ℓ_1-norm minimization

Charles J. Garfinkle

Problem 1. *Find matrices $\mathbf{B}$ and vectors $\overline{\mathbf{x}}^{(1)}, \ldots, \overline{\mathbf{x}}^{(N)}$ solving:*

$$\min \sum_{i=1}^{N} \|\overline{\mathbf{x}}^{(i)}\|_0 \quad \text{subject to} \quad \|\mathbf{z}^{(i)} - \mathbf{B}\overline{\mathbf{x}}^{(i)}\|_2 \leq \eta_0, \text{ for all } i \tag{1}$$

by solving:

$$\min \sum_{i=1}^{N} \|\overline{\mathbf{x}}^{(i)}\|_1 \quad \text{subject to} \quad \|\mathbf{z}^{(i)} - \mathbf{B}\overline{\mathbf{x}}^{(i)}\|_2 \leq \eta_1, \text{ for all } i. \tag{2}$$

Proof for $k = 1$. Since the only 1-uniform hypergraph with the SIP is $[m]$, which is obviously regular, we require only $\mathbf{x}^{(i)} = c_i \mathbf{e}_i$ for $i \in [m]$, with $c_i \neq 0$ to guarantee linear independence. While we have yet to define C_1 generally, in this case we may set $C_1 = 1/\min_{\ell \in [m]} |c_\ell|$.

Fix $\mathbf{A} \in \mathbb{R}^{n \times m}$ satisfying $L_2(\mathbf{A}) > 0$, since here we have $2\mathcal{H} = \binom{[m]}{2}$, and suppose some matrix $\mathbf{B}$ and vectors $\overline{\mathbf{x}}^{(i)} \in \mathbb{R}^{\overline{m}}$ have $\|\mathbf{A}\mathbf{x}^{(i)} - \mathbf{B}\overline{\mathbf{x}}^{(i)}\|_2 \leq \varepsilon$ for all i. Note that $\overline{\mathbf{x}}^{(i)} \neq 0$, since otherwise we would reach the following contradiction: $\|\mathbf{A}_i\|_2 \leq C_1 |c_i| \|\mathbf{A}_i\|_2 \leq \|\mathbf{A}\mathbf{x}_i\|_2 \leq C_1 \varepsilon \leq (?) < \min_{\ell \in [m]} \|\mathbf{A}_\ell\|_2$.

Let $\overline{\mathbf{x}}^{(i)} = \sum_{j=1}^{m} \overline{c}_j^{(i)} \mathbf{e}_j$ and let $\pi : [m] \to [\overline{m}]$ be the map $\pi(i) = \arg\max_j \overline{c}_j^{(i)}$. By the triangle inequality,

$$\|c_i \mathbf{A}_i - \overline{c}_{\pi(i)}^{(i)} \mathbf{B}_{\pi(i)}\|_2 - \|\sum_{k \neq \pi(i)} \mathbf{B}_k \overline{c}_k^{(i)}\|_2 \leq \|\mathbf{A}\mathbf{x}^{(i)} - \mathbf{B}\overline{\mathbf{x}}^{(i)}\|_2 \leq \varepsilon \tag{3}$$

Hence,

$$\begin{aligned}
\|c_i \mathbf{A}_i - \overline{c}_{\pi(i)}^{(i)} \mathbf{B}_{\pi(i)}\|_2 &\leq \varepsilon + \|\sum_{k \neq \pi(i)} \mathbf{B}_k \overline{c}_k^{(i)}\|_2 && (4)\\
&\leq \varepsilon + \|\mathbf{B}\|_2 \|\overline{\mathbf{x}}_{\pi(i)=0}^{(i)}\|_1 && (5)\\
&= \varepsilon + \|\mathbf{A}\|_2 \|\overline{\mathbf{x}}_{\pi(i)=0}^{(i)}\|_1 && (6)\\
& && (7)
\end{aligned}$$

We now show that π is injective (in particular, a permutation if $\overline{m} = m$). Suppose that $\pi(i) = \pi(j) = \ell$ for some $i \neq j$ and ℓ. Then we have:

$$\begin{aligned}
(|\overline{c}_\ell^{(i)}| + |\overline{c}_\ell^{(j)}|)\left(\varepsilon + \|\mathbf{A}\|_2 \|\overline{\mathbf{x}}_{\pi(i)=0}^{(i)}\|_1\right) &\geq |\overline{c}_\ell^{(i)}| \|c_j \mathbf{A}_j - \overline{c}_\ell^{(j)} \mathbf{B}_\ell\|_2 + |\overline{c}_\ell^{(j)}| \|c_i \mathbf{A}_i - \overline{c}_\ell^{(i)} \mathbf{B}_\ell\|_2 \\
&\geq \|\mathbf{A}(\overline{c}_\ell^{(i)} c_j \mathbf{e}_j - \overline{c}_\ell^{(j)} c_i \mathbf{e}_i)\|_2 \\
&\geq \sqrt{2} L_2(\mathbf{A}) \|\overline{c}_\ell^{(i)} c_j \mathbf{e}_j - \overline{c}_\ell^{(j)} c_i \mathbf{e}_i\|_2 \\
&\geq L_2(\mathbf{A}) (|\overline{c}_\ell^{(i)}| + |\overline{c}_\ell^{(j)}|) \min_{\ell \in [m]} |c_\ell|,
\end{aligned}$$

hence, assuming $\|\mathbf{A}\|_2 = 1$,

$$\varepsilon \geq L_2(\mathbf{A}) \min_{\ell \in [m]} |c_\ell| - \|\overline{\mathbf{x}}_{\pi(i)=0}^{(i)}\|_1$$

contradicting our assumed upper bound on ε. Hence, the map π is injective and so $\overline{m} \geq m$.

Letting $\mathbf{P}$ and $\mathbf{D}$ be the $\overline{m} \times \overline{m}$ permutation and invertible diagonal matrices with, respectively, columns $\mathbf{e}_{\pi(i)}$ and $\frac{\overline{c}_i}{c_i} \mathbf{e}_i$ for $i \in [m]$ (otherwise, $\mathbf{e}_i$ for $i \in [\overline{m}] \setminus [m]$), we may rewrite (4) to see that for all $i \in [m]$:

$$\|\mathbf{A}_i - (\mathbf{BPD})_i\|_2 = \|\mathbf{A}_i - \frac{\overline{c}_i}{c_i} \mathbf{B}_{\pi(i)}\|_2 \leq \frac{\varepsilon + |c_i|}{|c_i|} \leq C_1 \varepsilon + 1.$$

$\square$