lil' Lemma

C. Garfinkle

Lemma 1. *Suppose $E \subseteq 2^{[m]}$ is a hypergraph with the SIP and let $d_1 \geq d_2 \geq \ldots \geq d_m$ be the degree sequence of nodes in E. If for some $\bar{m}$ the map $\pi : E \to 2^{[\bar{m}]}$ has $\sum_{S\in E} |\pi(S)| = \sum_{S \in E} |S|$ and:*

$$|\cap \pi(F)| \leq |\cap F| \text{ for all } F \subseteq E, \tag{1}$$

then $\bar{m} \geq \sum_i d_i/d_1$ and the association $i \mapsto \cap\pi(F(i))$ defines an injective map from J to $[\bar{m}]$ for some $J \in \binom{[m]}{p}$, where p (provided it exists) is the largest positive integer in $\{1, \ldots, m\}$ satisfying:

$$\sum_{i=\ell}^{m} d_i > (\bar{m} + 1 - \ell)(d_\ell - 1) \text{ for all } \ell \leq p. \tag{2}$$

If $\bar{m} < \sum_i d_i/(d_1 - 1)$ then $p \geq 1$. In particular, if $\bar{m} = m$ and E is regular, then π is induced by a permutation on $[m]$.

Proof. Consider the collection of pairs: $T_1 := \{(i, S) : i \in \pi(S), S \in E\}$, which number $|T_1| = \sum_{S\in E} |\pi(S)| = \sum_{S\in E} |S| = \sum_{i\in[m]} d_i$. Note that assumption (1) implies $\bar{m} \geq |T_1|/d_1$, since otherwise pigeonholing the elements of T_1 with respect to their set of possible first indices $[\bar{m}]$ would lead us to conclude that there are more than d_1 sets in E sharing a common element.

By (2) we have $|T_1| > \bar{m}(d_1 - 1)$, which implies, again by the pigeonhole principle, that there must be at least d_1 elements of T_1 sharing the same first index. By (1), the intersection of the set F_1 consisting of their second indices is non-empty. As $d_1 \geq d_i$ for all i, and E satisfies the SIP, it must be that the sets in F_1 intersect at a singleton. Since $\cap\pi(F_1)$ is non-empty, applying (1) again implies $\cap\pi(F_1) = \{i_1\}$ for some $i_1 \in [m]$.

If $p = 1$ then we are done. Otherwise, define $T_2 := T_1 \setminus \{(i_1, S) \in T_1 : S \in E\}$, which contains $|T_2| = |T_1| - d_1 = \sum_{i=2}^{m} d_i$ ordered pairs having $\bar{m} - 1$ distinct first indices. By (2) we have $|T_2| > (\bar{m} - 1)(d_2 - 1)$ and reiterating the above arguments produces a (necessarily) distinct index i_2. Iterating the arguments p times yields the set of singletons $J = \{\cap F_1, \ldots, \cap F_p\} \subseteq [m]$. ☐

Remark 1 (Regular hypergraphs)**.** *Note that if E is d-regular then $\sum_{i=\ell}^{m} d = (m - \ell + 1)d$ and we have $\bar{m} \geq \sum_i d_i/d_1 \geq m$ while (2) becomes $\ell < \bar{m} - (\bar{m} - m)d + 1$ for all $\ell \leq p$. Hence, it suffices to know that $p = \bar{m} - (\bar{m} - m)d$ is positive to know that (2) is satisfied, which is true if an only if $\bar{m} < md/(d + 1)$. Hence we must have $m \leq \bar{m} < md/(d - 1)$ for the conclusion of the lemma to hold for some non-empty $J \subseteq [m]$. We therefore have $J = [m]$ when $\bar{m} = m$ and E is regular.*

Remark 2 (General hypergraphs)**.** *For what range of values of $\bar{m}$ can we guarantee there exists some $p \in \{1, \ldots, m\}$ in general? If $\bar{m} < \sum_i d_i/(d_1 - 1)$ then we can peel off at least one index for any hypergraph, i.e. $p \geq 1$. Can we find a lower bound on $\bar{m}$ in general, though (not just for regular hypergraphs)? The difficulty is we have to make sure (2) holds for* all *values of $\ell \leq p$, which was easy for regular hypergraphs (by the resulting transitivity of (2)). My thought is that in general the lower bound $\bar{m} \geq \sum_i d_i/d_1$ which *has* to hold given (1) might not correspond to what condition (2), which comes from our crappy pigeonholing, gives us as a constraint on $\bar{m}$ when we make the substitution $p = m$ (as was the case for regular hypergraphs). Could we have $\bar{m} < m$ but still recover a positive p number of indices?*

Question 1. *Do we only need E to satisfy the SIP on the p nodes of highest degree, those we end up isolating, for all of Chaz' Thm. to run through? That would be good!*