Stat 495 Lecture Notes

Ch. 4 Ex. of biased MLE

Let $Y_1, Y_2, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$. Then

MLE for $\mu = \bar{Y}$ and MLE for $\sigma^2 = \frac{1}{n}\sum(y_i-\mu)^2$

plug-in $\bar{y}$ for μ $\Rightarrow$ $\hat{\sigma}^2 = \frac{1}{n}\sum(y_i-\bar{y})^2$ is biased for σ^2

Instead, we use $s^2 = \frac{1}{n-1}\sum(y_i-\bar{y})^2 = \frac{n}{n-1}\hat{\sigma}^2$ which is unbiased.

Recall MLEs are a fn of sufficient statistics.

Fisher Information

Suppose X is a RV with pdf $f(x|\theta)$.
Suppose the log likelihood is twice differentiable wrt θ. Ie. $\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}$ exists.

Then the Fisher information is defined to be

$$I(\theta) = E_\theta\left\{\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2\right\}$$

but is usually found via the related computational formula:

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right]$$

For a random sample, $I_n(\theta) = nI(\theta)$.

Relationship to MLEs

$X_1, X_2, \ldots, X_n$ RS pdf $f(x|\theta)$. Let $\hat{\theta}_n$ be MLE of θ
Fisher showed that for large n,

$$\hat{\theta}_n \sim N\left(\theta, \frac{1}{nI(\theta)}\right)$$

Cramer - Rao Lower Bound (Information Bound)

Let $\tilde{\theta}$ be an unbiased est. of θ from an iid sample.
Then $Var_\theta(\tilde{\theta}) \geq \frac{1}{nI(\theta)}$.

Many unbiased estimators do not achieve the lower bound (ie. the $=$).
If it does, you get an efficient estimator. (even if it is a biased estimator)

Ex.
Suppose $X_1, \ldots, X_n$ is RS from Poisson (θ),
ie. $f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$, $x = 0, 1, 2, \ldots$ and $\theta > 0$

Find MLE for θ. Verify unbiased. Find CR LB.
Does MLE achieve lower bound on variance?

Solution. $f_n(x|\theta) = \frac{e^{-\theta n}\theta^{\sum x_i}}{\prod x_i!}$

$$\ell(\underset{\sim}{x}|\theta) = -\theta n + \sum X_i \log\theta - \log \prod X_i!$$

$$\ell'(\underset{\sim}{x}|\theta) = -n + \frac{\sum X_i}{\theta} = 0 \Rightarrow \frac{\sum X_i}{\theta} = n \rightarrow \frac{\sum X_i}{n} = \hat{\theta}$$

$\hat{\theta} = \bar{X}$

$$E(\bar{X}) = \frac{nE(X)}{n} = E(X) = \theta \quad \text{unbiased}$$

Now, to get Cramer Rao lower bound, need $I(\theta)$.

Stat 495 Lecture Notes

Ch. 4 Can get $I(\theta)$ from just 1 obs.

$$\ell(x|\theta) = -\theta + X \log \theta - \log X!$$

$\frac{d}{d\theta}$

$$\ell'(x|\theta) = -1 + \frac{X}{\theta} - 0 = X\theta^{-1} - 1$$

$$\ell''(x|\theta) = -X\theta^{-2} = \frac{-X}{\theta^2}$$

Now, $I(\theta) = -E_\theta\left[\frac{-X}{\theta^2}\right] = \frac{1}{\theta^2} E_\theta(X) = \frac{1}{\theta}$

$n I(\theta) = \frac{n}{\theta}$ So, lower bound on Variance is $\frac{\theta}{n}$.

$1/nI(\theta)$.

Does MLE achieve?

$V(\bar{X}) = \frac{n V(X)}{n^2} = \frac{n\theta}{n^2} = \frac{\theta}{n}$. Yes, the MLE here is unbiased & efficient. MVUE.