Q5

Proof:

(1) Let f and g be arbitrary functions from $\mathbb{Z}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. Suppose that $g \in \omega(f)$. That is,

$$\forall M > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (g(n) \geq Mf(n))$$

We wish to show that $g + f \in \omega(f)$, that is,

$$\forall M\prime > 0 \quad \exists n_0\prime > 0 \quad \forall n > n_0\prime \quad (g(n) + f(n) \geq M\prime f(n)).$$

(2) Let $M\prime > 0$ be arbitrary and let $M = M\prime - 1$.

(3) Let $n_0 > 0$ be so that $\forall n > n_0$, $g(n) \geq Mf(n)$. By (1), such an n_0 exists.

(4) Let $n_0\prime = n_0$. Let $n > n_0\prime$ be arbitrary. Then $n > n_0$, so

$$g(n) + f(n) \geq Mf(n) + f(n), \text{ by } (3)$$

$$= (M\prime - 1)f(n) + f(n), \text{ by } (2)$$

$$= M\prime f(n) - f(n) + f(n)$$

$$= M\prime f(n)$$

as needed.

(5) Thus, $g \in \omega(f) \implies g + f \in \omega(f)$.