We can already see this may be possible by examining the case $k = 1$. Consider the dataset generated as in (1) in the noiseless case $\eta = 0$, i.e.:

$$\mathbf{z}_i = \mathbf{A}\mathbf{x}_i, \quad i = 1, \dots, N, \tag{1}$$

with the additional constraint that $\|\mathbf{A}\mathbf{e}_i\|_2 = 1$ for all $i \in [m]$. Consider the following "convexified" version of Prob. 2:

Problem 1. *Find a $n \times m$ matrix $\mathbf{B}$ with $\|\mathbf{B}\mathbf{e}_i\|_2 = 1$ for all $i \in [m]$ and vectors $\overline{\mathbf{x}}_1, \dots, \overline{\mathbf{x}}_N$ solving:*

$$\min \sum_{i=1}^{N} \|\overline{\mathbf{x}}_i\|_1 \quad \textit{subject to} \quad \mathbf{z}_i = \mathbf{B}\overline{\mathbf{x}}_i, \ \textit{for all } i. \tag{2}$$

Proposition 1. *Fix $c > 0$. If $\mathbf{x}_i = c\mathbf{e}_i$ for $i = 1, \dots, m$, then every solution to Prob. 1 satisfies $\mathbf{A} = \mathbf{B}\mathbf{P}$ and $\mathbf{x}_i = \mathbf{P}^\top \overline{\mathbf{x}}_i$ for some $m \times m$ permutation matrix $\mathbf{P}$.*

Proof. Fix $i \in [m]$. Writing $\overline{\mathbf{x}}_i = \sum_{j=1}^m \overline{c}_j^{(i)} \mathbf{e}_j$, we have:

$$c = \|c\mathbf{A}\mathbf{e}_i\|_2 = \|\mathbf{B}\overline{\mathbf{x}}_i\|_2 = \|\sum_{j=1}^{m} \overline{c}_j^{(i)} \mathbf{B}\mathbf{e}_j\|_2 \leq \sum_{j=1}^{m} |\overline{c}_j^{(i)}| \|\mathbf{B}\mathbf{e}_j\|_2 = \|\overline{\mathbf{x}}_i\|_1 \tag{3}$$

So $\|\overline{\mathbf{x}}_i\|_1 \geq c$ for all $i \in [m]$. Therefore $\sum_{i=1}^m \|\overline{\mathbf{x}}_i\|_1 \geq mc$. But since $\mathbf{B} = \mathbf{A}$ and $\overline{\mathbf{x}}_i = \mathbf{x}_i$ ($i = 1, \dots, m$) satisfy the constraints of the minimization problem, we must have $\sum_{i=1}^m \|\overline{\mathbf{x}}_i\|_1 \leq \sum_{i=1}^m \|\mathbf{x}_i\|_1 = mc$ also. Thus $\sum_{i=1}^m \|\overline{\mathbf{x}}_i\|_1 = mc$. Since again $\|\overline{\mathbf{x}}_i\|_1 \geq c$ for all $i \in [m]$, we must have $\|\overline{\mathbf{x}}_i\|_1 = c$ for all $i \in [m]$.

Recalling (3) we therefore have $c = \|\mathbf{B}\overline{\mathbf{x}}_i\|_2 \leq \|\overline{\mathbf{x}}_i\|_1 = c$, with equality only when $\overline{c}_j^{(i)}\mathbf{B}\mathbf{e}_j$ are colinear. This would be the case either if $\overline{\mathbf{x}}_i$ is 1-sparse, in which case we may apply Thm. 1 to guarantee both dictionary and code recovery, or $\mathbf{B}$ has colinear columns. In the latter case, the same guarantees hold for a suitable submatrix of $\mathbf{B}$ containing one representative column from every colinear set (note that since $\|\mathbf{B}\mathbf{e}_j\| = 1$ for all $j \in [m]$, these columns are identical up to a sign). $\square$