COMP 2080 Summer 2024 - Assignment 1

- **Name**: Connor Langan
- **Student Number**: 7993941

Question 1

$\forall p \in \mathbb{N} \quad \forall q \in \mathbb{N} \quad p \neq q \ (P(p) \ \wedge \ P(q)) \implies \forall r \in \mathbb{N} \ ((\frac{q^p - 1}{q - 1}) \neq (\frac{p^q - 1}{p - 1}) \times r)$

Question 2

(a)

Proof:

(1) We wish to disprove $\exists a > 1, \exists M > 0, \forall x \in \mathbb{R}(a^x \geq M)$, where $a, M \in \mathbb{R}$. It suffices to prove the negation. Which is:

$$\forall a > 1, \forall M > 0, \exists x \ (a^x < M)$$

(2) Let $a, M \in \mathbb{R}$ be arbitrary, such that $a > 1$ and $M > 0$.

(3) Let $x = \log_a(M - 1)$

(4) Then we have,

$$a^x = a^{\log_a(M-1)} = M - 1 < M$$
$$\implies a^x < M$$

as needed.

(5) Thus, since $a, M \in \mathbb{R}$, were both arbitrary, we conclude that $\forall a > 1, \forall M > 0, \exists x \ (a^x < M)$.

(b)

Proof:

(1) We wish to prove $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}((y - x > 1) \implies (\exists n \in \mathbb{Z}(x < n < y)))$.

(2) Let $x, y \in \mathbb{R}$ be arbitrary, and assume that $y - x > 1$.

(3) Notice that either $x \notin \mathbb{Z}$, or $x \in \mathbb{Z}$. We proceed by cases.

(4) Note that from (2), $y - x > 1 \implies x + 1 < y$

(5) Case 1: If $x \notin \mathbb{Z}$, let $n = \lceil x \rceil$.

Now, $n = \lceil x \rceil > x$, so, $n > x$. And by (4), $n = \lceil x \rceil < x + 1 < y$. So, $n < y$.

Therefore, $x < n < y$, as needed.

(6) Case 2: If $x \in \mathbb{Z}$, let $n = x + 1$.

Now, $n = x + 1 > x$, so, $n > x$. And by (4), $n = x + 1 < y$, so $n < y$.

Therefore, $x < n < y$, as needed.

(7) In either case, since x and y were arbitrary, $x < n < y$, as needed. Thus,

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}((y - x > 1) \implies (\exists n \in \mathbb{Z}(x < n < y)))$$

Question 3

Proof:

(1) Let P(n) be the predicate that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$. We will prove using mathematical induction that $\forall n \in \mathbb{N}, n \geq 2$, P(n).

(2) Base Case: When $n = 2$, we have

$$\sum_{k=2}^{2} \frac{1}{k^2 - 1} = \frac{1}{2^2 - 1} = \frac{1}{3} \quad and \quad \frac{3}{4} - \frac{2(2) + 1}{2(2)(2 + 1)} = \frac{3}{4} - \frac{5}{12} = \frac{9}{12} - \frac{5}{12} = \frac{1}{3}$$

Thus, P(2) is true.

(3) Induction Step: Let $n \in \mathbb{N}$ be arbitrary and assume P(n). We will prove P(n+1), which states

$$\sum_{k=2}^{n+1} \frac{1}{k^2 - 1} = \frac{3}{4} - \frac{2(n + 1) + 1}{2(n + 1)((n + 1) + 1)}$$

(4) We have,

$$\sum_{k=2}^{n+1} \frac{1}{k^2 - 1} = (\sum_{k=2}^{n} \frac{1}{k^2 - 1}) + \frac{1}{(n + 2)^2 - 1}$$

$$= \frac{3}{4} - \frac{2n + 1}{2n(n + 1)} + \frac{1}{(n + 2)^2 - 1} \quad , by \quad I.H$$

$$= \frac{3}{4} - \frac{2n + 1}{2n^2 + 2n} + \frac{1}{n^2 + 2n}$$

$$= \frac{3}{4} - \frac{2n + 1}{2n(n + 1)} + \frac{1}{n(n + 2)}$$

$$= \frac{3}{4} - \frac{(2n + 1)(n + 2)}{2n(n + 1)(n + 2)} + \frac{2(n + 1)}{2n(n + 1)(n + 2)}$$

$$= \frac{3}{4} - \frac{2n^2 + 5n + 2}{2n(n + 1)(n + 2)} + \frac{2(n + 1)}{2n(n + 1)(n + 2)}$$

$$= \frac{3}{4} + \frac{-2n^2 - 5n - 2 + 2n + 2}{2n(n + 1)(n + 2)}$$

$$= \frac{3}{4} + \frac{-2n^2 - 3n}{2n(n+1)(n+2)}$$
$$= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$
$$= \frac{3}{4} - \frac{2(n+1)+1}{2(n+1)((n+1)+2)}$$

as needed.

(5) Thus, since P(2) is true, and P(n) $\implies$ P(n+1), by the principle of mathematical induction, we conclude that $\forall n \geq 2, n \in \mathbb{N}$, P(n).

Question 4

Proof:

(1) Define the sequence of numbers a_n for all positive integers as follows:

$$a_1 = -10, a_2 = 2, and \quad \forall n \geq 3, a_n = a_{n-1} + 12a_{n-2}$$

(2) Let P(n) be the predicate that $a_n = 2(-3)^n - 4^n$. We will prove using strong mathematical induction that $\forall n \in \mathbb{N}$, P(n).

(3) Base Case: When $n = 1$ we have $a_1 = -10$ and $2(-3)^1 - 4^1 = -10$. Thus P(1) is true.

(4) Base Case: When $n = 2$ we have $a_2 = 2$ and $2(-3)^2 - 4^2 = 18 - 16 = 2$. Thus P(2) is true.

(5) Induction Step: Let $n \geq 2$ be arbitrary, and assume that $\forall k \leq n$,

$$a_k = 2(-3)^k - 4^k$$

We will prove P(n+1), which states

$$a_{n+1} = 2(-3)^{n+1} - 4^{n+1}$$

(6) We have,

$$a_{n+1} = a_n + 12a_{n-1}$$
$$a_{n+1} = 2(-3)^n - 4^n + 12(2(-3)^{n-1} - 4^{n-1}) \quad ,by \quad I.H$$
$$a_{n+1} = 2(-3)^n - 4^n + 24(-3)^{n-1} - 12(4)^{n-1}$$
$$a_{n+1} = 2(-3)^n - 4^n - 8(-3)^n - 3(4)^n$$
$$a_{n+1} = -6(-3)^n - 4(4)^n$$
$$a_{n+1} = 2(-3)^{n+1} - 4^{n+1}$$

as needed.

(7) Thus, since P(1) and P(2) are true, and $\forall k \leq n$, P(k) $\implies$ P(n+1), by the principle of strong mathematical induction, we conclude that $\forall n \in \mathbb{N}, a_n = 2(-3)^n - 4^n$.