Stat 495 Lecture Notes

Ch. 2 ~~Ch~~ Highlights from book notes

Neyman-Pearson Review

$\mathcal{H}_0: \theta = \theta_0$ $\quad \mathcal{H}_1: \theta = \theta_1$ $\quad X_1, \dots, X_n$ RS $f(x|\theta)$

$f_0(\underset{\sim}{x})$ $\quad f_1(\underset{\sim}{x})$

Let δ denote a test.

$\alpha(\delta) = P(\text{Reject } \mathcal{H}_0 \mid \theta = \theta_0)$

$\beta(\delta) = P(\text{Do not reject } \mathcal{H}_0 \mid \theta = \theta_1)$

Power $= 1-\beta$

So $\alpha \uparrow, \beta \downarrow$, Power $\uparrow$

Suppose we fix α_0 and want min $\beta(\delta)$

Let δ^* be a test with form: $k > 0$

Reject $\mathcal{H}_0$ if $\dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} > k$, with $\alpha(\delta^*) = \alpha_0$.

NP Lemma.

If δ is another test such that $\alpha(\delta) \le \alpha(\delta^*)$

then $\beta(\delta) \ge \beta(\delta^*)$. (Can remove $=$)

δ^* is most powerful @ α_0 and uses likelihood ratio!

Ex. Bernoulli (p) $\quad \alpha(\delta) = 0.05$ $\quad$ RS. $X_1, \dots, X_n$ $\quad \mathcal{H}_0: p = 0.2$ $\quad \mathcal{H}_A: p = 0.4$

$f_0(\underset{\sim}{x}) = (0.2)^{\sum x_i}(0.8)^{n-\sum x_i}$ and

$f_1(\underset{\sim}{x}) = (0.4)^{\sum x_i}(0.6)^{n-\sum x_i}$

$$\frac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} = \frac{(4/10)^{\sum x_i}(6/10)^{n-\sum x_i}}{(2/10)^{\sum x_i}(8/10)^{n-\sum x_i}} = \left(\frac{6}{8}\right)^n \left(\frac{4 \cdot 8}{2 \cdot 6}\right)^{\sum x_i} =$$

$$\left(\frac{3}{4}\right)^n \left(\frac{8}{3}\right)^{\sum x_i}$$ is our likelihood ratio

So, we reject $\mathcal{H}_0$ if $\frac{f_1(x)}{f_0(x)} > k$. $\Rightarrow$

$$k < (3/4)^n (8/3)^{\Sigma X_i} \Rightarrow$$

$$(4/3)^n k < (8/3)^{\Sigma X_i} \Rightarrow \text{ take logs}$$

$$n \log 4/3 + \log k < \Sigma X_i \log 8/3 \Rightarrow$$

Reject if $\Sigma X_i > \dfrac{n \log 4/3 + \log k}{\log 8/3} = k'$

Now we apply the lemma. We want $\alpha = 0.05$ or some value and β to be a min. Under $\mathcal{H}_0$, $p = 0.2$.

$$\Sigma X_i \mid p = 0.2 \sim \text{Binomial}(n, 0.2)$$

Fix an n so we can examine cutoffs. $n = 10$

$P(\Sigma X_i > 3 \mid p = 0.2) = 0.1209$
$P(\Sigma X_i > 4 \mid p = 0.2) = 0.0328$ $\rightarrow$ could use either of these

No test @ $\alpha = 0.05$.

Can develop further to Uniformly Most Powerful tests
UMP's exist if dist has a monotone likelihood ratio (MLR)

In general, $f_n(\underset{\sim}{x} \mid \theta)$ has a MLR in $r(\underset{\sim}{x})$ if $\forall$ pairs of θ values $\theta_1, \theta_2 \in \Omega \ni \theta_1 < \theta_2$, $\dfrac{f_n(x \mid \theta_2)}{f_n(x \mid \theta_1)}$ depends on data only through $r(\underset{\sim}{x})$ and is $\uparrow$ as a fn of $r(\underset{\sim}{x})$

MLRs exist for dists in exponential family (more in Ch. 5)

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Ch. 2 Verify MLR for the Bernoulli example

$$f_n(\underset{\sim}{x} \mid p) = p^{\Sigma X_i} (1-p)^{n-\Sigma X_i} \qquad \text{Let } p_1 < p_2.$$

$$\frac{f_n(\underset{\sim}{x} \mid p_2)}{f_n(\underset{\sim}{x} \mid p_1)} = \left[\frac{p_2(1-p_1)}{p_1(1-p_2)}\right]^{\Sigma X_i} \left(\frac{1-p_2}{1-p_1}\right)^n$$

depends on data only through $r(\underset{\sim}{x}) = \Sigma X_i$
and it is an $\uparrow$ fn of ΣX_i if $0 < p_1 < p_2 < 1$

So, we could return to our example to say

Reject H_0 if $\Sigma X_i > 4$ ie. $\Sigma X_i \geq 5$ is UMP @
$\alpha = 0.0328$ for $H_0: p = 0.2$ vs. $H_A: p = 0.4$
$\Rightarrow p > 0.2$

Pivotal Statistic

Textbook shows 2-sample t example
Will show 1-sample t example for CI for μ

Suppose $X_1, \ldots, X_n$ is RS $N(\mu, \sigma^2)$ and note we have statistics

$$\bar{X}_n = \frac{\Sigma X_i}{n} \quad \text{and} \quad S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

Now we define $Z = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ and $Y = S_n^2/\sigma^2$

You can show that Y and Z must be $\perp$, and

$$Y \sim \chi^2(n-1) \text{ and } Z \sim N(0,1).$$

Now we construct $U = \frac{Z}{(Y/(n-1))^{1/2}}$. $U \sim t(n-1)$ proven by Fisher

Rewrite:

$$U = \frac{n^{1/2}(\bar{X}_n - \mu)}{\left(\frac{S_n^2}{n-1}\right)^{1/2}} \longrightarrow \text{pivot!}$$

Now, consider $Z = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$. If you don't know σ, use plug-in MLE!

Plug-in ex. 1

$$\hat{\sigma} = (S_n^2/n)^{1/2} \rightarrow \text{so plug in.}$$

Note: biased!

$$Z' = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}} = \left(\frac{n}{n-1}\right)^{1/2} U$$

this is a scalar shift of U, so $Z' \sim t(n-1)$ too.

Plug-in ex. 2

Use $s = \sigma' = \left[\frac{S_n^2}{n-1}\right]^{1/2}$ instead of σ in Z.

result

$$Z'' = \frac{\sqrt{n}(\bar{X}_n - \mu)}{s} \sim t(n-1) \quad \text{pivot!}$$

$$= U$$

$\Rightarrow$ use to create a CI for μ. Let $c = t_{\alpha/2}$ upper $\alpha/2$ th cutoff $t(n-1)$

$$P(-c < Z'' < c) = 1-\alpha \quad \rightarrow \text{helps solve for } c$$

$$P\left(-c < \frac{\sqrt{n}(\bar{X}_n - \mu)}{s} < c\right) = P\left(-\frac{cs}{\sqrt{n}} < \bar{X}_n - \mu < \frac{cs}{\sqrt{n}}\right)$$

$$= P(\bar{X}_n - cs/\sqrt{n} < \mu < \bar{X} + cs/\sqrt{n}) = 1-\alpha$$

$\bar{X} \pm t^*_{\alpha/2}\, s/\sqrt{n}$ is $(1-\alpha)100\%$ CI for μ

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Ch. 3 Bayesian Conjugate Family Example Poisson/Gamma

Let $X_1, \ldots, X_n$ be a RS from a Poisson dist for which θ is unknown $(\theta > 0)$, but has a prior dist. of Gamma(α, β) with $\alpha, \beta > 0$.

Parameterization or reverse

Then the posterior dist of $\theta \mid X_i$, $i = 1, \ldots, n$ is Gamma$(\alpha + \sum_{i=1}^{n} X_i, \beta + n)$.

Pf. Hint: Use proportionality. Let $y = \sum X_i$.

$$f_n(\underset{\sim}{x} \mid \theta) \propto e^{-n\theta} \theta^{y}$$
$$g(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0$$

You may be used to $\lambda = 1/\theta$, just a reparameterization for Gamma.

Then remember $\overset{\text{posterior}}{g(\theta \mid \underset{\sim}{x})} \propto f_n(\underset{\sim}{x} \mid \theta)\, g(\theta) \Rightarrow$

$$g(\theta \mid \underset{\sim}{x}) \propto \theta^{\alpha + y - 1} e^{-\theta(n+\beta)} \sim \text{Gamma}(\alpha + \underset{\sum X_i}{Y}, \beta + n).$$

Ex. Application

You are sampling Poisson obs and you want posterior variance to be 0.01 or less.
Suppose prior is Gamma$(\alpha = 3, \beta = 6)$.

Then the posterior for θ will be Gamma$(Y + 3, n + 6)$.

Next, recall variance of Gamma is $\dfrac{\alpha}{\beta^2}$ $\Rightarrow$ Your text has other parameterization

posterior variance is $V = \dfrac{y+3}{(n+6)^2}$ sample sequentially until $V \le 0.01$.

For hypothesis testing, can use Bayes Factors
- can find evidence in favor of $\mathcal{H}_0$

For CIs, use credible intervals, where you can say

$$P(\_\_ < \theta < \_\_) = 0.95, \text{ etc.}$$

Gamma Dist.

Ch. 5 Textbook $\text{Gamma}(r, \sigma)$

$$f(x) = \frac{x^{r-1} e^{-x/\sigma}}{\sigma^r \Gamma(r)} \qquad E(X) = \sigma r \qquad V(X) = \sigma^2 r$$

Dobrow textbook: $\text{Gamma}(r, \lambda) = \text{Gamma}(\alpha, \beta)$

☆ used here

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad E(X) = \frac{\alpha}{\beta} \qquad V(X) = \alpha/\beta^2$$