Question 5

Find a function $f(n)$ that counts the number of times that $fnc()$ is called as a function with the input n.

(1) Assume $n \geq 3$ and $\exists a \in \mathbb{N}, n = 3^a$.

(2) From the loop invariant, we know that $x = 3^r$.

(3) The outer loop starts at $x = 3$ and is then multiplied by 3 each iteration until it is executed for the last time when $x = n$, since $\exists a \in \mathbb{N}, n = 3^a$ from (1). So, $x = 3, 9, 27, 81, ..., 3^a$. Similarly, r starts at $r = 1$ and increases by one each iteration, and by (2), $x = 3^r \implies r = \log_3 x$, so we get $x = 3^1, 3^2, 3^3, ..., 3^a$, and thus $r = 1, 2, 3, ..., a$. But, from (1), $n = 3^a \implies a = \log_3(n)$. So the number of iterations from the outer loop is

$$\sum_{r=1}^{log_3 n} (1)$$

(4) Now, from (2), since $x = 3^r$, clearly x must be odd.

(5) Next, the inner loop starts at $j = 0$ and increases by 2 until it executes for the last time when $j = x - 1$, since j is even and from (4), x is odd. So, $j = 0, 2, 4, .., x - 1$. But, from (2), we get $j = 0, 2, 4, ..., 3^r - 1$. Now, let $k = j/2$, then $k = 0, 1, 2, 3, ..., (3^r - 1)/2$. Thus, the number of iterations of the inner loop is

$$\sum_{k=0}^{\frac{3^r-1}{2}} (1) = \frac{3^r - 1}{2} - 0 + 1 = \frac{3^r + 1}{2}$$

(6) Finally, since the loops are nested, and $fnc()$ is called once every inner loop iteration, the number of times $fnc()$ is called is

$$\sum_{r=1}^{log_3 n} (\frac{3^r + 1}{2})$$

$$= \frac{1}{2}[\sum_{r=1}^{log_3 n} (3^r) + \sum_{r=1}^{log_3 n} (1)]$$

$$= \frac{1}{2}[\frac{3^{\log_3(n)+1} - 1}{2} - 3^0 + \log_3(n)]$$

$$= \frac{1}{2}[\frac{(3^{\log_3(n)})(3^1) - 1}{2} - 1 + \log_3(n)]$$

$$= \frac{1}{2}[\frac{3n-1}{2} - 1 + \log_3(n)]$$

$$= \frac{1}{2}[\frac{3n-3}{2} + \log_3(n)]$$

(7) Thus, a fucntion $f(n)$ that counts the number of times that $fnc()$ is called as a funciton with the input n is

$$f(n) = \frac{1}{2}[\frac{3n-3}{2} + \log_3(n)]$$