Question 4

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1. //Pre: n >= 1
2. x <-- 2*n
3. sum <-- 0
4. //LoopInv: x is even
5. while(x > 0)
6.      y <-- 0
7.      while(y < x)
8.          sum <-- sum + 3*y
9.          y <-- y + 1
10.     x <-- x - 2
```

(a) Find $f(n)$, the number of steps that this program executes as a function of n.

For the outer loop, x starts at $2n$ and decreases by 2 until it it executes its final iteration when $x = 2$, since x is even from the loop invariant. So, $x = 2n, 2n-2, 2n-4, ..., 4, 2$. Let $k = x/2$, then k starts at n and decreases by 1 until $k = 1$. Since addition is commutative, the sum from n decreasing until 1 is equal to the sum from 1 increasing until n.

For the inner loop, y starts at 0 and increases by 1 until it it executes its final iteration when $y = x - 1 = 2k - 1$ since $x = 2k$. So, $y = 1, 2, 3, ..., 2k-2, 2k-1$. So, the number of steps exectuted by the inner loop is

$$\sum_{y=0}^{2k-1}(3) + 1 = 6k + 1$$

, since lines 7, 8, and 9 are executed each iteration, and then we add 1 for the final time the loop condition is checked and evaluates to false.

Then, the number of steps executed by the entire program is

$$\sum_{k=1}^{n}(3 + 6k + 1) + 1 + 2$$

Since lines 5, 6, and 10 are executed once per outer loop iteration, the inner loop is executed once per outer loop iteration, and then add the final check of the loop condition when it evaluates to false as well as when lines 2 and 3 are executed at the beginning of the program. We can further simplify this to

$$= 6\sum_{k=1}^{n}(k) + \sum_{k=1}^{n}(4) + 3$$

$$= \frac{6n(n+1)}{2} + 4n + 3$$

$$= \frac{6n^2 + 6n + 8n + 6}{2}$$

$$= 3n^2 + 7n + 3$$

Thus, a function $f(n)$ that calculates the number of steps that this program executes as a function of n is

$$f(n) = 3n^2 + 7n + 3$$

(b) Calculate the value of the variable *sum* when the program terminates as a function of the input n.

We can utilise our previous formulas and modify them to count the value of the variable *sum* instead. To start, the inner loop is the only time the value sum is modiifed, where $sum = sum + 3y$. So, we can then adjust our inner loop in the formula and keep the values for the summations the same. This gives us

$$\sum_{k=1}^{n}(\sum_{y=0}^{2k-1}(3y))$$

$$= 3\sum_{k=1}^{n}(\sum_{y=0}^{2k-1}(y)$$

$$= 3\sum_{k=1}^{n}(\frac{(2k-1)(2k)}{2})$$

$$= 3\sum_{k=1}^{n}(2k^2 - k)$$

$$= 3[\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}]$$

$$= \frac{2n(n+1)(2n+1) - 3n(n+1)}{2}$$

$$= \frac{2n(n+1)(2n+1) - 3n(n+1)}{2}$$

$$= \frac{(2n^2 + 2n)(2n + 1) - 3n^2 - 3n}{2}$$

$$= \frac{4n^3 + 4n^2 + 2n^2 + 2n - 3n^2 - 3n}{2}$$

$$= \frac{4n^3 + 3n^2 - n}{2}$$

Thus, a function $g(n)$ that calculates the value of the variable *sum* as a function of the input n is

$$g(n) = \frac{4n^3 + 3n^2 - n}{2}$$