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ME M311: Computational Methods to Viscous Flows

Video Assignment 01

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Problem Statement: Watch all the video lectures, and derive the governing equation for the kinetic energy, defined as $k = \frac{1}{2}(u^2 + v^2 + w^2)$. The technique involves multiplying the x-momentum equation by the u velocity, y-momentum by the v velocity, and z-momentum equation by the w velocity, and eventually add all three equations up. Show your derivation process.

Solution:

We must start this derivation with the momentum equations as mentioned above. First, we have the x-momentum equation:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \quad (1)$$

Next, we have the y-momentum equation:

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My} \quad (2)$$

Finally, we have the z-momentum equation:

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + S_{Mz} \quad (3)$$

We can now multiply each equation by its respective velocity component, doing so yields the following for x-momentum:

$$\rho u \frac{Du}{Dt} = -u \frac{\partial p}{\partial x} + u \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + u S_{Mx} \quad (4)$$

For y-momentum:

$$\rho v \frac{Dv}{Dt} = -v \frac{\partial p}{\partial y} + v \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + v S_{My} \quad (5)$$

And for z-momentum:

$$\rho w \frac{Dw}{Dt} = -w \frac{\partial p}{\partial z} + w \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + w S_{Mz} \quad (6)$$

We can now add the three equations. Starting with the left hand side, we get:

$$\rho \left(u \frac{Du}{Dt} + v \frac{Dv}{Dt} + w \frac{Dw}{Dt} \right) \quad (7)$$

Now, we have already defined kinetic energy, $k = \frac{1}{2}(u^2 + v^2 + w^2)$. If we take the material derivative of k, we are left with:

$$\frac{Dk}{Dt} = \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) + \frac{D}{Dt} \left(\frac{1}{2} v^2 \right) + \frac{D}{Dt} \left(\frac{1}{2} w^2 \right) \quad (8)$$

Applying the chain rule to each term leaves us with the following three equations:

$$\frac{D}{Dt} \left(\frac{1}{2} u^2 \right) = u \frac{Du}{Dt} \quad (9)$$

$$\frac{D}{Dt} \left(\frac{1}{2} v^2 \right) = v \frac{Dv}{Dt} \quad (10)$$

$$\frac{D}{Dt} \left(\frac{1}{2} w^2 \right) = w \frac{Dw}{Dt} \quad (11)$$

Therefore, we have:

$$\frac{Dk}{Dt} = u \frac{Du}{Dt} + v \frac{Dv}{Dt} + w \frac{Dw}{Dt} \quad (12)$$

And we can re-write the left hand side of our equation as:

$$\rho \frac{Dk}{Dt} \quad (13)$$

Now, we can turn our attention to the right hand side of our equation. Combining all of the pressure terms yeilds:

$$-u \frac{\partial p}{\partial x} - v \frac{\partial p}{\partial y} - w \frac{\partial p}{\partial z} \quad (14)$$

This can also be expressed as:

$$-\left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} \right) = -\vec{u} \cdot \nabla p \quad (15)$$

We can now combine all of the viscous stress terms. Recalling that from x-momentum we have:

$$u \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \quad (16)$$

From y-momentum we have:

$$v \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \quad (17)$$

And from z-momentum we have:

$$w \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \quad (18)$$

Finally, we can combine the source terms to get:

$$u S_{Mx} + v S_{My} + w S_{Mz} \quad (19)$$

We can recognize this as the dot product of the velocity vector and the source term vector, therefore we can write:

$$uS_{Mx} + vS_{My} + wS_{Mz} = \vec{u} \cdot \vec{S}_M \quad (20)$$

Finally, we can combine all aforementioned terms to get the governing equation for kinetic energy:

$$\begin{aligned} \rho \frac{Dk}{Dt} = & -\vec{u} \cdot \nabla p \\ & + u \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \\ & + v \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \\ & + w \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \\ & + \vec{u} \cdot \vec{S}_M \end{aligned} \quad (21)$$