Question 2

(a)

(i)

$$T(n) = 4T(n-2) - 2^n$$

$$T(n-2) = 4T(n-4) - 2^{n-2}$$

$$T(n-4) = 4T(n-6) - 2^{n-4}$$

(ii)

$$T(n) = 4T(n-2) - 2^n, \quad j = 1$$

$$= 4[4T(n-4) - 2^{n-2}] - 2^n$$

$$= 4^2T(n-4) - 2^2 \cdot 2^{n-2} - 2^n$$

$$= 4^2T(n-4) - 2 \cdot 2^n, \quad j = 2$$

$$= 4^2[4T(n-6) - 2^{n-4}] - 2 \cdot 2^n$$

$$= 4^3T(n-6) - 2^4 \cdot 2^{n-4} - 2 \cdot 2^n$$

$$= 4^3T(n-6) - 3 \cdot 2^n, \quad j = 3$$

In general, for an arbitrary j, we guess that

$$T(n) = 4^jT(n-2j) - j \cdot 2^n$$

The input to $T(n)$ is valid when $n \geq 0$, by definition. So, we have valid input for $T(n-2j)$ when $n - 2j \geq 0$, or when $j \leq n/2$, since n is even. But j must be at least zero, so the range of j values that apply to the conjecture are $j = [0, n/2], j \in \mathbb{Z}$.

(b)

We reach the lowest base case when the input to $T(n-2j)$ is 0. So, the lowest base case when $n-2j=0$, or when $j=n/2$. So, let's plug this into our conjecture. We have

$$T(n) = 4^j T(n-2j) - j \cdot 2^n$$

$$= (2^2)^{n/2} T(n-2(n/2)) - n/2 \cdot 2^n$$

$$= 2^n T(0) - n/2 \cdot 2^n$$

$$= 2^n(1-n/2)$$

$$= 2^{n-1}(2-n)$$

So, we conjecture that a closed-form solution is $T(n) = 2^{n-1}(2-n), \forall n \geq 0$.

(c)

Proof

(1) Let P(n) be the predicate that $T(n) = 2^{n-1}(2-n)$. We will prove P(n) for all $n \geq 0$.

(2) Base Case: When $n=0$, $T(n)=T(0)=1$ by definition, and $2^{0-1}(2-0) = \frac{1}{2} \cdot 2 = 1$, so P(0) is true.

(3) Base Case: When $n=1$, $T(n)=T(1)=1$ by definition, and $2^{1-1}(2-1) = 1 \cdot 1 = 1$, so P(1) is true.

(4) Induction Step: Let $n \geq 2$ be arbitrary and assume that $\forall k < n$, P(k). We will show that P(n) follows.

(5) We have,

$$T(n) = 4T(n-2) - 2^n, \text{ since } n \geq 2$$

$$= 4 \cdot 2^{n-3}(2-n+2) - 2^n, \text{ by I.H}$$

$$= 2^{n-1}(4-n) - 2^n$$

$$= 2^{n-1}(4-n) - 2 \cdot 2^{n-1}$$

$$= 2^{n-1}(4-n-2)$$

$$= 2^{n-1}(2-n)$$

as needed.

(6) Thus, by strong mathematical induction, $\forall n \geq 0$, P(n) is true, so a closed-form solution is that $T(n) = 2^{n-1}(2-n), \forall n \geq 0$.