A Tale of Two Dictionary Learning Problems

by

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Abstract

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Learning optimal dictionaries for sparse coding has exposed characteristic sparse features of many natural signals. However, universal guarantees of the uniqueness and stability of such features in the presence of noise are lacking. This work presents very general conditions guaranteeing when dictionaries yielding the sparsest encodings of a dataset are unique and stable with respect to measurement or modeling error. The stability constants are explicit and computable; as such, there is an effective procedure sufficient to affirm if a proposed solution to the dictionary learning problem is unique within bounds commensurate with the noise. Beyond the extension of existing results to the noisy regime, a theory of combinatorial designs for sparse supports is introduced to demonstrate that some or all generating dictionary elements are recoverable from noisy data even if the dictionary fails to satisfy the spark condition, its size is overestimated, or only a polynomial number of distinct supports appear in the encoded data. Importantly, the guarantees derived here assume no constraints on the recovered dictionary beyond a natural upper bound on its size. The work closes with some remaining open challenges to seed future research directions.

Contents

Contents i

List of Figures ii

1 Introduction 1
1.1 The dictionary learning problem(s) . . . 3
1.2 Outline of the thesis . . . 4

2 Results 5
2.1 Introduction . . . 5
2.2 Combinatorial designs . . . 5
2.3 Uniqueness theorems . . . 6
2.4 Discussion . . . 9

3 Proofs 11
3.1 Introduction . . . 11
3.2 Proving the case $k = 1$ . . . 12
3.3 The main lemma . . . 12
3.4 Proving the general case $k < m$ . . . 14
3.5 Discussion . . . 16

4 Discussion 19
4.1 Conclusions . . . 19
4.2 Future directions . . . 20

5 Appendix: Proving the main lemma 26

Bibliography 29

List of Figures

2.1 **Learning a dictionary from increasingly noisy data**. The (unraveled) basis elements of the 8×8 discrete cosine transform (DCT) form the 64 columns of the left-most matrix above. Three increasingly imprecise dictionaries (columns reordered to best match original) are recovered by FastICA [28] trained on data generated from 8-sparse linear combinations of DCT elements corrupted with additive noise (increasing from left to right). 8

4.1 **Learning a dictionary from an arbitrary number of samples.** Probability of successful dictionary and code recovery (as per Thm. 1) for a number of samples N given as a fraction of the deterministic sample complexity $N = |\mathcal{H}|[(k-1)\binom{m}{k} + 1]$ when the support set hypergraph $\mathcal{H}$ is the set of m consecutive intervals of length k in a cyclic order on $[m]$. Each plot has k ranging from 2 to $m-1$ (the case $k = 1$ requires $N = m$), with lighter grey lines corresponding to larger k. Successful recovery is nearly certain with far fewer samples than the deterministic sample complexity. 24

4.2 **Concentration of the constant C_2.** Distribution of $C_2(\mathbf{A}, \mathcal{H})$ computed for 1.33x overcomplete generic unit-norm dictionaries $\mathbf{A} \in \mathbb{R}^{n\times m}$ (i.e. with $n = 3m/4$) when the support set hypergraph $\mathcal{H}$ consists of the rows and columns formed by arranging the elements of $[m]$ into a square grid (i.e. $m = k^2$). The distribution becomes more concentrated as m grows. 25

Chapter 1

Introduction

It is a longstanding practice in the field of signal processing to describe signals as linear combinations of elementary waveforms from a pre-specified "dictionary". When this dictionary forms a basis for the signal space, every signal has a unique decomposition into these atomic components. In the simplest case, the basis is orthonormal and the representational coefficient scaling a given elementary waveform is merely the inner product of that waveform with the signal.

Until recently, bases have been the default form of signal representation due largely to their simplicity. For many signal analysis tasks, however, there is no one basis expressive enough to reveal clearly all of the relevant features of the signal. For example, a signal can be decomposed into its constituent frequencies via the Fourier transform, a linear change of basis. If our signal can be either a sine wave or a delta function, neither the standard basis nor the Fourier basis can capture one case as effectively as it can the other.

The need for greater freedom of expression led to the development of redundant signal representations utilizing overcomplete dictionaries containing more atoms than there are dimensions of the signal. In this case, there are infinitely many ways in which a signal may be decomposed into its constituent components, and the intention to seek the most informative such representation as measured by some task-specific cost function.

A popular approach to the design of overcomplete dictionaries has been to seek one with respect to which every signal in the signal class of interest has a sparse representation; that is, it can be represented, or at least well-approximated, as a combination of only a few dictionary elements from the bunch. Concluding our example, the union of the standard basis with the Fourier basis is an over-complete dictionary with respect to which both sines and delta functions (or any superposition thereof) have the sparsest possible representations as combinations of its elementary waveforms.

Early approaches to sparse coding assumed, as was assumed in our recurring example, a model of the signal class from which a suitable sparsifying dictionary could then be derived. While such dictionaries are typically characterized by an analytic formulation and a fast implicit implementation, these models tend to be over-simplistic when applied to natural phenomena.

In contrast, an alternative approach to dictionary design is conditioned on the assumption that the structure of complex natural phenomena can be more accurately extracted directly from a training dataset of signals, a process referred to as dictionary learning. In the seminal work [38], the authors trained a dictionary for sparse representation of small image patches collected from a number of images of the natural environment. Remarkably, the relatively simple algorithm they applied produced dictionary elements sharing qualitative similarities with linear filters estimated from response properties of simple-cell neurons in mammalian visual cortex, which until then were more weakly described by the analytic Gabor filters (see also [26, 7, 22]). Curiously, these waveforms (e.g., Gabor'-like wavelets) appear in dictionaries learned by a variety of algorithms trained over different natural image datasets, suggesting that learned features in natural signals may, in some sense, be canonical [11].

Sparse coding is a modern signal processing technique that views each of N observed n-dimensional signal samples as a (noisy) linear combination of at most k elementary waveforms drawn from a "dictionary" of size $m \ll N$ (see [51] for a comprehensive review). Optimizing dictionaries subject to this and related sparsity constraints has revealed seemingly characteristic sparse structure in several signal classes of current interest (e.g., in vision [49]). Of particular note are the seminal works in the field [38, 26, 7, 22], which discovered that dictionaries optimized for coding small patches of "natural" images share qualitative similarities with linear filters estimated from response properties of simple-cell neurons in mammalian visual cortex. Curiously, these waveforms (e.g., "Gabor" wavelets) appear in dictionaries learned by a variety of algorithms trained over different natural image datasets, suggesting that learned features in natural signals may, in some sense, be canonical [11].

Motivated by these discoveries and more recent work relating compressed sensing [14] to a theory of information transmission through random wiring bottlenecks in the brain [29], this thesis concerns itself with the following question: when is a dictionary indeed identifiable from data? Answers to this question may also have implications in practice wherever an appeal is made to latent sparse structure of data (e.g., forgery detection [25, 39]; brain recordings [30, 1, 33]; and gene expression [50]). While several algorithms have recently been proposed to provably recover unique dictionaries under specific conditions (see [45, Sec. I-E] for a summary of the state-of-the-art), few theorems can be invoked to justify the consistency of inference under this model of data more broadly. To my knowledge, a universal guarantee of the uniqueness and stability of learned dictionaries and the sparse representations they induce over noisy data has yet to appear in the literature.

In this work, it is proven very generally that uniqueness and stability is a typical property of learned dictionaries. More specifically, it is demonstrated that matrices injective on a sparse domain are identifiable from $N = m(k-1)\binom{m}{k} + m$ noisy linear combinations of k of their m columns up to an error that is linear in the noise (Thm. 1). In fact, provided $n \geq \min(2k, m)$, in almost all cases the problem is well-posed, as per Hadamard [21], given a sufficient amount of data (Thm. 3 and Cor. 2). These guarantees also hold for the related (and perhaps more commonly posed, e.g. [42]) optimization problem seeking a dictionary minimizing the average number of elementary waveforms required to reconstruct each sample of the dataset (Thm. 2). To practical benefit, the derivations impose no restrictions on learned dictionaries (e.g., that they, too, be injective over some sparse domain) beyond an upper bound on dictionary size, which is necessary in any case to avoid trivial solutions (e.g., allowing $m = N$).

1.1 The dictionary learning problem(s)

Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ be a matrix with columns $\mathbf{A}_j$ ($j = 1, \ldots, m$) and let dataset Z consist of measurements:

$$\mathbf{z}_i = \mathbf{A}\mathbf{x}_i + \mathbf{n}_i, \quad i = 1, \ldots, N, \tag{1.1}$$

for *k-sparse* $\mathbf{x}_i \in \mathbb{R}^m$ having at most $k < m$ nonzero entries and *noise* $\mathbf{n}_i \in \mathbb{R}^n$, with bounded norm $\|\mathbf{n}_i\|_2 \leq \eta$ representing our worst-case uncertainty in measuring the product $\mathbf{A}\mathbf{x}_i$. Let us first consider the following decidable[1] formulation of the dictionary learning problem.

Problem 1. *Find a dictionary matrix* $\mathbf{B}$ *and* k*-sparse codes* $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_N$ *that satisfy* $\|\mathbf{z}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \eta$ *for all* $i = 1, \ldots, N$.

Note that every solution to Prob. 1 represents infinitely many equivalent alternatives $\mathbf{BPD}$ and $\mathbf{D}^{-1}\mathbf{P}^\top\overline{\mathbf{x}}_1, \ldots, \mathbf{D}^{-1}\mathbf{P}^\top\overline{\mathbf{x}}_N$ parametrized by a choice of permutation matrix $\mathbf{P}$ and invertible diagonal matrix $\mathbf{D}$. Identifying these ambiguities (labelling and scale) yields a single orbit of solutions represented by any particular set of elementary waveforms (the columns of $\mathbf{B}$) and their associated sparse coefficients (the entries of $\overline{\mathbf{x}}_i$) that reconstruct each data point $\mathbf{z}_i$.

Previous theoretical work addressing the noiseless case $\eta = 0$ (e.g., [35, 18, 2, 23]) for matrices $\mathbf{B}$ having exactly m columns has shown that a solution to Prob. 1, when it exists, is unique up to such relabeling and rescaling provided the $\mathbf{x}_i$ are sufficiently diverse and $\mathbf{A}$ satisfies the *spark condition*:

$$\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2 \implies \mathbf{x}_1 = \mathbf{x}_2, \quad \text{for all } k\text{-sparse } \mathbf{x}_1, \mathbf{x}_2, \tag{1.2}$$

which is necessary to guarantee the uniqueness of arbitrary k-sparse $\mathbf{x}_i$. These results can be generalized to the practical setting $\eta > 0$ by considering the following natural notion of stability with respect to measurement error.

Definition 1. *Fix* $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_N\} \subset \mathbb{R}^n$. *We say* Y *has a* ***k-sparse representation in*** $\mathbb{R}^m$ *if there exists a matrix* $\mathbf{A}$ *and* k*-sparse* $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^m$ *such that* $\mathbf{y}_i = \mathbf{A}\mathbf{x}_i$ *for all* i. *This representation is* ***stable*** *if for every* $\delta_1, \delta_2 \geq 0$, *there exists some* $\varepsilon = \varepsilon(\delta_1, \delta_2)$ *that is strictly positive for positive* δ_1 *and* δ_2 *such that if* $\mathbf{B}$ *and* k*-sparse* $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_N \in \mathbb{R}^m$ *satisfy:*

$$\|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \varepsilon(\delta_1, \delta_2), \quad \textit{for all } i = 1, \ldots, N,$$

[1] Note that Prob. 1 is decidable for rational inputs $\mathbf{z}_i$ [24] since the statement that it has a solution can be expressed as a logical sentence in the theory of algebraically closed fields, and this theory has quantifier elimination [5].

then there is some permutation matrix $\mathbf{P}$ *and invertible diagonal matrix* $\mathbf{D}$ *such that for all* i, j*:*

$$\|\mathbf{A}_j - (\mathbf{BPD})_j\|_2 \leq \delta_1 \text{ and } \|\mathbf{x}_i - \mathbf{D}^{-1}\mathbf{P}^\top \overline{\mathbf{x}}_i\|_1 \leq \delta_2. \tag{1.3}$$

To see how Prob. 1 motivates Def. 1, suppose that Y has a stable k-sparse representation in $\mathbb{R}^m$ and fix δ_1, δ_2 to be the desired accuracies of recovery in (1.3). Consider any dataset Z generated as in (1.1) with $\eta \leq \frac{1}{2}\varepsilon(\delta_1, \delta_2)$. Using the triangle inequality, it follows that any $n \times m$ matrix $\mathbf{B}$ and k-sparse $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_N$ solving Prob. 1 are necessarily within δ_1 and δ_2 of the original dictionary $\mathbf{A}$ and codes $\mathbf{x}_1, \ldots, \mathbf{x}_N$, respectively.[2]

The main result of this work is a very general uniqueness theorem (Thm. 1) directly implying (Cor. 1), which guarantees that sparse representations of a dataset Z are unique up to noise whenever generating dictionaries $\mathbf{A}$ satisfy a spark condition on supports and the original sparse codes $\mathbf{x}_i$ are sufficiently diverse (e.g., Fig. 2.1). Moreover, an explicit, computable $\varepsilon(\delta_1, \delta_2)$ that is linear in desired accuracy δ_1, and essentially so in δ_2, is defined in (2.4).

The next chapter gives formal statements of these findings. The same guarantees are then extended (Thm. 2) to the following alternate formulation of the dictionary learning problem, which seeks to minimize the total number of nonzero entries in sparse codes.

Problem 2. *Find matrices* $\mathbf{B}$ *and vectors* $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_N$ *solving:*

$$\min \sum_{i=1}^{N} \|\overline{\mathbf{x}}_i\|_0 \quad \text{subject to} \quad \|\mathbf{z}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \eta, \text{ for all } i. \tag{1.4}$$

Surprisingly, the development of Thm. 1 is general enough to provide some uniqueness and stability even when generating $\mathbf{A}$ do not fully satisfy (1.2) and recovery dictionaries $\mathbf{B}$ have more columns than $\mathbf{A}$. Moreover, the approach incorporates a theory of combinatorial designs for the sparse supports of generating codes $\mathbf{x}_i$ that should be of independent interest. These results are then adapted to apply to dictionaries and codes drawn from arbitrary (continuous) probability distributions (Cor. 2).

1.2 Outline of the thesis

TODO

[2] We mention that the different norms in (1.3) reflect the distinct meanings typically ascribed to the dictionary and sparse codes in modeling data.

Chapter 2

Results

2.1 Introduction

TODO

2.2 Combinatorial designs

A precise statement of the results requires first the identification of some combinatorial criteria on the supports[1] of sparse vectors. Let $\{1, \ldots, m\}$ be denoted $[m]$, its power set $2^{[m]}$, and $\binom{[m]}{k}$ the set of subsets of $[m]$ of size k. A *hypergraph* on vertices $[m]$ is simply any subset $\mathcal{H} \subseteq 2^{[m]}$. Let us say that $\mathcal{H}$ is k-*uniform* when $\mathcal{H} \subseteq \binom{[m]}{k}$. The *degree* $\deg_{\mathcal{H}}(i)$ of a node $i \in [m]$ is the number of sets in $\mathcal{H}$ that contain i, and we say $\mathcal{H}$ is *regular* when for some r we have $\deg_{\mathcal{H}}(i) = r$ for all i (given such an r, we say $\mathcal{H}$ is r-*regular*). Let us also write $2\mathcal{H} := \{S \cup S' : S, S' \in \mathcal{H}\}$. The following class of structured hypergraphs is a key ingredient in this work.

Definition 2. *Given $\mathcal{H} \subseteq 2^{[m]}$, the **star** $\sigma(i)$ is the collection of sets in $\mathcal{H}$ containing i. We say $\mathcal{H}$ has the **singleton intersection property (SIP)** when $\cap\sigma(i) = \{i\}$ for all $i \in [m]$.*

Next, a quantitative generalization of the spark condition (1.2) to combinatorial subsets of supports is given. The *lower bound* of an $n \times m$ matrix $\mathbf{M}$ is the largest α with $\|\mathbf{M}\mathbf{x}\|_2 \geq \alpha\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^m$ [20]. By compactness of the unit sphere, every injective linear map has a positive lower bound; hence, if $\mathbf{M}$ satisfies (1.2), then submatrices formed from $2k$ of its columns or less have strictly positive lower bounds.

The lower bound of a matrix is generalized below in (2.1) by restricting it to the spans of certain submatrices[2] associated with a hypergraph $\mathcal{H} \subseteq \binom{[m]}{k}$ of column indices. Let $\mathbf{M}_S$

[1] Recall that a vector $\mathbf{x}$ is said to be *supported* in S when $\mathbf{x} \in \text{span}\{\mathbf{e}_j : j \in S\}$, with $\mathbf{e}_j$ forming the standard column basis.

[2] See [48] for an overview of the related "union of subspaces" model.

denote the submatrix formed by the columns of a matrix $\mathbf{M}$ indexed by $S \subseteq [m]$ (setting $\mathbf{M}_\emptyset := \mathbf{0}$). In the sections that follow, let $\boldsymbol{\mathcal{M}}_S$ denote the column-span of a submatrix $\mathbf{M}_S$, and $\boldsymbol{\mathcal{M}}_\mathcal{G}$ to denote $\{\boldsymbol{\mathcal{M}}_S\}_{S\in\mathcal{G}}$. Define:

$$L_\mathcal{H}(\mathbf{M}) := \min\left\{\frac{\|\mathbf{M}_S\mathbf{x}\|_2}{\sqrt{k}\|\mathbf{x}\|_2} : S \in \mathcal{H},\ \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^{|S|}\right\}, \tag{2.1}$$

writing also L_k in place of $L_\mathcal{H}$ when $\mathcal{H} = \binom{[m]}{k}$.[3] As explained above, compactness implies that $L_{2k}(\mathbf{M}) > 0$ for all $\mathbf{M}$ satisfying (1.2). Clearly, $L_{\mathcal{H}'}(\mathbf{M}) \geq L_\mathcal{H}(\mathbf{M})$ whenever $\mathcal{H}' \subseteq \mathcal{H}$, and similarly any k-uniform $\mathcal{H}$ satisfying $\cup\mathcal{H} = [m]$ has $L_2 \geq L_{2\mathcal{H}} \geq L_{2k}$ (letting $L_{2k} := L_m$ whenever $2k > m$).

2.3 Uniqueness theorems

Deterministic Uniqueness Theorems

We are now in a position to state the main result, though for expository purposes the quantity C_1 will be leftundefined until Chap. 3. All results below assume real matrices and vectors.

Theorem 1. *If an $n \times m$ matrix $\mathbf{A}$ satisfies $L_{2\mathcal{H}}(\mathbf{A}) > 0$ for some r-regular $\mathcal{H} \subseteq \binom{[m]}{k}$ with the SIP, and k-sparse $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^m$ include more than $(k-1)\binom{m}{k}$ vectors in general linear position[4] supported in each $S \in \mathcal{H}$, then the following recovery guarantees hold for $C_1 > 0$ given by* (3.11).

Dictionary Recovery: *Fix $\varepsilon < L_2(\mathbf{A})/C_1$.[5] If an $n \times \overline{m}$ matrix $\mathbf{B}$ has, for every $i \in [N]$, an associated k-sparse $\overline{\mathbf{x}}_i$ satisfying $\|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \varepsilon$, then $\overline{m} \geq m$, and provided that $\overline{m}(r-1) < mr$, there is a permutation matrix $\mathbf{P}$ and an invertible diagonal matrix $\mathbf{D}$ such that:*

$$\|\mathbf{A}_j - (\mathbf{BPD})_j\|_2 \leq C_1\varepsilon, \quad \textit{for all } j \in J, \tag{2.2}$$

for some $J \subseteq [m]$ of size $m - (r-1)(\overline{m} - m)$.

Code Recovery: *If, moreover, $\mathbf{A}_J$ satisfies* (1.2) *and $\varepsilon < L_{2k}(\mathbf{A}_J)/C_1$, then $(\mathbf{BP})_J$ also satisfies* (1.2) *with $L_{2k}(\mathbf{BP}_J) \geq (L_{2k}(\mathbf{A}_J) - C_1\varepsilon)/\|\mathbf{D}_J\|_1$, and for all $i \in [N]$:*

$$\|(\mathbf{x}_i)_J - (\mathbf{D}^{-1}\mathbf{P}^\top\overline{\mathbf{x}}_i)_J\|_1 \leq \left(\frac{1 + C_1\|(\mathbf{x}_i)_J\|_1}{L_{2k}(\mathbf{A}_J) - C_1\varepsilon}\right)\varepsilon, \tag{2.3}$$

[3] In compressed sensing literature, $1 - \sqrt{k}L_k(\mathbf{M})$ is the asymmetric lower restricted isometry constant for $\mathbf{M}$ with unit ℓ_2-norm columns [8].

[4] Recall that a set of vectors sharing support S are in *general linear position* when any $|S|$ of them are linearly independent.

[5] Note that the condition $\varepsilon < L_2(\mathbf{A})/C_1$ is necessary; otherwise, with $\mathbf{A} = \mathbf{I}$ (the identity matrix) and $\mathbf{x}_i = \mathbf{e}_i$, the matrix $\mathbf{B} = \left[\mathbf{0}, \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{e}_3, \ldots, \mathbf{e}_m\right]$ and sparse codes $\overline{\mathbf{x}}_i = \mathbf{e}_2$ for $i = 1, 2$ and $\overline{\mathbf{x}}_i = \mathbf{e}_i$ for $i \geq 3$ satisfy $\|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \varepsilon$ but nonetheless violate (2.2).

where subscript $(\cdot)_J$ *here represents the subvector formed from restricting to coordinates indexed by* J*.*

In words, Thm. 1 says that the smaller the regularity r of the original support hypergraph $\mathcal{H}$ or the difference $\overline{m} - m$ between the assumed and actual number of elements in the latent dictionary, the more columns and coefficients of the original dictionary $\mathbf{A}$ and sparse codes $\mathbf{x}_i$ are guaranteed to be contained (up to noise) in the appropriately labelled and scaled recovered dictionary $\mathbf{B}$ and codes $\overline{\mathbf{x}}_i$, respectively.

In the important special case when $\overline{m} = m$, the theorem directly implies that $Y = \{\mathbf{A}\mathbf{x}_1, \ldots, \mathbf{A}\mathbf{x}_N\}$ has a stable k-sparse representation in $\mathbb{R}^m$, with inequalities (1.3) guaranteed in Def. 1 for the following worst-case error ε:

$$\varepsilon(\delta_1, \delta_2) := \min \left\{ \frac{\delta_1}{C_1}, \frac{\delta_2 L_{2k}(\mathbf{A})}{1 + C_1 \left(\delta_2 + \max_{i \in [N]} \|\mathbf{x}_i\|_1\right)} \right\}. \tag{2.4}$$

Since sparse codes in general linear position are straightforward to produce with a "Vandermonde" construction (i.e., by choosing columns of the matrix $[\gamma_i^j]_{i,j=1}^{k,N}$, for distinct nonzero γ_i), we have the following direct consequence of Thm. 1.

Corollary 1. *Given any regular hypergraph* $\mathcal{H} \subseteq \binom{[m]}{k}$ *with the SIP, there are* $N = |\mathcal{H}| \left[(k-1)\binom{m}{k} + 1\right]$ *vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^m$ *such that every matrix* $\mathbf{A}$ *satisfying spark condition* (1.2) *generates* $Y = \{\mathbf{A}\mathbf{x}_1, \ldots, \mathbf{A}\mathbf{x}_N\}$ *with a stable* k*-sparse representation in* $\mathbb{R}^m$ *for* $\varepsilon(\delta_1, \delta_2)$ *given by* (2.4)*.*

One can easily verify that for every $k < m$ there are regular k-uniform hypergraphs $\mathcal{H}$ with the SIP besides the obvious $\mathcal{H} = \binom{[m]}{k}$. For instance, take $\mathcal{H}$ to be the k-regular set of consecutive intervals of length k in some cyclic order on $[m]$. In this case, a direct consequence of Cor. 1 is rigorous verification of the lower bound $N = m(k-1)\binom{m}{k} + m$ for sufficient sample size from the introduction. Special cases allow for even smaller hypergraphs. For example, if $k = \sqrt{m}$, then a 2-regular k-uniform hypergraph with the SIP can be constructed as the $2k$ rows and columns formed by arranging the elements of $[m]$ into a square grid.

It should be stressed here that framing the problem in terms of hypergraphs will allowed us to show, unlike in previous research on the subject, that the matrix $\mathbf{A}$ need not necessarily satisfy (1.2) to be recoverable from data. As an example, let $\mathbf{A} = [\mathbf{e}_1, \ldots, \mathbf{e}_5, \mathbf{v}]$ with $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5$ and take $\mathcal{H}$ to be all consecutive pairs of indices $1, \ldots, 6$ arranged in cyclic order. Then for $k = 2$, the matrix $\mathbf{A}$ fails to satisfy (1.2) while still obeying the assumptions of Thm. 1 for dictionary recovery.

A practical implication of Thm. 1 is the following: there is an effective procedure sufficient to affirm if a proposed solution to Prob. 1 is indeed unique (up to noise and inherent ambiguities). One need simply check that the matrix and codes satisfy the (computable) assumptions of Thm. 1 on $\mathbf{A}$ and the $\mathbf{x}_i$. In general, however, there is no known efficient procedure. A brief discussion on this point is deferred until later.

A less direct consequence of Thm. 1 is the following uniqueness and stability guarantee for solutions to Prob. 2.

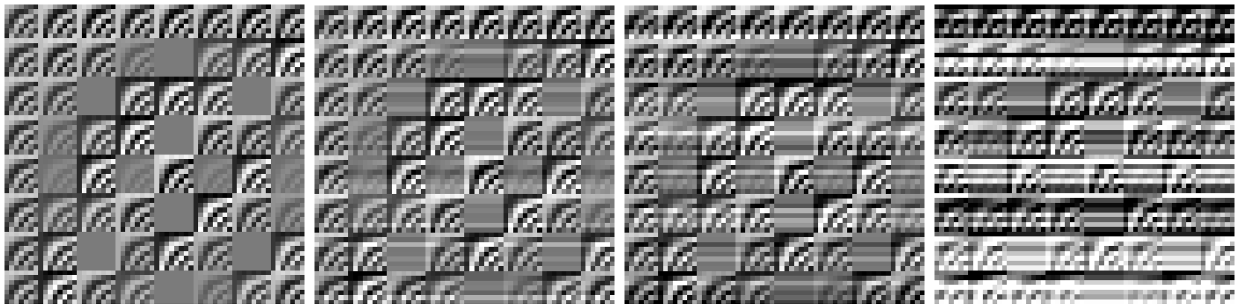

Figure 2.1: **Learning a dictionary from increasingly noisy data**. The (unraveled) basis elements of the 8×8 discrete cosine transform (DCT) form the 64 columns of the left-most matrix above. Three increasingly imprecise dictionaries (columns reordered to best match original) are recovered by FastICA [28] trained on data generated from 8-sparse linear combinations of DCT elements corrupted with additive noise (increasing from left to right).

Theorem 2. *Fix a matrix* $\mathbf{A}$ *and vectors* $\mathbf{x}_i$ *satisfying the assumptions of Thm. 1, only now with over* $(k-1)\left[\binom{\overline{m}}{k} + |\mathcal{H}|k\binom{\overline{m}}{k-1}\right]$ *vectors supported in general linear position in each* $S \in \mathcal{H}$*. Every solution to Prob. 2 (with* $\eta = \varepsilon/2$*) satisfies recovery guarantees* (2.2) *and* (2.3) *when the corresponding bounds on* η *are met.*

Probabilistic uniqueness theorems

Another extension of Thm. 1 can be derived from the following algebraic characterization of the spark condition. Letting $\mathbf{A}$ be the $n \times m$ matrix of nm indeterminates A_{ij}, the reader may work out why substituting real numbers for the A_{ij} yields a matrix satisfying (1.2) if and only if the following polynomial evaluates to a nonzero number:

$$f(\mathbf{A}) := \prod_{S \in \binom{[m]}{2k}} \sum_{S' \in \binom{[n]}{2k}} (\det \mathbf{A}_{S',S})^2,$$

where for any $S' \in \binom{[n]}{2k}$ and $S \in \binom{[m]}{2k}$, the symbol $\mathbf{A}_{S',S}$ denotes the submatrix of entries A_{ij} with $(i,j) \in S' \times S$.[6]

Since f is analytic, having a single substitution of a real matrix $\mathbf{A}$ satisfying $f(\mathbf{A}) \neq 0$ implies that the zeroes of f form a set of (Borel) measure zero. Such a matrix is easily constructed by adding rows of zeroes to a $\min(2k, m) \times m$ Vandermonde matrix as mentioned previously, so that every sum in the product defining f above is strictly positive. Thus, almost every $n \times m$ matrix with $n \geq \min(2k, m)$ satisfies (1.2).

[6]The large number of terms in this product is likely necessary given that deciding whether or not a matrix satisfies the spark condition is NP-hard [47].

It turns out that a similar phenomenon applies to datasets of vectors with a stable sparse representation. Briefly, following the same procedure as in [23, Sec. IV], for $k < m$ and $n \geq \min(2k, m)$, we may consider the "symbolic" dataset $Y = \{\mathbf{A}\mathbf{x}_1, \ldots, \mathbf{A}\mathbf{x}_N\}$ generated by an indeterminate $n \times m$ matrix $\mathbf{A}$ and m-dimensional k-sparse vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$ indeterminate within their supports, which form a regular hypergraph $\mathcal{H} \subseteq \binom{[m]}{k}$ satisfying the SIP. Restricting $(k-1)\binom{m}{k} + 1$ indeterminate $\mathbf{x}_i$ to each support in $\mathcal{H}$, and letting $\mathbf{M}$ be the $n \times N$ matrix with columns $\mathbf{A}\mathbf{x}_i$, it can be checked that when $f(\mathbf{M}) \neq 0$ for a substitution of real numbers for the indeterminates, all of the assumptions on $\mathbf{A}$ and the $\mathbf{x}_i$ in Thm. 1 are satisfied. We therefore have the following.

Theorem 3. *There is a polynomial in the entries of $\mathbf{A}$ and the $\mathbf{x}_i$ that evaluates to a nonzero number only when Y has a stable k-sparse representation in $\mathbb{R}^m$. In particular, almost all substitutions impart to Y this property.*

To extend this observation to arbitrary probability distributions, note that if a set of p measure spaces has all measures absolutely continuous with respect to the standard Borel measure on $\mathbb{R}$, then the product measure is also absolutely continuous with respect to the standard Borel product measure on $\mathbb{R}^p$ (e.g., see [15]). This fact combined with Thm. 3 implies the following.[7]

Corollary 2. *If the indeterminate entries of $\mathbf{A}$ and the $\mathbf{x}_i$ are drawn independently from probability distributions absolutely continuous with respect to the standard Borel measure, then Y has a stable k-sparse representation in $\mathbb{R}^m$ with probability one.*

Thus, drawing the dictionary and supported sparse coefficients from any continuous probability distribution almost always generates data with a stable sparse representation.

2.4 Discussion

Let us conclude this chapter with some comments on the optimality of these results. The linear scaling for ε in (2.4) is essentially optimal (e.g., see [3]), but a basic open problem remains: how many samples are necessary to determine the sparse coding model? These results demonstrate that sparse codes $\mathbf{x}_i$ drawn from only a polynomial number of k-dimensional subspaces permit stable identification of the generating dictionary $\mathbf{A}$. This lends some legitimacy to the use of the model in practice, where data in general are unlikely (if ever) to exhibit the exponentially many possible k-wise combinations of dictionary elements required by (to my knowledge) all previously published results.

Consequently, if k is held fixed or if the size of the support set of reconstructing codes is polynomial in $\overline{m}$ and k, then a practical (polynomial) amount of data suffices to identify the dictionary.[8] Reasons to be skeptical that this holds in general, however, can be found in

[7] We refer the reader to [23] for a more detailed explanation of these arguments.

[8] In the latter case, a reexamination of the pigeonholing argument in the proof of Thm. 1 requires a polynomial number of samples distributed over a polynomial number of supports.

[47, 46]. Even so, in the next chapter it will be discussed how probabilistic guarantees can in fact be made for any number of available samples.

Chapter 3

Proofs

3.1 Introduction

It is instructive to begin the proof of Thm. 1 by showing how dictionary recovery (2.2) already implies sparse code recovery (2.3) when $\mathbf{A}$ satisfies (1.2) and $\varepsilon < L_{2k}(\mathbf{A})/C_1$. We shall temporarily assume (without loss of generality) that $\overline{m} = m$, so as to omit an otherwise requisite subscript $(\cdot)_J$ around certain matrices and vectors. By definition of L_{2k} in (2.1), and noting that $\sqrt{k}\|\mathbf{v}\|_2 \geq \|\mathbf{v}\|_1$ for k-sparse $\mathbf{v}$, we have for all $i \in [N[$:

$$\begin{aligned}
\|\mathbf{x}_i - \mathbf{D}^{-1}\mathbf{P}^\top\overline{\mathbf{x}}_i\|_1 &\leq \frac{\|\mathbf{BPD}(\mathbf{x}_i - \mathbf{D}^{-1}\mathbf{P}^\top\overline{\mathbf{x}}_i)\|_2}{L_{2k}(\mathbf{BPD})} \\
&\leq \frac{\|(\mathbf{BPD} - \mathbf{A})\mathbf{x}_i\|_2 + \|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2}{L_{2k}(\mathbf{BPD})} \\
&\leq \frac{C_1\varepsilon\|\mathbf{x}_i\|_1 + \varepsilon}{L_{2k}(\mathbf{BPD})},
\end{aligned} \tag{3.1}$$

where the first term in the numerator above follows from the triangle inequality and (2.2).

It remains for us to bound the denominator. For any $2k$-sparse $\mathbf{x}$, we have by the triangle inequality:

$$\begin{aligned}
\|\mathbf{BPD}\mathbf{x}\|_2 &\geq \|\mathbf{A}\mathbf{x}\|_2 - \|(\mathbf{A} - \mathbf{BPD})\mathbf{x}\|_2 \\
&\geq \sqrt{2k}(L_{2k}(\mathbf{A}) - C_1\varepsilon)\|\mathbf{x}\|_2,
\end{aligned}$$

We therefore have that $L_{2k}(\mathbf{BPD}) \geq L_{2k}(\mathbf{A}) - C_1\varepsilon > 0$, and (2.3) then follows from (3.1). The reader may also verify that $L_{2k}(\mathbf{BP}) \geq L_{2k}(\mathbf{BPD})/\|\mathbf{D}\|_1$.

The heart of the matter is therefore (2.2), which we shall now establish beginning with the important special case of $k = 1$.

3.2 Proving the case $k = 1$

Since the only 1-uniform hypergraph with the SIP is $[m]$, which is obviously regular, we require only $\mathbf{x}_i = c_i \mathbf{e}_i$ for $i \in [m]$, with $c_i \neq 0$ to guarantee linear independence. While we have yet to define C_1 generally, in this case we may set $C_1 = 1/\min_{\ell \in [m]} |c_\ell|$ so that $\varepsilon < L_2(\mathbf{A}) \min_{\ell \in [m]} |c_\ell|$.

Proof of Thm. 1 for $k = 1$. Fix $\mathbf{A} \in \mathbb{R}^{n \times m}$ satisfying $L_2(\mathbf{A}) > 0$, since here we have $2\mathcal{H} = \binom{[m]}{2}$, and suppose some $\mathbf{B}$ and 1-sparse $\overline{\mathbf{x}}_i \in \mathbb{R}^{\overline{m}}$ have $\|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \varepsilon < L_2(\mathbf{A})/C_1$ for all i. Then, there exist $\overline{c}_1, \ldots, \overline{c}_m \in \mathbb{R}$ and a map $\pi : [m] \to [\overline{m}]$ such that:

$$\|c_i \mathbf{A}_i - \overline{c}_i \mathbf{B}_{\pi(i)}\|_2 \leq \varepsilon, \quad \text{for } i \in [m]. \tag{3.2}$$

Note that $\overline{c}_i \neq 0$, since otherwise we would reach the following contradiction: $\|\mathbf{A}_i\|_2 \leq C_1 |c_i| \|\mathbf{A}_i\|_2 \leq C_1 \varepsilon < L_2(\mathbf{A}) \leq L_1(\mathbf{A}) = \min_{i \in [m]} \|\mathbf{A}_i\|_2$.

Let us now show that π is injective (in particular, a permutation if $\overline{m} = m$). Suppose that $\pi(i) = \pi(j) = \ell$ for some $i \neq j$ and ℓ. Then, $\|c_i \mathbf{A}_i - \overline{c}_i \mathbf{B}_\ell\|_2 \leq \varepsilon$ and $\|c_j \mathbf{A}_j - \overline{c}_j \mathbf{B}_\ell\|_2 \leq \varepsilon$, and we have:

$$\begin{aligned}
(|\overline{c}_i| + |\overline{c}_j|)\varepsilon &\geq |\overline{c}_i| \|c_j \mathbf{A}_j - \overline{c}_j \mathbf{B}_\ell\|_2 + |\overline{c}_j| \|c_i \mathbf{A}_i - \overline{c}_i \mathbf{B}_\ell\|_2 \\
&\geq \|\mathbf{A}(\overline{c}_i c_j \mathbf{e}_j - \overline{c}_j c_i \mathbf{e}_i)\|_2 \\
&\geq \sqrt{2} L_2(\mathbf{A}) \|\overline{c}_i c_j \mathbf{e}_j - \overline{c}_j c_i \mathbf{e}_i\|_2 \\
&\geq L_2(\mathbf{A}) \left(|\overline{c}_i| + |\overline{c}_j|\right) \min_{\ell \in [m]} |c_\ell|,
\end{aligned}$$

contradicting our assumed upper bound on ε. Hence, the map π is injective and so $\overline{m} \geq m$.

Letting $\mathbf{P}$ and $\mathbf{D}$ be the $\overline{m} \times \overline{m}$ permutation and invertible diagonal matrices with, respectively, columns $\mathbf{e}_{\pi(i)}$ and $\frac{\overline{c}_i}{c_i} \mathbf{e}_i$ for $i \in [m]$ (otherwise, $\mathbf{e}_i$ for $i \in [\overline{m}] \setminus [m]$), we may rewrite (3.2) to see that for all $i \in [m]$:

$$\|\mathbf{A}_i - (\mathbf{BPD})_i\|_2 = \|\mathbf{A}_i - \frac{\overline{c}_i}{c_i} \mathbf{B}_{\pi(i)}\|_2 \leq \frac{\varepsilon}{|c_i|} \leq C_1 \varepsilon.$$

□

3.3 The main lemma

An extension of the proof to the general case $k < m$ requires some additional tools to derive the general expression (3.11) for C_1. These include a generalized notion of distance (Def. 3) and angle (Def. 4) between subspaces as well as a stability result in combinatorial matrix analysis (Lem. 1), which is the main lemma doing most of the heavy lifting in the proof of Thm. 1.

Definition 3. *For $\mathbf{u} \in \mathbb{R}^m$ and vector spaces $U, V \subseteq \mathbb{R}^m$, let $dist(\mathbf{u}, V) := \min\{\|\mathbf{u} - \mathbf{v}\|_2 : \mathbf{v} \in V\}$ and define:*

$$d(U, V) := \max_{\mathbf{u} \in U, \ \|\mathbf{u}\|_2 \leq 1} dist(\mathbf{u}, V). \tag{3.3}$$

Note the following facts about d. Clearly,

$$U' \subseteq U \implies d(U', V) \leq d(U, V). \tag{3.4}$$

From [31, Ch. 4 Cor. 2.6], we also have:

$$d(U, V) < 1 \implies \dim(U) \leq \dim(V), \tag{3.5}$$

and from [37, Lem. 3.2]:

$$\dim(U) = \dim(V) \implies d(U, V) = d(V, U). \tag{3.6}$$

The required stability result in combinatorial matrix analysis is the following.

Lemma 1. *If an $n \times m$ matrix $\mathbf{A}$ has $L_{2\mathcal{H}}(\mathbf{A}) > 0$ for some r-regular $\mathcal{H} \subseteq \binom{[m]}{k}$ with the SIP, then the following holds for $C_2 > 0$ given by* (3.10)*:*

Fix $\varepsilon < L_2(\mathbf{A})/C_2$. If for some $n \times \overline{m}$ matrix $\mathbf{B}$ and map $\pi : \mathcal{H} \mapsto \binom{[\overline{m}]}{k}$,

$$d(\boldsymbol{\mathcal{A}}_S, \boldsymbol{\mathcal{B}}_{\pi(S)}) \leq \varepsilon, \quad \text{for } S \in \mathcal{H}, \tag{3.7}$$

then $\overline{m} \geq m$, and provided $\overline{m}(r-1) < mr$, there is a permutation matrix $\mathbf{P}$ and invertible diagonal $\mathbf{D}$ such that:

$$\|\mathbf{A}_i - (\mathbf{BPD})_i\|_2 \leq C_2 \varepsilon, \quad \text{for } i \in J, \tag{3.8}$$

for some $J \subseteq [m]$ of size $m - (r-1)(\overline{m} - m)$.

The constant C_2 (a function of $\mathbf{A}$ and $\mathcal{H}$) will be presented relative to a quantity used in [10] to analyze the convergence of the "alternating projections" algorithm for projecting a point onto the intersection of subspaces. This quantity is incorporated into the following definition, which we shall refer to in the proof of Lem. 3 in the Appendix; specifically, it will be used to bound the distance between a point and the intersection of subspaces given an upper bound on its distance from each subspace.

Definition 4. *For a collection of real subspaces $\mathcal{V} = \{V_i\}_{i=1}^{\ell}$, define $\xi(\mathcal{V}) := 0$ when $|\mathcal{V}| = 1$, and otherwise:*

$$\xi^2(\mathcal{V}) := 1 - \max \prod_{i=1}^{\ell-1} \sin^2 \theta \left(V_i, \cap_{j>i} V_j\right), \tag{3.9}$$

where the maximum is taken over all ways of ordering the V_i and the angle $\theta \in (0, \frac{\pi}{2}]$ is defined implicitly as [10, Def. 9.4]:

$$\cos\theta(U, W) := \max\left\{ |\langle \mathbf{u}, \mathbf{w} \rangle| : {\mathbf{u} \in U \cap (U \cap W)^\perp, \ \|\mathbf{u}\|_2 \leq 1 \atop \mathbf{w} \in W \cap (U \cap W)^\perp, \ \|\mathbf{w}\|_2 \leq 1} \right\}.$$

Note that $\theta \in (0, \frac{\pi}{2}]$ implies $0 \leq \xi < 1$, and that $\xi(\mathcal{V}') \leq \xi(\mathcal{V})$ when $\mathcal{V}' \subseteq \mathcal{V}$.[1]

The constant $C_2 > 0$ of Lem. 1 can now be expressed as:

$$C_2(\mathbf{A}, \mathcal{H}) := \frac{(r+1)\max_{j\in[m]} \|\mathbf{A}_j\|_2}{1 - \max_{\mathcal{G}\in\binom{\mathcal{H}}{r+1}} \xi(\boldsymbol{\mathcal{A}}_\mathcal{G})}. \tag{3.10}$$

3.4 Proving the general case $k < m$

We may now define the constant $C_1 > 0$ of Thm. 1 in terms of C_2. Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^m$, let $\mathbf{X}$ denote the $m \times N$ matrix with columns $\mathbf{x}_i$ and let $I(S)$ denote the set of indices i for which $\mathbf{x}_i$ is supported in S. Define:

$$C_1(\mathbf{A}, \mathcal{H}, \{\mathbf{x}_i\}_{i=1}^N) := \frac{C_2(\mathbf{A}, \mathcal{H})}{\min_{S\in\mathcal{H}} L_k(\mathbf{A}\mathbf{X}_{I(S)})}. \tag{3.11}$$

Given the assumptions of Thm. 1 on $\mathbf{A}$ and the $\mathbf{x}_i$, this expression for C_1 is well-defined[2] and yields an upper bound on ε consistent with that proven sufficient in the case $k = 1$ considered at the beginning of this chapter.[3]

The practically-minded reader should note that the explicit constants C_1 and C_2 are effectively computable: the denominator of C_1 involves a quantity L_k that may be calculated as the smallest singular value of a certain matrix, while computing the quantity ξ in the denominator of C_2 involves computing "canonical angles" between subspaces, which reduces again to an efficient singular value decomposition. There is no known fast computation of L_k in general, however, since even $L_k > 0$ is NP-hard [47], although efficiently computable bounds have been proposed (e.g., via the "mutual coherence" of a matrix [12]); alternatively, fixing k yields polynomial complexity. Moreover, calculating C_2 requires an exponential number of queries to ξ unless r is held fixed, too (e.g., the "cyclic order" hypergraphs described above have $r = k$). Thus, as presented, C_1 and C_2 are not efficiently computable in general.

Proof of Thm. 1 for $k < m$. We shall find a map $\pi : \mathcal{H} \to \binom{[m]}{k}$ for which the distance $d(\boldsymbol{\mathcal{A}}_S, \boldsymbol{\mathcal{B}}_{\pi(S)})$ is controlled by ε for all $S \in \mathcal{H}$. Applying Lem. 1 then completes the proof.

Fix $S \in \mathcal{H}$. Since there are more than $(k-1)\binom{m}{k}$ vectors $\mathbf{x}_i$ supported in S, by the pigeonhole principle there must be some $\overline{S} \in \binom{[m]}{k}$ and a set of k indices $K \subseteq I(S)$ for which all $\overline{\mathbf{x}}_i$ with $i \in K$ are supported in $\overline{S}$. It also follows[4] from $L_{2\mathcal{H}}(\mathbf{A}) > 0$ and the general

[1] We acknowledge the counter-intuitive property: $\theta = \pi/2$ when $U \subseteq W$.

[2] To see this, fix $S \in \mathcal{H}$ and k-sparse $\mathbf{c}$. Using the definitions, we have $\|\mathbf{A}\mathbf{X}_{I(S)}\mathbf{c}\|_2 \geq \sqrt{k}L_\mathcal{H}(\mathbf{A})\|\mathbf{X}_{I(S)}\mathbf{c}\|_2 \geq kL_\mathcal{H}(\mathbf{A})L_k(\mathbf{X}_{I(S)})\|\mathbf{c}\|_2$. Thus, $L_k(\mathbf{A}\mathbf{X}_{I(S)}) \geq \sqrt{k}L_\mathcal{H}(\mathbf{A})L_k(\mathbf{X}_{I(S)}) > 0$, since $L_\mathcal{H}(\mathbf{A}) \geq L_{2\mathcal{H}}(\mathbf{A}) > 0$ and $L_k(\mathbf{X}_{I(S)}) > 0$ by general linear position of the $\mathbf{x}_i$.

[3] When $\mathbf{x}_i = c_i\mathbf{e}_i$, we have $C_2 \geq 2\|\mathbf{A}_i\|_2$ and the denominator in (3.11) becomes $\min_{i\in[m]} |c_i|\|\mathbf{A}_i\|_2$; hence, $C_1 \geq 2/\min_{i\in[m]} |c_i|$.

[4] See footnote 2.

linear position of the $\mathbf{x}_i$ that $L_k(\mathbf{AX}_K) > 0$; that is, the columns of the $n \times k$ matrix $\mathbf{AX}_K$ form a basis for $\boldsymbol{\mathcal{A}}_S$.

Fixing $\mathbf{y} \in \boldsymbol{\mathcal{A}}_S \setminus \{\mathbf{0}\}$, there then exists $\mathbf{c} = (c_1, \ldots, c_k) \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $\mathbf{y} = \mathbf{AX}_K\mathbf{c}$. Setting $\overline{\mathbf{y}} = \mathbf{B}\overline{\mathbf{X}}_K\mathbf{c}$, which is in $\boldsymbol{\mathcal{B}}_{\overline{S}}$, we have by triangle inequality:

$$\begin{aligned}\|\mathbf{y} - \overline{\mathbf{y}}\|_2 &= \|(\mathbf{AX}_K - \mathbf{B}\overline{\mathbf{X}}_K)\mathbf{c}\|_2 \leq \varepsilon\|\mathbf{c}\|_1 \leq \varepsilon\sqrt{k}\|\mathbf{c}\|_2 \\ &\leq \frac{\varepsilon}{L_k(\mathbf{AX}_K)}\|\mathbf{y}\|_2,\end{aligned}$$

where the last inequality uses (2.1). From Def. 3:

$$d(\boldsymbol{\mathcal{A}}_S, \boldsymbol{\mathcal{B}}_{\overline{S}}) \leq \frac{\varepsilon}{L_k(\mathbf{AX}_K)} \leq \frac{\varepsilon}{L_k(\mathbf{AX}_{I(S)})} \leq \varepsilon\frac{C_1}{C_2}. \tag{3.12}$$

Finally, apply Lem. 1 with $\varepsilon < L_2(\mathbf{A})/C_1$ and $\pi(S) := \overline{S}$. □

Before moving on to the proof of Thm. 2, let us briefly revisit the discussion on sample complexity from the end of the previous chapter. While an exponential number of samples may very well prove to be necessary in the deterministic or almost-certain case, our proof of Thm. 1 can be extended to hold with some probability for *any* number of samples by alternative appeal to a probabilistic pigeonholing at the point early in the proof where the (deterministic) pigeonhole principle is applied to show that for every $S \in \mathcal{H}$, there exist k vectors $\mathbf{x}_i$ supported on S whose corresponding $\overline{\mathbf{x}}_i$ all share the same support.[5] Given insufficient samples, this argument has some less-than-certain probability of being valid for each $S \in \mathcal{H}$. Nonetheless, simulations with small hypergraphs demonstrate that successful recovery is nearly certainly even when N is only a fraction of the deterministic sample complexity (see Fig. 4.2).

Proof of Thm. 2. We shall bound the number of k-sparse $\overline{\mathbf{x}}_i$ from below and then apply Thm. 1. Let n_p be the number of $\overline{\mathbf{x}}_i$ with $\|\overline{\mathbf{x}}_i\|_0 = p$. Since the $\mathbf{x}_i$ are all k-sparse, by (1.4) we have:

$$\sum_{p=0}^{\overline{m}} pn_p = \sum_{i=0}^{N} \|\overline{\mathbf{x}}_i\|_0 \leq \sum_{i=0}^{N} \|\mathbf{x}_i\|_0 \leq kN$$

Since $N = \sum_{p=0}^{\overline{m}} n_p$, we then have $\sum_{p=0}^{\overline{m}}(p-k)n_p \leq 0$. Splitting the sum yields:

$$\sum_{p=k+1}^{\overline{m}} n_p \leq \sum_{p=k+1}^{\overline{m}} (p-k)n_p \leq \sum_{p=0}^{k}(k-p)n_p \leq k\sum_{p=0}^{k-1} n_p, \tag{3.13}$$

[5] A famous example of such an argument is the counter-intuitive "birthday paradox", which demonstrates that the probability of two people having the same birthday in a room of twenty-three is greater than 50%.

demonstrating that the number of vectors $\overline{\mathbf{x}}_i$ that are *not* k-sparse is bounded above by how many are $(k-1)$-sparse.

Next, observe that no more than $(k-1)|\mathcal{H}|$ of the $\overline{\mathbf{x}}_i$ share a support $\overline{S}$ of size less than k. Otherwise, by the pigeonhole principle, there is some $S \in \mathcal{H}$ and a set of k indices $K \subseteq I(S)$ for which all $\mathbf{x}_i$ with $i \in K$ are supported in S; as argued previously, (3.12) follows. It is simple to show that $L_2(\mathbf{A}) \leq \max_j \|\mathbf{A}_j\|_2$, and since $0 \leq \xi < 1$, the right-hand side of (3.12) is less than one for $\varepsilon < L_2(\mathbf{A})/C_1$. Thus, by (3.5) we would have the contradiction $k = \dim(\boldsymbol{\mathcal{A}}_S) \leq \dim(\boldsymbol{\mathcal{B}}_{\overline{S}}) \leq |\overline{S}| < k$.

The total number of $(k-1)$-sparse vectors $\overline{\mathbf{x}}_i$ thus cannot exceed $|\mathcal{H}|(k-1)\binom{\overline{m}}{k-1}$. By (3.13), no more than $|\mathcal{H}|k(k-1)\binom{\overline{m}}{k-1}$ vectors $\overline{\mathbf{x}}_i$ are not k-sparse. Since for every $S \in \mathcal{H}$ there are over $(k-1)\left[\binom{\overline{m}}{k} + |\mathcal{H}|k\binom{\overline{m}}{k-1}\right]$ vectors $\mathbf{x}_i$ supported there, it must be that more than $(k-1)\binom{\overline{m}}{k}$ of them have corresponding $\overline{\mathbf{x}}_i$ that are k-sparse. The result now follows from Thm. 1, noting by the triangle inequality that $\|\mathbf{A}\mathbf{x}_i - \mathbf{B}\overline{\mathbf{x}}_i\| \leq 2\eta$ for $i = 1, \ldots, N$. $\square$

3.5 Discussion

The absence of any assumptions at all about dictionaries that solve Prob. 1 was a major technical hurdle in proving Thm. 1. This very general guarantee was sought because of the practical difficulty of ensuring that an algorithm maintain a dictionary satisfying the spark condition (1.2) at each iteration, which (to my knowledge) has been an explicit or implicit assumption of all previous works except [23]; indeed, even certifying a dictionary has this property is NP-hard [47].

Several results in the literature had to be combined to extend the guarantees derived in [23] into the noisy regime. The main challenge was to generalize Lem. 1 to the case where the k-dimensional subspaces spanned by corresponding submatrices $\mathbf{A}_S$ and $\mathbf{B}_{\pi(S)}$ are assumed to be“close” but not identical. Unlike in [23] where an inductive argument could be applied to the noiseless case, here it had to be explicitly demonstrated that this proximity relation is propagated through iterated intersections right down to the spans of the dictionary elements themselves. Lem. 3 was designed to encapsulate this fact, proven by appeal to a convergence guarantee for an alternating projections algorithm first proposed by von Neumann. This result, combined with a little known fact (3.6) about the distance metric between subspaces, make up the more obscure components of the deduction.

The proof of Lem. 1 diverges most significantly from the approach taken in [23] by way of Lem. 4, which utilizes a combinatorial design for support sets (the “singleton intersection property”) to reduce the deterministic sample complexity by an exponential factor. This constitutes a significant advance toward legitimizing dictionary learning in practice, since data must otherwise exhibit the exponentially many possible k-wise combinations of dictionary elements required by (to my knowledge) all previously published results; although an exponential number of samples per support is still required (unless k is held fixed). The issue is that the map $\pi : \mathcal{H} \to \binom{[m]}{k}$ is surjective only when $\mathcal{H}$ is taken to be $\binom{[m]}{k}$, in which case one may proceed by induction as in [23], freely choosing supports in the codomain of π to intersect over $(k-1)$ indices to then map back to some corresponding set of $(k-1)$ indices at the intersection of supports in the domain. Here, for $\mathcal{H} \subset \binom{[m]}{k}$, a bijection between indices had to instead be established by pigeonholing the image of π under constraints imposed by the SIP, which was formulated specifically for this purpose. It just so happened that this more general argument for a non-surjective π constrained by the SIP applied just as well to the situation where the number of dictionary elements m is over-estimated (i.e. $\overline{m} > m$), in which case a one-to-one correspondence can be guaranteed between a subset of columns of $\mathbf{A}$ and $\mathbf{B}$ of a size simply expressed in terms of the width of each matrix and the regularity of $\mathcal{H}$.

One of the mathematically significant achievements in [23] was to break free from the constraint that the recovery matrix $\mathbf{B}$ satisfy the spark condition in addition to the generating dictionary $\mathbf{A}$. Here, it has been demonstrated that, in fact, even $\mathbf{A}$ need not satisfy the spark condition! Rather, $\mathbf{A}$ need only be injective on the union of subspaces with supports forming a regular k-uniform hypergraph satisfying the SIP (a distinguishing example is given in Sec. 2). This relaxation of constraints inspired the definition of the restricted matrix lower bound $L_{\mathcal{H}}$ in (2.1), which generalizes the well-known (see footnote 3) restricted matrix lower bound L_k to be in terms of a hypergraph $\mathcal{H}$, and an interesting object for further study in its own right.

To reiterate, the methods applied here to prove Thm. 1 yield the following results beyond a straightforward extension of those in [23] to the noisy case:

1. A reduction in deterministic sample complexity: To identify the $n \times m$ generating dictionary $\mathbf{A}$, it is required in [23] that $k\binom{m}{k}$ data points be sampled from each of the $\binom{m}{k}$ subspaces spanned by subsets of k columns of $\mathbf{A}$. It is shown here that in fact it suffices to sample from at most m such subspaces (see Cor. 1).

2. An extension of guarantees to the case where the number of dictionary elements is unknown: The results of [23] only apply to the case where the matrix $\mathbf{B}$ has the same number of columns as $\mathbf{A}$. It is shown here that if $\mathbf{B}$ has at least as many columns as $\mathbf{A}$ then it contains (up to noise) a subset of the columns of $\mathbf{A}$.

3. Relaxed requirements (no spark condition) on the generating matrix $\mathbf{A}$: Rather, $\mathbf{A}$ need only be injective on the union of subspaces with supports that form a regular uniform hypergraph satisfying the SIP.

As it seemed to benefit a reviewer of this work, a brief discussion on how the deterministic sample complexity $N = |\mathcal{H}\left[|(k-1)\binom{m}{k}+1\right]$ derived here compares to those listed in the the top two rows listed in Table I of [23] may be in order. To be clear, the theory developed here is strictly more general, since $\mathcal{H}$ can always be taken to be $\binom{[m]}{k}$. The point of difference in this comparison is the assumed set of supports for sparse codes, which is always $\binom{[m]}{k}$ in [23], whereas here it can be assumed to be any regular k-uniform hypergraph that satisfies the SIP. By row,

I. The result here improves upon the listed $k\binom{m}{k}^2$ by an exponential factor, since for every $k < m$ there exists a regular k-uniform hypergraph $\mathcal{H}$ with $|\mathcal{H}| = m$ satisfying the SIP.

II. The authors in [23] have applied measure-theoretic arguments to achieve $(k+1)\binom{m}{k}$ with almost-certainty (i.e. with probability one), a factor of m reduction over that for which certainty can alternatively be guaranteed here.

Chapter 4

Discussion

4.1 Conclusions

The main motivation for this work was the observation that characteristic sparse representations emerge from sparse coding models trained over a variety of natural scene datasets by a variety of learning algorithms. The theorems proven here provide some insight into this phenomenon by establishing very general conditions under which identification of the model parameters is not only possible but also robust to measurement and modeling error.

The guarantees concerning the identification of a dictionary and corresponding sparse codes of minimal average support size (Thm. 2), which is the optimization problem of most interest to practitioners (Prob. 2), are to my knowledge the first of their kind in both the noise-free and noisy domains. It has been shown here that, given sufficient data, this problem reduces to an instance of Prob. 1 to which the main result (Thm. 1) then applies: every dictionary and corresponding set of sparse codes consistent with the data are equivalent up to inherent relabeling/scaling ambiguities and a discrepancy (error) that scales linearly with the noise. In fact, in almost all cases these problems are well-posed given a sufficient amount of data (Thm. 3 and Cor. 2). Furthermore, the derived scaling constants are explicit and computable; as such, there is an effective procedure that suffices to affirm if a proposed solution to these problems is indeed unique up to noise and inherent ambiguities, although it is not efficient in general.

While the extension from exact recovery to the noisy stability of dictionary learning is significant, the fact that the analysis relies on metrics of the data that are not feasible to compute limits its impact to the scientific community beyond computer science and applied mathematics. What sets the main results of this work apart from the vast majority of results in the field is that they are deterministic, and do not depend on any kind of assumption about the particular random distribution from which the sparse supports, coefficients, or dictionary entries are drawn (though Cor. 2 makes a sweeping statement applicable to all continuous distributions). Consequently, the inferences of those applying dictionary learning methods to inverse problems in their research are justified only in principle; but this is unavoidably the case for NP-hard problems.

Indeed, theoretical validation makes little practical difference if the methodology is already in widespread use, while practical criteria establishing whether the data models obtained by practitioners are optimal or not would have very high impact. To this end, the work has been laid out for those wanting to derive statistical criteria for inference with respect to more domain-specific structured dictionaries and codes (i.e. estimate C_1), and reduced by half for those hoping to prove the consistency of any dictionary learning algorithm (i.e. prove convergence to within $\varepsilon(\delta_1, \delta_2)$ given in (2.4)).

Nonetheless, a main reason for the sustained interest in dictionary learning as an unsupervised method for data analysis seems to be the assumed well-posedness of parameter identification in the model, confirmation of which forms the core of these findings. Several groups have applied compressed sensing to signal processing tasks; for instance, in MRI analysis [36], image compression [13], and even the design of an ultrafast camera [17]. It is only a matter of time before these systems incorporate dictionary learning to encode and decode signals (e.g., in a device that learns structure from motion [32]), just as scientists have used sparse coding to uncover latent structure in data (e.g., forgery detection [25, 39]; brain recordings [30, 1, 33]; and gene expression [50]). As uniqueness guarantees with minimal assumptions apply to all areas of data science and engineering that utilize learned sparse structure, assurances offered by these theorems give hope that different devices and algorithms may learn equivalent representations given enough data from statistically identical systems.[1]

Within the field of theoretical neuroscience in particular, dictionary learning for sparse coding and related methods have recovered characteristic components of natural images [38, 27, 7, 22] and sounds [6, 44, 9] that reproduce response properties of cortical neurons. The results of this work suggest that this correspondence could be due to the "universality" of sparse representations in natural data, an early mathematical idea in neural theory [41]. Furthermore, they justify the soundness of one of the few hypothesized theories of bottleneck communication in the brain [29]: that sparse neural population activity is recoverable from its noisy linear compression through a randomly constructed (but unknown) wiring bottleneck by any biologically plausible unsupervised sparse coding method that solves Prob. 1 or 2 (e.g., [42, 43, 40]).[2]

4.2 Future directions

There are many challenges left open by this work. First and foremost, it should be stressed that all conditions stated here which guarantee the uniqueness and stability of sparse representations have only been shown sufficient; it remains open to work out a set of necessary conditions on all fronts, be it on the number of required samples per support, the structure of

[1] To contrast with the current hot topic of "Deep Learning", there are few such uniqueness guarantees for these models of data; moreover, even small noise can dramatically alter their output [19].

[2] We refer the reader to [16] for more on interactions between dictionary learning and neuroscience.

support set hypergraphs, or the tolerable signal-to-noise ratio for a bounded recovery error. It is also worth stressing that the deterministic conditions derived here must accommodate always the worst possible cases. It would be of great practical benefit to see how drastically all conditions can be relaxed by requiring less-than-certain guarantees, as (for instance) exhibited in the discussion on probabilistic pigeonholing following the proof of Thm. 1. In a similar vein, the tolerable signal-to-noise ratio can be reduced by considering the probability that noise sampled from a concentrated isotropic distribution will point in a harmful direction, which may be especially low in high-dimensional spaces or for certain support set hypergraphs.

Another interesting remaining challenge is to work out for which special cases it is efficient to check that a solution to Prob. 1 or 2 is unique up to noise and inherent ambiguities. Considering that the sufficient conditions detailed here are in general NP-hard to compute, are the necessary conditions also hard to compute? Are Probs. 1 and 2 then also hard (e.g., see [46])? Since Prob. 2 is intractable in general (i.e. including the noiseless case), but efficiently solvable by convex relaxation when the matrix $\mathbf{A}$ is known and has a large enough lower bound over sparse domains [14], is there a version of Thm. 2 that lays down general conditions under which Prob. 2 can be solved efficiently in full by similar means?

I briefly expand on some of these directions below. It is my hope that these remaining challenges pique the interest of the community, and that practical guidelines can be established using the theoretical tools showcased here to support researchers applying sparse coding techniques.

Signal-to-Noise Ratio

A concern raised in peer review of this work was the typical size of the constant C_1, which sets the tolerable signal-to-noise ratio for dictionary and code recovery up to an acceptable error. Referring to the definition of this constant in (3.11), the reader should note that the denominator involves L_k, a standard quantity in the field of compressed sensing (the "restricted isometry constant", see footnote 3), which is known to be reasonable for many random distributions generating dictionaries $\mathbf{A}$ and sparse codes $\mathbf{x}_i$ [4]. The numerator C_2, on the other hand, incorporates the more obscure quantity ξ defined in (4), which is computed from the "Friedrichs angle" between certain spans of subsets of the columns of $\mathbf{A}$. Simulations for small (pseudo-)randomly generated dictionaries $\mathbf{A}$ suggest nonetheless that the constant C_2 is likely reasonable in general as well (at least, for the case where $m = k^2$ and $\mathcal{H}$ is taken to be the set of rows and columns formed by arranging the elements of $[m]$ into a square grid; see Fig. 4.2). These observations motivate the following conjecture:

Conjecture 1. *For all* $t > 0$*,*

$$Pr[|C_2 - \mathbb{E}[C_2]| > t] \to 0 \quad \textit{as } k \to \infty \textit{ and } k/m \to 0$$

provided the assumptions of Thm. 1 are satisfied.

Sample Complexity

It is possible to tighten the pigeonholing argument in the proof of Thm. 1 and thereby reduce the deterministic sample complexity without recourse to uncertainty (as previously suggested). The argument as presented iterates over supports $S \in \mathcal{H}$, in each case determining a corresponding support $\overline{S} \in \binom{[\overline{m}]}{k}$ without consideration of previously matched support pairs; and yet the assumption $L_{2\mathcal{H}}(\mathbf{A}) > 0$ implies that no two supports in $\mathcal{H}$ can map to the same $\overline{S}$. The number of bins to pigeonhole into thus decreases every iteration, though this is a drop in a bucket of exponential size. It would be interesting to see how much the deterministic sample complexity can be reduced by imposing these constraints holistically, given the specific structure of the hypergraph $\mathcal{H}$.

Incidentally, there is also room to breathe in the restrictions on $\mathcal{H}$. Already, the results of this work motivate the following question, which is only one among many combinatorial problems brought to mind by the SIP (Def. 2):

Question 1. *Fix integers m and $k < m$. What is the smallest regular hypergraph $\mathcal{H} \subseteq \binom{[m]}{k}$ satisfying the SIP?*

A close examination of the proof of Lemma 4 (see the Appendix) reveals, however, that $\mathcal{H}$ need not be regular so long as it satisfies a constraint on the sequence of node degrees compatible with the iterative argument. It is then natural to wonder: what are the necessary constraints on $\mathcal{H}$, and what is the smallest hypergraph satisfying these constraints for given m and k?

Opening ourselves up to uncertainty, we can furthermore ask:

Question 2. *Fix $k < m$. What is the probability that a random subset of $\binom{[m]}{k}$ is regular and satisfies the SIP?*

Within the realm of uncertain guarantees, we can also elaborate on the probabilistic pigeonholing strategy outlined in the discussion following the proof of Thm. 1. The problem is to count the number of ways in which vectors supported in $S \in \mathcal{H}$ can be partitioned among supports in $\binom{\overline{m}}{k}$ without allocating k or more to any individual one (in which case the logic of the proof fails to imply the result; we are interested in the probability that it doesn't). These are integer solutions to the problem $\sum_i n_i = N$ subject to $n_i < k$ for all i, where $i = 1, \ldots, \binom{\overline{m}}{k}$. Following closely the exposition in [34], it appears there is no closed formula for this problem, but the number of solutions can be computed in a number of operations independent of N. Writing $p = \binom{\overline{m}}{k}$, the number is the coefficient of X^N in the polynomial $(1 + X + \ldots + X^{k-1})^p$. Written as a rational function of X,

$$(1 + X + \ldots + X^{k-1})^p = \left(\frac{1 - X^k}{1 - X}\right)^p = \frac{\left(1 - X^k\right)^p}{(1 - X)^p}$$

the coeffiecient of X^i in the numerator is zero unless i is a multiple qk of k, in which case it is $(-1)^q\binom{p}{q}$, and the coefficient of X^j in the inverse of the denominator is $(-1)^j\binom{-p}{j} = \binom{j+p-1}{j}$, which is zero unless $j \geq 0$ and otherwise equal to $\binom{j+p-1}{p-1}$. It remains to sum over all $i + j = N$, which gives:

$$n_{\text{fails}} = \sum_{q=0}^{\min(p,N/k)} (-1)^q \binom{p}{q} \binom{N - qk + p - 1}{p - 1}$$

where the summation is truncated to ensure that $N - qk \geq 0$ (the condition $j \geq 0$) and has at most $p + 1 = \binom{m}{k} + 1$ terms.

The total number of ways to pigeonhole is $n_{\text{total}} = \binom{N+p-1}{p-1}$, and so the probability of full recovery is $(1 - n_{\text{fails}}/n_{\text{total}})^{|\mathcal{H}|}$. The curves computed in this way in Fig. 4.2 suggest that, while it may very well be impossible to exorcise exponentiality from the number of required samples in the deterministic or almost-certain case, perhaps it is possible with high-probability by one way or another. Informally,

Question 3. *While fixing k yields polynomial deterministic sample complexity in m (see Cor. 1), is there some more general probablistic sense (perhaps for some restricted class of hypergraphs) by which sample complexity is polynomial in both m and k?*

Dictionary learning via ℓ_1-norm minimization

A commonly applied workaround to the intractability of Prob. 2 (see [46]) is to swap out the ℓ_0-norm in (1.4) for ℓ_1-norm, thereby transforming the inference of sparse $\overline{\mathbf{x}}_i$ for a given $\mathbf{B}$ into a convex optimization solvable by a linear program. A major advance in compressive sensing was the discovery that (1.4) can in fact be solved in this way for fixed $\mathbf{B}$ provided $L_{2k}(\mathbf{B})$ is large enough [14]. The current work provides conditions on the generating dictionary $\mathbf{A}$ and k-sparse codes $\mathbf{x}_i$ under which *all* matrices $\mathbf{B}$ (of bounded column-norm) that solve Prob. 2 have $L_{2k}(\mathbf{B})$ bounded from below; specifically, $L_{2k}(\mathbf{B}) \geq (L_{2k}(\mathbf{A}) - C_1\varepsilon) / \|\mathbf{D}\|_1$ in the case where $\mathbf{A}$ satisfies (1.2). Thus, there is some noise bound inside of which all solutions to Prob. 2 are solutions to the convexified problem as well. It remains to determine practical constraints that would exclude alternative dictionaries which may solve the convefixied problem without solving Prob. 2.

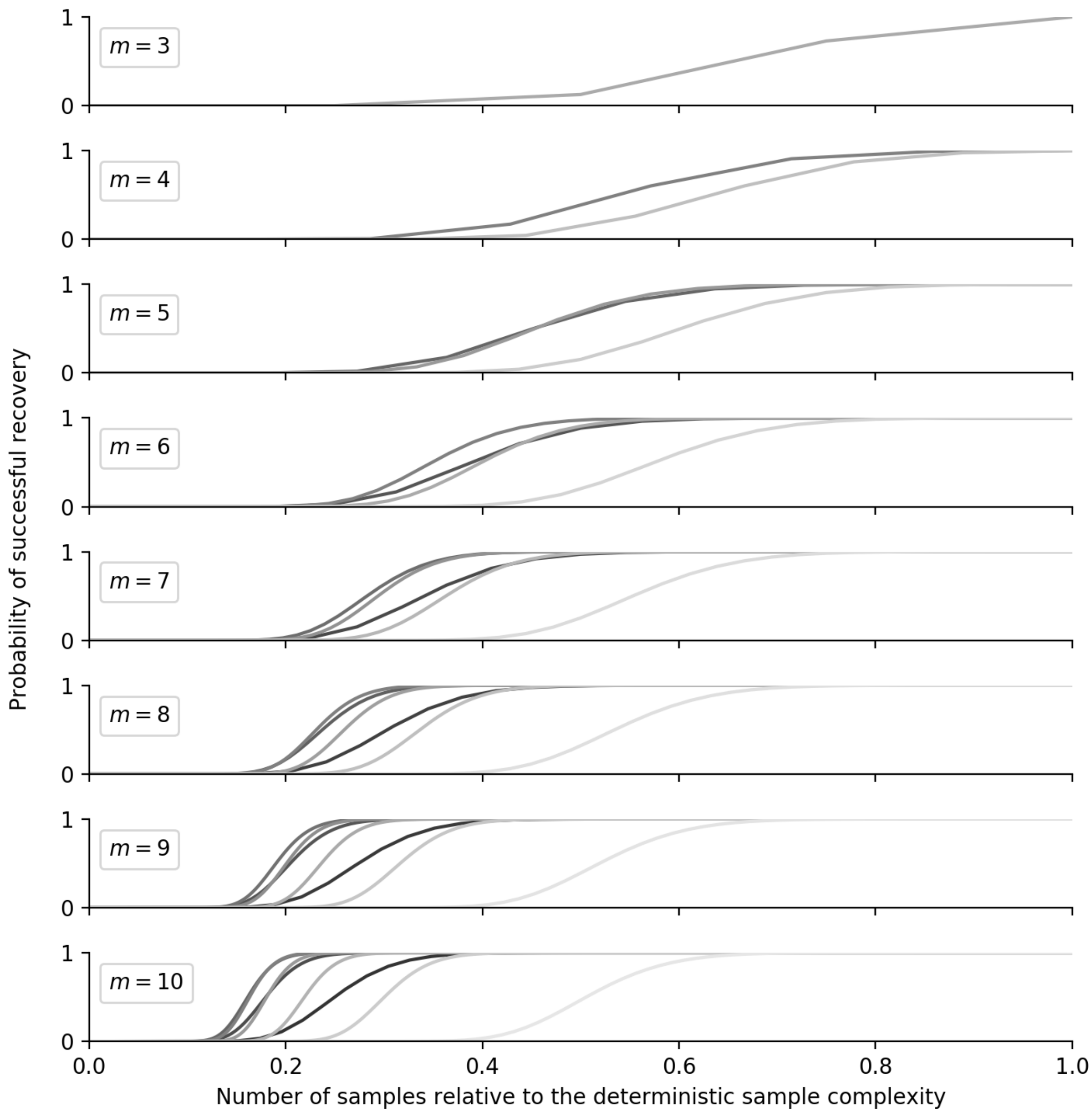

Figure 4.1: **Learning a dictionary from an arbitrary number of samples.** Probability of successful dictionary and code recovery (as per Thm. 1) for a number of samples N given as a fraction of the deterministic sample complexity $N = |\mathcal{H}|[(k-1)\binom{m}{k}+1]$ when the support set hypergraph $\mathcal{H}$ is the set of m consecutive intervals of length k in a cyclic order on $[m]$. Each plot has k ranging from 2 to $m-1$ (the case $k=1$ requires $N=m$), with lighter grey lines corresponding to larger k. Successful recovery is nearly certain with far fewer samples than the deterministic sample complexity.

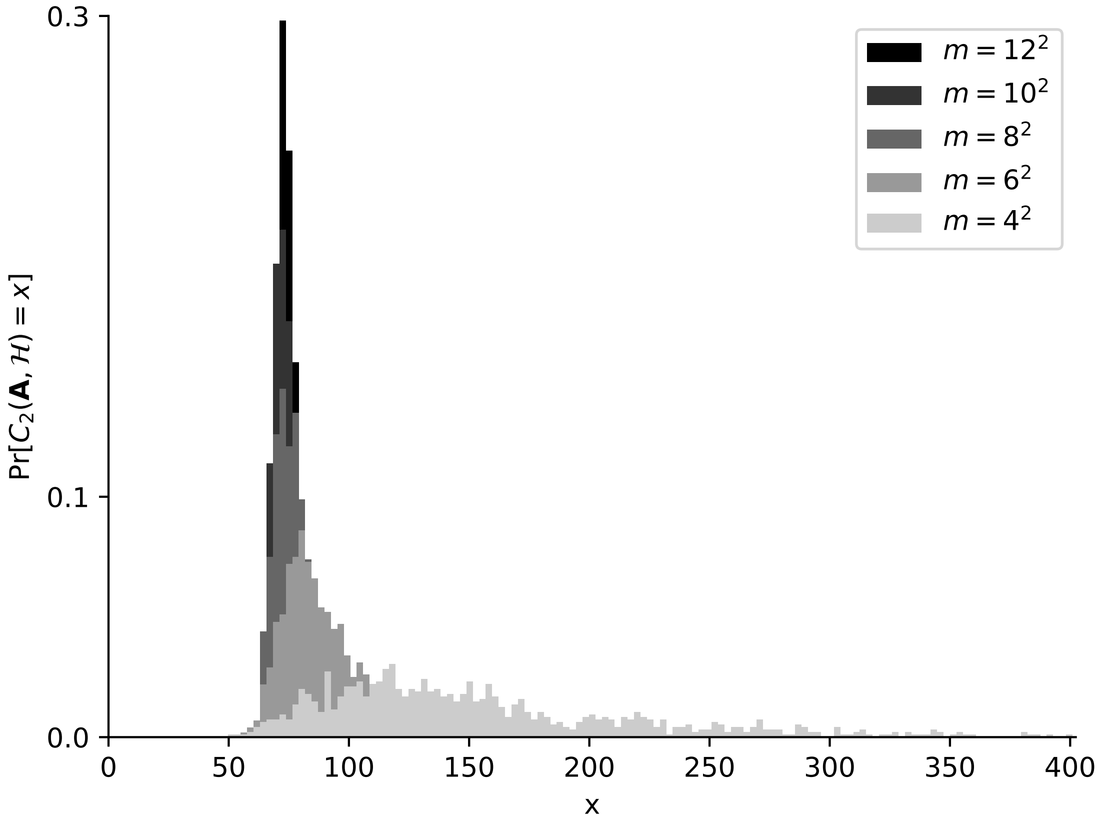

Figure 4.2: **Concentration of the constant C_2.** Distribution of $C_2(\mathbf{A},\mathcal{H})$ computed for 1.33x overcomplete generic unit-norm dictionaries $\mathbf{A} \in \mathbb{R}^{n\times m}$ (i.e. with $n = 3m/4$) when the support set hypergraph $\mathcal{H}$ consists of the rows and columns formed by arranging the elements of $[m]$ into a square grid (i.e. $m = k^2$). The distribution becomes more concentrated as m grows.

Chapter 5

Appendix: Proving the main lemma

We shall prove Lem. 1 after the following auxiliary lemmas.

Lemma 2. *If $f : V \to W$ is injective, then $f\left(\cap_{i=1}^{\ell} V_i\right) = \cap_{i=1}^{\ell} f\left(V_i\right)$ for any $V_1, \ldots, V_\ell \subseteq V$. ($f(\emptyset) := \emptyset$.)*

Proof. By induction, it is enough to prove the case $\ell = 2$. Clearly, for any map f, if $w \in f(U \cap V)$ then $w \in f(U)$ and $w \in f(V)$; hence, $w \in f(U) \cap f(V)$. If $w \in f(U) \cap f(V)$, then $w \in f(U)$ and $w \in f(V)$; thus, $w = f(u) = f(v)$ for some $u \in U$ and $v \in V$, implying $u = v$ by injectivity of f. It follows that $u \in U \cap V$ and $w \in f(U \cap V)$. □

In particular, if a matrix $\mathbf{A}$ satisfies $L_{2\mathcal{H}}(\mathbf{A}) > 0$, then taking V to be the union of subspaces consisting of vectors with supports in $2\mathcal{H}$, we have $\boldsymbol{\mathcal{A}}_{\cap\mathcal{G}} = \cap \boldsymbol{\mathcal{A}}_{\mathcal{G}}$ for all $\mathcal{G} \subseteq \mathcal{H}$.

Lemma 3. *Let $\mathcal{V} = \{V_i\}_{i=1}^k$ be a set of two or more subspaces of $\mathbb{R}^m$, and set $V = \cap\mathcal{V}$. For $\mathbf{u} \in \mathbb{R}^m$, we have (recall Defs. 3 & 4):*

$$dist(\mathbf{u}, V) \leq \frac{1}{1 - \xi(\mathcal{V})} \sum_{i=1}^{k} dist(\mathbf{u}, V_i). \tag{5.1}$$

Proof. Recall the projection onto the subspace $V \subseteq \mathbb{R}^m$ is the mapping $\Pi_V : \mathbb{R}^m \to V$ that associates with each $\mathbf{u}$ its unique nearest point in V; i.e., $\|\mathbf{u} - \Pi_V \mathbf{u}\|_2 = \text{dist}(\mathbf{u}, V)$. By repeatedly applying the triangle inequality, we have:

$$\begin{aligned} \|\mathbf{u} - \Pi_V \mathbf{u}\|_2 &\leq \|\mathbf{u} - \Pi_{V_k}\mathbf{u}\|_2 + \|\Pi_{V_k}\mathbf{u} - \Pi_{V_k}\Pi_{V_{k-1}}\mathbf{u}\|_2 \\ &\quad + \cdots + \|\Pi_{V_k}\Pi_{V_{k-1}} \cdots \Pi_{V_1}\mathbf{u} - \Pi_V \mathbf{u}\|_2 \\ &\leq \sum_{\ell=1}^{k} \|\mathbf{u} - \Pi_{V_\ell}\mathbf{u}\|_2 + \|(\Pi_{V_k} \cdots \Pi_{V_1} - \Pi_V)\mathbf{u}\|_2, \end{aligned} \tag{5.2}$$

where we have also used that the spectral norm of the orthogonal projections Π_{V_ℓ} satisfies $\|\Pi_{V_\ell}\|_2 \leq 1$ for all ℓ.

It remains to bound the second term in (5.2) by $\xi(\mathcal{V})\|\mathbf{u} - \Pi_V \mathbf{u}\|_2$. First, note that $\Pi_{V_\ell}\Pi_V = \Pi_V$ and $\Pi_V^2 = \Pi_V$, so we have $\|(\Pi_{V_k} \cdots \Pi_{V_1} - \Pi_V)\mathbf{u}\|_2 = \|(\Pi_{V_k} \cdots \Pi_{V_1} - \Pi_V)(\mathbf{u} - \Pi_V \mathbf{u})\|_2$. Consequently, inequality (5.1) follows from [10, Thm. 9.33]:

$$\|\Pi_{V_k}\Pi_{V_{k-1}} \cdots \Pi_{V_1}\mathbf{x} - \Pi_V \mathbf{x}\|_2 \leq z\|\mathbf{x}\|_2, \quad \text{for all } \mathbf{x}, \tag{5.3}$$

with $z^2 = 1 - \prod_{\ell=1}^{k-1}(1 - z_\ell^2)$ and $z_\ell = \cos\theta\left(V_\ell, \cap_{s=\ell+1}^{k} V_s\right)$ (recall θ from Def. 4), after substituting $\xi(\mathcal{V})$ for z and rearranging terms. $\square$

Lemma 4. *Fix an r-regular hypergraph $\mathcal{H} \subseteq 2^{[m]}$ satisfying the SIP. If the map $\pi : \mathcal{H} \to 2^{[\overline{m}]}$ has $\sum_{S\in\mathcal{H}} |\pi(S)| \geq \sum_{S\in\mathcal{H}} |S|$ and:*

$$|\cap\, \pi(\mathcal{G})| \leq |\cap\, \mathcal{G}|, \quad \textit{for } \mathcal{G} \in \binom{\mathcal{H}}{r} \cup \binom{\mathcal{H}}{r+1}, \tag{5.4}$$

then $\overline{m} \geq m$; and if $\overline{m}(r-1) < mr$, the map $i \mapsto \cap_{S\in\sigma(i)}\pi(S)$ is an injective function to $[\overline{m}]$ from some $J \subseteq [m]$ of size $m - (r-1)(\overline{m} - m)$ (recall σ from Def. 2).

Proof. Consider the following set: $T_1 := \{(i, S) : i \in \pi(S), S \in \mathcal{H}\}$, which numbers $|T_1| = \sum_{S\in\mathcal{H}} |\pi(S)| \geq \sum_{S\in\mathcal{H}} |S| = \sum_{i\in[m]} \deg_{\mathcal{H}}(i) = mr$ by r-regularity of $\mathcal{H}$. Note that $|T_1| \leq \overline{m}r$; otherwise, pigeonholing the tuples of T_1 with respect to their $\overline{m}$ possible first elements would imply that more than r of the tuples in T_1 share the same first element. This cannot be the case, however, since then some $\mathcal{G} \in \binom{\mathcal{H}}{r+1}$ formed from any $r+1$ of their second elements would satisfy $\cap\pi(\mathcal{G}) \neq 0$; hence, $|\cap\,\mathcal{G}| \neq 0$ by (5.4), contradicting r-regularity of $\mathcal{H}$. It follows that $\overline{m} \geq m$.

Suppose now that $\overline{m}(r-1) < mr$, so that $p := mr - \overline{m}(r-1)$ is positive and $|T_1| \geq \overline{m}(r-1) + p$. Pigeonholing T_1 into $[\overline{m}]$ again, there are at least r tuples in T_1 sharing some first element; that is, for some $\mathcal{G}_1 \subseteq \mathcal{H}$ of size $|\mathcal{G}_1| \geq r$, we have $|\cap\,\pi(\mathcal{G}_1)| \geq 1$ and (by (5.4)) $|\cap\,\mathcal{G}_1| \geq 1$. Since no more than r tuples of T_1 can share the same first element, we in fact have $|\mathcal{G}_1| = r$. It follows by r-regularity that $\mathcal{G}_1$ is a star of $\mathcal{H}$; hence, $|\cap\,\mathcal{G}_1| = 1$ by the SIP and $|\cap\,\pi(\mathcal{G}_1)| = 1$ by (5.4).

If $p = 1$, then we are done. Otherwise, define $T_2 := T_1 \setminus \{(i, S) \in T_1 : i = \cap\pi(\mathcal{G}_1)\}$, which contains $|T_2| = |T_1| - r \geq (\overline{m} - 1)(r-1) + (p-1)$ ordered pairs having $\overline{m} - 1$ distinct first indices. Pigeonholing T_2 into $[\overline{m} - 1]$ and repeating the above arguments produces the star $\mathcal{G}_2 \in \binom{\mathcal{H}}{r}$ with intersection $\cap\mathcal{G}_2$ necessarily distinct (by r-regularity) from $\cap\mathcal{G}_1$. Iterating this procedure p times in total yields the stars $\mathcal{G}_i$ for which $\cap\mathcal{G}_i \mapsto \cap\pi(\mathcal{G}_i)$ defines an injective map to $[\overline{m}]$ from $J = \{\cap\mathcal{G}_1, \ldots, \cap\mathcal{G}_p\} \subseteq [m]$. $\square$

Proof of Lem. 1. Let us begin by showing that $\dim(\boldsymbol{\mathcal{B}}_{\pi(S)}) = \dim(\boldsymbol{\mathcal{A}}_S)$ for all $S \in \mathcal{H}$. Note that since $\|\mathbf{A}\mathbf{x}\|_2 \leq \max_j \|\mathbf{A}_j\|_2\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_1 \leq \sqrt{k}\|\mathbf{x}\|_2$ for all k-sparse $\mathbf{x}$, by (2.1) we have $L_2(\mathbf{A}) \leq \max_j \|\mathbf{A}_j\|_2$ and therefore (as $0 \leq \xi < 1$) the right-hand side of (3.7) is less than one. From (3.5), we have $|\pi(S)| \geq \dim(\boldsymbol{\mathcal{B}}_{\pi(S)}) \geq \dim(\boldsymbol{\mathcal{A}}_S) = |S|$, the final equality holding by injectivity of $\mathbf{A}_S$. As $|\pi(S)| = |S|$, the claim follows. Note, therefore, that $\mathbf{B}_{\pi(S)}$ has full-column rank for all $S \in \mathcal{H}$.

We shall next demonstrate that (5.4) holds. Fixing $\mathcal{G} \in \binom{\mathcal{H}}{r} \cup \binom{\mathcal{H}}{r+1}$, it suffices to show that $d(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}, \boldsymbol{\mathcal{A}}_{\cap\mathcal{G}}) < 1$, since by (3.5) we then have $|\cap\pi(\mathcal{G})| = \dim(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}) \leq \dim(\boldsymbol{\mathcal{A}}_{\cap\mathcal{G}}) = |\cap\mathcal{G}|$, with equalities from the full column-ranks of $\mathbf{A}_S$ and $\mathbf{B}_{\pi(S)}$ for all $S \in \mathcal{H}$.[1] Observe that $d(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}, \boldsymbol{\mathcal{A}}_{\cap\mathcal{G}}) \leq d\left(\cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})}, \cap\boldsymbol{\mathcal{A}}_{\mathcal{G}}\right)$ by (3.4), since trivially $\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})} \subseteq \cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})}$ and also $\boldsymbol{\mathcal{A}}_{\cap\mathcal{G}} = \cap\boldsymbol{\mathcal{A}}_{\mathcal{G}}$ by Lem. 2. Recalling Def. 3 and applying Lem. 3 yields:

$$\begin{aligned} d\left(\cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})}, \cap\boldsymbol{\mathcal{A}}_{\mathcal{G}}\right) &\leq \max_{\mathbf{u}\in\cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})},\ \|\mathbf{u}\|_2\leq 1} \sum_{S\in\mathcal{G}} \frac{\operatorname{dist}\left(\mathbf{u}, \boldsymbol{\mathcal{A}}_S\right)}{1-\xi(\boldsymbol{\mathcal{A}}_{\mathcal{G}})} \\ &= \sum_{S\in\mathcal{G}} \frac{d\left(\cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})}, \boldsymbol{\mathcal{A}}_S\right)}{1-\xi(\boldsymbol{\mathcal{A}}_{\mathcal{G}})}, \end{aligned}$$

passing the maximum through the sum. Since $\cap\boldsymbol{\mathcal{B}}_{\pi(\mathcal{G})} \subseteq \boldsymbol{\mathcal{B}}_{\pi(S)}$ for all $S \in \mathcal{G}$, by (3.4) the numerator of each term in the sum above is bounded by $d\left(\boldsymbol{\mathcal{B}}_{\pi(S)}, \boldsymbol{\mathcal{A}}_S\right) = d\left(\boldsymbol{\mathcal{A}}_S, \boldsymbol{\mathcal{B}}_{\pi(S)}\right) \leq \varepsilon$, with the equality from (3.6) since $\dim(\boldsymbol{\mathcal{B}}_{\pi(S)}) = \dim(\boldsymbol{\mathcal{A}}_S)$. Thus, altogether:

$$d(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}, \boldsymbol{\mathcal{A}}_{\cap\mathcal{G}}) \leq \frac{|\mathcal{G}|\varepsilon}{1-\xi(\boldsymbol{\mathcal{A}}_{\mathcal{G}})} \leq \frac{C_2\varepsilon}{\max_j \|\mathbf{A}_j\|_2}, \tag{5.5}$$

recalling the definition of C_2 in (3.10). Lastly, since $C_2\varepsilon < L_2(\mathbf{A}) \leq \max_j \|\mathbf{A}_j\|_2$, we have $d(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}, \boldsymbol{\mathcal{A}}_{\cap\mathcal{G}}) \leq 1$ and therefore (5.4) holds.

Applying Lem. 4, the association $i \mapsto \cap_{S\in\sigma(i)}\pi(S)$ is an injective map $\overline{\pi} : J \to [\overline{m}]$ for some $J \subseteq [m]$ of size $m-(r-1)(\overline{m}-m)$, and $\mathbf{B}_{\overline{\pi}(i)} \neq \mathbf{0}$ for all $i \in J$ since the columns of $\mathbf{B}_{\pi(S)}$ are linearly independent for all $S \in \mathcal{H}$. Letting $\overline{\varepsilon} := C_2\varepsilon/\max_i \|\mathbf{A}_i\|_2$, it follows from (3.6) and (5.5) that $d\left(\boldsymbol{\mathcal{A}}_i, \boldsymbol{\mathcal{B}}_{\overline{\pi}(i)}\right) = d\left(\boldsymbol{\mathcal{B}}_{\overline{\pi}(i)}, \boldsymbol{\mathcal{A}}_i\right) \leq \overline{\varepsilon}$ for all $i \in J$. Setting $c_i := \|\mathbf{A}_i\|_2^{-1}$ so that $\|c_i\mathbf{A}\mathbf{e}_i\|_2 = 1$, by Def. 3 for all $i \in J$:

$$\min_{\overline{c}_i\in\mathbb{R}} \|c_i\mathbf{A}\mathbf{e}_i - \overline{c}_i\mathbf{B}\mathbf{e}_{\overline{\pi}(i)}\|_2 \leq d\left(\boldsymbol{\mathcal{A}}_i, \boldsymbol{\mathcal{B}}_{\overline{\pi}(i)}\right) \leq \overline{\varepsilon},$$

for $\overline{\varepsilon} < L_2(\mathbf{A})\min_{i\in[m]}|c_i|$. But this is exactly the supposition in (3.2), with J and $\overline{\varepsilon}$ in place of $[m]$ and ε, respectively. The same arguments of the case $k=1$ in Sec. 3 can then be made to show that for any $\overline{m}\times\overline{m}$ permutation and invertible diagonal matrices $\mathbf{P}$ and $\mathbf{D}$ with, respectively, columns $\mathbf{e}_{\overline{\pi}(i)}$ and $\frac{\overline{c}_i}{c_i}\mathbf{e}_i$ for $i \in J$ (otherwise, $\mathbf{e}_i$ for $i \in [\overline{m}]\setminus J$), we have $\|\mathbf{A}_i - (\mathbf{BPD})_i\|_2 \leq \overline{\varepsilon}/|c_i| \leq C_2\varepsilon$ for all $i \in J$. $\square$

[1]Note that if ever $\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})} \neq /bfseries0$ while $\cap\mathcal{G} = \emptyset$, we would have $d(\boldsymbol{\mathcal{B}}_{\cap\pi(\mathcal{G})}, \mathbf{0}) = 1$. However, that leads to a contradiction.

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