# The Standard Map

## Dynamical Systems MATH A4500

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# Definitions and Characteristics Overview

The standard map, $S$, is an area-preserving map defined on the torus where $S : \mathbb{R}^2/2\pi\mathbb{Z} \hookleftarrow$ is given by

$$f \equiv p_{n+1} = p_n + K \sin\theta_n$$

$$g \equiv \theta_{n+1} = \theta_n + p_{n+1}$$

and parameter $K \in \mathbb{R}^+$. The value of $k$ determines the magnitude of nonlinearity featured in the system. The map $S$ is an example of a Hamiltonian conservative system, in that it neither contracts nor expands by iteration. This property is shown by computing the determinant of its Jacobian matrix:

$$det\ J = \begin{vmatrix} \frac{\partial x}{\partial\theta_n} & \frac{\partial x}{\partial p_n} \\ \frac{\partial y}{\partial\theta_n} & \frac{\partial y}{\partial p_n} \end{vmatrix} = 1$$

# Symmetries of The Standard Map

For $K > 0$, the symmetries associated to the standard map help determine the dynamics of $S$.
For example, by observing that $S$ is periodic, this implies its projection has a discrete translation symmetry, $S \circ T_{m,0} = T_{m,0} \circ S$, for $T_{m,n}(x, y) = (x + m, y + n)$, the translation by an integer vector $(m, n)$.
Similarly, $S$ also has a discrete vertical translation symmetry. These symmetries allow $S$ to be thought of acting on the torus. Consequently, symmetric orbits are invariant under these conditions.

# Periodic and Quasiperiodic Orbits

The two difference equations of $S$ describe the orbit's behavior for $(\theta_n, p_n)$ over the two-dimensional phase-space, and is uniquely determined by the initial condition $(\theta_0, p_0)$. By setting $K = 0$, we see that $\theta_n = \theta_0$. Then $p_n$ is a linear equation and $S$ is integrable. Here, orbits behave periodically or quasiperiodically depending on their initial conditions, since if $p_0 = \frac{m}{n}$ for some $m, n$ integers, then the orbit has period $n$.
Otherwise, for irrational $p_0$, the orbit will be quasiperiodic and dense in the torus. Therefore, these orbits must lie on the invariant torus.

# Approaching Chaos

The range of map behavior is demonstrated in the following plots, which trace the evolution of a number of trajectories for three different $K = \{0, 1, 1.9\}$. Generally speaking, circles correspond to regular orbits and absence of structure indicates chaos. These graphs show an important feature of Hamiltonian chaos - the coexistence of regions of regular and chaotic motion. (Diakonova)

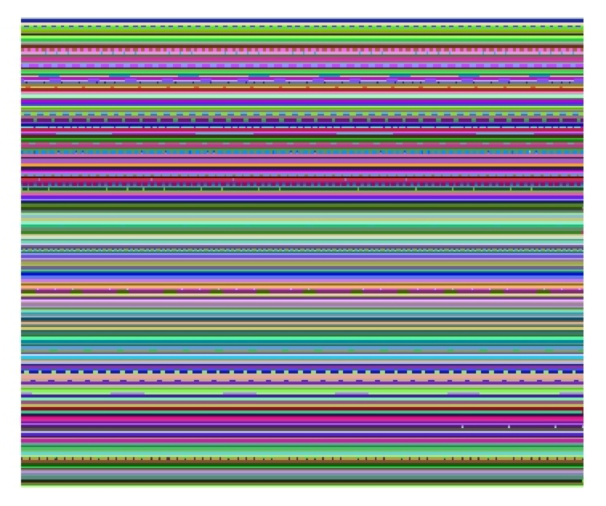

Figure: $K = 0$, 1000 random points on the square $[0, 2\pi] \times [0, 2\pi]$ and uses each point as the starting point to draw 1000 iterations of the standard map

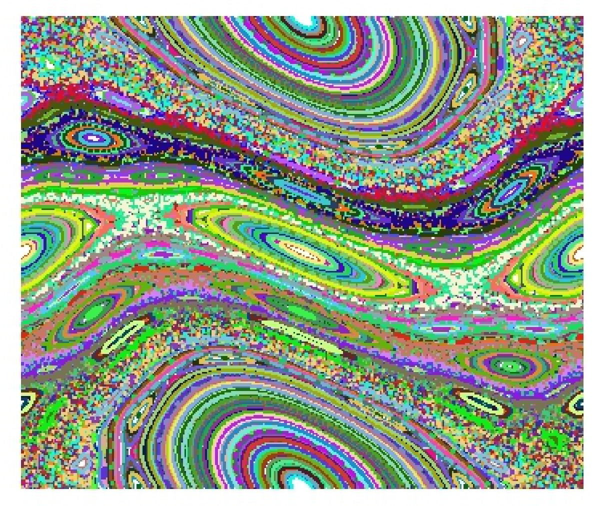

Figure: $K = 1$, 1000 random points on the square $[0, 2\pi] \times [0, 2\pi]$ and uses each point as the starting point to draw 1000 iterations of the standard map

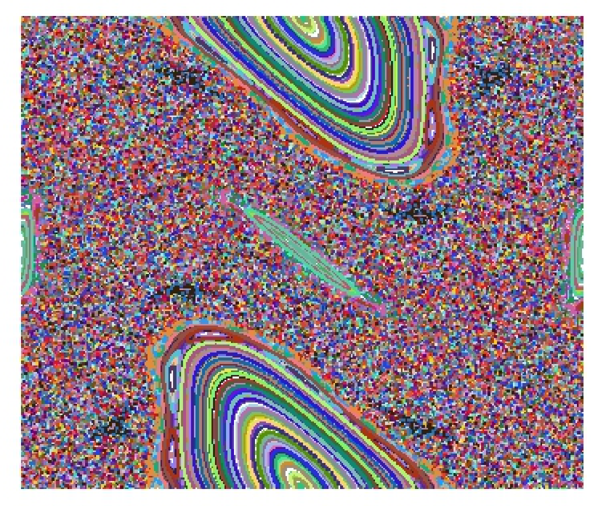

Figure: $K = 1.9$, 1000 random points on the square $[0, 2\pi] \times [0, 2\pi]$ and uses each point as the starting point to draw 1000 iterations of the standard map

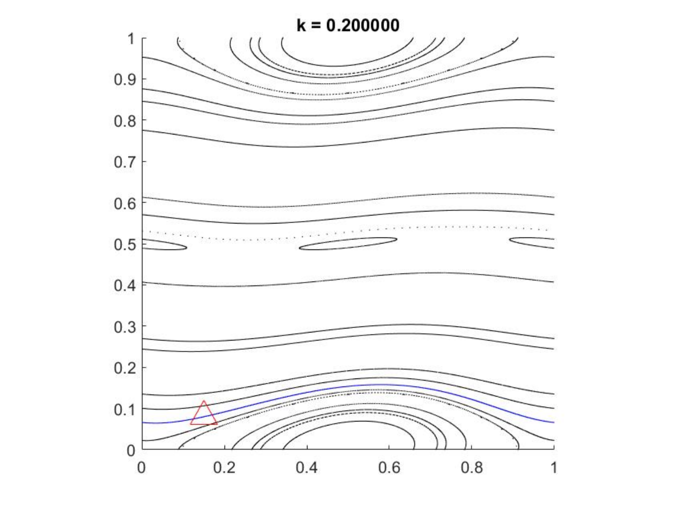


Figure: $K = .02$, surfaces of section for standard map and trajectories with m orbits and randomized initial conditions. The red triangle marker locates the stable manifold

# Final Observations

As K increases, we know that the nonlinearity of the system increases. The size of the islands will likewise increase. We can see the region where invariant tori exist is engulfed as $K$ increases.

However, not all of the orbits are part of the invariant torus. We see that there are ellipses encircling the period-1 fixed point at $(0.5, 0)$ for $K = 0.2$. These circles are known as islands. The circles dont rotate the torus and are bounded by a separatrix curve. The separatrix is defined as the boundary which separates two modes of behavior in a dynamic system. (Tobin)

An interesting feature of the standard map is the renormalization behavior it displays over time. Return maps show that points near periodic regions structurally resemble the cycles of larger islands.

# References


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