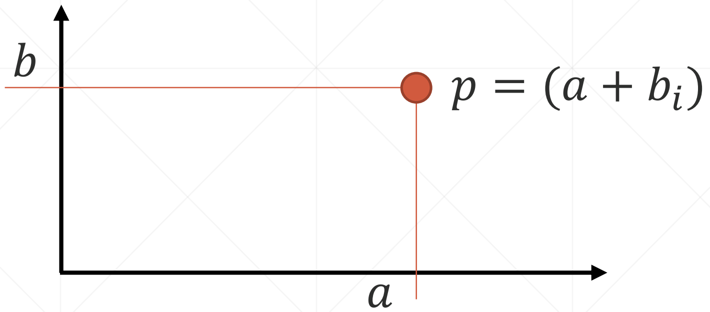
Quarternion

- 1. Quarternion
- 2. Quarternion for rotation
- 3. ZYX Euler angles to quaternions

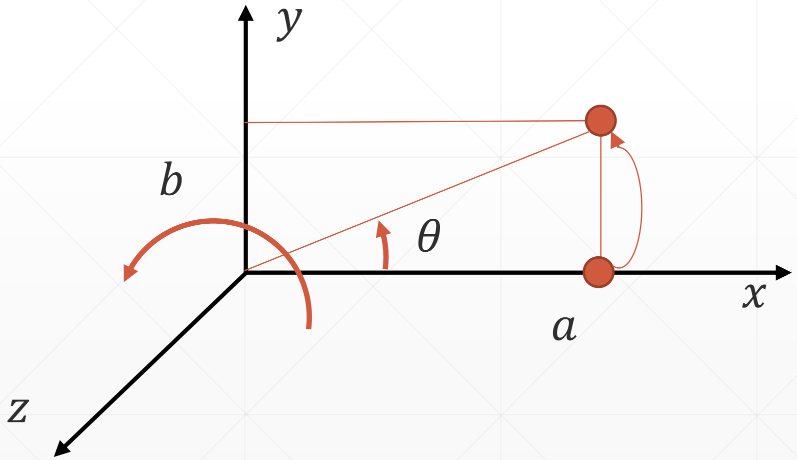
1. Quarternion



$$e^{i\theta} = cos\theta + isin\theta$$

$quaternions: extends\ the\ complex\ numbers$

$in\ 1843\ , by\ mathematiciam\ william\ rowan\ hamilton$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$rotation\ about\ Z - axis$

$on - dimensional\ rotation$

how about three dimensional rotations?

Quarternions as

$$q = p_0 + q_1 i + q_2 j + q_3 k$$
$$= (p_0, \vec{q})$$
$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \qquad ji = -k$$

1. Quarternion

properties

1) $(p_1, \vec{s_1}) + (p_2, \vec{s_2}),\quad \vec{s_1} = s_{11}i + s_{12}j + s_{13}k,\quad \vec{s_2} = s_{21}i + s_{22}j + s_{23}k$

$= (p_1 + p_2\,, (s_{11} + s_{21})i + (s_{12} + s_{22})j + (s_{13} + s_{23})k)$

2) $(p_1, \vec{s_1})(p_2, \vec{s_2})$

$= (p_1 p_2 - s_1 \cdot s_2, \qquad p_1\vec{s_2} + p_2\vec{s_1} + \vec{s_1} \times \vec{s_2})$

3) *conjugate*

let $q = q_0 + q_1 i + q_2 j + q_3 k$

$q^* = q_0 - q_1 i - q_2 j - q_3 k$

norm $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

$qq^* = (q_0 + q_1 i + q_2 j + q_3 k)(q_0 - q_1 i - q_2 j - q_3 k)$

$= (q_0 q_0 - q_1 q_1 i^2 - q_2 q_2 j^2 - q_3 q_3 k^2)$
$+(-q_0 q_1 + q_1 q_0 - q_2 q_3 + q_3 q_2)i$
$+(-q_0 q_2 + q_2 q_0 + q_1 q_3 - q_3 q_1)j$
$+(-q_0 q_3 + q_3 q_0 - q_1 q_2 + q_2 q_1)k$

$= (q_0^2 + q_1^2 + q_2^2 + q_2^3) = |q|^2$

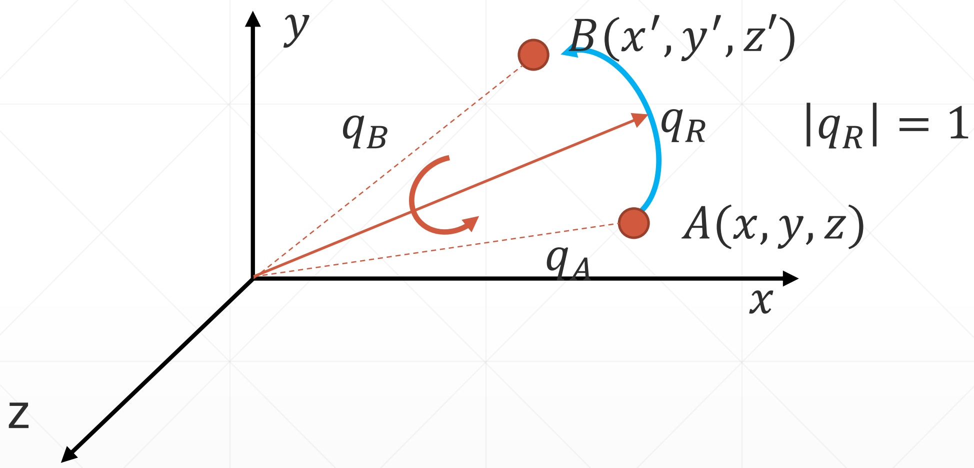
$\therefore q^{-1} = \frac{q^*}{|q|^2}$

2. Quarternion for rotation

$vector\ in\ 3D\ space$: $pure$ Quarternion

$$q = 0 + xi + yi + zk$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad R = ?$$



$$q_B = q_R\, q_A\, q_R^*$$
$$= (q_0 + q_1 i + q_2 j + q_3 k)(xi + yj + zk)(q_0 - q_1 i - q_2 j - q_3 k)$$

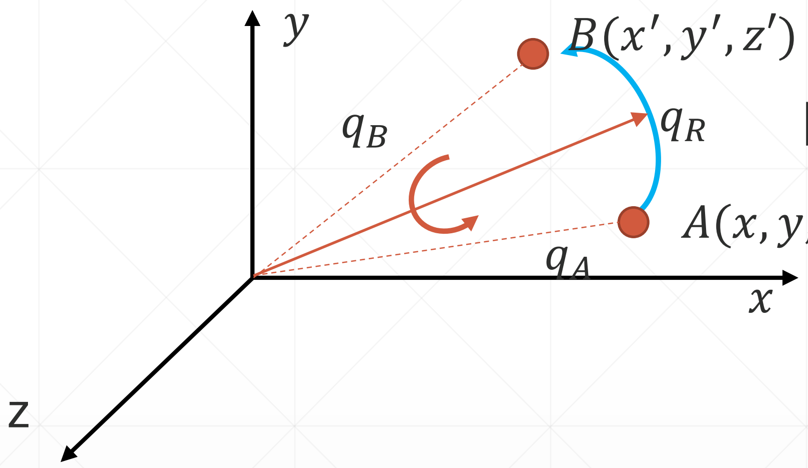
$$= [x(q_0^2 + q_1^2 - q_2^2 - q_3^2) + 2y(q_1 q_2 - q_0 q_3) + 2z(q_0 q_2 + q_1 q_3)]i$$
$$+[2x(q_0 q_3 + q_1 q_2) + y(q_0^2 - q_1^2 + q_2^2 - q_3^2) + 2z(q_2 q_3 - q_0 q_1)]j$$
$$+[2x(q_1 q_3 - q_0 q_2) + 2y(q_0 q_1 + q_2 q_3) + z(q_0^2 - q_1^2 - q_2^2 + q_3^2)]k$$

$$\triangleq R \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2. Quarternion for rotation

$vector\ in\ 3D\ space$: $pure$ Quarternion

$$q = 0 + xi + yi + zk$$



$$|q_R| = 1$$

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$since\ q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

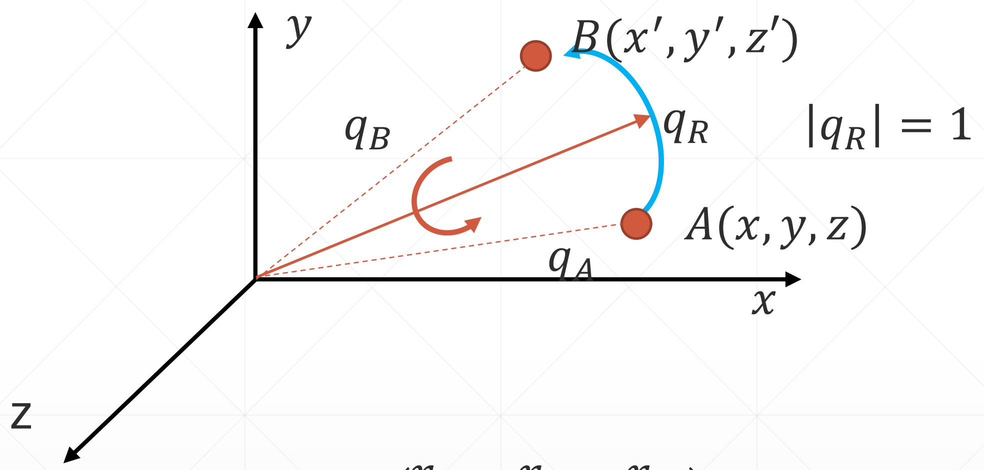
$$R = 2\begin{bmatrix} q_0^2 + q_1^2 - 0.5 & q_1q_2 - q_0q_3 & q_0q_2 + q_1q_3 \\ q_0q_3 + q_1q_2 & q_0^2 + q_2^2 - 0.5 & q_2q_3 - q_0q_1 \\ q_1q_3 - q_0q_2 & q_0q_1 + q_2q_3 & q_0^2 + q_3^2 - 0.5 \end{bmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

2. Quarternion for rotation

$vector\ in\ 3D\ space$: $pure$ Quarternion

$$q = 0 + xi + yi + zk$$



$$R\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$R = 2\begin{bmatrix} q_0^2 + q_1^2 - 0.5 & q_1q_2 - q_0q_3 & q_0q_2 + q_1q_3 \\ q_0q_3 + q_1q_2 & q_0^2 + q_2^2 - 0.5 & q_2q_3 - q_0q_1 \\ q_1q_3 - q_0q_2 & q_0q_1 + q_2q_3 & q_0^2 + q_3^2 - 0.5 \end{bmatrix}$$

a rotational matrix to a quaternion

$$\begin{aligned} Trace\ (R) &= r_{11} + r_{22} + r_{33} \\ &= 2(3q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1.5) \\ &= 2\left(3q_0^2 + (1 - q_0^2) - \frac{3}{2}\right) \\ &= 4q_0^2 - 1 \end{aligned}$$

$$\therefore |q_0| = \sqrt{\frac{Trace(R) + 1}{4}}$$

$$r_{11} = 2(q_0^2 + q_1^2 - 0.5)$$

$$= 2\left(\frac{Trace(R) + 1}{4} + q_1^2 - 0.5\right)$$

$$\therefore |q_1| = \sqrt{\frac{r_{11}}{2} + \frac{1 - Trace(R)}{4}}$$

2. Quarternion for rotation

$similarly$

$$|q_2| = \sqrt{\frac{r_{22}}{2} + \frac{1 - Trace(R)}{4}}$$

$$|q_3| = \sqrt{\frac{r_{33}}{2} + \frac{1 - Trace(R)}{4}}$$

$From\ Euler\ angle\ to\ a\ quarternion$

$$R_\psi = Rot(z, \psi)$$

$$= \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

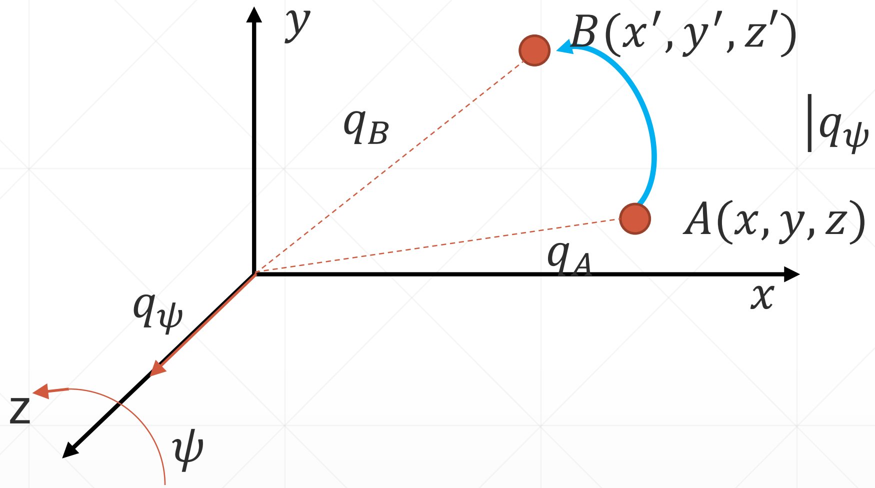
$$|q_0| = \sqrt{\frac{Trace(R) + 1}{4}} = \sqrt{\frac{2cos\psi + 2}{4}} = \cos\frac{\psi}{2}$$

$$\bigstar\ (\cos\frac{\psi}{2}\cos\frac{\psi}{2} - \sin\frac{\psi}{2}\sin\frac{\psi}{2}) = \cos 2 \cdot \frac{\psi}{2}$$

$$\cos\psi = \cos^2\frac{\psi}{2} - \left(1 - \cos^2\frac{\psi}{2}\right) = 2\cos^2\frac{\psi}{2} - 1$$

$$\therefore\ \cos\frac{\psi}{2} = \sqrt{\frac{cos\psi + 1}{2}}$$

2. Quarternion for rotation



$|q_\psi| = 1$

$$|q_1| = |q_2| = \sqrt{\frac{r_{11}}{2} + \frac{1 - Trace(R)}{4}}$$

$$= \sqrt{\frac{\cos\psi}{2} + \frac{1 - (2\cos\psi + 1\,)}{4}} = 0$$

$$|q_3| = \sqrt{\frac{r_{33}}{2} + \frac{1 - Trace(R)}{4}} = \sin\frac{\psi}{2}$$

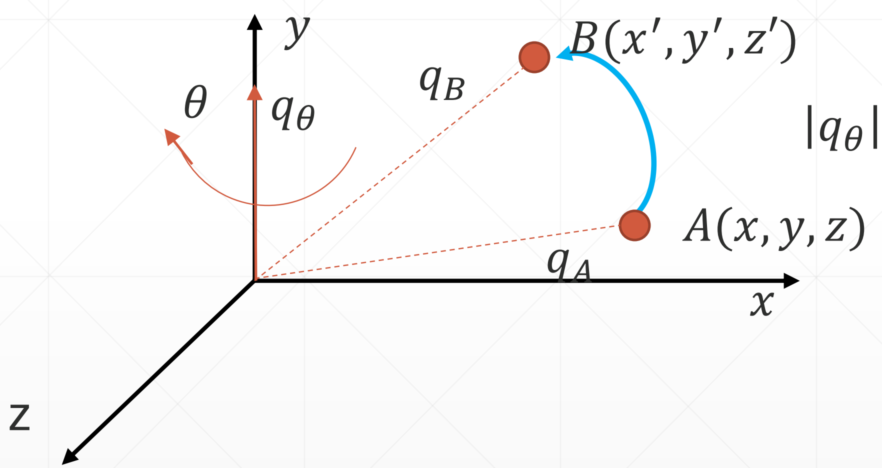
$\therefore$ quarternion $q_\psi = \cos\frac{\psi}{2} + \sin\frac{\psi}{2}\ k$

$\Rightarrow$ $rotaion\ about\ z$ axis $by\ \psi$

2. Quarternion for rotation

Y-axis rotation by θ

$$R_\theta = Rot(y,\theta) = \begin{bmatrix} cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & cos\theta \end{bmatrix}$$



$|q_\theta| = 1$

$$|q_0| = \sqrt{\frac{Trace(R)+1}{4}} = \sqrt{\frac{2cos\theta+2}{4}} = \cos\frac{\theta}{2}$$

$$|q_1| = 0$$

$$|q_2| = \sqrt{\frac{r_{11}}{2} + \frac{1-Trace(R)}{4}}$$

$$= \sin\frac{\theta}{2}$$

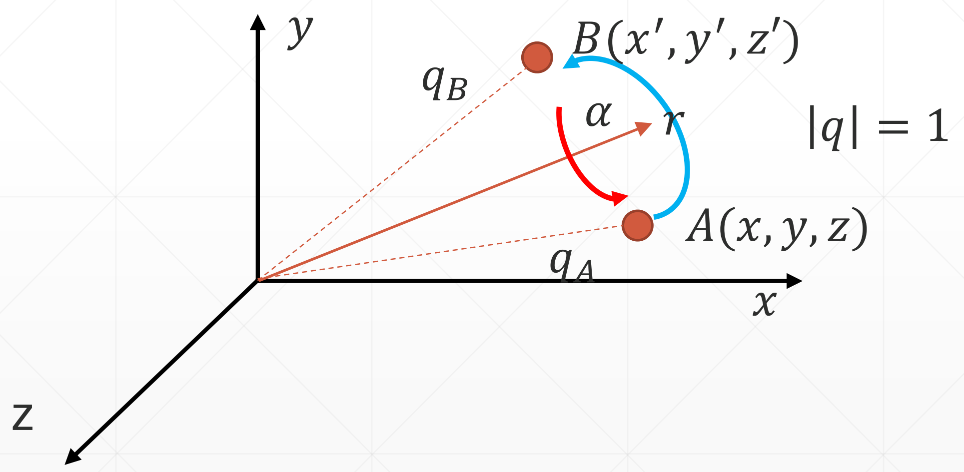
$$|q_3| = 0$$

∴ quarternion $q_\theta = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} j$

$\Rightarrow$ $rotaion\ about\ y$ axis $by\ \theta$

2. Quarternion for rotation

For all 3D rotation $quartenion$
$q = \left(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\frac{r}{|r|}\right)$
$=>$ point $q_B = q q_A q^*$ is determined which is rotated from point q_A
by a rotation about axis r by α angle



3. ZYX Euler angles to quaternions

$$zyx\ Euler\ angles\ to\ quaternions : moving\ frame\ based$$

$$R_{zyx} = Rot(z,\psi)\, Rot(y,\theta) Rot\,(x,\phi)$$
$$= \begin{pmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{pmatrix}$$
$$= (q_0, q_1 i + q_2 j + q_3 k) = q_z(\psi) q_y(\theta) q_x(\phi)$$

$$where$$
$$q_0 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\ \cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\ \sin\frac{\phi}{2}$$
$$q_1 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\ \sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\ \cos\frac{\phi}{2}$$
$$q_2 = \cos\frac{\psi}{2}\sin\frac{\theta}{2}\ \cos\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\ \sin\frac{\phi}{2}$$
$$q_3 = \sin\frac{\psi}{2}\cos\frac{\theta}{2}\ \cos\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\ \sin\frac{\phi}{2}$$

3. ZYX Euler angles to quaternions

$by\ q_z(\psi) q_z(\theta)\ q_z(\phi)$

$when$

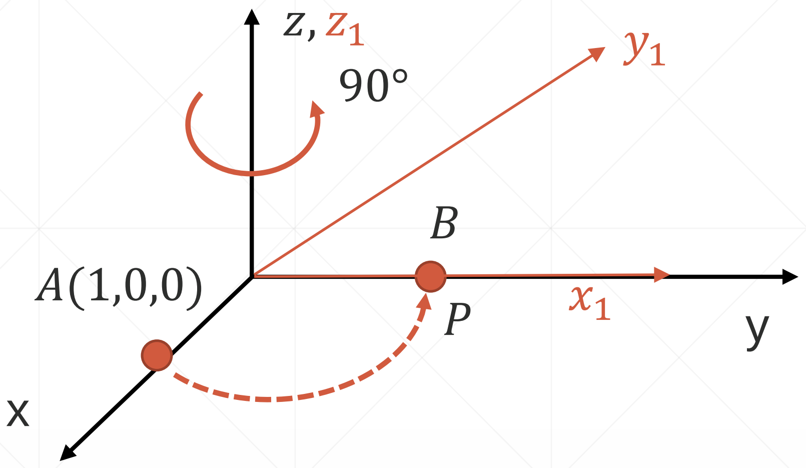
$$q_z(\psi) = \left(\cos\frac{\psi}{2}, 0,0, \sin\frac{\psi}{2}\right) = \left(\cos\frac{\psi}{2}, \sin\frac{\psi}{2}k\right)$$

$$q_y(\theta) = \left(\cos\frac{\theta}{2}, 0, \sin\frac{\theta}{2}, 0\right) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}j\right)$$

$$q_x(\phi) = \left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2}, 0,0\right) = \left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2}i\right)$$

3. ZYX Euler angles to quaternions

Example



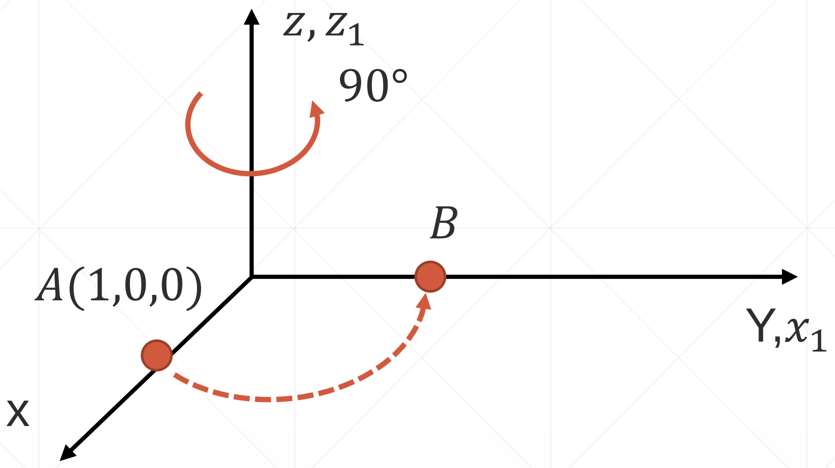
By rotation matrix

$$P_0 = Rot(z, 90°)P_1$$
$$= \begin{pmatrix} c90 & -s90 & 0 \\ s90 & c90 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

By quaternion

$$Rot(z, 90°) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$|q_0| = \sqrt{\frac{Trace(Rot)+1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

3. ZYX Euler angles to quaternions



$$|q_1| = \sqrt{\frac{r_{11}}{2} + \frac{1 - Trace(R)}{4}} = 0$$

$$|q_2| = \sqrt{\frac{r_{22}}{2} + \frac{1 - Trace(R)}{4}} = 0$$

$$|q_3| = \sqrt{\frac{r_{33}}{2} + \frac{1 - Trace(R)}{4}} = \sqrt{\frac{1}{2} + \frac{1-1}{4}} = \frac{\sqrt{2}}{2}$$

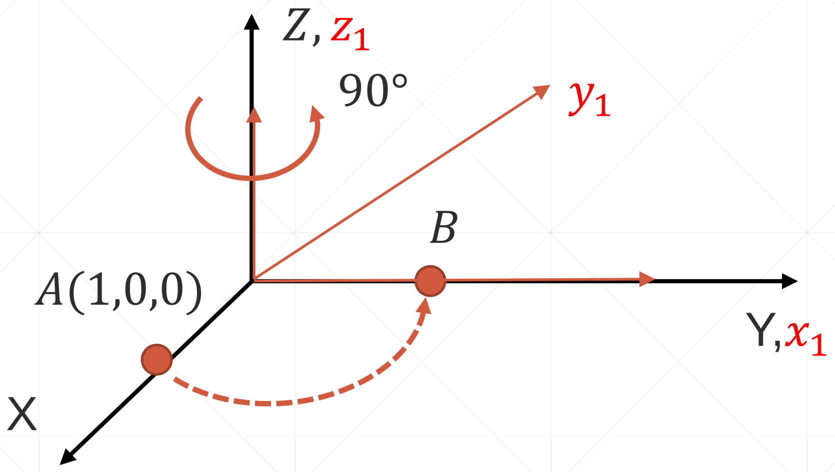
$$\therefore q_R = q_0 + q_1 i + q_2 j + q_3 k$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} k$$

$$or = (\cos\frac{90°}{2}, \sin\frac{90°}{2} k)$$

$$unit\ vector, |q_R| = 1$$

3. ZYX Euler angles to quaternions



$$q_B = q_R q_A q_R^* \quad , q_A = (1,0,0)$$

$$q_R = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}k\right)$$

$$q_R^* = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}k\right)$$

$$q_B = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}k\right)(0, i)\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}k\right) = \left(0, \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j\right)\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}k\right)$$

$$= \left(0 + 0, 0 + \frac{1}{2}i + \frac{1}{2}j + \left(\frac{1}{2}j - \frac{1}{2}i\right)\right) = (0, j) = (0,0,1,0)$$

$\therefore q_B = (0,1,0)$ after rotation