Solution

ME 608 Fluid Dynamics
Mid-Term Exam II — 29 April 2016

Closed books, closed notes, no calculators. Formula sheet handed out with exam can be used.

Name:

Problem 1 (20 points) An airfoil profile to be used on a wind turbine blade is tested in a wind tunnel. The prototype (full-scale) airfoil section has a chord length of 2m and moves at 50 m/s relative to the wind, the wind tunnel model has a chord length of 0.4m. The temperature for the model test is the same as for prototype operation.

(a) What criterion should be used to obtain dynamic similarity in the experiment? (Write out the criterion in an equation.)

(b) What velocity should the wind tunnel be operated at to have dynamic similarity for the scale model? (assuming tests at atmospheric pressure.)

(c) The wind tunnel available for this test can only be operated at 100 m/s, but it can also be pressurized. What pressure should be used in the wind tunnel to have dynamic similarity? (Assume the air behaves like an ideal gas.)

(a) Reynolds number should be the same for model and prototype.

$$Re_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} = Re_p$$

(b)
$$\underline{\underline{V_m}} = \underbrace{\frac{L_p}{L_m}}_{=\frac{2m}{0.4m}=5} \cdot \underbrace{\frac{\rho_p}{\rho_m}}_{=1} \cdot \underbrace{\frac{\mu_m}{\mu_p}}_{=1} \cdot V_p = 5 \cdot v_p = 5 \cdot 50 \frac{m}{s} = \underline{\underline{250 \frac{m}{s}}}$$

(c) now $V_m = 100 \frac{m}{s}$, but $\rho(p)$ variable. With ideal gas law $p = \rho R T$ or $\rho = \frac{P}{RT}$

$$\therefore \frac{V_p}{V_m} \cdot \frac{L_p}{L_m} \cdot \underbrace{\frac{\mu_m}{\mu_p}}_{=1} \cdot \frac{p_p}{p_m} \cdot \underbrace{\frac{R_m}{R_p}}_{=1} \cdot \underbrace{\frac{T_m}{T_p}}_{=1} = 1$$

$$\therefore \underline{\underline{\frac{p_m}{p_p}}} = \underbrace{\frac{V_p}{V_m}}_{=\frac{50}{100}=\frac{1}{2}} \cdot \underbrace{\frac{L_p}{L_m}}_{=5} = \frac{1}{2} \cdot 5 = \underline{\underline{2.5}} \qquad \text{or: } \underline{\underline{p_m = 2.5\, p_p}}$$

Problem 2 (50 points)

Consider a steady, laminar, fully developed, incompressible flow in a channel of height $2H$ between two infinite plates, c.f. Figure 1. The flow is due to a pressure gradient that is applied in the streamwise direction, $dp/dx < 0$.

Obtain expressions for the following quantities (per unit channel depth into the drawing plane):

(a) the velocity profile u and v (Hint: show that v must be zero),

(b) the shear stress distribution,

(c) the volumetric flow rate and

(d) the average velocity.

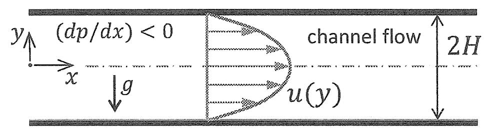


Figure 1: Sketch of channel flow between two horizontal plates.

(a) velocity profile

(1) between "infinite plates": 2-D $\Rightarrow w=0, \ \frac{\partial}{\partial z}=0$

(2) incompressible flow: $\nabla\cdot\vec{v}=0$

(3) steady flow: $\frac{\partial}{\partial t}=0$

(4) fully developed flow: velocity profile does not change in flow direction $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{\partial w}{\partial x}=0$

(5) gravity acts in -y direction only.

continuity: $\nabla\cdot\vec{v} = \overset{(4)}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial v}{\partial y} + \overset{(1)}{\cancel{\frac{\partial w}{\partial z}}} = 0 \quad \therefore \frac{\partial v}{\partial y}=0 \Leftrightarrow v=\text{const.}$

w/ kinematic boundary condition: $v(y=H)=0 \ \therefore \ \text{const.}=0 \ \therefore \boxed{v=0}$ (6)

x-momentum: $\rho\left(\overset{(3)}{\cancel{\frac{\partial u}{\partial t}}} + u\overset{(4)}{\cancel{\frac{\partial u}{\partial x}}} + \overset{(6)}{\cancel{v}}\frac{\partial u}{\partial y} + \overset{(1)}{\cancel{w}}\overset{(1)}{\cancel{\frac{\partial u}{\partial z}}}\right) = -\frac{\partial p}{\partial x} + \overset{(5)}{\cancel{\rho g_x}} + \mu\left(\overset{(4)}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \overset{(1)}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right)$

$$\boxed{\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}} \quad (7) \quad \text{integrate } 2x$$

$$\frac{du}{dy} = \frac{1}{\mu}\left(\frac{dp}{dx}\right)y + C_1 \quad (8)$$

$$u(y) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)y^2 + C_1 y + C_2 \quad (9)$$

boundary conditions:

$u(y=H)=0$: $\quad 0 = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)H^2 + C_1 H + C_2 \quad (10)$

$u(y=-H)=0$: $\quad 0 = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)H^2 - C_1 H + C_2 \quad (11)$

(10) − (11): $C_1 = 0$

(10): $C_2 = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right)H^2$

(space for solution of Problem 2)

$$\therefore \quad \boxed{u(y) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[y^2 - H^2\right] = -\frac{H^2}{2\mu}\left(\frac{dp}{dx}\right)\left[1 - \left(\frac{y}{H}\right)^2\right]}$$

(b) shear stress distribution (w/ (8)): $\tau = \mu \frac{du}{dy} = \left(\frac{dp}{dx}\right) y$

(c) volumetric flow rate (per unit channel depth W)

$$\frac{\dot{V}}{W} = \int_{-H}^{H} u(y)\, dy = \int_{-H}^{H} \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[y^2 - H^2\right] dy = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[\frac{1}{3}y^3 - H^2 y\right]_{-H}^{H} =$$

$$= \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[\frac{H^3}{3} - H^3 - \left(-\frac{H^3}{3} + H^3\right)\right] = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[-\frac{4}{3}H^3\right] =$$

$$= -\frac{2}{3\mu}\left(\frac{dp}{dx}\right) H^3$$

units check: $\left[\frac{1}{\cancel{\frac{Ns}{m^2}}} \frac{\cancel{\frac{N}{m^2}}}{m} m^3\right] = \left[\frac{\frac{m^3}{s}}{m}\right]$ ✓

(d) $u_{avg} = \frac{\dot{V}}{A} = \frac{\dot{V}/W}{A/W} = \frac{\dot{V}/W}{2H} = -\frac{1}{3\mu}\left(\frac{dp}{dx}\right) H^2$

units check: $\left[\frac{1}{\cancel{\frac{Ns}{m^2}}} \frac{\cancel{\frac{N}{m^2}}}{\cancel{m}} m^{\cancel{2}}\right] = \left[\frac{m}{s}\right]$ ✓

Note: The y-momentum eqn. is not needed for answers (a) – (d). If we do consider it, we find the following:

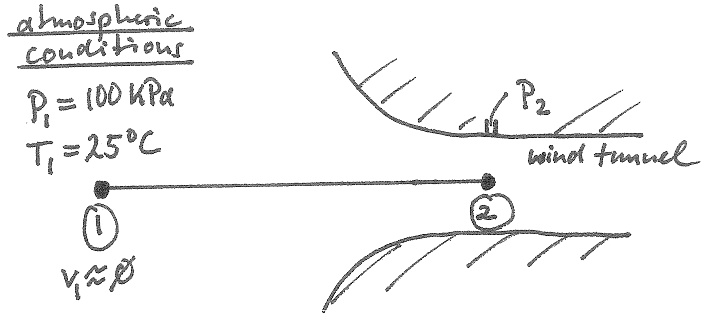
y-momentum: $\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$

with (6), $v=0$: $\quad \frac{\partial p}{\partial y} = \rho g \quad$ or: $p(y) = p_0 + \rho g y$ pressure in y-direction varies hydrostatically

Problem 3 (30 points) An open-circuit wind tunnel draws in air from the atmosphere through a well-contoured nozzle. In the test section, where the flow is straight and nearly uniform, a static pressure tap is drilled into the tunnel wall. A manometer connected to the tap shows that static pressure in the tunnel is 50 mm of water below atmospheric. Assume that the air is incompressible and behaves like an ideal gas, and is at 25°C, 100 kPa (absolute).

Calculate the air speed in the wind-tunnel test section.

State all assumptions you are making. ($\rho_{H_2O} = 1000\ kg/m^3$, $g = 9.81\ m/s^2$, $R_{air} = 287\ J/kgK$)



assumptions:

- flow from outside (atmospheric condition) into wind tunnel can be treated as ~~a~~ inviscid flow (neglect friction)
- points 1 and 2 as sketched above are on a streamline
- incompressible
- steady state
- flow speed at 1 is negligible (or: $P_{atm} = P_{stagnation}$)
- flow at 2 is uniform
- hydrostatic pressure differences can be neglected, e.g. between streamline and where pressure is measured
- air behaves as ideal gas

Bernoulli: $\cancel{\frac{V_1^2}{2}} + \frac{P_1}{\rho} + \cancel{gz_1} = \frac{V_2^2}{2} + \frac{P_2}{\rho} + \cancel{gz_2}$

$\therefore\ V_2 = \sqrt{\frac{2(P_1 - P_2)}{\rho_{air}}}$ w/ $\rho_{air} = \frac{P}{RT} = \frac{100{,}000\ \frac{N}{m^2}}{287\ \frac{Nm}{kg\,K}\,(25 + 273.15)\,K}$

$= 1.169\ \frac{kg}{m^3}$

Δp is "50mm H_2O": $(P_1 - P_2) = \rho_{H_2O}\, g\, h_{H_2O} = 1000\ \frac{kg}{m^3}\ 9.81\ \frac{m}{s^2} \cdot 50 \times 10^{-3}\ m$

$= 490.5\ Pa$ (or: $\frac{N}{m^2}$)

$\therefore\ V_2 = \sqrt{\frac{2 \cdot 490.5\ \frac{kg\,m}{s^2\,m^2}}{1.169\ \frac{kg}{m^3}}} = 28.973\ \frac{m}{s} = 29.0\ \frac{m}{s}$