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ME 5311: Computational Methods to Viscous Flows
Video Assignment 01
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Problem Statement: Watch all the video lectures, and derive the governing equation for the kinetic energy, defined as $k = \frac{1}{2}\left(u^2 + v^2 + w^2\right)$. The technique involves multiplying the x-momentum equation by the u velocity, y-momentum by the v velocity, and z-momentum equation by the w velocity, and eventually add all three equations up. Show your derivation process.

Solution:

We must start this derivation with the momentum equations as mentioned above. First, we have the x-momentum equation:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \tag{1}$$

Next, we have the y-momentum equation:

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My} \tag{2}$$

Finally, we have the z-momentum equation:

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + S_{Mz} \tag{3}$$

We can now multiply each equation by its respective velocity component, doing so yields the following for x-momentum:

$$\rho u \frac{Du}{Dt} = -u\frac{\partial p}{\partial x} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + uS_{Mx} \tag{4}$$

For y-momentum:

$$\rho v \frac{Dv}{Dt} = -v\frac{\partial p}{\partial y} + v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + vS_{My} \tag{5}$$

And for z-momentum:

$$\rho w \frac{Dw}{Dt} = -w\frac{\partial p}{\partial z} + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + wS_{Mz} \tag{6}$$

We can now add the three equations. Starting with the left hand side, we get:

$$\rho\left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right) \tag{7}$$

Now, we have already defined kinetic energy, $k = \frac{1}{2}\left(u^2 + v^2 + w^2\right)$. If we take the material derivative of k, we are left with:

$$\frac{Dk}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}u^2\right) + \frac{D}{Dt}\left(\frac{1}{2}v^2\right) + \frac{D}{Dt}\left(\frac{1}{2}w^2\right) \tag{8}$$

Applying the chain rule to each term leaves us with the following three equations:

$$\frac{D}{Dt}\left(\frac{1}{2}u^2\right) = u\frac{Du}{Dt} \tag{9}$$

$$\frac{D}{Dt}\left(\frac{1}{2}v^2\right) = v\frac{Dv}{Dt} \tag{10}$$

$$\frac{D}{Dt}\left(\frac{1}{2}w^2\right) = w\frac{Dw}{Dt} \tag{11}$$

Therefore, we have:

$$\frac{Dk}{Dt} = u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt} \tag{12}$$

And we can re-write the left hand side of our equation as:

$$\rho\left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right) = \rho\frac{Dk}{Dt} \tag{13}$$

Now, we can turn our attention to the right hand side of our equation. Combining all of the pressure terms yields:

$$-u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} - w\frac{\partial p}{\partial z} \tag{14}$$

This can also be expressed as:

$$-\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) = -\vec{u}\cdot\nabla p \tag{15}$$

We can now combine all of the viscous stress terms. Recalling that from x-momentum we have:

$$u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \tag{16}$$

From y-momentum we have:

$$v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \tag{17}$$

And from z-momentum we have:

$$w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) \tag{18}$$

Finally, we can combine the source terms to get:

$$uS_{Mx} + vS_{My} + wS_{Mz} \tag{19}$$

We can recognize this as the dot product of the velocity vector and the source term vector, therefore we can write:

$$uS_{Mx} + vS_{My} + wS_{Mz} = \vec{u} \cdot \vec{S}_M \tag{20}$$

Finally, we can combine all aforementioned terms to get the governing equation for the kinetic energy:

$$\boxed{\begin{aligned} \rho\frac{Dk}{Dt} = & -\vec{u} \cdot \nabla p \\ & + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \\ & + v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \\ & + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) \\ & + \vec{u} \cdot \vec{S}_M \end{aligned}} \tag{21}$$

As mentioned in "An Introduction to Computational Fluid Dynamics: The Finite Volume Method" by H.K. Versteeg and W. Malalasekera, deriving the governing equation for the kinetic energy allows us to separate the kinetic/mechanical contributions (effects of pressure forces, viscous stresses, and related source terms) to the total energy of the fluid from the contributions of internal energy, heat transfer, viscous dissipation, and other sources. Doing so also allows us to obtain expressions for quantities such as internal energy, i or temperature, T.