Question 2

(a)

Proof:

(1) We wish to disprove $\exists a > 1, \exists M > 0, \forall x \in \mathbb{R}(a^x \geq M)$, where $a, M \in \mathbb{R}$. It suffices to prove the negation. Which is:

$$\forall a > 1, \forall M > 0, \exists x \ (a^x < M)$$

(2) Let $a, M \in \mathbb{R}$ be arbitrary, such that $a > 1$ and $M > 0$.

(3) Let $x = \log_a(M - 1)$

(4) Then we have,

$$a^x = a^{\log_a(M-1)} = M - 1 < M$$
$$\implies a^x < M$$

as needed.

(5) Thus, since $a, M \in \mathbb{R}$, were both arbitrary, we conclude that $\forall a > 1, \forall M > 0, \exists x \ (a^x < M)$.

(b)

Proof:

(1) We wish to prove $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}((y - x > 1) \implies (\exists n \in \mathbb{Z}(x < n < y)))$.

(2) Let $x, y \in \mathbb{R}$ be arbitrary, and assume that $y - x > 1$.

(3) Notice that either $x \notin \mathbb{Z}$, or $x \in \mathbb{Z}$. We proceed by cases.

(4) Note that from (2), $y - x > 1 \implies x + 1 < y$

(5) Case 1: If $x \notin \mathbb{Z}$, let $n = \lceil x \rceil$.

Now, $n = \lceil x \rceil > x$, so, $n > x$. And by (4), $n = \lceil x \rceil < x + 1 < y$. So, $n < y$.

Therefore, $x < n < y$, as needed.

(6) Case 2: If $x \in \mathbb{Z}$, let $n = x + 1$.

Now, $n = x + 1 > x$, so, $n > x$. And by (4), $n = x + 1 < y$, so $n < y$.

Therefore, $x < n < y$, as needed.

(7) In either case, since x and y were arbitrary, $x < n < y$, as needed. Thus,

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}((y - x > 1) \implies (\exists n \in \mathbb{Z}(x < n < y)))$$