Question 4

(a)

The probability that bad(a, b) returns the correct answer can be split into two scenarios. Particularly, notice that weightedCoin(a) has a $\frac{a-1}{a}$ probability of returning 0 and a $\frac{1}{a}$ probability of returning 1.

Case 1: When weightedCoin(a) returns 0, the correct answer $a + b$ is returned.

Case 2: When weightedCoin(a) returns 1, random(b) is called, with a $\frac{1}{b}$ probability of returning b, in which case the function would return $a + b$, the correct answer.

So, the overall probability that bad(a, b) returns the correct answer is the probability of case 1 occuring plus the probability that case 2 occurs AND returns the correct answer. Thus, the probability that bad(a, b) returns the correct answer is

$$\frac{a-1}{a} + (\frac{1}{a})(\frac{1}{b})$$

(b)

In case 1, the function can only return $a + b$. In case 2, the function and return $a + 1, a + 2, ..., a + b$, each with a probability of $\frac{1}{b}$. So from this information, and the previously found probability of returing $a + b$, we derive the following probability distrubution:

X	a+1	a+2	...	a + b - 1	a + b
P(X)	$(\frac{1}{a})(\frac{1}{b})$	$(\frac{1}{a})(\frac{1}{b})$	...	$(\frac{1}{a})(\frac{1}{b})$	$\frac{a-1}{a} + (\frac{1}{a})(\frac{1}{b})$

Now, we can calucalte the expected value of X, $E(X)$.

$$E(X) = \sum_{k=1}^{b-1}((a + k)(\frac{1}{a})(\frac{1}{b})) + (a + b)(\frac{a-1}{a} + (\frac{1}{a})(\frac{1}{b}))$$

$$= (\frac{1}{a})(\frac{1}{b})\sum_{k=1}^{b}(a + k) + (a + b)(\frac{a-1}{a})$$

$$= (\frac{1}{a})(\frac{1}{b})(ab + \frac{b(b+1)}{2}) + (a - 1) + (\frac{b(a-1)}{a})$$

$$= 1 + \frac{b+1}{2a} + a - 1 + \frac{2b(a-1)}{2a}$$

$$= \frac{b + 1 + 2ba - 2b}{2a} + a$$

$$= \frac{2ba - b + 1}{2a} + a$$

$$= a + b - \frac{b}{2a} + \frac{1}{2a}$$