Solving Polynomial Systems With Special Structure

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Abstract

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The general problem of solving systems of polynomial equations has a rich history and progression. We study a collection of problems that give rise to systems of equations with nice structure. We exploit such special structure using a variety of techniques, thereby solving the original problems.

Our tools are diverse, involving algebraic combinatorics, semigroup algebras, Brouwer degree theory, fixed-point methods, and the theory of Gröbner bases, to name a few.

Professor Bernd Sturmfels
Dissertation Committee Chair

To my parents

Contents

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Chapter 1

Outline of Techniques

Before embarking on our thesis journey, we present here the collection of techniques, tools, and definitions that underlie the proofs of our main results. They are listed here primarily because readers might find them useful for other applications.

1.1 Gröbner Bases

1.2 Brouwer Mapping Degree and Fixed-Point Theory

In this section, we give a brief overview of degree theory, and some of its main implications. The bulk of this discussion is material taken from [13, 16]. First we introduce some notation. Let U be a bounded open subset of $\mathbb{R}^m$. We denote the set of r-times differentiable functions from U ($\overline{U}$) to $\mathbb{R}^m$ by $C^r(U, \mathbb{R}^m)$ ($C^r(\overline{U}, \mathbb{R}^m)$) (when $r = 0$, $C^r(U, \mathbb{R}^m)$ is the set of continuous functions). The *identity function* id satisfies $\mathrm{id}(\mathbf{x}) = \mathbf{x}$. If $f \in C^1(U, \mathbb{R}^m)$, then the *Jacobi matrix* of f at a point $\mathbf{x} \in U$ is

$$f'(\mathbf{x}) = \left(\frac{\partial f_j}{\partial x_i}(\mathbf{x}) \right)_{1 \leq i,j \leq m}$$

and the *Jacobi determinant* (or simply *Jacobian*) of f at $\mathbf{x}$ is

$$J_f(\mathbf{x}) = \det f'(\mathbf{x}).$$

The set of *regular values* of f is

$$\mathrm{RV}(f) = \left\{ \mathbf{y} \in \mathbb{R}^m : \forall \mathbf{x} \in f^{-1}(\mathbf{y}),\ J_f(\mathbf{x}) \neq 0 \right\}$$

and for $\mathbf{y} \in \mathbb{R}^m$, we set

$$D^r_{\mathbf{y}}(\overline{U}, \mathbb{R}^m) = \left\{ f \in C^r(\overline{U}, \mathbb{R}^m) : \mathbf{y} \notin f(\partial U) \right\}.$$

A function $\deg : D^0_{\mathbf{y}}(\overline{U}, \mathbb{R}^m) \to \mathbb{R}$ which assigns to each $f \in D^0_{\mathbf{y}}(\overline{U}, \mathbb{R}^m)$ and $\mathbf{y} \in R^m$ a real number $\deg(f, U, \mathbf{y})$ will be called a *degree* if it satisfies the following conditions.

(1) $\deg(f, U, \mathbf{y}) = \deg(f - \mathbf{y}, U, 0)$ (*translation invariance*).

(2) $\deg(\mathrm{id}, U, \mathbf{y}) = 1$ if $\mathbf{y} \in U$ (*normalization*).

(3) If U_1 and U_2 are open, disjoint subsets of U such that $\mathbf{y} \notin f(\overline{U} \setminus (U_1 \cup U_2))$, then $\deg(f, U, \mathbf{y}) = \deg(f, U_1, \mathbf{y}) + \deg(f, U_2, \mathbf{y})$ (*additivity*).

(4) If $H(t) = tf + (1-t)g \in D^0_{\mathbf{y}}(\overline{U}, \mathbb{R}^m)$ for all $t \in [0, 1]$, then $\deg(f, U, \mathbf{y}) = \deg(g, U, \mathbf{y})$ (*homotopy invariance*).

Motivationally, one should think of a degree map as somehow "counting" the number of solutions to $f(\mathbf{x}) = \mathbf{y}$. Condition (1) reflects that the solutions to $f(\mathbf{x}) = \mathbf{y}$ are the same as those of $f(\mathbf{x}) - \mathbf{y} = 0$, and since any multiple of a degree will satisfy (1), (3), and (4), the condition (2) is a normalization. Additionally, (3) is natural since it requires deg to be additive with respect to components.

Of course, we need a theorem guaranteeing that a degree even exists.

Theorem 1. *There is a unique degree* deg*. Moreover,* $\deg(\cdot, U, \mathbf{y}) : D^0_{\mathbf{y}}(\overline{U}, \mathbb{R}^m) \to \mathbb{Z}$.

When functions are differentiable, a degree can be calculated explicitly in terms of Jacobians at solutions to the equation $f(\mathbf{x}) = \mathbf{y}$.

Theorem 2. *Suppose that* $f \in D^1_{\mathbf{y}}(\overline{U}, \mathbb{R}^m)$ *and* $\mathbf{y} \in \mathrm{RV}$*. Then a degree satisfies*

$$\deg(f, U, \mathbf{y}) = \sum_{\mathbf{x} \in f^{-1}(\mathbf{y})} \operatorname{sgn} J_f(\mathbf{x}),$$

where this sum is finite and we adopt the convention that $\sum_{\mathbf{x} \subseteq \emptyset} = 0$.

1.3 Algebraic Combinatorics and Generating Functions

1.4 Well-Quasi-Orderings

A **quasi-ordering** on a set S is a binary relation $\leq$ on S which is reflexive and transitive. A **quasi-ordered set** is a pair $(S, \leq)$ consisting of a set S and a quasi-ordering $\leq$ on S. (If no confusion is possible, we will omit $\leq$ from the notation, and just call S a quasi-ordered set.) If in addition the relation $\leq$ is anti-symmetric, then $\leq$ is called an **ordering** on the set S, and $(S, \leq)$ (or S) is called an **ordered set.** A quasi-ordering $\leq$ on a set S induces an ordering on the set $S/{\sim} = \{a/{\sim} : a \in S\}$ of equivalence classes of the equivalence relation $x \sim y \iff x \leq y \;\&\; y \leq x$ on S in a natural way. If x and y are elements of a quasi-ordered set $(S, \leq)$, we write as usual $x \leq y$ also as $y \geq x$, and we write $x < y$ if $x \leq y$ and $y \not\leq x$.

Suppose that $(S, \leq_S)$ is a quasi-ordered set. The restriction $\leq_U := \leq_S \cap (U \times U)$ of the quasi-ordering $\leq_S$ makes $U \subseteq S$ into a quasi-ordered set. If $\leq_S$ is an ordering, then so is $\leq_U$. Let $(T, \leq_T)$ be another quasi-ordered set. The cartesian product $S \times T$ of S and T can be made into a quasi-ordered set by means of the **product quasi-ordering** $(x, y) \leq_{S\times T} (x', y') \iff x \leq_S x'$ and $y \leq_T y'$. If $\leq_S$ and $\leq_T$ are orderings, then so is $\leq_{S\times T}$. Taking $S = T$ and repeating this construction yields the product quasi-ordering on S^n.

Example 1. We consider $\mathbb{N} = \{0, 1, 2, \dots\}$ as an ordered set with its usual ordering, and we equip $\mathbb{N}^n$ with the product ordering.

Final segments and antichains

A **final segment** of a quasi-ordered set $(S, \leq)$ is a subset $F \subseteq S$ which is closed upwards: $x \leq y \wedge x \in F \Rightarrow y \in F$, for all $x, y \in S$. We construe the set $\mathcal{F}(S)$ of final segments of S as an ordered set, with the ordering given by *reverse* inclusion. Given a subset M of S, the set $\big\{y \in S : \exists x \in M \; (x \leq y)\big\}$ is a final segment of S, the final segment **generated by** M**.** An **antichain** of S is a subset $A \subseteq S$ such that $x \not\leq y$ and $y \not\leq x$ for all $x \not\sim y$ in A.

Well-quasi-orderings

A quasi-ordered set S is **well-founded** if there is no infinite strictly decreasing sequence $x_0 > x_1 > \cdots$ in S. A quasi-ordered set S is **well-quasi-ordered** if it is well-founded, and every antichain of S is finite. The following characterization of well-quasi-orderings is classical. (See, e.g., [**?**].) An infinite sequence $s_0, s_1, \ldots$ in S such that $s_i \leq s_j$ for some indices $i < j$ is called **good**, and **bad** otherwise.

Proposition 1. *The following are equivalent, for a quasi-ordered set S:*

(1) *S is a well-quasi-ordering.*

(2) *Every infinite sequence in S is good.*

(3) *Every infinite sequence in S contains an infinite increasing subsequence.*

(4) *Any final segment of S is finitely generated.*

(5) *$\big(\mathcal{F}(S), \supseteq\big)$ is well-founded (i.e., the ascending chain condition with respect to inclusion holds for final segments of S).* $\square$

Suppose $(S, \leq_S)$ and $(T, \leq_T)$ are well-quasi-ordered sets. Then the induced quasi-ordering on a subset of S is a well-quasi-ordering, and the cartesian product $S \times T$ of S and T is well-quasi-ordered. (Easily seen, e.g., using the equivalence of (1) and (3) above.) Inductively, it follows that the product quasi-ordering on S^n is a well-quasi-ordering, for every $n > 0$. (With $S = \mathbb{N}$ this is known as "Dickson's Lemma".)

Chapter 2

Infinite Permutation Ideals in Countable Polynomial Rings are Noetherian

2.1 Finiteness result

Let Ω be the set of monomials in indeterminates $p_1, p_2, \ldots$ (including the "monomial" 1). Order these monomials lexicographically with $p_1 < p_2 < \cdots$. The following fact is immeadiate.

Lemma 1. *The ordering $\leq$ is a well-ordering.*

Proof. Let S be any nonempty set of monomials and $m = p_1^{a_1} \cdots p_n^{a_n} \in S$. It suffices to show that $T = \{q \in S : q < m\}$ has a smallest element. If $q = p_1^{b_1} \cdots p_t^{b_t} \in T$ with $b_t \neq 0$, then $t \leq n$. In particular, q only involves the variables $p_1, \ldots, p_n$. But it is well-known that a set of monomials in the variables $p_1, \ldots, p_n$ with lexicographic order has a least element ("Dickson's Lemma"). $\square$

Consider the $\mathbb{C}$-algebra $R = \mathbb{C}[p_1, p_2, \ldots]$ generated by Ω with the natural ring operations. Also, let $R\mathfrak{S}_\infty$ be the (right) group ring with multiplication $r\sigma \cdot s\tau = rs(\sigma\tau)$, and view R as an $R\mathfrak{S}_\infty$-module in the standard way. Suppose one has an ideal I of R

invariant under $R\mathfrak{S}_\infty$; that is, $\mathfrak{S}_\infty I \subseteq I$ (so that $R\mathfrak{S}_\infty I \subseteq I$). A natural question is the following.

Question 1. Is I finitely generated as a $R\mathfrak{S}_\infty$-module?

Before answering this question in the affirmative, we gather some preliminary results. Given the set of monomials Ω, we define a quasi-ordering as follows:

Definition 1.

$$P \preceq Q \quad :\Longleftrightarrow \quad \begin{cases} P \le Q, \textit{ there exist } \sigma \in \mathfrak{S}_\infty \textit{ and a} \\ \textit{monomial } m \in \Omega \textit{ with } Q = m\sigma P, \\ \textit{and } U \le P \Rightarrow m\sigma U \le Q \end{cases}$$

Example 2. As an example of this quasi-ordering, consider the following chain,

$$p_1^2 \preceq p_1 p_2^2 \preceq p_1^3 p_2 p_3^2.$$

To verify the first inequality, notice that $p_1 p_2^2 = p_1 \sigma(p_1^2)$, in which σ is the transposition (12). If $U = p_1^{u_1} \cdots p_n^{u_n} \le p_1^2$, then it follows that $n = 1$ and $u_1 \le 2$. In particular, $p_1 \sigma U = p_1 p_2^{u_1} \le p_1 p_2^2$.

For the second relationship, we have that $p_1^3 p_2 p_3^2 = p_1^3 \tau(p_1 p_2^2)$, in which τ is the cycle (123). Additionally, if $U = p_1^{u_1} \cdots p_n^{u_n} \le p_1 p_2^2$, then $n = 2$ and $u_2 \le 2$. Also, if $u_2 = 2$, then $u_1 \le 1$. It follows, therefore, that $p_1^3 \tau U = p_1^3 p_2^{u_1} p_3^{u_2} \le p_1^3 p_2 p_3^2$. $\square$

Although this definition appears technical, the reason for its introduction will become clear in the proof of the theorem above. Since our ordering of monomials is linear ($u \le v \iff uw \le vw$ for all monomials u, v, w), this last condition may be rewritten as $U \le P \Rightarrow \sigma U \le \sigma P$. Let us see that it is a quasi-ordering. First notice that $P \preceq P$ since we may take $m = 1$ and σ, the identity permutation. Next, suppose that $P \preceq Q \preceq T$. Then, there exist permutations σ, τ and monomials m_1, m_2 such that $Q = m_1 \sigma P$, $T = m_2 \tau Q$. In particular, $T = m_2(\tau m_1)(\tau\sigma P)$. Additionally, if $U \le P$, then $m_1 \sigma U \le Q$ so that $m_2 \tau(m_1 \sigma U) \le T$. It follows that $m_2(\tau m_1)(\tau \sigma U) \le T$, which is the requirement for transitivity.

We shall also prove that $\preceq$ is a well-quasi-ordering. We begin with some preliminary lemmas. In what follows, it will be convenient to represent monomials $p_1^{a_1} \cdots p_n^{a_n}$ as vectors $(a_1, \ldots, a_n, 0, \ldots)$, and we will move freely from one representation to the other.

Lemma 2. *Suppose that* $(a_1, \ldots, a_n, 0, \ldots) \preceq (b_1, \ldots, b_t, 0, \ldots)$*. Then, for any* $c \in \mathbb{N}$, $(a_1, \ldots, a_n, 0, \ldots) \preceq (c, b_1, \ldots, b_t, 0, \ldots)$.

Proof. First notice that we may assume that $n \le t$ since $(a_1, \ldots, a_n, 0, \ldots) \le (b_1, \ldots, b_t, 0, \ldots)$. Let $(c_1, c_2, \ldots, c_{t+1}, 0, \ldots) = (c, b_1, \ldots, b_t, 0, \ldots)$, in which $c_1 = c$ and $c_i = b_{i-1}$ for $i > 1$. Since $(a_1, \ldots, a_n, 0, \ldots) \preceq (b_1, \ldots, b_t, 0, \ldots)$, there exists a permutation σ such that $p_{\sigma(1)}^{a_1} \cdots p_{\sigma(n)}^{a_n} \mid p_1^{b_1} \cdots p_t^{b_t}$. Let τ be the cyclic permutation $\tau = (123 \cdots (t+1))$. Then, $\hat{\sigma} = \tau\sigma$ is such that $p_{\hat{\sigma}(1)}^{a_1} \cdots p_{\hat{\sigma}(n)}^{a_n} \mid p_1^{c_1} \cdots p_{t+1}^{c_{t+1}}$.

Next, suppose that $U = p_1^{u_1} \cdots p_n^{u_n} \le p_1^{a_1} \cdots p_n^{a_n} = P$. By assumption, $\sigma U \le \sigma P$, so that

$$\hat{\sigma} U = p_{\sigma(1)+1}^{u_1} \cdots p_{\sigma(n)+1}^{u_n} \le p_{\sigma(1)+1}^{a_1} \cdots p_{\sigma(n)+1}^{a_n} = \hat{\sigma} P$$

since we are using lex order. Finally, the condition $p_1^{a_1} \cdots p_n^{a_n} \le p_1^{c_1} \cdots p_{t+1}^{c_{t+1}}$ is obvious, completing the proof. $\square$

Lemma 3. *If* $(a_1, \ldots, a_n, 0, \ldots) \preceq (b_1, \ldots, b_t, 0, \ldots)$ *and* $a, b \in \mathbb{N}$ *are such that* $a \le b$, *then* $(a, a_1, \ldots, a_n, 0, \ldots) \preceq (b, b_1, \ldots, b_t, 0, \ldots)$.

Proof. As before, we may assume that $n \le t$. Since we are using lex order, it is clear that $(a, a_1, \ldots, a_n, 0, \ldots) \le (b, b_1, \ldots, b_t, 0, \ldots)$. Let $\sigma \in \mathfrak{S}_t$ be such that $p_{\sigma(1)}^{a_1} \cdots p_{\sigma(n)}^{a_n} \mid p_1^{b_1} \cdots p_t^{b_t}$, and let τ be the cyclic permutation $\tau = (12 \cdots (t+1))$. Setting $\hat{\sigma} = \tau\sigma\tau^{-1}$, we have

$$\begin{aligned} \hat{\sigma}\left(p_1^a p_2^{a_1} \cdots p_{n+1}^{a_n}\right) &= \tau\sigma\left(p_{t+1}^a p_1^{a_1} \cdots p_n^{a_n}\right) \\ &= \tau\left(p_{t+1}^a p_{\sigma(1)}^{a_1} \cdots p_{\sigma(n)}^{a_n}\right) \\ &= p_1^a p_{\sigma(1)+1}^{a_1} \cdots p_{\sigma(n)+1}^{a_n}. \end{aligned}$$

It is easily seen that this last expression divides $p_1^b p_2^{b_1} \cdots p_{t+1}^{b_t}$.

Finally, suppose that $U = p_1^{u_1} \cdots p_{n+1}^{u_{n+1}} \le p_1^a p_2^{a_1} \cdots p_{n+1}^{a_n} = P$. Then, since we are using lex order, it follows that

$$p_2^{u_2} \cdots p_n^{u_{n+1}} \le p_2^{a_1} \cdots p_{n+1}^{a_n}.$$

By assumption, however, this implies that $\sigma\tau^{-1} p_2^{u_2} \cdots p_{n+1}^{u_{n+1}} \le \sigma\tau^{-1} p_2^{a_1} \cdots p_{n+1}^{a_n}$ and thus $\hat{\sigma} p_2^{u_2} \cdots p_{n+1}^{u_{n+1}} \le \hat{\sigma} p_2^{a_1} \cdots p_{n+1}^{a_n}$. It follows that if $\hat{\sigma} p_2^{u_2} \cdots p_{n+1}^{u_{n+1}} \ne \hat{\sigma} p_2^{a_1} \cdots p_{n+1}^{a_n}$, we must have $\hat{\sigma}U < \hat{\sigma}P$. On the other hand, if $p_2^{u_2} \cdots p_{n+1}^{u_{n+1}} = p_2^{a_1} \cdots p_{n+1}^{a_n}$, then $u_1 \leq a$, in which case we still have $\hat{\sigma}U \leq \hat{\sigma}P$ (since $\hat{\sigma}p_1 = p_1$). This completes the proof. $\square$

We finally have enough to prove the well-quasi-ordering of $\preceq$.

Theorem 3. *The quasi-order relation $\preceq$ is a well-quasi-ordering.*

Proof. The proof uses some ideas from Nash-Williams' proof [14] of a result of Higman [9]. Assume for the sake of contradiction that $m^{(1)}, m^{(2)}, \ldots$ is a bad sequence in Ω, in which $m^{(i)} = (m_1^{(i)}, m_2^{(i)}, \ldots)$. Given a monomial $m \in \Omega$, we let $j(s)$ denote the smallest index j such that $m_j = m_{j+1} = \cdots$. We may assume that the bad sequence is chosen in such a way that for every i, $j(m^{(i)})$ is *minimal* among the $j(m)$, where m ranges over all elements of Ω with the property that $m^{(1)}, m^{(2)}, \ldots, m^{(i-1)}, m$ can be continued to a bad sequence in Ω. Clearly, we have $j(m^{(i)}) > 1$ for all i. Let $t^{(i)} = (m_2^{(i)}, m_3^{(i)}, \ldots) \in \Omega$ so that $j(t^{(i)}) = j(m^{(i)}) - 1$, for all i. Now consider the sequence $m_1^{(1)}, m_1^{(2)}, \ldots, m_1^{(i)}, \ldots$ of elements of $\mathbb{N}$. Since $\mathbb{N}$ is well-ordered, there is an infinite sequence $1 \leq i_1 < i_2 < \cdots$ of indices such that $m_1^{(i_1)} \leq m_1^{(i_2)} \leq \cdots$. By minimality of $m^{(1)}, m^{(2)}, \ldots$, the sequence $m^{(1)}, m^{(2)}, \ldots, m^{(i_1-1)}, t^{(i_1)}, t^{(i_2)}, \ldots$ is good; that is, there exist $j < i_1$ and k with $m^{(j)} \preceq t^{(i_k)}$, or $k < l$ with $t^{(i_k)} \preceq t^{(i_l)}$. In the first case we have $m^{(j)} \preceq m^{(i_k)}$ by Lemma 2; and in the second case, $m^{(i_k)} \preceq m^{(i_l)}$ by Lemma 3. This contradicts the badness of our sequence $m^{(1)}, m^{(2)}, \ldots$. $\square$

If $f \in R$, we define the *leading monomial* of f, $\mathrm{lm}(f)$, to be the largest monomial occurring in f with respect to $\leq$. We now come to our main result.

Theorem 4. *Let I be an ideal of $R = \mathbb{C}[p_1, p_2, \ldots]$ such that $\mathfrak{S}_\infty I \subseteq I$. Then, I is finitely generated as a $R\mathfrak{S}_\infty$-module.*

Proof. Let I be an ideal of R with $\mathfrak{S}_\infty I \subseteq I$ that is not finitely generated. Define a sequence of elements of I as follows: f_1 is an element of I with minimal leading monomial; f_{i+1} is an element of $I \setminus \langle f_1, \ldots, f_i \rangle_{R\mathfrak{S}_\infty}$ with minimal leading monomial. Of course, if $f_1 \in \mathbb{C}$, then $I = R$ or $I = (0)$ is finitely generated.

It is clear that $\mathrm{lm}(f_i) \leq \mathrm{lm}(f_{i+1})$. If $\mathrm{lm}(f_i) = \mathrm{lm}(f_{i+1})$, then for a suitable $a \in \mathbb{C}$, $f_{i+1} - af_i \in I \setminus \langle f_1, \ldots, f_i \rangle_{R\mathfrak{S}_\infty}$ has a smaller leading term, a contradiction. Thus, $\mathrm{lm}(f_i) < \mathrm{lm}(f_{i+1})$.

We, therefore, obtain a sequence of (strictly) increasing monomials $m_i = \mathrm{lm}(f_i)$. By the well-quasi-ordering theorem above, there exist $i < j$ such that $m_i \preceq m_j$. Let σ be the permutation inducing $m_i \preceq m_j$. Then, $m_j = m\sigma m_i$ for some monomial m. Examine the element

$$g = f_{i+1} - am\sigma f_i \in I \setminus \langle f_1, \ldots, f_i \rangle_{R\mathfrak{S}_\infty},$$

in which $a \in \mathbb{C}$ is chosen so that the leading term of f_{i+1} cancels. The leading monomial of g cannot come from any of the (necessarily, strictly smaller) monomials occurring in f_{i+1} by our choice of f_{i+1}.

On the other hand, any monomial occurring in $m\sigma f_i$ is of the form $m\sigma u$ for u occurring in f_i. Since $u < \mathrm{lm}(f_i) = m_i$, the relation $m_i \preceq m_j$ implies that $m\sigma u < m_j$. In particular, g again has a smaller leading term than f_{i+1}. This contradiction finishes the proof. $\square$

Chapter 3

Logarithmic Derivatives

3.1 Statement of Results

Chapter 4

Cyclic Resultants

4.1 Statement of Results

Chapter 5

Symmetric Word Equations in Positive Definite Letters

5.1 Statement of Results

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