Question 1

(a)

$$T(n) = nT(n-1) + n!$$

$$T(n-1) = (n-1)T(n-2) + (n-1)!$$

$$T(n-2) = (n-2)T(n-3) + (n-2)!$$

(b)

$$T(n) = nT(n-1) + n!, \quad j = 1$$

$$= n[(n-1)T(n-2) + (n-1)!] + n!$$

$$= n(n-1)T(n-2) + 2n!, \quad j = 2$$

$$= n(n-1)[(n-2)T(n-3) + (n-2)!] + 2n!$$

$$= n(n-1)(n-2)T(n-3) + 3n!, \quad j = 3$$

In general, for an arbitrary j, we guess that

$$T(n) = \frac{n!}{(n-j)!}T(n-j) + j \cdot n!$$

The input to $T(n)$ is valid when $n \geq 0$ by definition. So, we have valid input for $T(n-j)$ when $n - j \geq 0$, or when $j \leq n$. But, j must be at least zero, so the range of j value that apply to the conjecture are $j = [0, n], j \in \mathbb{Z}$.

(c)

We reach a base case when the input to $T(n-j)$ is 0. So, a base case when $n - j = 0$, or when $j = n$. So, let's plug this into our conjecture to find a closed-form solution. We have

$$T(n) = \frac{n!}{(n-j)!}T(n-j) + n \cdot n!$$

$$= \frac{n!}{(n-n)!}T(n-n) + n \cdot n!$$

$$= \frac{n!}{0!}T(0) + n \cdot n!$$

$$= n! + n \cdot n!$$

$$= n!(n+1)$$

$$= (n+1)!$$

So, we conjecture that a closed-form solution is $T(n) = (n+1)!, \forall n \geq 0$. Let's prove it.

Proof

(1) Let P(n) be the predicate that $T(n) = (n+1)!$. We will show that P(n) is true for all $n \geq 0$.

(2) Base Case: When $n = 0$, $T(0) = 1$ by definition, and $(0+1)! = 1$, so P(0) is true.

(3) Induction Step: Let $n \geq 1$ be arbitrary and assume that P(n-1) is true.

(4) By definition, we have that

$$T(n) = nT(n-1) + (n-1)!, \text{ since } n \geq 1$$

$$= n \cdot n! + n!, \text{ by I.H}$$

$$= n!(n+1)$$

$$= (n+1)!,$$

as needed.

(5) Thus, by mathematical induciton, $\forall n \geq 0$, P(n) is true, so a closed-form solution is that $T(n) = (n+1)!, \forall n \geq 0$.