Q1

Proof:

(1) Let $M > 0$ and $n_0 > 0$ be arbitrary. Let $n = \max\{\ n_0 + 1\ ,\ \frac{M4^5}{6} + 2,\ 6\}$.

(2) By (1), $n \geq n_0 + 1$, so $n > n_0$.

(3) By (1), $n > 6$, so $n > n - 2 > n - 3 > n - 4 > \cdots > 4 \geq 4$. And, $n - 5 \geq 1$.

(4) By (1), $n \geq \frac{M4^5}{6} + 2$, so $n > \frac{M4^5}{6} + 1$. Then we have

$$n > \frac{M4^5}{6} + 1$$

$$\implies 6n > M4^5 + 6$$

$$\implies 6n - 6 > M4^5$$

$$\implies 6n(n-1) > Mn4^5$$

$$\implies n(n-1)(3)(2)(1) > Mn4^5$$

(5) Now,

$$n! = n(n-1)(n-2)\cdots(5)(4)(3)(2)(1)$$

$$> Mn4^5 \cdot (n-2)(n-3)\cdots(5)(4), \quad \textit{by} \quad \textit{step} \quad (4)$$

$$\geq Mn4^5 \cdot 4 \cdot 4 \cdots 4 \cdot 4, \quad \textit{by} \quad (3),$$

where we match the $n - 5$ factors from $(n - 2)$ to 4, to the $n - 5$ factors of 4's. Note that by (3), $n \geq 6$, so $n - 5 \geq 1$, so there is always at least one factor. Continuing we have

$$= Mn4^5 \cdot 4^{n-5}$$

$$= Mn4^n,$$

as needed.

(6) Therefore, $\forall M > 0 \quad \forall n_0 > 0 \quad \exists n > n_0 \quad (n! > Mn4^n)$. Thus, by contraposition, $n! \notin O(n(4^n))$.