Department of Computer Science
University of Manitoba

COMP 2080 Summer 2024 – Assignment 4

This assignment is due by 11:59pm on **Monday June 3**.

Your assignment will be submitted in Crowdmark and *not* in UM Learn. A Crowdmark invitation will be sent to your U of M email.

Proofs will be graded on both correctness and presentation. Clearly explain all of your steps.

The total number of points is 25.

Questions

1. This question refers to the following recurrence relation:

$$T(n) = \begin{cases} 1 & \text{, if } n = 0 \\ nT(n-1) + n! & \text{, if } n \geq 1. \end{cases}$$

   (a) **[2 marks]** Assume that n is much larger than 1. Use the recurrence relation to write $T(n)$ in terms of $T(n-1)$, then in terms of $T(n-2)$, then in terms of $T(n-3)$.

   (b) **[2 marks]** Conjecture an expression for $T(n)$ in terms of $T(n-j)$, for an arbitrary j. For what range of j values does your conjecture apply?

   You might find it convenient to use product notation:

$$\prod_{k=1}^{n} f(k) = f(1) \cdot f(2) \cdot \cdots \cdot f(n)$$

   A potentially useful fact: $\displaystyle\prod_{k=1}^{n} k = n!$.

   (c) **[3 marks]** Using your answer to part (b), conjecture a closed-form solution for $T(n)$ as a function of n. Prove that your conjecture is correct using mathematical induction.

2. This question refers to the following recurrence relation:

$$T(n) = \begin{cases} 1 & \text{, if } n = 0 \\ 1 & \text{, if } n = 1 \\ 4T(n-2) - 2^n & \text{, if } n \geq 2. \end{cases}$$

(a) For part (a), consider the case where n is even **only**. You do not need to repeat any of the work for the case where n is odd.

(i) [**2 marks**] Assume that n is much larger than 1. Use the recurrence relation to write $T(n)$ in terms of $T(n-2)$, then in terms of $T(n-4)$, then in terms of $T(n-6)$.

(ii) [**2 marks**] Conjecture an expression for $T(n)$ in terms of $T(n-2j)$, for an arbitrary value of j. For what range of j values does your conjecture apply?

(b) [**1 mark**] Using your answer to part (a), conjecture a closed-form solution for $T(n)$ as a function of n. Your answer should not contain any summation notation (i.e., you should evaluate any summation notation that is present).

(c) [**2 marks**] The closed-form solution to this recurrence relation is

$$T(n) = 2^{n-1}(2-n), \ \forall n \geq 0.$$

Prove that this solution is correct using strong mathematical induction. Note that this formula holds for both even and odd n. A proof by cases is not necessary (and you will lose marks for using one).

3. This question refers to the following code. Assume that the `/` symbol represents integer division: $a/b = \lfloor \frac{a}{b} \rfloor$. Let `A` be an integer array.

```
// pre: (A.length >= 1) ∧ (0 <= j,k < A.length)
void recursiveFnc(int[] A, int j, int k)
    if (j < k-6)
        localFnc(A, j, k)
        recursiveFnc(A, (3*j+k+1)/4, (j+k+1)/2 - 1)
        recursiveFnc(A, (j+3*k+3)/4, k)
```

Let $n = k - j + 1$. Assume that `localFnc(A, j, k)` performs some operation on the entries $A[j], \ldots, A[k]$ inclusive and that, if this function acts on n entries, its cost is $n^{1/2}$.

(a) **[3 marks]** Let $T(n)$ be the cost of `recursiveFnc()` as a function of the input size n. Find the recurrence relation that defines $T(n)$.

Use the simplified conventions for counting steps that are described and used in the materials on recurrence relations. Namely, a fixed number of steps that does not depend on the size of n can be approximated by 1 and a local cost function $f(n)$ can be approximated by its largest term.

You will need to use floor and/or ceiling functions. The only variable should be n: the indices j and k should not appear. Once you have eliminated j and k in favour of n, you do not need to simplify your expressions any further.

For the remainder of question 3, assume that $n = 4^r$ for some integer $r \geq 0$.

(b) **[3 marks]** In the special case where $n = 4^r$ for some integer $r \geq 0$, show that your recurrence relation from part (a) can be simplified to the following form:

$$T(n) = \begin{cases} c & \text{, if } n \leq d \\ aT(\frac{n}{b}) + f(n) & \text{, if } n > d. \end{cases}$$

Hint: in the recursive case, is it possible that $r = 0$?

(c) **[3 marks]** Draw the recursion tree for your recurrence relation from part (b). Include at least three rows. Calculate the sum of each row. Determine the total number of rows, and calculate the sum of all the entries in the tree. Use that result to conjecture a simple function $g(n)$ so that $T(n) \in \Theta(g(n))$. You do not need to prove your conjecture.

(d) **[2 marks]** Explain which case of the Master Theorem applies to the recurrence relation from part (b). Use the Master Theorem to find a function $g(n)$ such that $T(n) \in \Theta(g(n))$, and confirm that you get the same answer as in part (c).