We can already see this may be possible by examining the case $k = 1$. Consider the dataset generated as in (1) in the noiseless case $\eta = 0$, i.e.:

$$\mathbf{z}_i = \mathbf{A}\mathbf{x}_i, \quad i = 1, \ldots, N, \tag{1}$$

with the additional constraint that $\|\mathbf{A}\mathbf{e}_i\|_2 = 1$ for all $i \in [m]$ and $\|\mathbf{A}\|_2 \leq 1$. Consider the following "convexified" version of Prob. **??**:

Problem 1. *Find a matrix* $\mathbf{B}$ *with* $\|\mathbf{B}\|_2 \leq 1$ *and vectors* $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_N$ *solving:*

$$\min \sum_{i=1}^{N} \|\overline{\mathbf{x}}_i\|_1 \quad \textit{subject to} \quad \mathbf{z}_i = \mathbf{B}\overline{\mathbf{x}}_i, \ \textit{for all } i. \tag{2}$$

Proposition 1. *Fix* $c > 1$. *If* $\mathbf{x}_i = c\mathbf{e}_i$ *for* $i = 1, \ldots, m$, *then every solution to Prob. 1 satisfies* $\mathbf{A} = \mathbf{B}\mathbf{P}$ *and* $\mathbf{x}_i = \mathbf{P}^\top \overline{\mathbf{x}}_i$ *for some* $m \times m$ *permutation matrix* $\mathbf{P}$.

Proof. For all $i \in [m]$, since $\mathbf{B}\overline{\mathbf{x}}_i = c\mathbf{A}\mathbf{e}_i$, we have:

$$c = c\|\mathbf{A}\mathbf{e}_i\|_2 = \|\mathbf{B}\mathbf{x}_i\|_2 \leq \|\mathbf{B}\|_2 \|\overline{\mathbf{x}}_i\|_2 \leq \|\overline{\mathbf{x}}_i\|_2 \leq \|\overline{\mathbf{x}}_i\|_1. \tag{3}$$

Since $\mathbf{A}$ and $c\mathbf{e}_i$ $(i = 1, \ldots, m)$ solve (2), we must have:

$$\sum_{i=1}^{m} \|\overline{\mathbf{x}}_i\|_1 \leq \sum_{i=1}^{m} \|c\mathbf{e}_i\|_1 = mc. \tag{4}$$

thus by (3) it must be the case that $\|\overline{\mathbf{x}}_i\|_2 = \|\overline{\mathbf{x}}_i\|_1 = c$. Now, writing $\overline{\mathbf{x}}_i = \sum_{j=1}^{m} \overline{c}_j \mathbf{e}_j$, we have:

$$\sum_{j=1}^{m} \overline{c}_j^2 = \left(\sum_{j=1}^{m} |\overline{c}_j| \right)^2 = \sum_{j=1}^{m} \overline{c}_j^2 + \sum_{j=1}^{m} |\overline{c}_j| \sum_{\ell \neq j} |\overline{c}_\ell| \tag{5}$$

We therefore have $\overline{c}_j \sum_{\ell \neq j} |\overline{c}_\ell| = 0$ for all $j \in [m]$. Since $\overline{\mathbf{x}}_i \neq 0$, we must have $\overline{c}_j \neq 0$ for at least some $j \in [m]$, in which case $c_\ell = 0$ for all $\ell \neq j$, and we can be sure $\overline{\mathbf{x}}_i$ is a 1-sparse vector. Since this applies for all $i \in [m]$, we may appeal to Thm. **??** (with $k = 1$). $\square$