Question 4

Proof:

(1) Define the sequence of numbers a_n for all positive integers as follows:

$$a_1 = -10, a_2 = 2, and \quad \forall n \geq 3, a_n = a_{n-1} + 12a_{n-2}$$

(2) Let P(n) be the predicate that $a_n = 2(-3)^n - 4^n$. We will prove using strong mathematical induction that $\forall n \in \mathbb{N}$, P(n).

(3) Base Case: When $n = 1$ we have $a_1 = -10$ and $2(-3)^1 - 4^1 = -10$. Thus P(1) is true.

(4) Base Case: When $n = 2$ we have $a_2 = 2$ and $2(-3)^2 - 4^2 = 18 - 16 = 2$. Thus P(2) is true.

(5) Induction Step: Let $n \geq 2$ be arbitrary, and assume that $\forall k \leq n$,

$$a_k = 2(-3)^k - 4^k$$

We will prove P(n+1), which states

$$a_{n+1} = 2(-3)^{n+1} - 4^{n+1}$$

(6) We have,

$$a_{n+1} = a_n + 12a_{n-1}$$
$$a_{n+1} = 2(-3)^n - 4^n + 12(2(-3)^{n-1} - 4^{n-1}) \quad , by \quad I.H$$
$$a_{n+1} = 2(-3)^n - 4^n + 24(-3)^{n-1} - 12(4)^{n-1}$$
$$a_{n+1} = 2(-3)^n - 4^n - 8(-3)^n - 3(4)^n$$
$$a_{n+1} = -6(-3)^n - 4(4)^n$$
$$a_{n+1} = 2(-3)^{n+1} - 4^{n+1}$$

as needed.

(7) Thus, since P(1) and P(2) are true, and $\forall k \leq n$, P(k) $\implies$ P(n+1), by the principle of strong mathematical induction, we conclude that $\forall n \in \mathbb{N}, a_n = 2(-3)^n - 4^n$.