Q1

Proof:

(1) Let $M > 0$ and $n_0 > 0$ be arbitrary. Let $n = \max\{\ n_0 + 1\ ,\ \frac{M4^5}{6} + 2,\ 6\}$.

(2) By (1), $n \geq n_0 + 1$, so $n > n_0$.

(3) By (1), $n \geq 6$, so $n > n-2 > n-3 > n-4 > \cdots > 5 > 4 \geq 4$. And, $n - 5 \geq 1$.

(4) By (1), $n \geq \frac{M4^5}{6} + 2$, so $n > \frac{M4^5}{6} + 1$. Then we have

$$n > \frac{M4^5}{6} + 1$$

$$\implies 6n > M4^5 + 6$$

$$\implies 6n - 6 > M4^5$$

$$\implies 6n(n-1) > Mn4^5$$

$$\implies n(n-1)(3)(2)(1) > Mn4^5$$

(5) Now,

$$n! = n(n-1)(n-2)\cdots(5)(4)(3)(2)(1)$$

$$> Mn4^5 \cdot (n-2)(n-3)\cdots(5)(4), \quad by \quad step \quad (4)$$

$$\geq Mn4^5 \cdot 4 \cdot 4 \cdots 4 \cdot 4, \quad by \quad (3),$$

where we match the $n-5$ factors from $(n-2)$ to 4, to the $n-5$ factors of 4's. Note that by (3), $n \geq 6$, so $n - 5 \geq 1$, so there is always at least one factor. Continuing we have

$$= Mn4^5 \cdot 4^{n-5}$$

$$= Mn4^n,$$

as needed.

(6) Therefore, $\forall M > 0 \quad \forall n_0 > 0 \quad \exists n > n_0 \quad (n! > Mn4^n)$. Thus, by contraposition, $n! \notin O(n(4^n))$.

Q2

Proof:

(1) Let $M = 1$, $n_0 = 2$, and $n > n_0$ be arbitrary.

(2) By (1), $n > 2$, so $\log_2 n > 1 \implies \frac{1}{\log_2 n} < 1$.

(3) By (1), $n > 2$, so $n > 1$.

(4) Now,

$$1 \geq M, \quad by \quad (1)$$

$$\implies (\log_2 n)^2(2-1) \geq M(\log_2 n)^2$$

$$\implies (\log_2 n)^2(2 - \frac{1}{\log_2 n}) \geq M(\log_2 n)^2, \quad \text{by (2)}$$

$$\implies 2(\log_2 n)^2 - \log_2 n \geq M(\log_2 n)^2$$

$$\implies 2(\log_2 n)^3 - \log_2 n \geq M(\log_2 n)^2, \quad \text{by (3)}$$

$$\implies 2(\log_2 n)^3 - \log_2 n + 2024 \geq M(\log_2 n)^2,$$

as needed.

(4) Therefore,

$$\exists M > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (2(\log_2 n)^3 - \log_2 n + 2024 \geq M(\log_2 n)^2).$$

Thus, $2(\log_2 n)^3 - \log_2 n + 2024 \in \Omega((\log_2 n)^2)$.

Q3

Proof:

(1) Let $M_1 = 2, M_2 = 3, n_0 = 1$, and $n > n_0$ be arbitrary.

(2) By (1), $n > 1$, so $2n > 1 \implies 2^{2n} > 2^1 \implies \frac{2}{2^{2n}} < 1$.

(3) First, we will show that $M_1 n \leq \log_2(4^n + 2)$. We have

$$M_1 \leq 2, \quad \text{by (1)}$$

$$\implies M_1 n \leq 2n$$

$$\implies M_1 n \leq \log_2(2^{2n})$$

$$\implies M_1 n \leq \log_2(2^{2n} + 2)$$

$$\implies M_1 n \leq \log_2(4^n + 2),$$

as needed.

(4) Next, we will show that $\log_2(4^n + 2) \leq M_2 n$. We have

$$3 \leq M_2, \quad by \quad (1)$$

$$\implies 3n \leq M_2 n$$

$$\implies 2n + n \leq M_2 n$$

$$\implies 2n + 1 \leq M_2 n, \quad by \quad (2)$$

$$\implies 1 \leq M_2 n - 2n$$

$$\implies \log_2(1 + 1) \leq M_2 n - 2n$$

$$\implies \log_2(1 + \frac{2}{2^{2n}}) \leq M_2 n + 0 - 2n, \quad by \quad (2)$$

$$\implies \log_2(1+\frac{2}{2^{2n}}) \le M_2 n + \log_2(1) - \log_2(2^{2n})$$

$$\implies \log_2((\frac{1}{2^{2n}})(2^{2n}+2)) \le M_2 n + \log_2(\frac{1}{2^{2n}})$$

$$\implies \log_2(\frac{1}{2^{2n}}) + \log_2(2^{2n}+2) \le M_2 n + \log_2(\frac{1}{2^{2n}})$$

$$\implies \log_2(4^n+2) \le M_2 n,$$

as needed.

(5) By (3) and (4),

$$\exists M_1 > 0 \quad \exists M_2 > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (M_1 n \le \log_2(4^n+2) \le M_2 n).$$

Thus, $\log_2(4^n+2) \in \Theta(n)$.

Q4

Proof:

(1) Let $M > 0$ be arbitrary. Let $n_0 = \max\{\ \frac{4}{M}\ , 2\}$. Let $n > n_0$ be arbitrary.

(2) By (1), $n > 2$, so $2n-2 > 2n-4 > \cdots > 6 > 4 \ge 4$.

(3) By (1) $n > \frac{4}{M}$. We will show that $4^n \le M(2n)!!$.

(4) We have

$$4 < Mn, \quad by \quad (3)$$

$$\implies 4 \cdot 4 < M4n$$

$$\implies 4 \cdot 4 \cdot 4 \cdots 4 \le M(2n)(2n-2)\cdots 6 \cdot 4 \cdot 2, \quad by \quad (2),$$

where the $n-2$ factors from $2n-2$ to 4 can be replaced by the $n-2$ factors of 4's. Note that by (2) $n > 2$, so $n \ge 3 \implies n-2 \ge 1$, so there is always at least one factor. Continuing we have

$$\implies 4^2 \cdot 4^{n-2} \le M(2n)!!$$

$$\implies 4^n \leq M(2n)!!,$$

as needed.

(5) Therefore, $\forall M > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (4^n \leq M(2n)!!)$. Thus, $4^n \in o((2n)!!)$.

Q5

Proof:

(1) Let f and g be arbitrary functions from $\mathbb{Z}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. Suppose that $g \in \omega(f)$. That is,

$$\forall M > 0 \quad \exists n_0 > 0 \quad \forall n > n_0 \quad (g(n) \geq Mf(n))$$

We wish to show that $g + f \in \omega(f)$, that is,

$$\forall M\prime > 0 \quad \exists n_0\prime > 0 \quad \forall n > n_0\prime \quad (g(n) + f(n) \geq M\prime f(n)).$$

(2) Let $M\prime > 0$ be arbitrary and let $M = M\prime - 1$.

(3) Let $n_0 > 0$ be so that $\forall n > n_0$, $g(n) \geq Mf(n)$. By (1), such an n_0 exists.

(4) Let $n_0\prime = n_0$. Let $n > n_0\prime$ be arbitrary. Then $n > n_0$, so

$$g(n) + f(n) \geq Mf(n) + f(n), \text{ by (3)}$$

$$= (M\prime - 1)f(n) + f(n), \text{ by (2)}$$

$$= M\prime f(n) - f(n) + f(n)$$

$$= M\prime f(n)$$

as needed.

(5) Thus, $g \in \omega(f) \implies g + f \in \omega(f)$.