CS23 Assignment Six

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1 We often define graph theory concepts using set theory. For example, given a graph $G = (V, E)$ and a vertex $v \in V$, we define

$$N(v) = \{u \in V : \{v, u\} \in E\}$$

We define $N[v] = N(v) \cup \{x\}$. The goal of this problem is to figure out what all this means.

a. **Let G be the graph with $V = \{a, b, c, d, e, f\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}.\{c, d\}, \{c, f\}, \{d, f\}, \{e, f\}\}$. Find $N(a)$, $N[a]$, $N(c)$, and $N[c]$.**

$N(a) = \{b, e\}$, $N[a] = \{a, b, e\}$, $N(c) = \{b, d, f\}$, and $N[c] = \{b, c, d, f\}$.

b. **What is the largest and smallest possible values for $|N(v)|$ and $|N[v]|$ for the graph in part (a)? Explain?**

a has two degrees, b and c each have three, d has two, e has three, and f has only one. Therefore, $|N(v)|$ can be at most 3 and $|N[v]|$ can be at most 4. $|N(v)|$ can be at least 1 and $|N[v]|$ can be at least 2.

c. **Give an example of a graph $G = (V, E)$ (probably different than the one above) for which $N[v] = V$ for some vertex $v \in V$. Is there a graph for which $N[v] = V$ for all $v \in V$? Explain.**

Suppose $V = \{a, b, c\}$ and $E = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. In this case, $N[a] = N[b] = N[c] = \{a, b, c\}$. In this graph, $N[v] = V$ for all vertices $v \in V$.

d. **Give an example of a graph $G = (V, E)$ for which $N(v) = \emptyset$ for some $v \in V$. Is there an example of such a graph for which $N[u] = V$ for some other $u \in V$ as well? Explain.**

Consider the case where $V = \{a, b, c\}$ and $E = \{b, c\}$. In this case, $N(v) = \emptyset$, as a has a degree of zero.

e. **Describe in words what $N(v)$ and $N[v]$ mean in general.**

$N(v)$ returns a set of all the vertices that vertex v is directly connected to. $N[v]$ returns $N(v) \cup \{v\}$, meaning the set of v and all the vertices it connects to.

2 Which of the following graphs are trees?

(a) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$

(b) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$

(c) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$

(d) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{d, e\}\}$

Trees satisfy two conditions, they do not have cycles and are connected. Graph (a) has a cycle (a, b, c, d, e, a), therefore it is not a tree. Graph (b) is a tree. Graph (c) a tree. Graph (d) is not a tree, as the graph is not fully connected.

3 For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree.

(a) $(4, 1, 1, 1, 1)$

(b) $(3, 3, 2, 1, 1)$

(c) $(2, 2, 2, 1, 1)$

(d) $(4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$

For (a), the node degrees must always be a tree. This is evident in problem 2c. For (b), (c), and (d), there's no way to form a tree graph.

4 Suppose you have a graph with v vertices and e edges that satisfies $v = e + 1$. Must the graph be a tree? Prove your answer?

The graph could be a tree, but it does not necessarily have to be. Take the graph $V = \{a, b, c, d\}$ with edges $E = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. The equation is still satisfied, but the graph is not a tree as vertex d has a degree of 0.

5 Prove that any graph with v vertices and e edges that satisfies $v > e + 1$ will not be connected.

Imagine the basic geometric shapes (line — $n = 2$, triangle — $n = 3$, square — $n = 4$, pentagon — $n = 5$, etc). Exlcuding a line (and a point for that matter), in order to completely represent these shapes using a graph, you will need n vertices and n edges. However, you can still have a connected graph if you take away one edge. This graph will have no cycles, so, for n vertices and $n - 1$ edges, you will have a tree graph. Take away one more edge and you no longer have a connected graph. Therefore, $v > e + 1$ infers that the graph will not be connected.

6 Let T be a rooted tree that contains vertices u, v, and w (among possibly others). Prove that if w is a descendant of both u and v, then u is a descendant of v or vice versa.

Since trees do not contain cycles, for a vertex to be a descendant of two other vertices, there must be a traceable path between all three. Suppose the path starts (or the root is) at u and ends and w. Then v must be in the middle, making both v and w descendants of u. The opposite can be said for the case where the graph starts at v.

7 Prove that every connected graph which is not itself a tree must have at last three different spanning trees.

Let's return to the basic shapes from problem 5 and analyze them. We start off with a line. In a complete line, there is n vertices and $n - 1$ edges with no cycles. Therefore, a line is a tree graph, and thus excluded from this problem. With a complete triangle, you have n vertices and n edges. If you take away one edge, you create a tree graph. There are three different edges you could remove, meaning there are three different possible tree graphs. Thus, there are 3 different spanning trees. For a square, the same logic applies, and there are 4 different spanning trees. For each subsequent shape, there will be n spanning trees. Thus, if this logic begins at a triangle ($n = 3$), every connected graph which is not itself a tree must have at least 3 different spanning trees.