Question 3

$E = 6a - b + 25$

Proof: That E is a loop measure

(1) Let $n \geq 1$ be arbitrary. We claim that $E_{n+1} \leq E_n - 1$.

(2) If the loop condition is being checked for the n+1st time, then it was previously checked for the n-th time and evaluated to true. When line 3 is reached for the n-th time, either $a_n > 0$ or $a_n \leq 0$. Proceed by cases.

(3) Case 1: $a_n > 0$. Then the condition on line 3 evaluates to true, so lines 4 and 5 are executed and line 7 is skipped. In line 4, a_{n+1} is assigned the value $a_n - 4$. In line 5, b_{n+1} is assigned the value $b_n - 20$. So,

$$E_{n+1} = 6a_{n+1} - b_{n+1} + 25$$

$$= 6a_n - 24 - b_n + 20 + 25$$

$$= 6a_n - b_n + 25 - 4$$

$$= E_n - 4 < E_n - 1$$

(4) Case 2: $a_n \leq 0$. Then the condition on line 3 evaluates to false, so lines 4 and 5 are skipped and line 7 is executed. In line 7, b_{n+1} is assigned the value $b_n + 3$ and a_{n+1} is not modified, so $a_{n+1} = a_n$. Now,

$$E_{n+1} = 6a_{n+1} - b_{n+1} + 25$$

$$= 6a_n - b_n - 3 + 25$$

$$= E_n - 3 < E_n - 1$$

(5) By (3) and (4), in all cases, $E_{n+1} \leq E_n - 1$.

(6) Now, suppose that $E \leq 0$. We claim that the loop condition evauluates to false. That is, $a \leq 0 \land b \geq 5$.

(7) By the given loop invariant, $a \geq -3$, and $b \leq 7 \implies -b \geq -7$, which together imply that

$$E = 6a - b + 25 \geq 6(-3) - 7 + 25 = 0$$

By supposition, $E \leq 0$, so $E = 0$. However, this can only occur if $a = -3$ and $b = 7$, so $a \leq 0$ and $b \geq 5$, as needed. Therefore, when $E \leq 0$, the loop evaluates to false.

(8) Thus, by (5) and (7), $E = 6a - b + 25$ is a loop measure.