Q4

Proof:

(1) Let $M > 0$ be arbitrary. Let $n_0 = \max\{\frac{4}{M}, 2\}$. Let $n > n_0$ be arbitrary.

(2) By (1), $n > 2$, so $2n - 2 > 2n - 4 > \cdots > 6 > 4 \geq 4$.

(3) By (1) $n > \frac{4}{M}$. We will show that $4^n \leq M(2n)!!$.

(4) We have

$$4 < Mn, \quad by \quad (3)$$

$$\implies 4 \cdot 4 < M4n$$

$$\implies 4 \cdot 4 \cdot 4 \cdots 4 \leq M(2n)(2n-2)\cdots 6 \cdot 4 \cdot 2, \quad by \quad (2),$$

where the $n-2$ factors from $2n-2$ to 4 can be replaced by the $n-2$ factors of 4's. Note that by (2) $n > 2$, so $n \geq 3 \implies n - 2 \geq 1$, so there is always at least one factor. Continuing we have

$$\implies 4^2 \cdot 4^{n-2} \leq M(2n)!!$$

$$\implies 4^n \leq M(2n)!!,$$

as needed.

(5) Therefore, $\forall M > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (4^n \leq M(2n)!!)$. Thus, $4^n \in o((2n)!!)$.