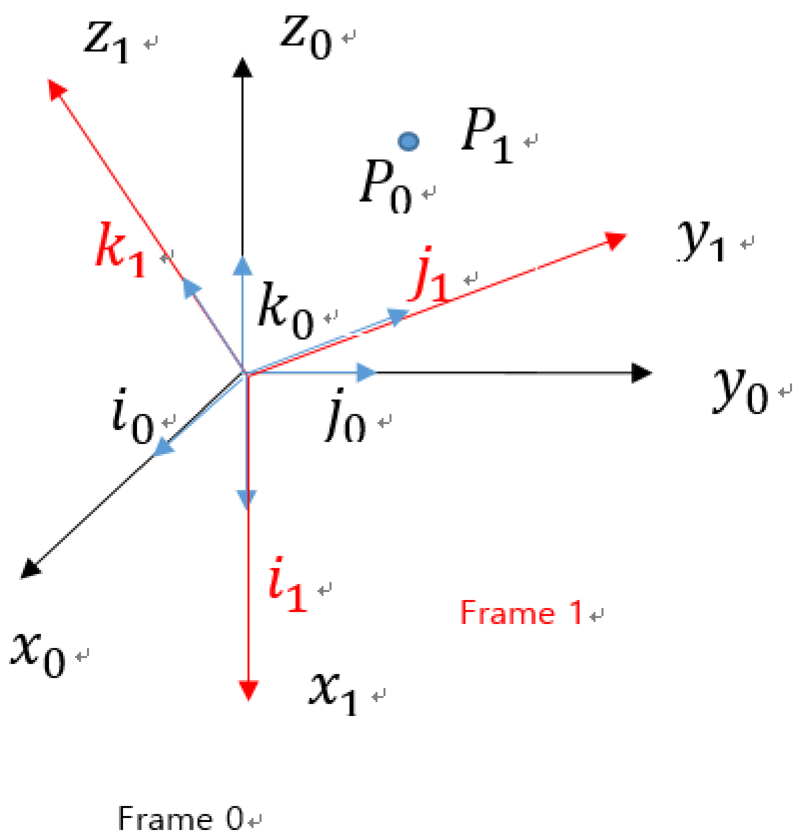
Chap. 2 Robot Kinematics

2.1 Coordinates transformation : Rotation



P_1 : Position vector point 1 with respective to Frame 1
P_0 : Position vector point 0 with respective to Frame 0

i_0, j_0, k_0 : Frame 0 unit vector
i_1, j_1, k_1: Frame 1 unit vector

R_0^1: Rotational Matrix between Frame 0 and Frame 1

$$R_0^1 = \begin{bmatrix} i_1 \cdot i_0 & j_1 \cdot i_0 & k_1 \cdot i_0 \\ i_1 \cdot j_0 & j_1 \cdot j_0 & k_1 \cdot j_0 \\ i_1 \cdot k_0 & j_1 \cdot k_0 & k_1 \cdot k_0 \end{bmatrix}$$

Position of point P with respective to frame 0 (P_0)
for the Point P with respective to frame 1 (P_1)

$$P_0 = R_0^1 \ P_1$$

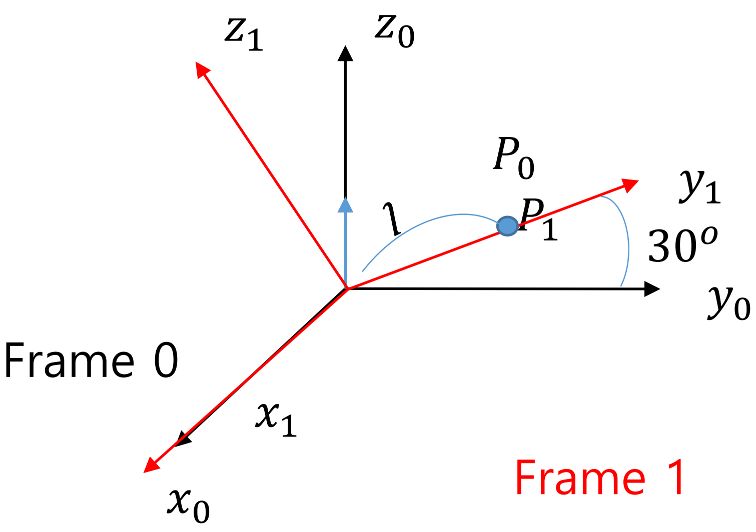
Reversely, it holds as

$$P_1 = R_1^0 \; P_0$$

where

$$R_1^0 = (R_0^1)^T = (R_0^1)^{-1}$$

e.g.



E.g. Point (P_0) of length l is rotated by 30 degrees about x_0. Determine the position of the point with respect to frame 0.
Here, point P_1 is on moving frame 1.
Using rotational matrix,

$$P_0 = R_0^1 \ P_1$$

where

$$R_0^1 = \begin{bmatrix} i_1 \cdot i_0 & j_1 \cdot i_0 & k_1 \cdot i_0 \\ i_1 \cdot j_0 & j_1 \cdot j_0 & k_1 \cdot j_0 \\ i_1 \cdot k_0 & j_1 \cdot k_0 & k_1 \cdot k_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(30^o) & -\sin(30^o) \\ 0 & \sin(30^o) & \cos(30^o) \end{bmatrix}$$

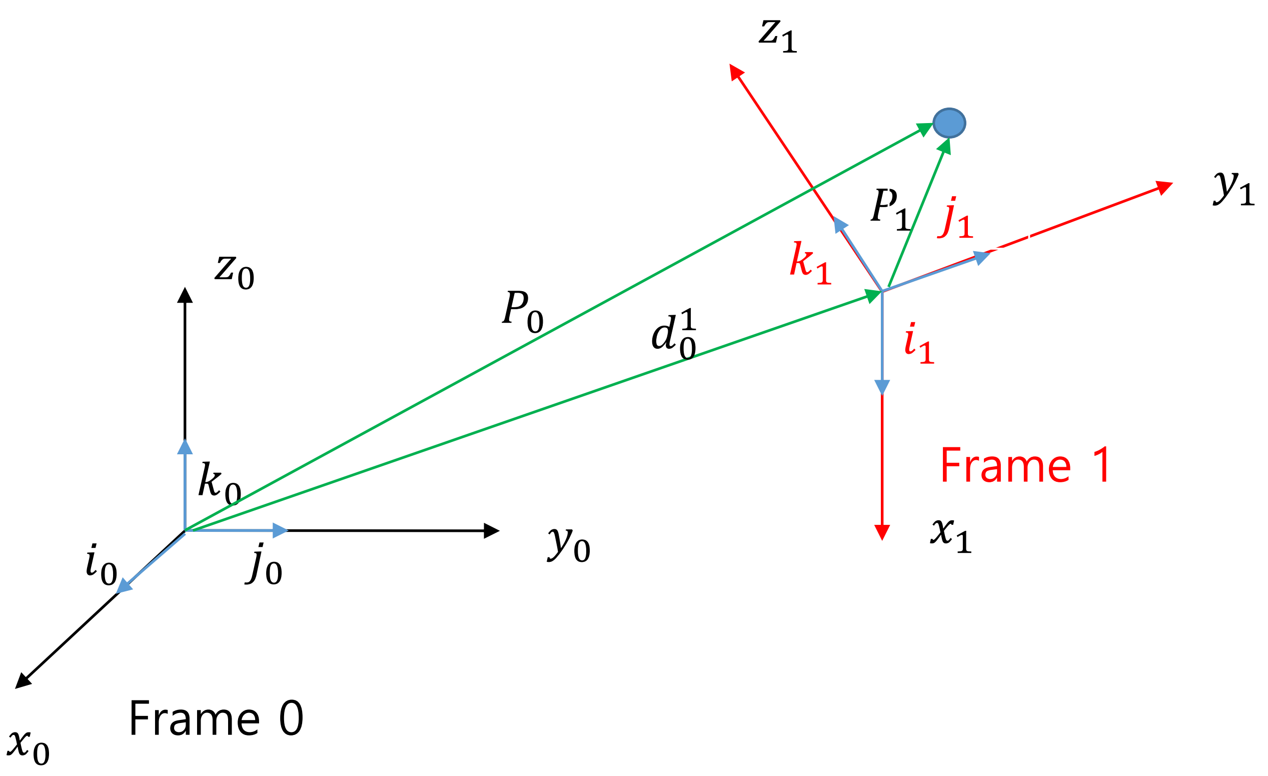
and

$$P_1 = \begin{pmatrix} 0 \\ l \\ 0 \end{pmatrix}$$

Thus,

$$P_0 = \begin{pmatrix} 0 \\ l\cos(30^o) \\ l\sin(30^o) \end{pmatrix}$$

2.2 Coordinates transformation (Rotation+Translation)



P_1 : Position vector of the point ● w.r.t Frame 1
P_0 : Position vector of the point ● w.r.t Frame 0

d_0^1 : Position vector of Frame 1 origin w.r.t Frame 0

If we express the point w.r.t Frame 0

$$P_0 = R_0^1 \; P_1 + d_0^1$$

where

$$d_0^1 = \begin{pmatrix} d_{x0} \\ d_{y0} \\ d_{z0} \end{pmatrix}$$

$$P_0 = R_0^1 \; P_1 + d_0^1 \; = R_0^1 \; P_1 + \begin{pmatrix} d_{x0} \\ d_{y0} \\ d_{z0} \end{pmatrix}$$

It can be modified as

$$P_0 = \begin{bmatrix} P_{x0} \\ P_{y0} \\ P_{z0} \\ 1 \end{bmatrix} = \begin{bmatrix} x_{1_{x0}} & y_{1_{x0}} & z_{1_{x0}} & P_{x0} \\ x_{1_{y0}} & y_{1_{y0}} & z_{1_{y0}} & P_{x0} \\ x_{1_{z0}} & y_{1_{z0}} & z_{1_{z0}} & P_{x0} \\ 0 & 0 & 0 & 1 \end{bmatrix} P_1$$

where

$$P_1 = \begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ 1 \end{bmatrix}$$

☞☞ *At* P_0 **and** P_1 put additional element of 1 at last row, making 4x1 vector

$x_{1_{x0}}$ **:** x_0-component of of x_1 axis, others are similarly defined

Using transformation matrix between Frame 0 and Frame (noa)

$$P_0 = \begin{bmatrix} n_x & o_x & a_x & P_{x0} \\ n_y & o_y & a_y & P_{y0} \\ nz & oz & az & P_{z0} \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{noa} = T_0^{noa} \; P_{noa}$$

T_0^{noa} : Transformation Matrix of frame (noa) w.r.t frame 0

Structure of transformation matrix

Rotation

$$\begin{bmatrix} n_x & o_x & a_x & P_{x0} \\ n_y & o_y & a_y & P_{y0} \\ n_z & o_z & a_z & P_{z0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

☞ point on space can be obtained if we know the transformation matrix between frames formed by rotation+ translation.

2.3 Combined transformation

- Combined Transformation relative to fixed frame or moving frame
- Combined Transformation relative to fixed frame

$$T_0^{noa} = T_2^{noa}\, T_1^2\, T_0^1$$

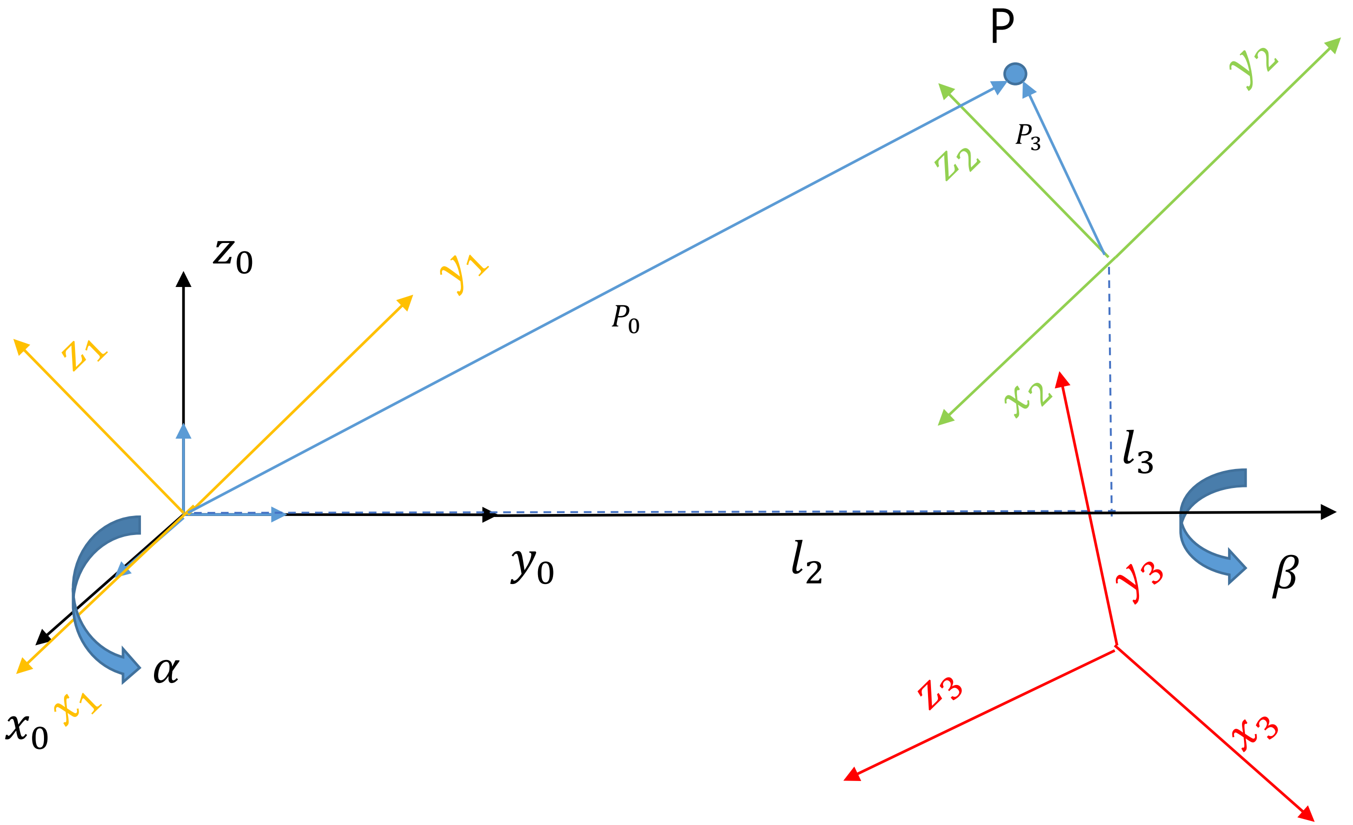
- Pre-multiplication order

- Combined Transformation relative to moving frame

$$T_0^{noa} = T_0^1\, T_1^2\, T_2^{noa}$$

-Post-multiplication order

Combined Transformation relative to fixed frame
e.g.

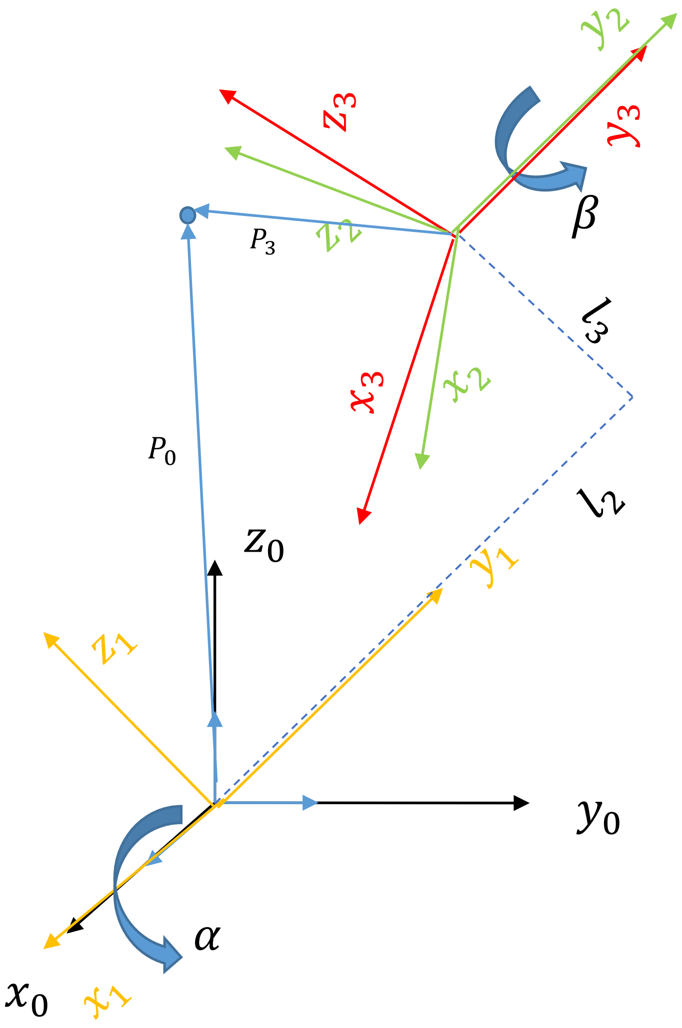


Frame 0

$$T_0^3 = T_2^3 T_1^2 T_0^1 = Rot(y_0, \beta) Trans(0, l_2, l_3) Rot(x_0, \alpha)$$

$$P_0 = T_0^3 \ P_3$$

Combined Transformation relative to moving frame
e.g.



$$T_0^3 = T_0^1 T_1^2 T_2^3 = Rot(x_0, \alpha) Trans(0, l_2, l_3) Rot(y_2, \beta)$$

$$P_0 = T_0^3 \; P_3$$

2.4 Skew symmetric matrix

- Skew symmetric matrix
- Def. Skew symmetric if and only if $S^T + S = 0$

  i.e. $s_{ij} + s_{ji} = 0, \;\; i \neq j = 1.2.3 \ldots\ldots$

  $s_{ii} = 0$

$$S = \begin{bmatrix} 0 & -s_1 & s_2 \\ s_1 & 0 & -s_3 \\ -s_2 & s_3 & 0 \end{bmatrix}$$

Define

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}, \; a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix},$$

In case $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Skew symmetric matrix

- **Properties on skew symmetric matrix, vector, and rotational matrix**

i) $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$, α, β*: scala*

ii) $S(a)p = a \times p$,

iii) $R(a \times b) = Ra \times Rb$, R is a rotational matrix

Since

$R(\theta)R(\theta)^T = I$

Differentiating with θ

$\frac{dR}{d\theta}R(\theta)^T + R(\theta)\frac{dR(\theta)^T}{d\theta} = 0$ (1)

If

$S = \frac{dR}{d\theta}R(\theta)^T$

then (1) becomes

$S^T + S = 0$;

Thus, $S = \frac{dR}{d\theta}R(\theta)^T$ is skew symmetric

Also, it holds

$\frac{dR}{d\theta} = SR(\theta)$

Skew symmetric matrix

- E.g. x-direction rotation (i-direction)

$$R_{x,\theta} = R(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$S = \frac{dR}{d\theta}R_{x,\theta}(\theta)^T$$

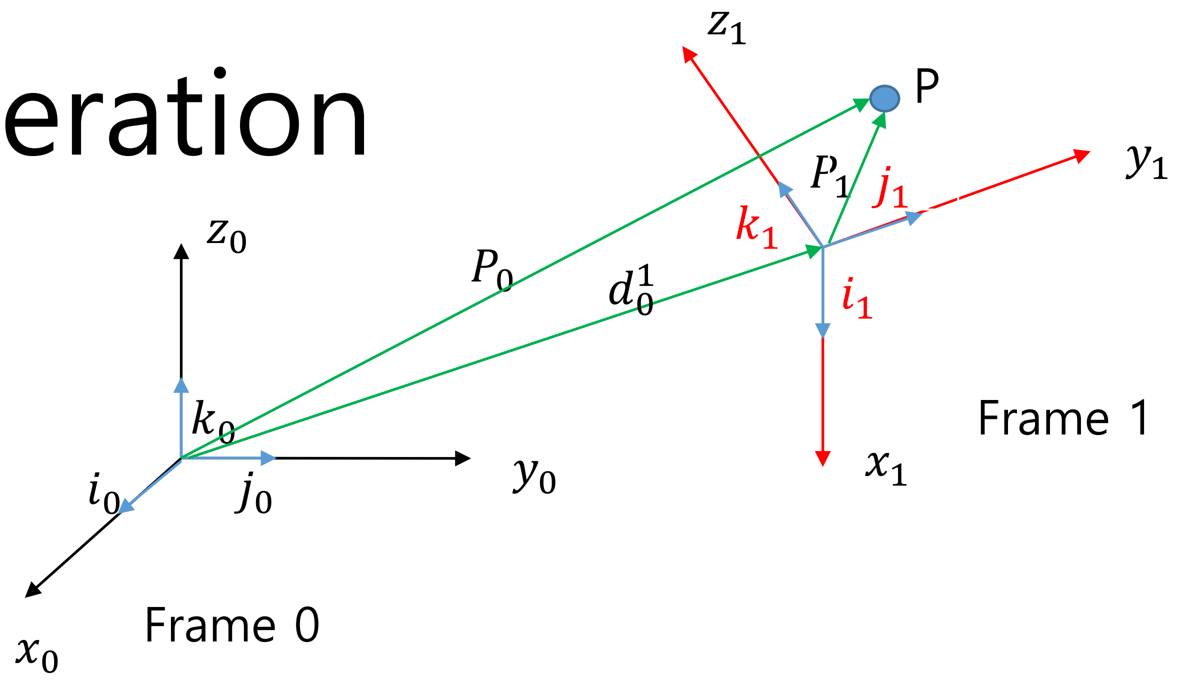
$$=\begin{bmatrix}0 & 0 & 0\\ 0 & -sin(\theta) & -\cos(\theta)\\ 0 & \cos(\theta) & -\sin(\theta)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} = S(i),\quad skew\ symmetric$$

And $\frac{dR_{x,\theta}}{d\theta} = S(i)R_{x,\theta}$

Similarly,

$$\frac{dR_{y,\theta}}{d\theta} = S(j)R_{y,\theta},\qquad \frac{dR_{z,\theta}}{d\theta} = S(k)R_{z,\theta}$$

2.5 Velocity and Acceleration



- **velocity**

$$\dot{R} = \frac{dR}{dt} = S(t)R(t), \qquad where \;\; S(t) = \frac{dR}{dt}R(t)^T$$

e.g. $R(t) = R_{x,\theta(t)}$

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \dot{\theta}S(i)R(\theta) = S(\dot{\theta}\, i)R(\theta)$$
$$= S(\omega_x)R(\theta)$$

ω_x: angular velocity vector about x-axis (i axis)

Velocity of point P w.r.t frame 1

$$P_0 = R_0^1 \; P_1 + d_0^1 \;\; => \;\; \dot{P}_0 = \dot{R}_0^1 P_1 + R_0^1 \dot{P}_1 + \dot{d}_0^1 = S(\omega_0^1)R_0^1 P_1 + R_0^1 \dot{P}_1 + \dot{d}_0^1$$
$$= \omega_0^1 \times R_0^1 P_1 + R_0^1 \dot{P}_1 + \dot{d}_0^1$$
$$= \omega_0^1 \times r + R_0^1 \dot{P}_1 + \dot{d}_0^1 \; ,$$

$r = R_0^1 \; P_1$: vector from o_1 to point p w.r.t frame 0

$\mathrm{v} = \dot{d}_0^1$: velocity of origin o_1 w.r.t frame 0

2.5 Velocity and Acceleration

- **acceleration**

Velocity of point P w.r.t frame 0

$$P_0 = R_0^1\ P_1 + d_0^1 \ => \ \dot{P}_0 = \omega_0^1 \times r + R_0^1 \dot{P}_1 + \dot{d}_0^1\ ,$$

Differentiating

$$\ddot{P}_0 = \dot{\omega}_0^1 \times r + \omega_0^1 \times \dot{r} + \dot{R}_0^1 \dot{P}_1 + R_0^1 \ddot{P}_1 + \ddot{d}_0^1$$

Here, $\dot{r} = \frac{d}{dt}(R_0^1\ P_1) = \dot{R}_0^1 P_1 + R_0^1 \dot{P}_1 = \omega_0^1 \times R_0^1 P_1 + R_0^1 \dot{P}_1$

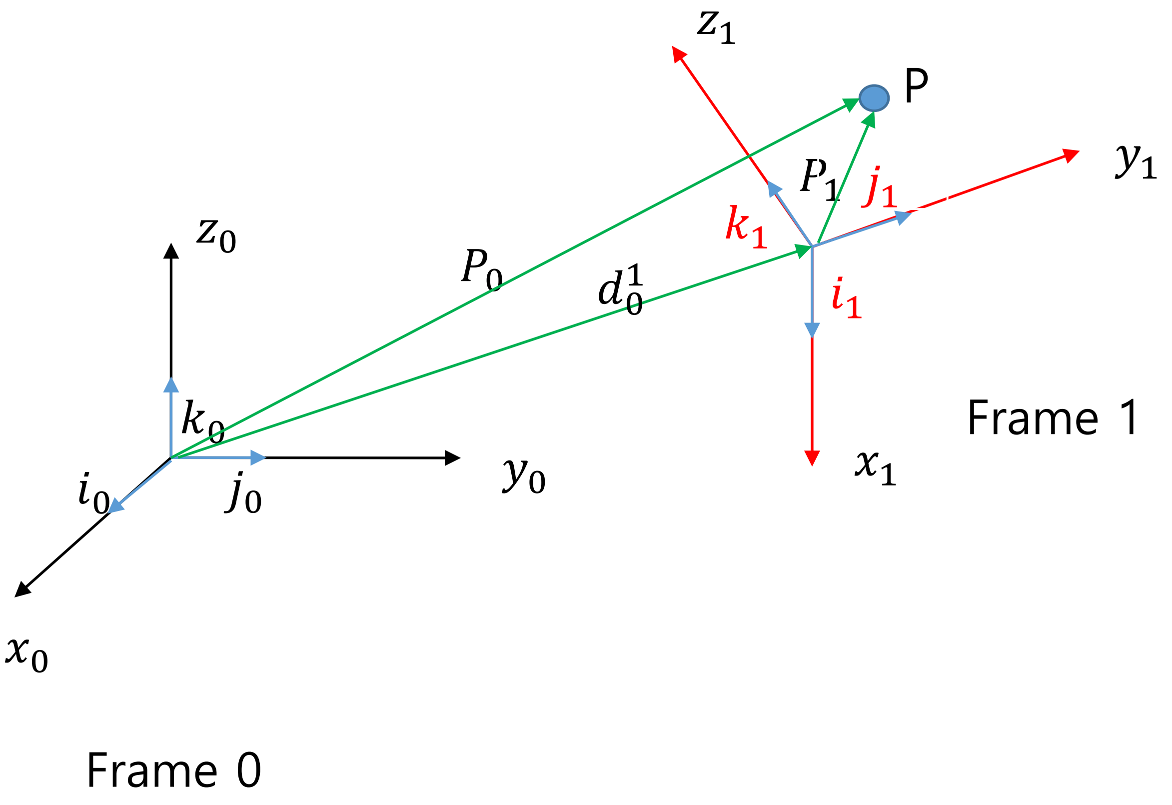
$$= \omega_0^1 \times r + R_0^1 \dot{P}_1$$

Thus,

$$\ddot{P}_0 = \dot{\omega}_0^1 \times r + \omega_0^1 \times \left(\omega_0^1 \times r + R_0^1 \dot{P}_1\right) + \dot{R}_0^1 \dot{P}_1 + R_0^1 \ddot{P}_1 + \ddot{d}_0^1$$

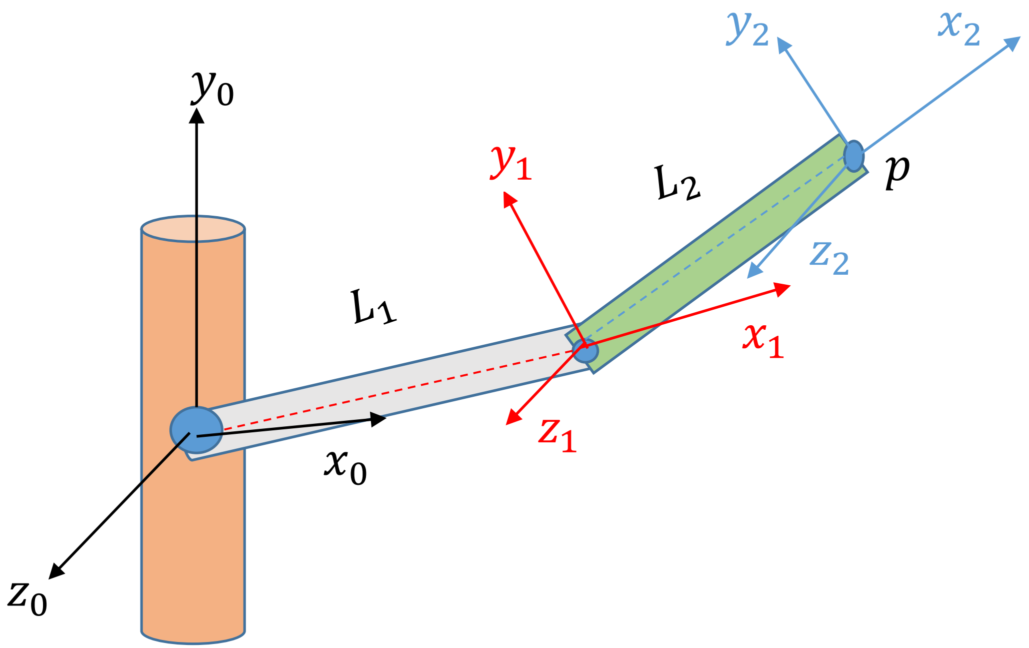
$$= \dot{\omega}_0^1 \times r + \omega_0^1 \times \left(\omega_0^1 \times r + R_0^1 \dot{P}_1\right) + \omega_0^1 \times R_0^1 \dot{P}_1 + R_0^1 \ddot{P}_1 + \ddot{d}_0^1$$

$$= \dot{\omega}_0^1 \times r + \omega_0^1 \times (\omega_0^1 \times r) + 2\omega_0^1 \times R_0^1 \dot{P}_1 + R_0^1 \ddot{P}_1 + \ddot{d}_0^1$$



2.5 Velocity and Acceleration

- E.g



$$R_0^1 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_1^2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Suppose p is fixed on frame 2

$$P_1 = R_1^2 \ P_2 + d_1^2$$
$$P_0 = R_0^1 \ P_1 + d_0^1$$

$$d_0^1 = \begin{bmatrix} L_1 cos(\theta_1) \\ L_1 sin(\theta_1) \\ 0 \end{bmatrix}, d_1^2 = \begin{bmatrix} L_2 cos(\theta_2) \\ L_2 sin(\theta_2) \\ 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$P_1 = R_1^2 \; P_2 + d_1^2$$
$$= \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L_2 cos(\theta_2) \\ L_2 sin(\theta_2) \\ 0 \end{bmatrix} = \begin{bmatrix} L_2 cos(\theta_2) \\ L_2 sin(\theta_2) \\ 0 \end{bmatrix}$$

$$P_0 = R_0^1 \; P_1 + d_0^1 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 cos(\theta_2) \\ L_2 sin(\theta_2) \\ 0 \end{bmatrix} + \begin{bmatrix} L_1 cos(\theta_1) \\ L_1 sin(\theta_1) \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 cos(\theta_1) + L_2 cos(\theta_1 + \theta_2) \\ L_1 sin(\theta_1) + L_2 sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Velocity

$\dot{P}_1 = \dot{R}_1^2 P_2 + R_1^2 \dot{P}_2 + \dot{d}_1^2$ (since $\dot{P}_2 = 0$)

$= \omega_1^2 \times R_1^2 P_2 + \dot{d}_1^2$

$$= \begin{bmatrix} -L_2\dot{\theta}_2\ c\theta_2 \\ L_2\dot{\theta}_2\ s\theta_2 \\ 0 \end{bmatrix},$$

$$\dot{P}_0 = \dot{R}_0^1 P_1 + R_0^1 \dot{P}_1 + \dot{d}_0^1$$

$$= \omega_0^1 \times R_0^1 P_1 + R_0^1 \dot{P}_1 + \dot{d}_0^1$$

$$= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} c\theta_1 & -\mathrm{s}\,\theta_1 & 0 \\ \mathrm{s}\,\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2\ c\theta_2 \\ L_2 s\theta_2 \\ 0 \end{bmatrix} + \begin{bmatrix} c\theta_1 & -\mathrm{s}\,\theta_1 & 0 \\ \mathrm{s}\,\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L_2\dot{\theta}_2\ c\theta_2 \\ L_2\dot{\theta}_2\ s\theta_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1\dot{\theta}_1\ s\theta_1 \\ L_1\dot{\theta}_1\ c\theta_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -L_2\dot{\theta}_1\ s(\theta_1+\theta_2) \\ L_2\dot{\theta}_1\ c(\theta_1+\theta_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -L_2\dot{\theta}_2 s(\theta_1+\theta_2) \\ L_2\dot{\theta}_2 c(\theta_1+\theta_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1\dot{\theta}_1\ s\theta_1 \\ L_1\dot{\theta}_1\ c\theta_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -L_2(\dot{\theta}_1+\dot{\theta}_2)s(\theta_1+\theta_2)\ -L_1\ \dot{\theta}_1\ s\theta_1 \\ L_2(\dot{\theta}_1+\dot{\theta}_2)c(\theta_1+\theta_2) + L_1\dot{\theta}_1\ c\theta_1 \\ 0 \end{bmatrix}$$

Combined angular velocity

$$P_1 = R_1^2\ P_2 + d_1^2$$
$$P_0 = R_0^1\ P_1 + d_0^1$$

Thus, $P_0 = R_0^1(R_1^2\ P_2 + d_1^2) + d_0^1 = R_0^1 R_1^2 P_2 + R_0^1 d_1^2 + d_0^1 = R_0^2\ P_2 + d_0^2$ ---------(1)

Taking derivative for R_0^2

$$\dot{R}_0^2 = \dot{R}_0^1\ R_1^2 + R_0^1\ \dot{R}_1^2$$

Also,

$$\dot{R}_0^2 = S(\omega_0^2) R_0^2 \qquad (2)$$

And

$$\dot{R}_0^1\ R_1^2 = S(\omega_0^1) R_0^1 R_1^2 = S(\omega_0^1) R_0^2$$
$$R_0^1 \dot{R}_1^2\ = R_0^1 S(\omega_1^2) R_1^2 = R_0^1 S(\omega_1^2) {R_0^1}^T R_0^1 R_1^2$$

Since

$$R_0^1 S(\omega_1^2) {R_0^1}^T = S(R_0^1 \omega_1^2)$$

$$R_0^1 S(\omega_1^2) {R_0^1}^T b$$
$$= R_0^1\ (\omega_1^2 \times {R_0^1}^T b\ )$$
$$= R_0^1\ \omega_1^2 \times R_0^1 {R_0^1}^T b$$
$$= R_0^1\ \omega_1^2 \times b$$
$$= S(R_0^1 \omega_1^2) b$$

$$\therefore\ R_0^1\ S(\omega_1^2) {R_0^1}^T = S(R_0^1 \omega_1^2)$$

Thus,

$$R_0^1 \dot{R}_1^2 = S(R_0^1 \omega_1^2)\ R_0^1 R_1^2 = S(R_0^1 \omega_1^2) R_0^2$$

Combining those

$$S(\omega_0^2) R_0^2 = S(\omega_0^1) R_0^2 + S(R_0^1 \omega_1^2) R_0^2$$

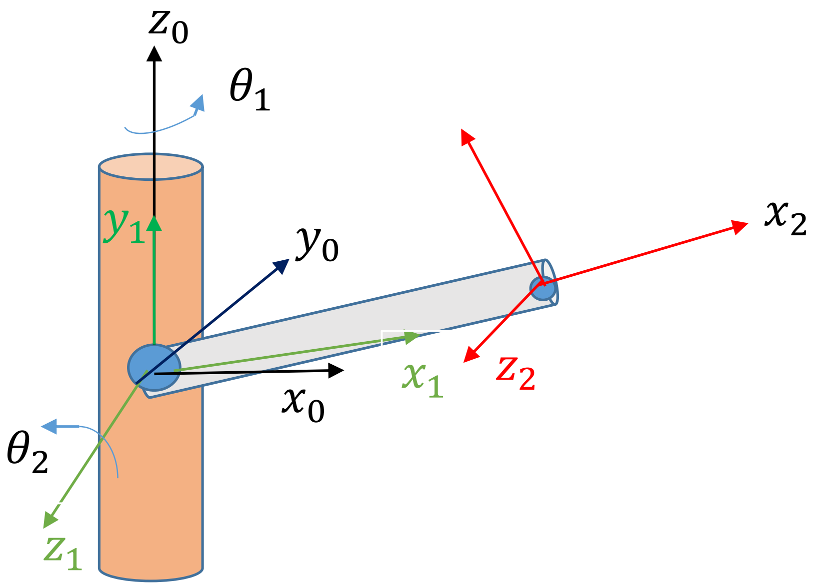
$$= S(\omega_0^1 + R_0^1 \omega_1^2)\ R_0^2 \qquad \text{------(3)}$$

From (2) and (3)

$$\omega_0^2 = \omega_0^1 + R_0^1 \omega_1^2$$

In general

$$\omega_0^n = \omega_0^1 + R_0^1 \omega_1^2 + R_0^2 \omega_2^3 + \cdots R_0^{n-1} \omega_{n-1}^n$$

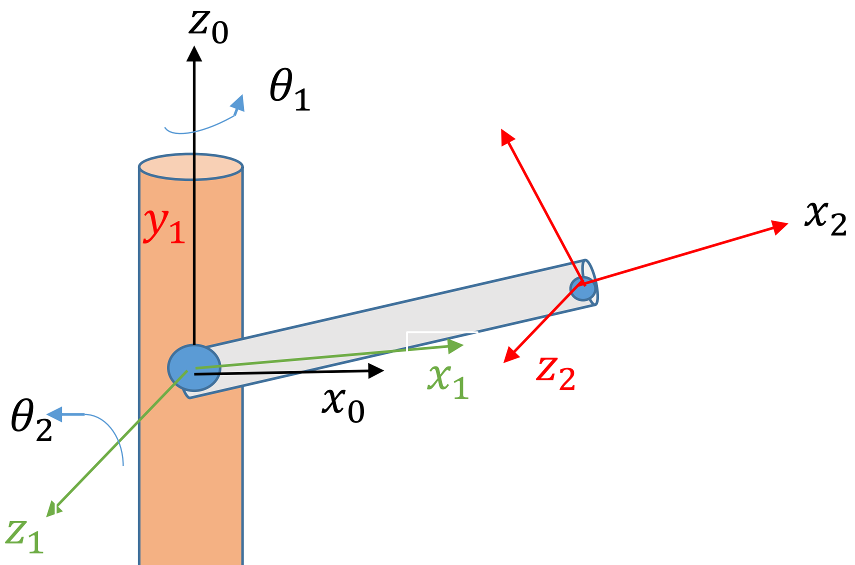


$$R_0^1 = \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) \\ sin(\theta_1) & 0 & -cos(\theta_1) \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_1^2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega_0^2 = \omega_0^1 + R_0^1 \omega_1^2$$

$$\omega_0^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad \omega_1^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$



$$R_0^1 = \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) \\ sin(\theta_1) & 0 & -cos(\theta_1) \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_1^2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega_0^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad \omega_1^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\omega_0^2 = \omega_0^1 + R_0^1\omega_1^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) \\ sin(\theta_1) & 0 & -cos(\theta_1) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_2 sin(\theta_1) \\ -\dot{\theta}_2 cos(\theta_1) \\ \dot{\theta}_1 \end{bmatrix}$$