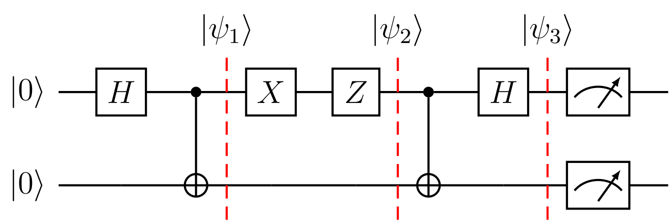
Superdense Coding

1 Quantum Circuit



2 State Vector

At first 2 qubits are entangled to form a bell state $|\phi^+\rangle$

$$
\begin{aligned}
|\psi_1\rangle &= |0\rangle \otimes |0\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= |\Phi^+\rangle
\end{aligned}
$$

There are in total 4 bell states, transformation between them is carried out with quantum gates

$$
\begin{aligned}
|\Phi^+\rangle &\xrightarrow{I} |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^+\rangle &\xrightarrow{X} |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Phi^+\rangle &\xrightarrow{Z} |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Phi^+\rangle &\xrightarrow{XZ} |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Each of these bell states are used to represent the 4 binary states

$$
\begin{aligned}
|\Phi^+\rangle &\rightarrow B_0 = 00 \\
|\Psi^+\rangle &\rightarrow B_1 = 01 \\
|\Phi^-\rangle &\rightarrow B_2 = 10 \\
|\Psi^-\rangle &\rightarrow B_3 = 11
\end{aligned}
$$

Here, the data to be transmitted is 11. Thus X and Z gates are used

$$
\begin{aligned}
|\psi_2\rangle &= |\Phi^+\rangle \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
&= \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle) \\
&= |\Psi^-\rangle
\end{aligned}
$$

Final step is to do bell basis measurement on the state vector and measure it to decode the sent data

$$\begin{aligned}
|\psi_3\rangle &= |\Psi^-\rangle \\
&= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |1\rangle\right) - \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |1\rangle\right)\right) \\
&= \frac{1}{2}(|01\rangle + |11\rangle - |01\rangle + |11\rangle) \\
&= |11\rangle
\end{aligned}$$