MAE 598 HW#4

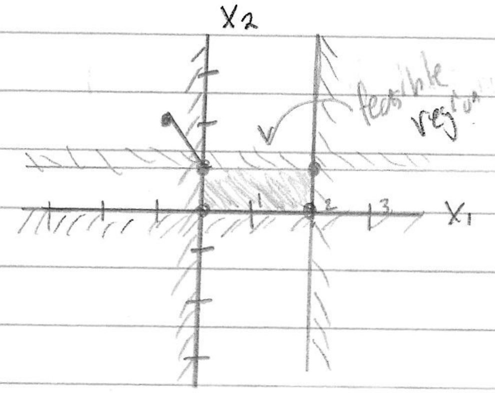
1. Sketch graphically the problem

$\min f(x) = (x_1+1)^2 + (x_2-2)^2$

s.t. $g_1 = x_1 - 2 \le 0$ $\quad g_3 = -x_1 \le 0$

$g_2 = x_2 - 1 \le 0$ $\quad g_4 = -x_2 \le 0$

$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$



graphically, solution is $x_1=0,\ x_2=1$

if $x_1-2=0$ then $\mu_1>0$ otherwise..

if $x_2-1=0$ then $\mu_2>0$ otherwise

if $-x_1=0$ then $\mu_3>0$ otherwise..

if $-x_2=0$ then $\mu_4>0$ otherwise...

For corner $x_1=0,\ x_2=1$; $\mu_1=0,\ \mu_2>0,\ \mu_3\ge0,\ \mu_4=0$

$$\nabla_x L = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

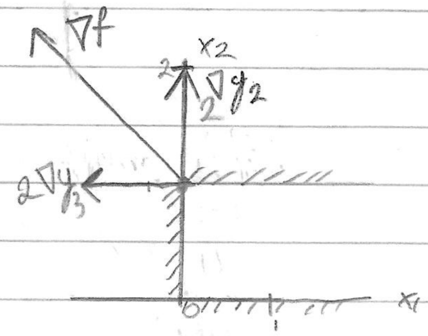
$$\nabla_x L = \begin{bmatrix} 2(0+1) + 0 - \mu_3 \\ 2(1-2) + \mu_2 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \mu_3 = 2 \ \checkmark \\ \mu_2 = 2 \ \checkmark \end{matrix}$$ Solution to problem

$$\nabla L_{xx} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \ \text{P.D.}$$

$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 + \mu_3 \nabla g_3 + \mu_4 \nabla g_4 = 0$

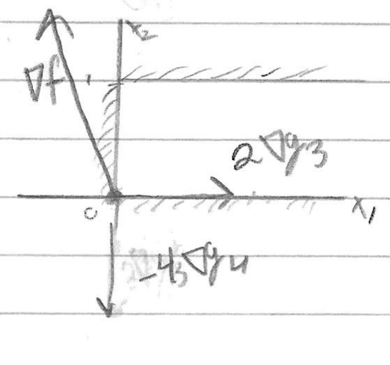
@ $x_1=0,\ x_2=1,\ \mu_1=\mu_4=0\ \mu_2=\mu_3=2$

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \nabla f = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



@ $x_1=0,\ x_2=0;\ \mu_1=\mu_2=0\ \mu_3=2\ \mu_4=-4$

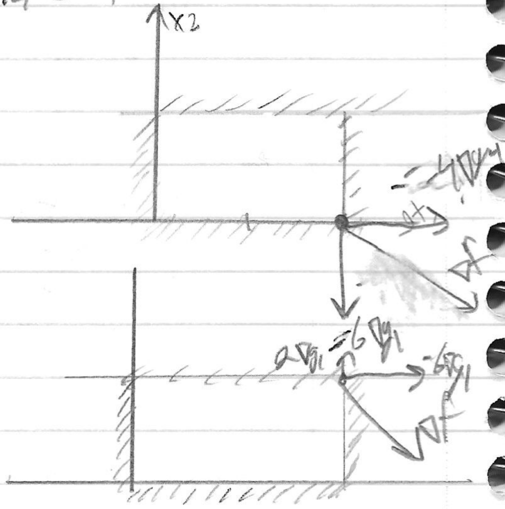
$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$@\ x_1 = 2, x_2 = 0 \;;\; \mu_1 > 0, \mu_2 = 0, \mu_3 = 0\ \mu_4 > 0$

$\mu_1 = -6 \qquad \mu_4 = -4$

$$\begin{bmatrix} 6 \\ -4 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$@\ x_1 = 2, x_2 = 1 \;;\; \mu_1 > 0, \mu_2 > 0, \mu_3 = 0 ; \mu_4 = 0$

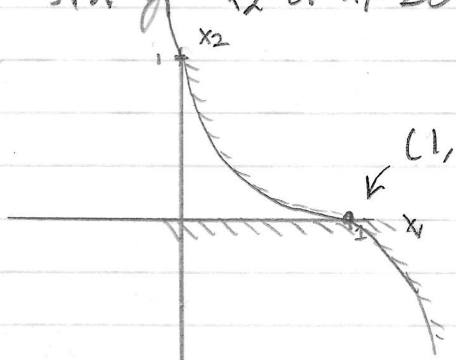
$\mu_1 = -6, \mu_2 = 2,$

$$\begin{bmatrix} 6 \\ -2 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. Graph the problem.

$\min f = -x_1$

s.t. $g_1 = x_2 - (1 - x_1)^3 \le 0 \quad g_2 = -x_2 \le 0$



(1, 0) is the solution graphically

$$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 3(1 - x_1)^2 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

@ (1, 0)

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;;\; \mu_1 = \mu_2 \text{ but } -1 \ne 0$$

<u>Not</u> a KKT Point

3. Find a local solution to the problem

$\max f = x_1x_2 + x_2x_3 + x_1x_3$

s.t. $h = x_1 + x_2 + x_3 - 3 = 0$

Lagrangian $\min f = -x_1x_2 - x_2x_3 - x_1x_3$

$$L = -x_1x_2 - x_2x_3 - x_1x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\nabla L = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 = x_3 = 1$$

$$\lambda = 2$$

$$dx^T L_{xx} dx = \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = -2dx_1dx_2 - 2dx_1dx_3 - 2dx_2dx_3$$

$$\frac{\partial h}{\partial x} dx = 0 \Rightarrow \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = 0 \Rightarrow dx_1 + dx_2 + dx_3 = 0 \text{ or } dx_1 = -dx_2 - dx_3$$

$$\begin{aligned} dx^T L_{xx} dx &= -2(-dx_2 - dx_3)dx_2 - 2(-dx_2 - dx_3)dx_3 - 2dx_2dx_3 \\ &= 2dx_2^2 + 2dx_2dx_3 + 2dx_2dx_3 + 2dx_3^2 - 2dx_2dx_3 \\ &= 2(dx_2^2 + dx_2dx_3 + dx_3^2) \\ &= 2\left[(dx_2 + \tfrac{1}{2}dx_3)^2 + \tfrac{3}{4}dx_3^2\right] \end{aligned}$$

which is always positive except when $x_1 = x_2 = x_3 = 0$

$\therefore$ $x_1 = x_2 = x_3 = 1$ is a local solution

5. $\min \sum^{N}\sum^{N}_{i} c_{ij}a_{ij}$ when a_{ij} is $\begin{cases} 1 & \text{if the route } ij \text{ is used} \\ 0 & \text{if it is not} \end{cases}$

s.t. $\sum^{N}_{j}\sum^{N}_{i} a_{ij} = N$