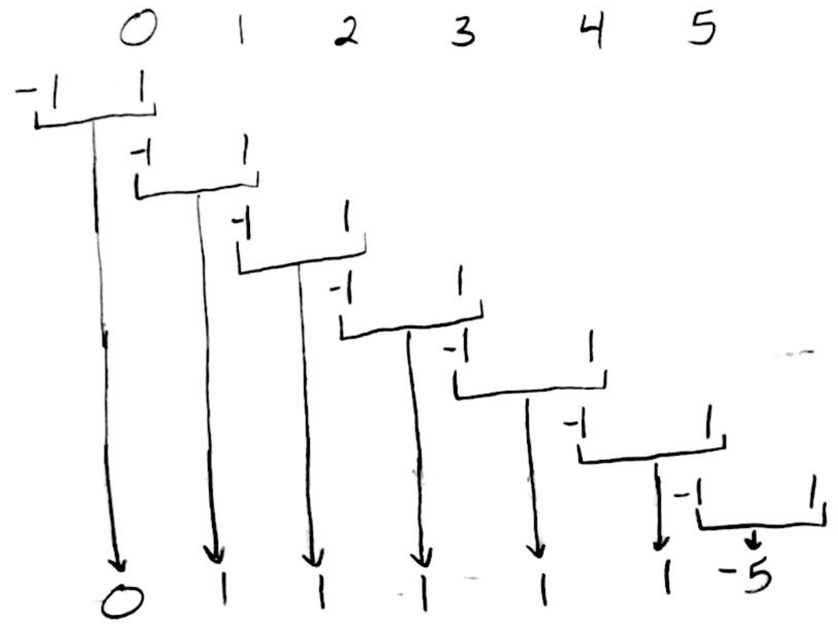
$$\boxed{\left(|a^H b| = 6.57\right) \leq \left(\|a\|_2 \|b\|_2 = 10.39\right)}$$

Problem 2

$$u = [1, -1]$$

$$v = [0, 1, 2, 3, 4, 5]$$

(a) $w = u * v$



flip u & pass it along v to get w

$$\boxed{w = [0 \ 1 \ 1 \ 1 \ 1 \ 1 \ -5]}$$

(b) $w = Uv$ $\quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ $\quad U \in \mathbb{R}^{7 \times 6}$

To get U, take the structure from part (a) & insert 0s where values are missing:

$$Uv = \overset{U}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}} \overset{v}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1-0 \\ 2-1 \\ 3-2 \\ 4-3 \\ 5-4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \end{bmatrix}$$

$\Rightarrow$ Ignoring the edge cases (0, -5), we see that this operation is calculating the difference b/t adjacent values. This is similar to a rate of change calculation (a derivative if continuous

(c) Given $w = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \end{bmatrix}$ find D such that $v = Dw$

$v \in \mathbb{R}^{6\times1}$ $w \in \mathbb{R}^{7\times1}$ $\Rightarrow D \in \mathbb{R}^{6\times7}$

$$\Rightarrow D_w = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \end{bmatrix}}_{w} = \begin{bmatrix} 0 \\ 0+1 \\ 0+1+1 \\ 0+1+1+1 \\ 0+1+1+1+1 \\ 0+1+1+1+1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

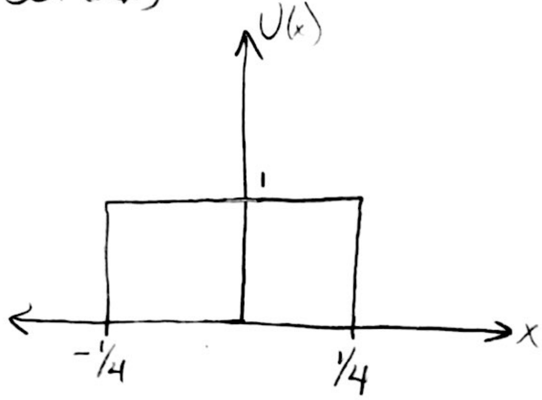
$\Rightarrow$ In this case, D "undoes" the convolution by progressively summing up the ones in w to get the values 0 to 5 in w. This generalized operation is a summing of adjacent terms, which in the continuous case is an integral. (This is as expected b/c as a deconvolution is the opposite of a convolution, an integral is the opposite of a derivative.)

(d)

$$DD^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Problem 3

(a) $U(x) = \text{rect}(2x)$



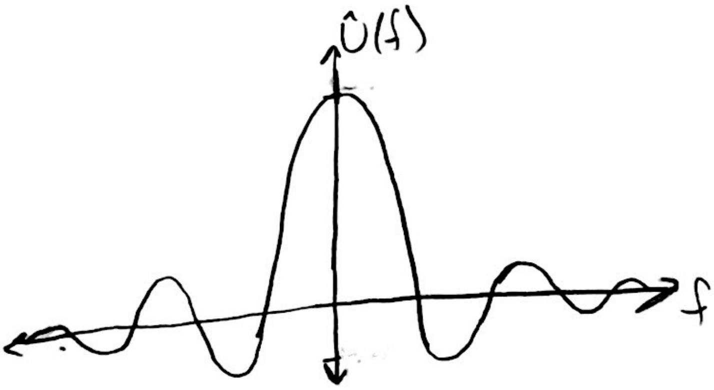
$$\hat{U}(x) = \int_{-\infty}^{\infty} U(x) e^{-2\pi i f x}\, dx = \int_{-1/4}^{1/4} \cancel{\text{rect}(2x)}^{\,1} e^{-2\pi i f x}\, dx$$

$$= \frac{1}{-2\pi i f}\left[e^{-2\pi i f x}\right]\Big|_{-1/4}^{1/4} = \frac{1}{-2\pi i f}\left(e^{-\frac{1}{2} i\pi f} - e^{\frac{1}{2} i\pi f}\right)$$

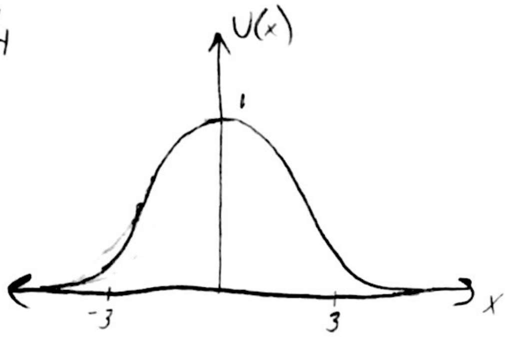
$$= \frac{1}{\pi f}\underbrace{\left(\frac{e^{\frac{1}{2} i\pi f} - e^{-\frac{1}{2} i\pi f}}{2i}\right)}_{\text{this is an identity for sin}} = \frac{1}{\pi f}\sin\left(\tfrac{\pi}{2} f\right)\left(\frac{\pi/2}{\pi/2}\right)$$

$\rightarrow$ Given $\text{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$

$$= \frac{\pi/2}{\pi}\,\frac{\sin\left(\frac{\pi}{2} f\right)}{\frac{\pi}{2} f} = \frac{1}{2}\,\text{sinc}\left(\tfrac{\pi}{2} f\right) \Rightarrow \boxed{\hat{U}(f) = \frac{1}{2}\,\text{sinc}\left(\frac{\pi}{2} f\right)}$$



(b) $U(x) = e^{-x^2/4}$



$$\hat{U}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/4} e^{-ifx}\, dx$$ ← in this case, it helps to define the fourier transform this way

← complete the square

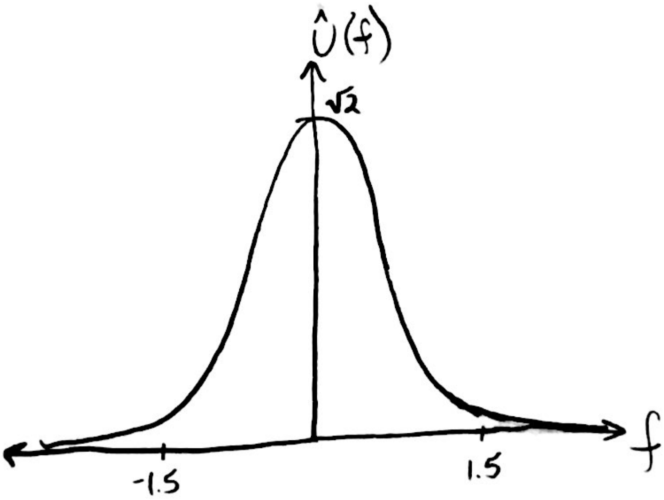
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}(x^2 + 4ifx - 4f^2) - f^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-f^2} \int_{-\infty}^{\infty} e^{-\frac{(x+2if)^2}{4}} dx$$ ← this integral can be calculated from

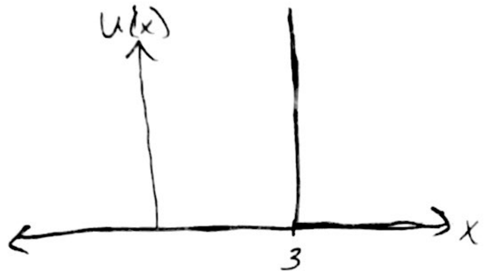
$$\int_{-\infty}^{\infty} e^{-\frac{(x+b)^2}{a}} dx = \sqrt{a\pi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-f^2} \left(\sqrt{4\pi}\right)$$

$$= \frac{2\cancel{\sqrt{\pi}}}{\sqrt{2}\cancel{\sqrt{\pi}}} e^{-f^2} \Rightarrow \boxed{\hat{U}(f) = \sqrt{2}\, e^{-f^2}}$$



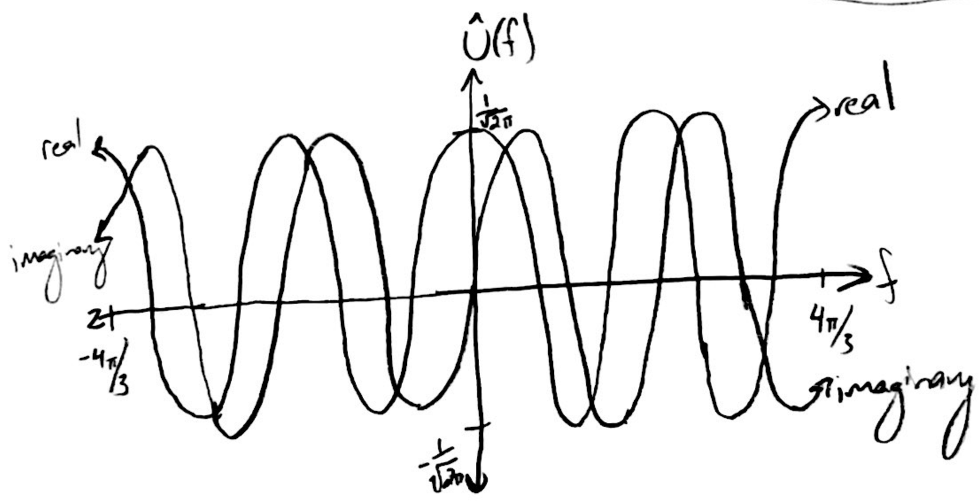
(c) $U(x) = \delta(x-3)$



$$\hat{U}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-3)\, e^{ifx}\, dx$$ → the integrand only has value where $x = 3$ so we can write

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-3)\, e^{3if}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{3if} \underbrace{\cancel{\int_{-\infty}^{\infty} \delta(x-3)\, dx}}_{=1} \Rightarrow \boxed{\hat{U}(f) = \frac{1}{\sqrt{2\pi}} e^{3if} = \frac{\cos(3f)}{\sqrt{2\pi}} + \frac{i\sin(3f)}{\sqrt{2\pi}}}$$



(d) $U(x) = \text{ReLU}(x) = x \cdot u(x)$ where $u(x)$ = step function

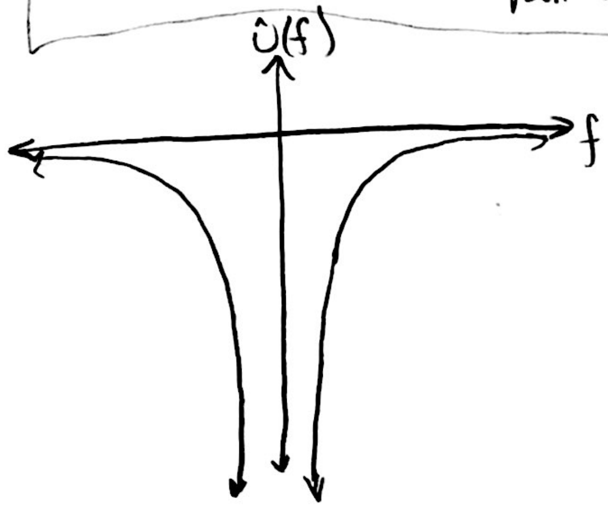
We see $U(x)$ is the multiplication of two functions $f(x) = x$ & $g(x) = u(x)$.
From the convolution theorem, we know that $\mathcal{F}[f(x)g(x)] = \mathcal{F}[f(x)] * \mathcal{F}[g(x)]$

We find that $\mathcal{F}[f(x)] = -i\sqrt{2\pi}\,\delta'(f)$

$$\mathcal{F}[g(x)] = \sqrt{\frac{\pi}{2}}\,\delta(f) + \frac{i}{\sqrt{2\pi}\, f}$$

If we convolve these Fourier transforms, we get:

$$\boxed{\mathcal{F}[U(x)] = \hat{U}(f) = -\frac{1}{\sqrt{2\pi}\, f^2} - i\sqrt{\frac{\pi}{2}}\,\delta'(f)}$$



(e) Prove $\mathcal{F}[u(x)v(x)] = \mathcal{F}[u(x)] * \mathcal{F}[v(x)]$

Given $u(x)$ and its Fourier transform $U(f)$
and $v(x)$ and its Fourier transform $V(\omega)$

We can write the functions $u(x)$ & $v(x)$ as the inverse transforms of their transforms:

$\mathcal{F}[u(x)] = U(f)$

$\mathcal{F}[v(x)] = V(f)$

$$u(x) = \int_{-\infty}^{\infty} U(\eta) e^{2\pi i \eta x} d\eta \qquad v(x) = \int_{-\infty}^{\infty} V(\omega) e^{2\pi i \omega x} d\omega$$

So $$u(x)v(x) = \int_{-\infty}^{\infty} U(\eta) e^{2\pi i \eta x} d\eta \cdot \int_{-\infty}^{\infty} V(\omega) e^{2\pi i \omega x} d\omega$$

Which we rewrite as

$$u(x)v(x) = \int_{-\infty}^{\infty} U(\eta) \underbrace{\int_{-\infty}^{\infty} V(\omega) e^{2\pi i(\omega+\eta)x} d\omega}_{} d\eta$$

for this integral we write $\omega = f - \eta$ ($d\omega = df$ since η is constant in the inner integral)

$$\Rightarrow u(x)v(x) = \int_{-\infty}^{\infty} U(\eta) \int_{-\infty}^{\infty} V(f-\eta) e^{2\pi i f x} df \, d\eta$$

Arrange the integrals again to get:

$$u(x)v(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} U(\eta) V(f-\eta) \, d\eta \right] e^{2\pi i f t} df$$

The inner integral is a convolution of the Fourier transforms $U(f)$ and $V(f)$, while the outer one is an inverse Fourier transform. So we can write this as:

$$u(x)v(x) = \mathcal{F}^{-1}\left[\mathcal{F}[u(x)] * \mathcal{F}[v(x)]\right]$$

then, taking the Fourier transform of both sides gives:

$$\mathcal{F}[u(x)v(x)] = \mathcal{F}[u(x)] * \mathcal{F}[v(x)]$$

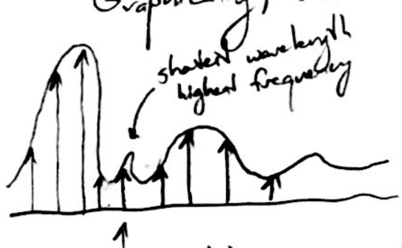
Problem 4

(a) In this case, we must compare the sampling frequency to the highest frequency in the field $U(x,y)$. According to the sampling theorem and Nyquist sampling, we must take at least two samples per wavelength of the highest frequency in order to fully recreate the field.

i.e.

$$\text{Sampling interval} < \frac{1}{2(\underset{\text{in sample}}{\text{Shortest Wavelength}})}$$

Graphically, this idea looks like this:



we only take one sample in this wavelength ⇒ we cannot recreate the field

We take at least two samples in the shortest wavelength ⇒ we can recreate the field

In the context of this problem, we need to check if

$$\underset{\text{(sampling interval)}}{\text{pixel pitch}} < \frac{1}{2\cdot\left(\begin{matrix}\text{max spatial}\\ \text{frequency}\end{matrix}\right)}$$

For a pixel pitch of 10 μm, we see

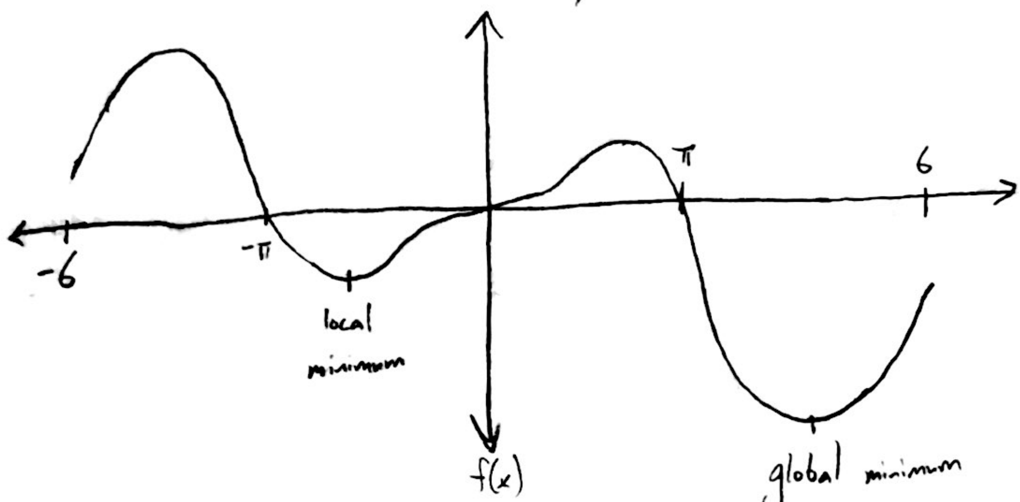
$$10\ \mu m \not< \frac{1}{2(0.1\mu m^{-1})} = 5\ \mu m$$

→ The inequality is not met, so a pixel pitch 10 μm cannot recreate the field

For a pixel pitch of 4 μm, we see:

$$4\,\mu m < \frac{1}{2(0.1\,\mu m^{-1})} = 5\,\mu m \Rightarrow$$ The inequality is met, so a pixel pitch of 4 μm can recreate the field

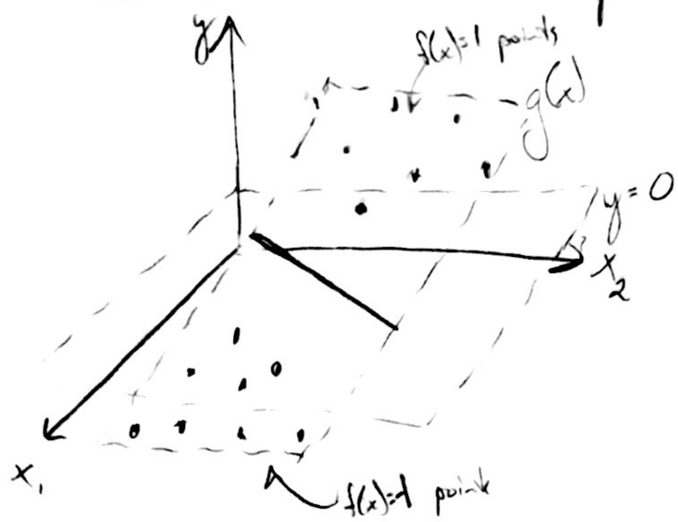
(b) Plot $f(x) = x^2 \sin x$ on $x \in [-6, 6]$



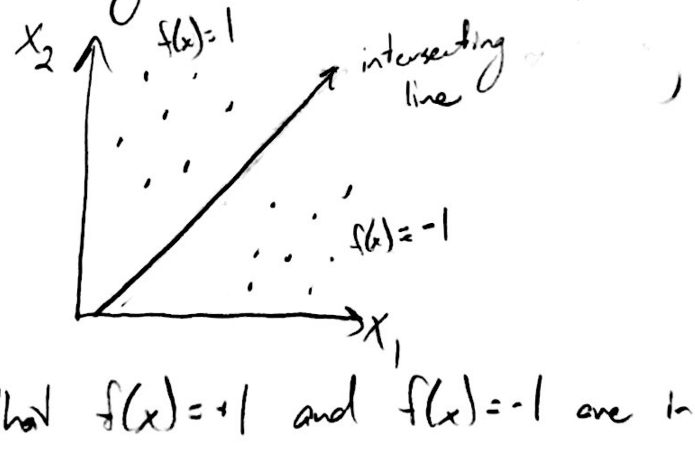
If we perform gradient descent, we are not guaranteed to reach the global minimum. For example, if we start with an $\pi < x < 6$, then gradient descent will take us down to the global minimum. But if we start with an $-\pi < x < 0$, then gradient descent will take us down the slope to the local minimum, not the global one.

(c) Given the function $f(x) = \text{sign}(\omega^T x)$ where $\omega = [3 \;\; 2 \;\; 1]$ are weights and $x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$ are data, we can think of the function $g(x) = \omega^T x$ as a plane of best fit through the points x and the function $f(x) = \text{sign}(g(x))$ separates these points onto the planes $y = +1$ & $y = -1$

As such, we can sketch the 3D space as:



As we see, these points are defined in 3D by 2 planes, by $g(x)=w^Tx$ and $y=0$ (functionally, this is the sign fxn).
Where these two planes intersect, they form a line, and because one of the two planes is $y=0$, this line must lie on that plane, i.e. it must lie in the x_1-x_2 plane. So we can look at these data projected on the x_1-x_2 plane and see that the intersecting line on that plane separates the data.



We see that $f(x)=+1$ and $f(x)=-1$ are in fact separated by a line.