Christopher Lang, Ph.D. — Research Proposal

cjlang@uwaterloo.ca https://cjlang96.github.io/
https://github.com/cjlang96

My research interests involve Lie theory, representation theory, moduli spaces, and gauge theory—specifically instantons and monopoles. I study gauge theories arising from particle physics that are interesting from a geometric and topological perspective. My results come from applying Lie theory to group actions on moduli spaces, obtaining representations of Lie algebras. Going forward, my main research directions involve generalizing my work to infinite dimensions, studying monopoles using topological recursion, and examining geodesic submanifolds of moduli spaces. Below, I describe my main results before going into my research directions in more detail.

Main Results

Continuous symmetries

Topological solitons are gauge-theoretic objects that often satisfy complicated non-linear constraints. These objects have a long history filled with many applications to physics and deep mathematical properties. In fact, the subject has matured to the point that it is the sole focus of multiple textbooks [MS04; Man22]. Examples of solitons help us better understand the properties of these real-world objects but are hard to find due to the non-linear constraints.

Skyrmions and magnetic monopoles are two examples of topological solitons modelling important real world phenomena. Skyrmions model nuclei, and by studying their dynamics, we can model important nuclear reactions like that of Boron Neutron Capture Therapy (BNCT), which is used to treat inoperable cancers. Magnetic monopoles are hypothetical particles that are sources of magnetic charge, a magnetic analogue to electrons. To this day, no one has observed a monopole in nature [Col23], however, it is predicted that they were plentiful in the early universe [Pre84]. In fact, due to the integrality of Chern numbers, the existence of a single magnetic monopole in the universe would explain the quantization of electric charge. Moreover, the lack of detection of magnetic monopoles in the universe was the impetus for the inflationary model of cosmology.

Broadly speaking, gauge theory involves two ingredients: objects (solutions to some equation) and a gauge action (a group action preserving solutions to said equation). In gauge theory, we only care about the moduli space: the set of objects modulo the gauge action. Symmetric objects are those whose equivalence class in the moduli space is fixed by the action of some symmetry group.

I have been studying continuous symmetries, which allow me to use Lie theory as well as differentiate the equations of symmetry, obtaining linear equations. These linear constraints are simple to solve and make solving the non-linear constraints easier.

Theorem 1 *[Lan24a, Theorem 1.1] Let $\mathcal{X}$ be a smooth manifold, $\mathcal{G}$ a compact Lie group, and $\mathcal{S}$ a compact, connected Lie group. Suppose that $\mathcal{G}$ and $\mathcal{S}$ act smoothly on $\mathcal{X}$ on the left and the two actions commute. We have that $[A] \in \mathcal{X}/\mathcal{G}$ is fixed by $\mathcal{S}$ if and only if there exists a Lie algebra homomorphism $\rho\colon \mathrm{Lie}(\mathcal{S}) \to \mathrm{Lie}(\mathcal{G})$ such that, for all $x \in \mathrm{Lie}(\mathcal{S})$,*

$$x.A + \rho(x).A = 0.$$

As compact Lie groups correspond to matrix Lie groups, this result reduces the problem of finding symmetric elements to a problem in representation theory. Additionally, this theorem not only provides a simple method for identifying symmetric elements, it produces all of them, providing a framework for classifying them.

By using Theorem 1, I have found many novel examples of topological solitons [Lan24c; Lan24b; Cha+22]. In particular, I have found a spherically symmetric $\mathrm{Sp}(4)$ hyperbolic monopole that vanishes nowhere [Lan24b]. This is surprising, as for $\mathrm{Sp}(1)$ hyperbolic monopoles, the number of zeros is related to the charge of the monopole. However, my novel monopoles demonstrate that this does not hold for higher rank structure groups, necessitating the study of these higher rank structure groups. This $\mathrm{Sp}(4)$ monopole is also in stark contrast to the family of $\mathrm{Sp}(1)$ monopoles that all vanish at the origin [Oli14, Appendix A].

One of the objects that I have studied using Theorem 1 is instantons (on $\mathbb{R}^4$). The holonomy of instantons closely approximates skyrmions [AM89]. This is useful, as instantons are relatively simpler to find, due to the Atiyah—Drinfeld—Hitchin—Manin (ADHM) theorem. Indeed, I have identified several novel instantons [Lan24c].

Due to the relationship between skyrmions and instantons, many authors began searching for symmetric instantons. This search utilized representations of finite groups or abelian groups and continues to be an active area [Whi22; CH22; Bec20; MS14; Coc14; AS13]. However, the justification of the use of these methods was very specific to instantons and not widely applicable. Similarly, work has been done using representations of finite groups to study symmetric monopoles [BD23; Bra11; BDE11; BE10]. My work differs by considering Lie groups (abelian and non-abelian) and is applicable to general cases, it is not restricted to specific solitons. In fact, it is not restricted to working solely on solitons.

As an example of an application of Theorem 1, I consider rotationally symmetric instantons. By the powerful ADHM theorem, instantons (solutions to the self-dual equations with finite action) correspond to ADHM data (quaternionic matrices satisfying some non-linear constraints). While ADHM data is much easier to find than instantons, the non-linear constraints still provide a challenge, especially when dealing with higher rank structure groups. By carefully handling problems arising from non-compactness, I obtained the following result.

Theorem 2 *[Lan24c, Theorem 3.6.1] A $\mathrm{Sp}(n)$ instanton with ADHM data $\hat{M} = \begin{bmatrix} L \\ M \end{bmatrix} \in \mathrm{Mat}(n+k, k, \mathbb{H})$ has rotational symmetry if and only if there is some Lie algebra homomorphism $\rho\colon \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{so}(k)$ such that for all $\upsilon, \omega \in \mathfrak{sp}(1)$,*

$$\nu M + [\rho(\nu, \omega), M] - M\omega = 0 \quad \text{and} \quad \left[\rho(\nu, \omega), \hat{M}^\dagger \hat{M}\right] = 0.$$

Theorem 3 *[Lan24c; Lan24b; Cha+22] I have obtained results similar to Theorem 2 for instantons with every kind of continuous conformal symmetry as well as axial and spherically symmetric hyperbolic and Euclidean monopoles.*

The following result shows the power of Theorem 1: it not only helps us find examples but also tells us when we have all of them. Note that I talk about instantons that live on $\mathbb{R}^4$, which are equivalent to instantons living on S^4 [Uhl82].

Corollary 1 *[Lan24c, Corollary 3.4.45] The basic instanton is the only instanton symmetric under all the isometries of the four-sphere.*

Symmetry breaking

Symmetry breaking is part of the topological classification of monopoles. While monopoles with arbitrary symmetry breaking is not a new concept [GNO77], the main focus of monopole research has been monopoles with structure group $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$, where the symmetry is always broken to $\mathrm{U}(1)$. In fact, much of the work on $\mathrm{SU}(n)$ monopoles—Euclidean and hyperbolic—has focused on those with maximal symmetry breaking, which is a generalization of the $\mathrm{SU}(2)$ case. For instance, work generalizing Braam–Austin's discrete Nahm equations to $\mathrm{SU}(n)$ hyperbolic monopoles with maximal symmetry breaking [Cha18; BA90].

Despite the focus on maximal symmetry breaking, some work was done on monopoles with minimal symmetry breaking. In particular, a specific moduli space of monopoles with minimal symmetry was examined in depth, providing an important example of a hyper-Kähler manifold [Dan93]. Except for work on these two extremal cases, maximal and minimal symmetry breaking, little was known about monopoles with arbitrary symmetry breaking.

Recently, there has been more interest in monopoles with arbitrary symmetry breaking. For instance, recent work has generalized the Nahm transform for $\mathrm{SU}(n)$ Euclidean monopoles with arbitrary symmetry breaking [CN22]. Previous work of mine with collaborators has used this Nahm transform to generate $\mathrm{SU}(n)$ Euclidean monopoles with continuous symmetries and neither maximal nor minimal symmetry breaking [Cha+22]. Additionally, it has been proven that the moduli space of monopoles with arbitrary symmetry breaking and a compact, connected Lie group for a structure group is not only composed of strata with dimension divisible by four, but is hyper-Kähler [Men24; Sán19]. This answered a long-standing conjecture [MS03, Conjecture 3.3].

For general compact, connected Lie groups, little is written about classifying symmetry breaking. These other cases are important as previous work of mine involves providing a framework for classifying $\mathrm{Sp}(n)$ hyperbolic monopoles with continuous symmetries [Lan24b]. Given the recent focus on arbitrary symmetry breaking for arbitrary structure groups, we needed to understand this phenomenon in general.

As an example of my work on symmetry breaking, the following result outlines the symmetry breaking for $\mathrm{Sp}(n)$ monopoles. Similar results provide a simple classification of symmetry breaking for all Lie groups with a classical, simple Lie algebra. I also outlined a method for doing the same for the exceptional simple Lie algebras, which provides a way to classify symmetry breaking for all compact, connected Lie groups [Lan24c].

Theorem 4 *[Lan24c, Theorem 2.3.37] Let $\Phi_\infty \in \mathfrak{sp}(n)$. Let $\alpha_1, \ldots, \alpha_n$ be the modulus of the right eigenvalues of Φ_∞. Let N be the number of distinct, non-zero values of α_i. Then the symmetry breaking of Φ_∞ is given by the group $\mathbb{Z}^N$.*

As another example of the importance of my work on symmetry breaking, through this examination, I discovered that the typical notion of minimal symmetry breaking for $\mathrm{SU}(n)$ monopoles is too restrictive. Indeed, minimal symmetry breaking occurs when $\Phi_\infty \in \mathfrak{su}(n)$ has two distinct eigenvalues and one has multiplicity one. The following result shows that monopoles where Φ_∞ has two distinct eigenvalues and any multiplicity have the same symmetry breaking as typical minimal symmetry breaking.

Theorem 5 *[Lan24c, Theorem 2.3.31] Let $\Phi_\infty \in \mathfrak{su}(n)$. Let N be the number of distinct eigenvalues of Φ_∞. Then the symmetry breaking of Φ_∞ is given by the group $\mathbb{Z}^{N-1}$.*

Research Program

My current focus is converting my thesis into papers to be published. Section 2.1 of my thesis has already been published, but the rest of the thesis contains three papers' worth of novel research. Afterwards, I have several ideas on where to take the research program that I developed over the course of my Ph.D. Below I list some of these directions that I am interested in. Not only is this list not exhaustive, I am very open to seeing it evolve as a result of discussions and collaborations with new colleagues.

Direction #1: Symmetries in infinite dimensions

Over the course of my Ph.D., I developed Theorem 1 and applied it to many gauge-theoretic objects: Euclidean monopoles, hyperbolic monopoles, and instantons. Going forward, I would like to expand the scope of this result and find applications to study moduli spaces outside of those found in gauge theory. As one example, I would like to extend my result to say something about when the spaces of objects and gauge transformations are infinite-dimensional. Subsequent projects could then focus on applying these results to various paradigms.

Although Theorem 1 has applications to many other gauge-theoretic objects, in general, when dealing with such objects, the space of objects and the gauge group are infinite-dimensional; only their moduli space is finite-dimensional. The aforementioned applications were made possible due to the ADHM and Nahm transforms, which provide a correspondence between the moduli space generated by these infinite-dimensional spaces and a moduli space generated by finite-dimensional spaces. As we are only interested in the moduli space, these correspondences allowed me to use the finite-dimensional replacements.

In general, we do not have such transforms and are stuck with the infinite-dimensional spaces. These spaces require an infinite-dimensional analogue to Theorem 1. Such a result would make finding other symmetric gauge-theoretic objects much simpler, which would provide us with many examples of these objects to study, where they are sorely lacking.

Direction #2: Spectral curves and topological recursion

One of the most interesting aspects of $\mathrm{SU}(2)$ monopoles is their relationships with other objects: $\mathrm{SU}(2)$ monopoles are in one-to-one correspondences with Nahm data, rational maps, and spectral curves. These alternative viewpoints have been crucial to understanding the moduli space of monopoles and finding examples of monopoles.

Recent work on monopoles, including my own, has focused on understanding those with higher rank structure groups [CN22; Cha+22; Men24; Sán19]. In particular, the Nahm transform has been generalized, providing a one-to-one correspondence between $\mathrm{SU}(n)$ monopoles with arbitrary symmetry breaking and Nahm data [CN22]. Other work has generalized the correspondence between spectral curves and $\mathrm{SU}(n)$ monopoles with maximal symmetry breaking [HM89]. The first step in this research direction is a project generalizing the correspondence between spectral curves and $\mathrm{SU}(n)$ monopoles with arbitrary symmetry breaking.

Topological recursion is a construction in algebraic geometry which studies invariants of spectral curves. It has been applied to study random matrices, Gromov–Witten invariants, enumerative geometry, and knot theory [EO15; EO07]. Additionally, it has been used to study Higgs bundles, which are related to monopoles, as they are both dimensional reductions of the self-dual equations [BH19]. After finding a correspondence between spectral curves and general $\mathrm{SU}(n)$ monopole, I want to use topological recursion to study monopoles via these spectral curves, obtaining invariants of these objects.

Direction #3: Examining geodesic submanifolds

As a result of my applications of Theorem 1, I have obtained many geodesic submanifolds of moduli spaces. Due to time constraints during my Ph.D., I was unable to study these geodesic submanifolds. Going forward, I would like to examine these spaces through multiple projects and learn more about their properties. Below, I discuss how these geodesic submanifolds can be used to study the dynamics of physical phenomena.

By studying the dynamics of topological solitons, we can examine how they interact. For instance, by studying the dynamics of skyrmions, we can model important nuclear reactions like that of the cancer therapy BNCT. Recall that instantons generate Skyrme fields closely approximating skyrmions. It turns out that geodesic motion on the moduli space of instantons closely approximates skyrmion dynamics [SS99]. By studying symmetric topological solitons, I have identified novel geodesic submanifolds of the moduli spaces of instantons [Lan24c]. By examining these submanifolds, I can not only approximate novel skyrmions, I can study their dynamics.

Euclidean monopole dynamics are closely approximated by geodesic motion on their moduli space [Stu94]. Historically, the focus of monopole research has been when the structure group is $\mathrm{SU}(2)$. This is primarily due to the fact that few examples were known for higher rank structure groups. I have identified novel geodesic submanifolds of the moduli spaces of Euclidean monopoles for higher rank structure groups [Cha+22]. While it is known that the moduli space is hyper-Kähler, an explicit metric for the moduli space has not been found [Men24]. In the absence of such a metric, motion is still difficult to understand. By using my submanifolds, I can study the motion of monopoles in more depth.

Unfortunately, the natural L^2 metric used for Euclidean monopoles diverges for hyperbolic monopoles. As such, a different metric is needed. The natural metric in the Euclidean case is nice as it is hyper-Kähler. Recent work has identified a hyperbolic analogue of hyper-Kähler geometry on the moduli space of $\mathrm{SU}(2)$ hyperbolic monopoles [FH24]. By generalizing this metric to higher rank structure groups, I can use my novel hyperbolic monopoles to compute this metric on geodesic submanifolds and examine its structure. Moreover, should the geodesic motion be shown to approximate monopole dynamics, then my novel examples can be used to examine the dynamics of hyperbolic monopoles.

Student Training

Many of the ideas I started to explore during my Ph.D. present opportunities for training students at various levels through supervised projects. Below are some examples.

Student project #1

The first project involves finding Euclidean monopoles with arbitrary charges and symmetry breaking. Based on my work with collaborators, I have identified novel spherically symmetric Euclidean monopoles [Cha+22]. Recent work has identified a hyper-Kähler structure on the moduli space of Euclidean monopoles [Men24]. However, we do not know that these moduli spaces are all non-empty. Indeed, in my collaborative work, we only identified monopoles with a specific kind of symmetry breaking. However, by Theorem 1, if there are spherically symmetric monopoles with arbitrary symmetry breaking, then they can be produced in the same manner as the aforementioned novel Euclidean monopoles. The production of these monopoles is only a matter of computation, albeit one that requires significant investment and supervision.

Student project #2

The second project involves proving the conjecture that I proposed in my thesis. The group of conformal symmetries on $\mathbb{R}^4$ is $\mathrm{SL}(2, \mathbb{H})$ and the symmetry group of an instanton is a subgroup of this group. In my thesis, I conjecture that the symmetry group of a non-flat instanton is conjugate to a subgroup of $\mathrm{Sp}(2) \subseteq \mathrm{SL}(2, \mathbb{H})$ [Lan24c]. We can decompose $\mathrm{SL}(2, \mathbb{H})$ as [FS16]

$$\mathrm{SL}(2, \mathbb{H}) = \mathrm{Sp}(2) \{\mathrm{diag}(\nu, 1/\nu) \mid \nu \in (0, 1]\} \, \mathrm{Sp}(2).$$

Thus, we need only check that up to some consistent conjugation, ν must be one for every element of the symmetry group. Unfortunately, I ran out of time to complete this investigation during my Ph.D.

Student project #3

The third project involves applying Theorem 1 to study symmetric calorons. Calorons are periodic instantons and are closely related to monopoles and skyrmions [Cor18a]. In fact, calorons can be used to construct skyrmions [Cor18b]. It is believed that understanding calorons may help solve the confinement / mass-gap problem, a Millenium Prize Problem [Gre11, § 8.5]. Calorons are related to Nahm data on a circle via the Nahm transform [CH10]. As such, we are able to apply Theorem 1 to this situation in order to generate examples of symmetric calorons, which have received recent study for finite symmetries [KNT21; Cor18c].

Bibliography

[AS13] J. P. Allen and P. M. Sutcliffe. "ADHM polytopes". *Journal of High Energy Physics* 2013.5 (2013), p. 63. ISSN: 1029-8479. DOI: 10.1007/JHEP05(2013)063. arXiv: 1302.4664.

[AM89] M. F. Atiyah and N. S. Manton. "Skyrmions from instantons". *Physics Letters B* 222.3-4 (1989), pp. 438–442. ISSN: 0370-2693. DOI: 10.1016/0370-2693(89)90340-7.

[BH19] David Baraglia and Zhenxi Huang. "Special Kähler geometry of the Hitchin system and topological recursion". *Advances in Theoretical and Mathematical Physics* 23.8 (2019), pp. 1981–2024. ISSN: 1095-0753. DOI: 10.4310/atmp.2019.v23.n8.a2. arXiv: 1707.04975.

[Bec20] M. Beckett. "Equivariant Nahm Transforms and Minimal Yang–Mills Connections". PhD Thesis. Durham, NC: Duke University, 2020. URL: https://hdl.handle.net/10161/20982.

[BA90] P. J. Braam and D. M. Austin. "Boundary values of hyperbolic monopoles". *Nonlinearity* 3.3 (1990), p. 809. ISSN: 0951-7715. DOI: 10.1088/0951-7715/3/3/012.

[Bra11] H. W. Braden. "Cyclic monopoles, affine Toda and spectral curves". *Comm. Math. Phys.* 308.2 (2011), pp. 303–323. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-011-1347-1.

[BDE11] H. W. Braden, A. D'Avanzo, and V. Z. Enolski. "On charge-3 cyclic monopoles". *Nonlinearity* 24.3 (2011), pp. 643–675. ISSN: 0951-7715,1361-6544. DOI: 10.1088/0951-7715/24/3/001.

[BD23] H. W. Braden and L. Disney-Hogg. "Towards a classification of charge-3 monopoles with symmetry". *Lett. Math. Phys.* 113.4 (2023), Paper No. 87, 40. ISSN: 0377-9017,1573-0530. DOI: 10.1007/s11005-023-01708-5.

[BE10] H. W. Braden and V. Z. Enolski. "On the tetrahedrally symmetric monopole". *Comm. Math. Phys.* 299.1 (2010), pp. 255–282. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-010-1081-0.

[Cha18] J. YC Chan. "Discrete Nahm equations for SU(N) hyperbolic monopoles". *Journal of Geometry and Physics* 132 (2018), pp. 239–256. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2018.01.012. arXiv: 1506.08736.

[Cha+22] B. Charbonneau, A. Dayaprema, **C. J. Lang**, Á. Nagy, and H. Yu. "Construction of Nahm data and BPS monopoles with continuous symmetries". *Journal of Mathematical Physics* 63.1 (2022). **Editor's Pick**, p. 013507. ISSN: 0022-2488. DOI: 10.1063/5.0055913. arXiv: 2102.01657.

[CH10] B. Charbonneau and J. Hurtubise. "The Nahm transform for calorons". *The many facets of geometry*. Oxford Univ. Press, Oxford, 2010, pp. 34–70. ISBN: 978-0-19-953492-0. DOI: 10.1093/acprof:oso/9780199534920.003.0004.

[CN22] B. Charbonneau and Á. Nagy. *On the construction of monopoles with arbitrary symmetry breaking*. 2022. arXiv: 2205.15246.

[Coc14] A. Cockburn. "Symmetric hyperbolic monopoles". *Journal of Physics A: Mathematical and Theoretical* 47.39 (2014), p. 395401. ISSN: 1751-8113. DOI: 10.1088/1751-8113/47/39/395401. arXiv: 1406.4720.

[Col23] ATLAS Collaboration. *Search for magnetic monopoles and stable particles with high electric charges in $\sqrt{s}$ =13 TeV pp collisions with the ATLAS detector*. 2023. DOI: 10.1007/jhep11(2023)112. arXiv: 2308.04835.

[Cor18a] J. Cork. "Calorons, symmetry, and the soliton trinity". PhD thesis. Leeds, England: University of Leeds, 2018. URL: https://etheses.whiterose.ac.uk/22097/.

[Cor18b] J. Cork. “Skyrmions from calorons”. *J. High Energy Phys.* 2018.11 (2018). ISSN: 1029-8479. DOI: 10.1007/jhep11(2018)137.

[Cor18c] J. Cork. “Symmetric calorons and the rotation map”. *J. Math. Phys.* 59.6 (2018), pp. 062902, 23. ISSN: 1089-7658. DOI: 10.1063/1.5017193.

[CH22] J. Cork and C. Halcrow. “ADHM skyrmions”. *Nonlinearity* 35.8 (2022), p. 3944. ISSN: 0951-7715. DOI: 10.1088/1361-6544/ac72e6. arXiv: 2110.15190.

[Dan93] A.S. Dancer. “Nahm’s equations and hyperkähler geometry”. *Communications in Mathematical Physics* 158.3 (1993), pp. 545–568. ISSN: 1432-0916. DOI: 10.1007/BF02096803.

[EO07] B. Eynard and N. Orantin. “Invariants of algebraic curves and topological expansion”. *Communications in Number Theory and Physics* 1.2 (2007), pp. 347–452. ISSN: 1931-4531. DOI: 10.4310/cntp.2007.v1.n2.a4. arXiv: math-ph/0702045.

[EO15] B. Eynard and N. Orantin. “Computation of Open Gromov–Witten Invariants for Toric Calabi–Yau 3-Folds by Topological Recursion, a Proof of the BKMP Conjecture”. *Communications in Mathematical Physics* 337.2 (2015), pp. 483–567. ISSN: 1432-0916. DOI: 10.1007/s00220-015-2361-5. arXiv: 1205.1103.

[FH24] G. Franchetti and D. Harland. *L^2 geometry of hyperbolic monopoles*. 2024. arXiv: 2408.07145.

[FS16] G. Franchetti and B. J. Schroers. “Adiabatic Dynamics of Instantons on S^4”. *Communications in Mathematical Physics* 353.1 (2016), pp. 185–228. ISSN: 1432-0916. DOI: 10.1007/s00220-016-2769-6. arXiv: 1508.06566.

[GNO77] P. Goddard, J. Nuyts, and D. Olive. “Gauge theories and magnetic charge”. *Nuclear Physics B* 125.1 (1977), pp. 1–28. ISSN: 0550-3213. DOI: 10.1016/0550-3213(77)90221-8.

[Gre11] J. Greensite. *An Introduction to the Confinement Problem*. 1st ed. Vol. 821. Lecture Notes in Physics. Berlin, Heidelberg: Springer Nature, 2011. ISBN: 9783642143823.

[HM89] J. Hurtubise and M. K. Murray. “On the construction of monopoles for the classical groups”. *Communications in Mathematical Physics* 122.1 (1989), pp. 35–89. DOI: 10.1007/bf01221407.

[KNT21] T. Kato, A. Nakamula, and K. Takesue. “Symmetric calorons of higher charges and their large period limits”. *J. Geom. Phys.* 162 (2021), Paper No. 104071, 20. ISSN: 0393-0440,1879-1662. DOI: 10.1016/j.geomphys.2020.104071.

[Lan24a] **C. J. Lang**. “Fixed points of Lie group actions on moduli spaces: A tale of two actions”. 2024. arXiv: 2412.06970.

[Lan24b] **C. J. Lang**. “Hyperbolic monopoles with continuous symmetries”. *Journal of Geometry and Physics* 203 (2024), p. 105258. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2024.105258. arXiv: 2310.10626.

[Lan24c] **C. J. Lang**. “Solitons with continuous symmetries”. Ph.D. Thesis. Waterloo, Canada: University of Waterloo, 2024. URL: https://hdl.handle.net/10012/20906.

[Man22] N. S. Manton. *Skyrmions – A Theory of Nuclei*. 1st. London: World Scientific, 2022. ISBN: 978-1-80061-249-5. DOI: 10.1142/q0368.

[MS04] N. S. Manton and P. M. Sutcliffe. *Topological solitons*. Cambridge monographs on mathematical physics. Cambridge: Cambridge University Press, 2004. ISBN: 978-0-511-61703-4. DOI: 10.1017/cbo9780511617034.

[MS14] N. S. Manton and P. M. Sutcliffe. “Platonic hyperbolic monopoles”. *Communications in Mathematical Physics* 325.3 (2014), pp. 821–845. ISSN: 1432-0916. DOI: 10.1007/s00220-013-1864-1. arXiv: 1207.2636.

[Men24] J. Mendizabal. “A hyper-Kähler metric on the moduli space of monopoles with arbitrary symmetry breaking”. *Annals of Global Analysis and Geometry* 66.4 (2024). ISSN: 1572-9060. DOI: 10.1007/s10455-024-09954-z. arXiv: 2306.03069.

[MS03] M. K. Murray and M. A. Singer. “A note on monopole moduli spaces”. *Journal of Mathematical Physics* 44.8 (2003), pp. 3517–3531. ISSN: 0022-2488. DOI: 10.1063/1.1590056. arXiv: math-ph/0302020.

[Oli14] G. Oliveira. “Monopoles on the Bryant–Salamon G_2 Manifolds”. *Journal of Geometry and Physics* 86 (2014), pp. 599–632. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2014.10.005. arXiv: 1310.7392.

[Pre84] J Preskill. “Magnetic Monopoles”. *Annual Review of Nuclear and Particle Science* 34.1 (1984), pp. 461–530. DOI: 10.1146/annurev.ns.34.120184.002333.

[Sán19] R. Sánchez Galán. “Monopoles in R^3”. Ph.D. Thesis. London, England: University College London, 2019. URL: https://discovery.ucl.ac.uk/id/eprint/10069278.

[SS99] M. A. Singer and P. M. Sutcliffe. “Symmetric instantons and Skyrme fields”. *Nonlinearity* 12.4 (1999), p. 987. ISSN: 0951-7715. DOI: 10.1088/0951-7715/12/4/315. arXiv: hep-th/9901075.

[Stu94] D. Stuart. “The geodesic approximation for the Yang–Mills–Higgs equations”. *Communications in Mathematical Physics* 166.1 (1994), pp. 149–190. ISSN: 1432-0916. DOI: 10.1007/BF02099305.

[Uhl82] K. K. Uhlenbeck. “Removable singularities in Yang–Mills fields”. *Communications in Mathematical Physics* 83.1 (1982), pp. 11–29. ISSN: 1432-0916. DOI: 10.1007/BF01947068.

[Whi22] S. Whitehead. “Integrality theorems for symmetric instantons”. M.Math Thesis. Waterloo, Ontario, Canada: University of Waterloo, 2022. URL: http://hdl.handle.net/10012/18487.