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ECMA 31340
PSET 3

Problem 1

$$Y_{i1} = X_i + \epsilon_{i1} \qquad E(X) = \mu, \ Var(X) = \sigma_x^2, \ X \perp \epsilon_1 \perp \epsilon_2$$
$$Y_{i2} = X_i + \epsilon_{i1} \qquad \mathbb{E}(\epsilon_1) = \mathbb{E}(\epsilon_2) = 0, \ Var(\epsilon_1) = \sigma_1^2, \ Var(\epsilon_2) = \sigma_2^2$$

$$\mu_n = \alpha \frac{1}{n} \sum_{i=1}^n Y_{1i} + (1-\alpha) \frac{1}{n} \sum_{i=1}^n Y_{2i}$$

1) $$\mathbb{E}[\mu_n] = \alpha \, \mathbb{E}[\tfrac{1}{n} \textstyle\sum_{i=1}^n (X_i + \epsilon_{i1})] + (1-\alpha) \tfrac{1}{n} \mathbb{E}[\textstyle\sum_{i=1}^n (X_i + \epsilon_{i2})]$$

$$= \alpha \, \mathbb{E}[X_i] + (1-\alpha) \mathbb{E}[X_i]$$

$$\boxed{\mu_n = \mathbb{E}[X_i]}$$

2) $$\lim_{n\to\infty} \mu = \lim_{n\to\infty} \alpha \frac{1}{n} \sum_{i=1}^n (X_i + \epsilon_{i1}) + \lim_{n\to\infty} (1-\alpha) \frac{1}{n} \sum_{i=1}^n (X_i + \epsilon_{i2})$$

By law of large numbers as $n \to \infty$ the sample average of $\frac{1}{n} \sum_{i=1}^n (X_i + \epsilon_{i1})$ converges to $\mathbb{E}[X]$ and with rearrangement μ_n is consistent.

5) No, without independence we cannot find unbiasdness.

Problem 1

3) $\begin{bmatrix} Y_{i1} \\ Y_{i2} \end{bmatrix} \sim \left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho_1 \\ \rho_1 & \sigma_2^2 \end{pmatrix} \right)$ $\Omega = \begin{bmatrix} \mathbb{E}[Y_{i1}^2] & \mathbb{E}[Y_{i1} \cdot Y_{i2}] \\ \mathbb{E}[Y_{i1} \cdot Y_{i2}] & \mathbb{E}[Y_{i2}^2] \end{bmatrix}$

$$\begin{bmatrix} X_i \\ \epsilon_{i1} \\ \epsilon_{i2} \end{bmatrix} \sim \left(\begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{pmatrix} \right)$$

$\rho_1 = \mathbb{E}[Y_{i1} \cdot Y_{i2}]$

$\mathbb{E}[(X_i + \epsilon_{i1}) \cdot (X_i + \epsilon_{i2})]$

$(1-\alpha)(1-\alpha)$

$1 - \alpha - \alpha + \alpha^2$

$\mathbb{E}[X_i^2 + X\epsilon_{i1} + X\epsilon_{i2} + \epsilon_{i1}\epsilon_{i2}]$

$\mathbb{E}[X_i^2] = \sigma_x^2 = Cov(Y_{i1}, Y_{i2})$

$$Var(X) = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2(\alpha)(1-\alpha)\sigma_x^2$$

$$\alpha^2\sigma_1^2 + (1 - 2\alpha + \alpha^2)\sigma_2^2 + 2(\alpha - \alpha^2)\sigma_x^2$$

$$\alpha^2\sigma_1^2 + \sigma_2^2 - 2\alpha\sigma_2^2 + \alpha^2\sigma_2^2 + 2\alpha\sigma_x^2 - 2\alpha^2\sigma_x^2$$

$$Var(X) = \alpha^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_x^2) + 2\alpha(\sigma_x^2 - \sigma_2^2) + \sigma_2^2$$

4) $0 = 2\alpha(\sigma_1^2 + \sigma_2^2 - 2\sigma_x^2) + 2\sigma_x^2 - 2\sigma_2^2$

$$\alpha = \frac{\sigma_2^2 - \sigma_x^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_x^2}$$

The larger the variance of X the more that both Y's will be used evenly.

Problem 2

1) $\begin{bmatrix} Y_{i1} \\ Y_{i2} \end{bmatrix} \sim \left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho_2 \\ \rho_1 & \sigma_2^2 \end{pmatrix} \right) \quad \Omega = \begin{bmatrix} \mathbb{E}[Y_{i1}^2] & \mathbb{E}[Y_{i1} \cdot Y_{i2}] \\ \rho_1 & \mathbb{E}[Y_{i2}] \end{bmatrix}$

$$\rho_1 = \mathbb{E}[Y_{i2} \cdot Y_{i1}]$$
$$\mathbb{E}[(X + \epsilon_{i1})(X + \epsilon_{i2})]$$
$$\mathbb{E}[X^2 + X\epsilon_{i1} + X\epsilon_{i2} + \epsilon_{i1}\epsilon_{i2}]$$

$$\rho_1 = \mathbb{E}[X^2] = \sigma_X^2 = 0 = Cov(Y_{i1}, Y_{i2})$$

$$\boxed{Y_1 \perp Y_2}$$

2) Since Y_1 and Y_2 are normal and independant they use the standard bivariat normal dist function

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi} exp\left\{-\frac{1}{2}[Y_1^2 + Y_2^2]\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^n (y_1-\mu)^2}{\sigma_1^2} + \frac{\sum_{i=1}^n (y_2-\mu)^2}{\sigma_2^2}\right]\right\}$$

$\log(f_{Y_1Y_2}(y_1, y_2 | \sigma_1, \sigma_2, \mu)$

$$= \boxed{\log\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) - \frac{1}{2}\left[\frac{\sum_{i=1}^n (y_1-\mu)^2}{\sigma_1^2} + \frac{\sum_{i=1}^n (y_2-\mu)^2}{\sigma_2^2}\right]}$$

3) Take derivative with respect to u; set equal to zer

$$\frac{dL}{du} = -\frac{\sum_{i=1}^n (y_1-\mu)^2}{2\sigma_1^2} - \frac{\sum_{i=1}^n (y_2-\mu)^2}{2\sigma_2^2} = 0$$

Problem 2

3) $$\frac{-\sum_{i=1}^{n}(Y_{i1}-\mu)^2}{\sigma_1^2} - \frac{\sum_{i=1}^{n}(Y_{i2}-\mu)^2}{\sigma_2^2} = 0$$

$$-\sigma_2^2 \sum_{i=1}^{n}(Y_{i1}-\mu)^2 - \sigma_1^2 \sum_{i=1}^{n}(Y_{i2}-\mu)^2 = 0$$

$$\frac{-\sigma_2^2}{\sigma_1^2} = \frac{\sum_{i=1}^{n}(Y_{i2}-\mu)^2}{\sum_{i=1}^{n}(Y_{i1}-\mu)^2}$$

Problem 3

1) For uniform distribution

$$\mathbb{E}(e^{t}) = \frac{e^{tb} - e^{ta}}{t(b-a)} \quad \text{for } t = 2,\ b = 1,\ a = 0$$

$$\mathbb{E}[U^{2}] = \frac{U^{2} - U^{0}}{2} = \frac{U^{2}}{2} - \frac{1}{2}$$

$\boxed{\text{False}}$

2) False, because of the intrinsic randomness of the sample the $\hat{\theta}$ estimate will never quite be unbiased.

3) $\mathbb{E}[X_i] = 1 \cdot p + (1-p)p + 3(1-p)^2 p + (1-p)^3 p + \dots$

$$\mathbb{E}[X_i] = p(1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots)$$

$$(1-p)\,\mathbb{E}[X_i] = p((1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 \dots)$$

$$\mathbb{E}[X_i] - p\,\mathbb{E}[X_i] = p((1-p) + (1-p)^2 \dots)$$

$$p\,\mathbb{E}[X_i] = p(1 + (1-p) + (1-p)^2 + (1-p)^3 \dots)$$

$$\mathbb{E}[X_i] = (1 + (1-p) + (1-p)^2 + (1-p)^3 \dots)$$

By series sum formula $\boxed{\text{True}}$