Q3

Proof:

(1) Let $M_1 = 2, M_2 = 3, n_0 = 1$, and $n > n_0$ be arbitrary.

(2) By (1), $n > 1$, so $2n > 1 \implies 2^{2n} > 2^1 \implies \frac{2}{2^{2n}} < 1$.

(3) First, we will show that $M_1 n \leq \log_2(4^n + 2)$. We have

$$M_1 \leq 2, \quad \text{by (1)}$$

$$\implies M_1 n \leq 2n$$

$$\implies M_1 n \leq \log_2(2^{2n})$$

$$\implies M_1 n \leq \log_2(2^{2n} + 2)$$

$$\implies M_1 n \leq \log_2(4^n + 2),$$

as needed.

(4) Next, we will show that $\log_2(4^n + 2) \leq M_2 n$. We have

$$3 \leq M_2, \quad by \quad (1)$$

$$\implies 3n \leq M_2 n$$

$$\implies 2n + n \leq M_2 n$$

$$\implies 2n + 1 \leq M_2 n, \quad by \quad (2)$$

$$\implies 1 \leq M_2 n - 2n$$

$$\implies \log_2(1 + 1) \leq M_2 n - 2n$$

$$\implies \log_2(1 + \frac{2}{2^{2n}}) \leq M_2 n + 0 - 2n, \quad by \quad (2)$$

$$\implies \log_2(1 + \frac{2}{2^{2n}}) \leq M_2 n + \log_2(1) - \log_2(2^{2n})$$

$$\implies \log_2((\frac{1}{2^{2n}})(2^{2n} + 2)) \leq M_2 n + \log_2(\frac{1}{2^{2n}})$$

$$\implies \log_2(\frac{1}{2^{2n}}) + \log_2(2^{2n} + 2) \leq M_2 n + \log_2(\frac{1}{2^{2n}})$$

$$\implies \log_2(4^n + 2) \leq M_2 n,$$

as needed.

(5) By (3) and (4),

$$\exists M_1 > 0 \quad \exists M_2 > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (M_1 n \leq \log_2(4^n + 2) \leq M_2 n).$$

Thus, $\log_2(4^n + 2) \in \Theta(n)$.