Question 3

Proof:

(1) Let P(n) be the predicate that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$. We will prove using mathematical induction that $\forall n \in \mathbb{N}, n \geq 2$, P(n).

(2) Base Case: When $n = 2$, we have

$$\sum_{k=2}^{2} \frac{1}{k^2-1} = \frac{1}{2^2-1} = \frac{1}{3} \quad and \quad \frac{3}{4} - \frac{2(2)+1}{2(2)(2+1)} = \frac{3}{4} - \frac{5}{12} = \frac{9}{12} - \frac{5}{12} = \frac{1}{3}$$

Thus, P(2) is true.

(3) Induction Step: Let $n \in \mathbb{N}$ be arbitrary and assume P(n). We will prove P(n+1), which states

$$\sum_{k=2}^{n+1} \frac{1}{k^2-1} = \frac{3}{4} - \frac{2(n+1)+1}{2(n+1)((n+1)+1)}$$

(4) We have,

$$\sum_{k=2}^{n+1} \frac{1}{k^2-1} = (\sum_{k=2}^{n} \frac{1}{k^2-1}) + \frac{1}{(n+1)^2-1}$$
$$= \frac{3}{4} - \frac{2n+1}{2n(n+1)} + \frac{1}{(n+1)^2-1} \quad , by \quad I.H$$
$$= \frac{3}{4} - \frac{2n+1}{2n^2+2n} + \frac{1}{n^2+2n}$$
$$= \frac{3}{4} - \frac{2n+1}{2n(n+1)} + \frac{1}{n(n+2)}$$
$$= \frac{3}{4} - \frac{(2n+1)(n+2)}{2n(n+1)(n+2)} + \frac{2(n+1)}{2n(n+1)(n+2)}$$
$$= \frac{3}{4} - \frac{2n^2+5n+2}{2n(n+1)(n+2)} + \frac{2(n+1)}{2n(n+1)(n+2)}$$
$$= \frac{3}{4} + \frac{-2n^2-5n-2+2n+2}{2n(n+1)(n+2)}$$
$$= \frac{3}{4} + \frac{-2n^2-3n}{2n(n+1)(n+2)}$$
$$= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$
$$= \frac{3}{4} - \frac{2(n+1)+1}{2(n+1)((n+1)+1)}$$

as needed.

(5) Thus, since P(2) is true, and P(n) $\implies$ P(n+1), by the principle of mathematical induction, we conclude that $\forall n \geq 2, n \in \mathbb{N}$, P(n).