MAE 598 HW #2

1) $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

2nd order Taylor's Expansion

$f(x_1, x_2) = f(1,1) + \nabla_x f|_{x_0}^T [x_0] + \frac{1}{2}[x_0]^T H|_{x_0} [x_0]$

$\nabla_x f|_{x_0}^T = [4x_1 - 4x_2, -4x_1 + 3x_2 + 1]$ @ $[1,1] = [0,0]$

$H|_{x_0} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$

$f(x_1, x_2) - f(1,1) = 0 + \frac{1}{2}[\partial x_1, \partial x_2] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$

$f(x_1, x_2) - f(1,1) = \frac{1}{2}[4\partial x_1 - 4\partial x_2, -4\partial x_1 + 3\partial x_2] \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$

$f(x_1, x_2) - f(1,1) = \frac{1}{2}(\partial x_1(4\partial x_1 - 4\partial x_2) + \partial x_2(-4\partial x_1 + 3\partial x_2))$

$f(x_1, x_2) - f(1,1) = \frac{1}{2}(4\partial x_1^2 - 4\partial x_1 \partial x_2 - 4\partial x_1 \partial x_2 + 3\partial x_2 \partial x_2)$

$f(x_1, x_2) - f(1,1) = \frac{1}{2}(4\partial x_1^2 - 8\partial x_1 \partial x_2 + 3\partial x_2^2) \boxed{=} (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2)$

$ac = 4$

$bd = 3$, for $a=1, c=4, b=3, d=1$

$ad + bc = 8$

$(d + 4b = 8)$, $d=1$ $b=3$

$\therefore f(x_1, x_2) - f(1,1) = \frac{1}{2}(\partial x_1 - 3\partial x_2)(4\partial x_1 - \partial x_2) < 0$

when $(\partial x_1 - 3\partial x_2) > 0$ and $(4\partial x_1 - \partial x_2) < 0$

or $((\partial x_1 - 3\partial x_2) < 0$ and $(4\partial x_1 - \partial x_2) > 0$

2) $x_1 + 2x_2 + 3x_3 = 1$ nearest to $(-1, 0, 1)^T$

$\min_{x_1 x_2 x_3} (x_1 - (-1))^2 + ((x_2 - 0)^2 + (x_3 - 1)^2$

To make unconstrained, $x_1 = 1 - 2x_2 - 3x_3$

so $\min_{x_2 x_3} (2 - 2x_2 - 3x_3)^2 + (x_2)^2 + (x_3 - 1)^2$

code ran and got $(\underset{x_1}{-1.07}, \underset{x_2}{-0.14}, \underset{x_3}{0.79})$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$ eigenvalues of Hessian are > 0 so function is convex

3) Prove a hyperplane is a complex set.

In $\mathbb{R}^n$ $a^T x = c$ for $x \in \mathbb{R}^n$

All points of $a^T x_i = c$ so $a^T x_1 = c$ and $a^T x_2 = c$

$\therefore$ $a^T x_1 = a^T x_2 = c$

from notes $a^T(\lambda x_1 + (1-\lambda)x_2) = c$

$\lambda a^T x_1 + (1-\lambda) a^T x_2 = \lambda \cdot c + (1-\lambda) \cdot c = c$

If the above is true a hyperplane is a convex set

4) $\min_p \max_k \{h(a_k^T p, I_t)\}$ $h(I, I_t) = \begin{cases} I_t/I & \text{if } I < I_t \\ I/I_t & \text{if } I \geq I_t \end{cases}$

subject to $0 \leq p_i \leq p_{max}$

a) $I = a_k^T P$ which is a linear function and by definition is convex

I_t/I is a reciprocal function which, by definition, is a convex function

thus h is a function of convex function and thus convex

$\max_k$ is comprised of a set of convex functions and thus is convex, the same is true for $\min_p$

So it is a convex problem

b) for any C_n^{10} for any n $\sum P(C_n^{10}) \leq p^*$

with only this constraint given we can't prove or disprove the problems uniqueness

c) $\sum_{i=1}^{n} P_i \leq 10$

P is not a linear equation, just a constant, and thus does not meet the requirement for a convex problem and thus won't have any solution.

5) $c^*(y) = \max_x \{xy - c(x)\}$

For a variable price, y, across a set of product product cost, c(x), for x amount of product

$\Downarrow$

$$c^*(y) = \max\{x_1 y - c(x_1)\}$$
$$c^*(y) = \max_x\{x_2 y - c(x_2)\}$$
$$\vdots$$
$$c^*(y) = \max_x\{x_n y - c(x_n)\}$$

Since x_i and $c(x_i)$ for any n are constant in this set, $x_i y - c(x_i)$ is a linear equation and by definition is convex.

Since we are taking the max of a set of convex functions the result will be convex