4.1

Friday, February 23, 2018 10:00 AM

Problem 4.1
(a) Using principles of buoyancy, estimate the density of the human body, using reasonable assumptions from your experience of swimming in fresh water, and from reports of how you "float" in the Dead Sea.
(b) Estimate the buoyancy force you experience in air, while walking around on the UNH campus.
(c) Estimate the buoyancy force you experience while hiking the last few steps to the peak after driving up the Mount Evans Scenic Byway outside of Denver, CO.

(a) density of the human body

- if you need a quick answer: $\boxed{\rho_{body} \simeq \rho_{H_2O}}$ (fresh water)
- a more refined estimate, based on experience w/ water:

$\rho_{H_2O}(T=4°C) = 1{,}000\ kg/m^3$

Do you float or sink? Densities of human body and water are clearly close, but most of us sink without swimming motions, i.e. $\rho > \rho_{H_2O}$

It is actually a bit more complicated:

$$\left.\begin{array}{l} \rho_{Bones} > \rho_{H_2O} \\ \rho_{fat} < \rho_{H_2O} \\ \rho_{\text{"other stuff"}} \simeq \rho_{H_2O} \\ \text{Lung capacity and whether} \\ \text{we hold our breath also matters} \end{array}\right\} \text{all these affect } \rho_{body}$$

$\rho_{Seawater}$ (30 ppt salinity) $\simeq 1025\ kg/m^3$

$\rho_{Dead\ Sea}$ (340 ppt salinity) $\simeq 1240\ kg/m^3$

Report: People easily float in the Dead Sea.

So a more refined estimate of your density may be:

$\boxed{\rho_{body} \simeq 1.05\,\rho_{H_2O}}$ (there will be some variability person-to-person)

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(b) <u>Estimate your buoyancy force in air on UNH campus</u>

UNH is at sealevel $\therefore$ $\underline{p_0} = 1$ atm $= \underline{1.013 \text{ bar}}$; $\underline{T_0 = 15°C}$

$$\rho = \frac{p}{RT} = \frac{101325\ \mathrm{N/m^2}}{287 \frac{\mathrm{Nm}}{\mathrm{kg\,K}} (15 + \underline{273.15})\,\mathrm{K}} = 1.22523\ \frac{\mathrm{kg}}{\mathrm{m^3}}$$ ← must use absolute T!

$$\therefore \quad \underline{\rho_{air}} = \underline{1.225}\ \mathrm{kg/m^3}$$

<u>Buoyancy force:</u> $F_B = \iint p\, dA = \iiint \rho_{fluid}\, g\, d\forall = \rho_{fluid}\, g\, \forall$

= weight of the fluid displaced.

$$\therefore \quad \underline{F_B} = \rho_{air}\, g\, \forall_{body} = \rho_{air}\, g\, \frac{m_{body}}{\rho_{body}}$$

m_{body} ← you know this, e.g. $m = 90$ kg

ρ_{body} ← we estimated this

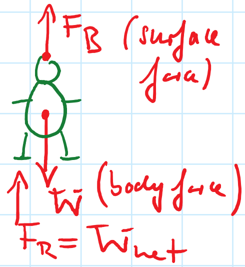
$$= 1.225 \frac{\cancel{\mathrm{kg}}}{\cancel{\mathrm{m^3}}} \cdot 9.81 \frac{\mathrm{m}}{\mathrm{s^2}} \cdot \frac{90\ \mathrm{kg}}{1.05 \frac{\cancel{\mathrm{kg}}}{\cancel{\mathrm{m^3}}}}$$

$$= \underline{1.03\ \mathrm{N}} \ \hat{=}\ \underline{0.232\ \mathrm{Lbf}}$$

(This means your "net weight" in air is ~ 1N less than your actual weight $W = mg$ due to buoyancy.)

$$W = m \cdot g = 90\ \mathrm{kg} \cdot 9.81 \frac{\mathrm{m}}{\mathrm{s}} = 883\ \mathrm{N}$$

$$W_{net} = W - F_B = 883 - 1 = 882\ \mathrm{N}$$



error in measurement when you "weigh" yourself in air

$$e = \frac{W_{net} - W}{W} = \frac{882 - 883}{883} = -0.00117 = \underline{-0.117\ \%}$$

(c) <u>Mt. Evans Scenic Byway</u>

→ highest paved road in North America

ends @ 4310 m (14,130 ft)
summit @ 4350 m (14,271 ft)

We need to calculate p and ρ at this altitude

U.S. Standard Atmosphere: temperature varies linearly with altitude

$$\boxed{T = T_0 - az} \quad \text{w/} \quad T_0 = 15°C, \quad a = 6.5\,°C/km$$

$$\frac{dp}{dz} = \rho g = \frac{p}{RT} g \iff \frac{dp}{p} = \frac{g}{R}\frac{1}{T(z)}\,dz$$

$$\int_{p_0}^{p} \frac{dp}{p} = \ln\frac{p}{p_0} = -\frac{g}{R}\int_{z_0}^{z} \frac{1}{T_0 - az}\,dz$$

$$\ln\frac{p}{p_0} = \frac{g}{aR}\ln\left(\frac{T_0 - az}{T_0}\right) = \frac{g}{aR}\ln\left(1 - \frac{az}{T_0}\right)$$

$$\therefore \quad p = p_0\left(1 - \frac{az}{T_0}\right)^{g/aR}$$

$$p = 101325\,\frac{N}{m^2}\left(1 - \frac{\frac{6.5}{1000}\frac{K}{m}\cdot 4350\,m}{(15+273.15\,K)}\right)^{\frac{9.81\,m/s^2}{\frac{6.5}{1000}\frac{K}{m}\,287\frac{J}{kgK}}} = 58\,863\,\frac{N}{m^2}$$

$$= 0.5886 \text{ bar} = 0.58 \text{ atm}$$

$$\rho_{air} = \frac{p}{RT} = 0.7118\,\frac{kg}{m^3}$$

$$F_B = \rho_{fluid}\, g\, V_{body} = \rho_{air}\, g\,\frac{m_{body}}{\rho_{body}} = 0.7118\cdot 9.81\,\frac{90}{1050} = 0.60\,N$$

$$= 0.135\,lbf$$

So: you will feel slightly "heavier" at altitude, but the difference is only about one Snickers bar. (~1.5 oz)

Problem 4.2

A gate connected to a concrete weight is used to keep water at a certain level inside a reservoir, as shown in the figure below. If the gate is 3m wide (into the drawing plane), what diameter sphere is just sufficient to keep the gate closed?

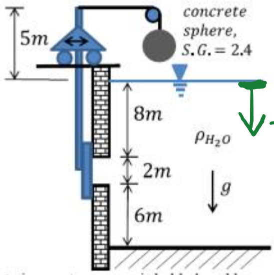


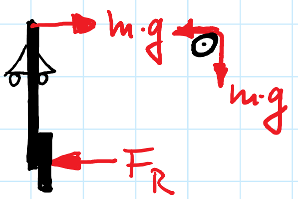
Figure 1: Sketch of gate in a water reservoir held closed by a concrete sphere weight.

$z_1 = 8m$

$\downarrow z \quad z_2 = 10m$

Assumptions:
- Fluid statics
- $\rho =$ const.
- no friction in wheel redirecting weight of sphere

FBD:



Gov. Equs: $\frac{dp}{dz} = \rho g$; $F_R = \int p\, dA$

$p = \rho g z$; $dA = W dz$

$$\therefore F_R = \int_{z_1}^{z_2} \rho g W z\, dz$$

$$= \frac{1}{2} \rho g W z^2 \Big|_{z_1}^{z_2} = \frac{1}{2} \rho g W (z_2^2 - z_1^2)$$

$$F_R = \frac{1}{2}\, 1{,}000 \frac{kg}{m^3}\, 9.81 \frac{m}{s^2}\, 3m\, (10^2 - 8^2)\, m^2$$

$$= 529{,}740\ N = 530\ kN$$

$\rho_{concrete} = S.G. \cdot \rho_{H_2O}$

$\Sigma F = 0$: $\quad F_R = m \cdot g = \rho V g = \rho \frac{4}{3} \pi R^3 g$

$$\therefore R = \frac{D}{2} = \left(\frac{3}{4} \frac{\frac{1}{2} \rho_{H_2O}\, g\, W (z_2^2 - z_1^2)}{S.G.\, \rho_{H_2O}\, g\, \pi} \right)^{\frac{1}{3}}$$

$R = 1.75\ m$ or $D = 3.5\ m$

4.3

Sunday, February 25, 2018 9:20 PM

Problem 4.3
A gate (Figure 1) is hinged at point B. It is 4m wide normal to the drawing plane. The pressure to the right of the gate and at the water surface is atmospheric pressure. Assume the gate is massless and find:
(a) The resultant force magnitude on the gate due to the water.
(b) The position of the resultant force relative to the free surface.
(c) The force F required at location A to hold the gate closed.

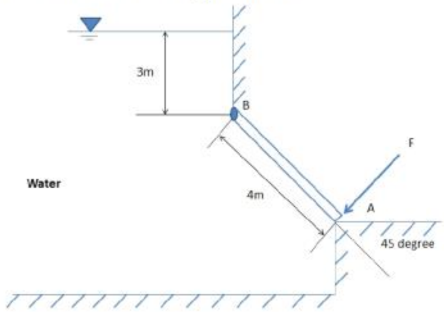
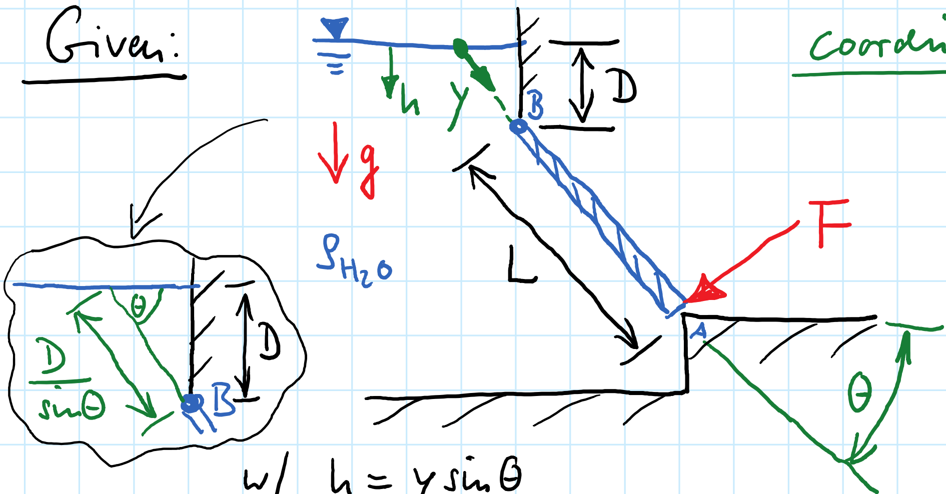


Figure 2: Sketch of hinged submerged gate.

Given:



w/ $h = y \sin\theta$

coordinate y: from water surface in direction of gate

$D = 3\,m$
$L = 4\,m$
$\theta = 45°$
$\rho = 1{,}000\ kg/m^3$
$W = 4\,m$
$g = 9.81\ m/s^2$

Assumptions:
- Static Fluid (no fluid motion)
- $\rho =$ const.
- P_{atm} acts on all sides (consider P_{gage} only)

Find:
(a) resultant force F_R
(b) position of F_R
(c) force F required to keep gate closed

Governing Equations: $\frac{dp}{dh} = \rho g$; $dF = \int p\, dA$;

$\Sigma M_R = 0$

Solution: $F_R = \int p\, dA$ w/ $p = \rho g h = \rho g y \sin\theta$

$dA = \bar{W}\, dy$

$$\therefore \quad F_R = \int_{D/\sin\theta}^{D/\sin\theta + L} \rho g y \sin\theta\, \bar{W}\, dy$$

$$\boxed{F_R = \frac{1}{2}\rho g \bar{W} \sin\theta \left[\left(\frac{D}{\sin\theta} + L\right)^2 - \left(\frac{D}{\sin\theta}\right)^2 \right]}$$

w/ numbers $\boxed{F_R = 692 \text{ kN}}$

(b) $$y' F_R = \int y\, dF = \int_{D/\sin\theta}^{D/\sin\theta + L} \rho g \sin\theta\, \bar{W}\, y^2\, dy$$

$$= \frac{1}{3}\rho g \bar{W} \sin\theta \left[\left(\frac{D}{\sin\theta} + L\right)^3 - \left(\frac{D}{\sin\theta}\right)^3 \right]$$

$$\therefore \quad \boxed{y' = \frac{2}{3} \; \frac{\left(\frac{D}{\sin\theta} + L\right)^3 - \left(\frac{D}{\sin\theta}\right)^3}{\left(\frac{D}{\sin\theta} + L\right)^2 - \left(\frac{D}{\sin\theta}\right)^2}}$$

w/ numbers $\boxed{y' = 6.456 \text{ m}}$

(c) $$\Sigma M_B = 0 = F_R \left(y' - \frac{D}{\sin\theta} \right) - F \cdot L$$

$$\therefore \quad \boxed{F = F_R \, \frac{y' - \frac{D}{\sin\theta}}{L} = 383 \text{ kN}}$$

ME 608 Fluid Dynamics
Martin Wosnik

Problem 4.4
A Tainter gate is a common type of spillway gate used to control water flow from dams, as shown in Figure 2. The gate width is $W = 15\ m$ (out of the page), the radius is $= 20\ m$, and the water depth is $D = 10\ m$. You can assume that the gate is massless and the pressure everywhere outside of the water is equal to atmospheric pressure p_{atm}. Determine the
(a) the magnitude,
(b) the direction, and
(c) the line of action
of the force from the water acting on the gate.

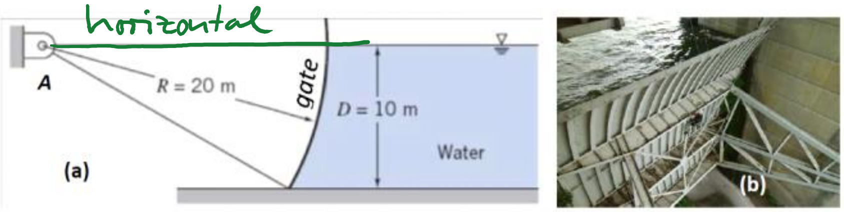
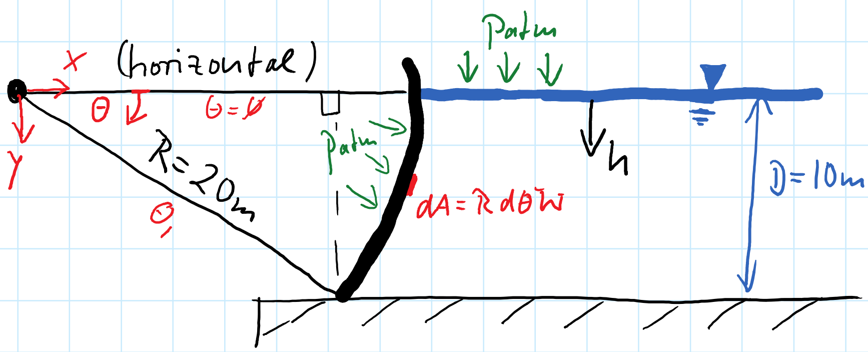
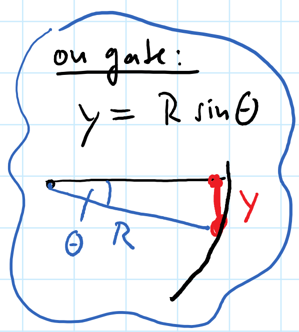


Figure 3: (a) Sketch of Tainter gate, anchored in point A, holding back water to the right; (b) example of a Tainter gate – with a worker performing maintenance work on the gate.

assumptions: static fluid
$\rho = \text{const.} = 1000\ \frac{kg}{m^3}$ (water)
p_{atm} everywhere outside of water → p_{gage}





governing equations:

$$p = \rho g h$$
$$\vec{F} = -\int p\, d\vec{A}$$

Note: The approach below – splitting the differential resultant force $d\vec{F}$ into components and integrating – always works.

See below for the simplified approach (c.f. lecture notes and Figure 3.8 in text book) in terms of projected areas.

(a) Force on gate: pressure on gate in terms of gate coordinates:
$p = \rho g y = \rho g R \sin\theta$

$$p = \rho g y = \underbrace{\rho g}\, R \sin\theta$$

$$\boxed{\vec{F} = -\int p\, d\vec{A}} = -\int \rho g R \sin\theta\, dA$$

w/ $dA = R\, d\theta \cdot W$

also: $\boxed{\vec{F} = F_x \cdot \hat{e}_x + F_y \cdot \hat{e}_y}$

$$d\vec{F} = \underbrace{dF\cos\theta}_{dF_x} \cdot \vec{e}_x + \underbrace{dF\sin\theta}_{dF_y} \cdot \hat{e}_y$$

integrate from: $\theta = 0$ to $\theta = \theta_1$ w/ $\sin\theta_1 = \frac{10}{20} = 0.5$

$\theta_1 = \sin^{-1}(0.5) = 30° = \frac{\pi}{6}$

- $$F_x = -\int_0^{\pi/6} \overbrace{\rho g R \sin\theta}^{= \rho g y}\ \overbrace{\cos\theta}^{x\text{-comp.}}\ \overbrace{R W\, d\theta}^{dA} = -\rho g R^2 W \int_0^{\pi/6} \sin\theta\cos\theta\, d\theta =$$

w/ $\int \sin\theta\cos\theta\, d\theta = -\frac{1}{2}\cos^2\theta + c_1 = \frac{1}{2}\sin^2\theta + c_2$

2nd solution will be easier to work with for given angles.

$$= -\rho g R^2 W \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/6} = -\rho g R^2 W \frac{1}{2}(0.5)^2$$

$$= -\frac{1}{8}\rho g R^2 W = -\frac{1}{8}\, 1000\frac{kg}{m^3}\, 9.81\frac{m}{s^2}\,(20m)^2 \cdot 15m = \left[\frac{kg\, m}{s^2}\right] \checkmark$$

check:

$$= -7.3575 \times 10^6\, N = -7.36 \times 10^6\, N$$

- $$F_y = -\int_0^{\pi/6} \overbrace{\rho g R \sin\theta}^{\rho g y}\ \overbrace{\sin\theta}^{y\text{-comp.}}\ \overbrace{R W\, d\theta}^{dA} = -\rho g R^2 W \int_0^{\pi/6} \sin^2\theta\, d\theta =$$

w/ $\int \sin^2\theta\, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C_1 = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C_2$

$$= -\rho g R^2 W \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]_0^{\pi/6} = -\frac{1}{2}\rho g R^2 W \left\{\left(\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right) - (0 - 0)\right\} =$$

$$= -\frac{1}{24}\rho g R^2 W\left(2\pi - 3\sqrt{3}\right) = -\frac{1}{8}\, 1000 \cdot 9.81 \cdot 20^2 \cdot 15\left(2\pi - 3\sqrt{3}\right)$$

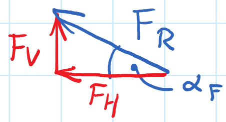
$$= -2.6659 \times 10^6\, N = -2.67 \times 10^6\, N$$

$$= -2.6659 \times 10^6\,N = -2.67 \times 10^6\,N$$

- $F_R = \left(F_x^2 + F_y^2\right)^{\frac{1}{2}} = 7.8256 \times 10^6\,N = 7.83 \times 10^6\,N$

(b) direction of force: since the gate is a circular arc, all elements dF of the force, and hence the resultant force, must pass through the pivot A.

(c) line of action:



1. $\tan\alpha_F = \dfrac{F_V}{F_H}$ ∴ $\alpha_F = \tan^{-1}\left(\dfrac{2.67}{7.36}\right) = 19.92°$ off horizontal

2. find x', y':

- $x' F_y = \int x'\, dF_y = \int_0^{\pi/6} -R\cos\theta\, \rho g R \sin\theta \sin\theta\, R\bar{w}\, d\theta =$

$$= -R^3 \rho g \bar{w} \int_0^{\pi/6} \cos\theta \sin^2\theta\, d\theta =$$ (substitute: $u = \sin\theta$)

integrate w/ $u = \sin\theta$ $\quad \frac{du}{d\theta} = \cos\theta \Leftrightarrow du = \cos\theta\, d\theta$

$$= * \int_0^{\pi/6} u^2\, du = * \left.\frac{u^3}{3}\right|_0^{\pi/6}$$ (substitute back) $=$

$$= -R^3 \rho g \bar{w}\, \frac{1}{3}\left(\frac{1}{2}\right)^3$$

∴ $x' = \dfrac{-R^{\cancel{3}} \cancel{\rho g} \cancel{\bar{w}} \cancel{\frac{1}{3}} \left(\cancel{\frac{1}{2}}\right)^3}{-\cancel{\frac{1}{24}} \cancel{\rho g} \cancel{R^2} \cancel{\bar{w}} \left(2\pi - 3\sqrt{3}\right)} = \dfrac{R}{2\pi - 3\sqrt{3}} = 18.3988\,m \approx 18.4\,m$

- $y' F_x = \int y'\, dF_x = \int_0^{\pi/6} -R\sin\theta\, \rho g R \sin\theta \cos\theta\, R\bar{w}\, d\theta =$

$$= -R^3 \rho g \bar{w} \int_0^{\pi/6} \sin^2\theta \cos\theta\, d\theta$$ (same integral as before)

$$= -\frac{1}{24} R^3 \rho g \bar{w}$$

∴ $y' = \dfrac{-\frac{1}{24} R^3 \rho g \bar{w}}{-\frac{1}{\ldots} R^2 \rho g \bar{w}} = \dfrac{1}{3} R = 6.67\,m$

$$\therefore\ y' = \frac{-\frac{1}{24} R^3 \rho g w}{-\frac{1}{8} R^2 \rho g w} = \frac{1}{3} R = \underline{6.67\ m}$$

<u>Note:</u> if we write: $\sin\left(\frac{\pi}{6}\right) = \frac{D}{R}$ (which is where $\frac{\pi}{6}$ came from!)

then $F_x = -\frac{1}{2} R^2 \rho g w \left(\frac{D}{R}\right)^2$

and $y' F_x = -\frac{1}{3} R^3 \rho g w \left(\frac{D}{R}\right)^3$

$$\therefore\ \underline{y'} = \frac{y' F_x}{F_x} = \underline{\frac{2}{3} D}$$

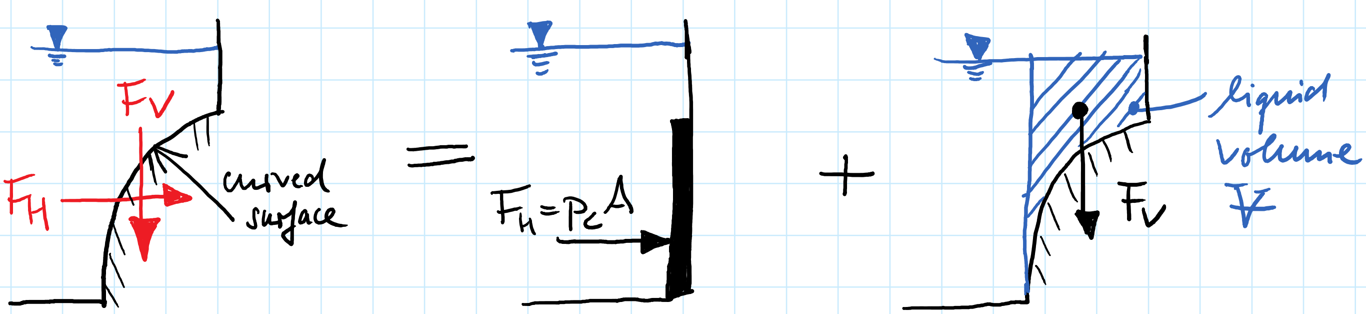
<u>Note:</u> this is true for all submerged surfaces!
$\Rightarrow$ point of action of resultant force is at $\frac{2}{3}$ depth!

Alternative approach for (a) to calculate force magnitude:

resultant force in the "ℓ" direction:

$$\boxed{F_{R_\ell} = \iint_{A_\ell} p\, dA_\ell}$$

where $dA_\ell = d\vec{A} \cdot \hat{n}_\ell$
(projection of the area element $d\vec{A}$ in the ℓ-direction)

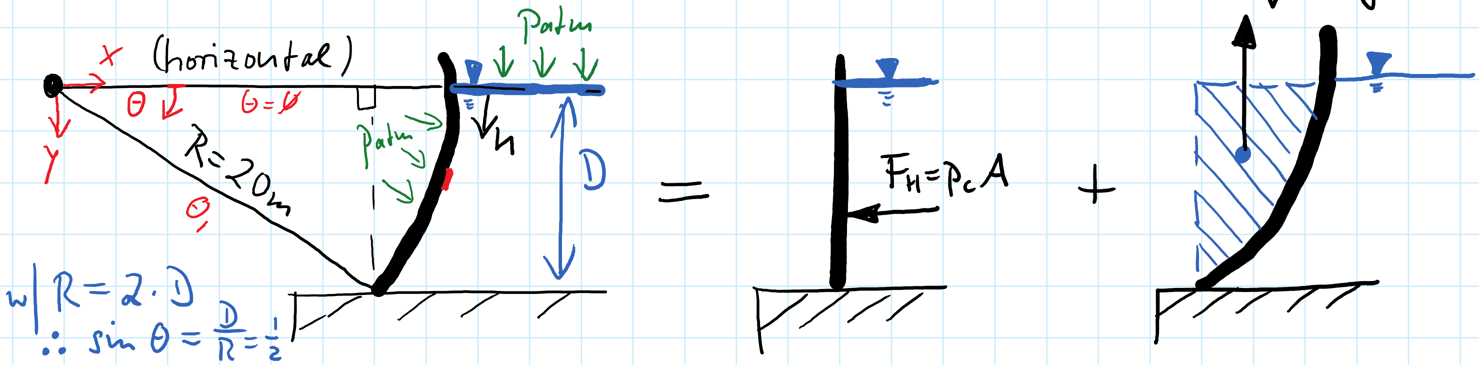


$$F_H = p_c A \qquad\qquad F_V = \rho g V$$

(a) <u>Apply to Tainter gate:</u>

$F_V = \rho g V$

(a) Apply to Tainter gate :

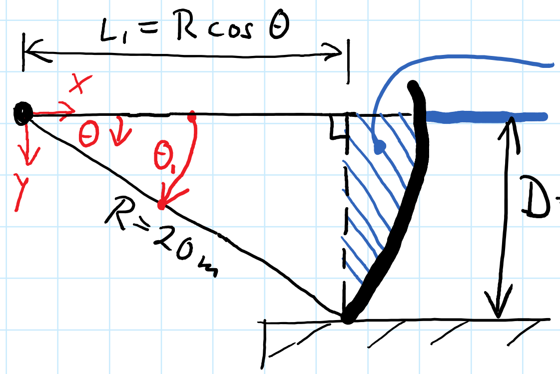


w| $R = 2 \cdot D$
$\therefore \sin\theta = \frac{D}{R} = \frac{1}{2}$
$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

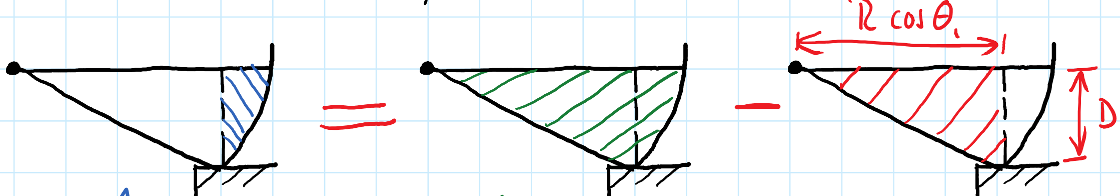
$$\underline{F_H = F_x} = \int_A p\,dA = \int_A \rho g h\,dA = W \int_0^D \rho g y\,dy = W \rho g \left[\frac{y^2}{2}\right]_0^D$$

$$= \frac{1}{2}\rho g W D^2 = \underline{\frac{1}{8}\rho g W R^2} \quad (q.e.d.)$$

$\underline{F_V = F_y}$



volume directly above surface (water is "missing", force must point upward)



A $\quad$ A_{total} (w/ $dA = r\,dr\,d\theta$) $\quad$ $A_{triangle}$

$$\underline{A} = \int_0^{\pi/6}\int_0^R r\,dr\,d\theta - \frac{1}{2} D R \cos\theta_1$$

$$= \frac{\pi}{6}\frac{1}{2}R^2 - \frac{1}{4}R^2\frac{\sqrt{3}}{2} \quad \left(w/\ D = \frac{R}{2}\right)$$

$$= \underline{\frac{1}{24}R^2\left(2\pi - 3\sqrt{3}\right)}$$

$$\therefore \underline{F_V = F_V} = -\rho g \forall = -\underline{\frac{1}{24}\rho g W R^2 \left(2\pi - 3\sqrt{3}\right)} \quad (q.e.d.)$$

$$\therefore\ F_V = F_V = -\rho g \forall = -\frac{1}{24}\rho g W R^2 \left(2\pi - 3\sqrt{3}\right) \qquad (q.e.d.)$$

minus sign: Force is pointing upward, since volume of water is "missing" directly above surface.

$= \frac{D}{2}$

Problem 4.5
Consider a submarine, modeled as a hollow circular cylinder of radius R and length L (into page), submerged to depth H (at top of submarine), c.f. Figure 3.

(a) Give an expression for the differential area element dA of the submarine in terms of angle θ.
(b) Give an expression for submergence h on the surface of the sub in terms of angle θ.
(c) Give an expression for the net pressure force exerted on the submarine, in terms of p_{atm}, submarine cabin pressure $p_{submarine}$, g, R and L.

To solve the problem, consider the vector equation for the pressure force: $\vec{F} = \iint_A -p\, d\vec{A}$.

(d) Why is the submergence depth H not needed to calculate the net pressure force?
(e) If H does not affect the net pressure force, explain the reasons for a maximum depth limit for submarines (Hint: This is a solid mechanics issue).
(f) How will varying $p_{submarine}$ change the maximum depth limit?

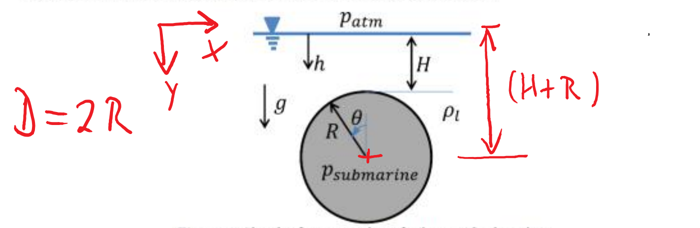
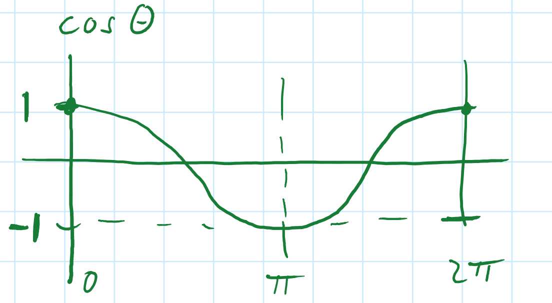


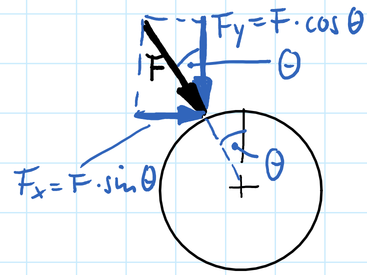
Figure 4: Sketch of cross-section of submerged submarine.



(a) differential area element $dA = R\, d\theta\, L$ [m^2]

(b) the submergence of a surface element of the submarine is best expressed from the center of the sub.

$$h(\theta) = (H+R) - R\cos\theta$$



(c) Net pressure force exerted on submarine

<u>Given:</u> p_{atm}, p_{sub}, ρ_l, D, L

$$\vec{F} = -\int p\, d\vec{A} \qquad w/\ p = p_{atm} + \rho g h$$

$$\vec{F} = \cancel{\vec{F}_x}^{\,0} + \vec{F}_y \qquad \vec{F}_x = 0 \text{ due to symmetry}$$

$$\underline{F_y} = \int d\vec{F}_y = -\int p\, dA \cos\theta =$$

$$= -\int_0^{2\pi} \left\{ p_{atm} + \rho g \left[(H+R) - R\cos\theta \right] \right\} R L \cos\theta\, d\theta$$

$$= -\int_0^{2\pi} \cancel{p_{atm} R L \sin\theta\, d\theta}^{\,=0} - \rho g R L \int_0^{2\pi} \left[(H+R)\cos\theta - R\cos^2\theta \right] d\theta$$

Note: $\int p\, dA = 0$ if $p = $ const.

$$= -\rho g R L \left[(H+R)\sin\theta + \frac{R}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) \right]_0^{2\theta}$$

$$= -\rho g R L \left[(H+R)\sin\theta + \frac{R}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$= -\rho g R L \left[(H+R)(0-0) + \frac{R}{2}\left(2\pi + 0 - (0+0)\right)\right]$$

$$= \underline{\underline{-\rho_\ell g R^2 \pi L}} = \underline{\underline{-\rho_\ell g V}}$$

= weight of water displaced! ✓

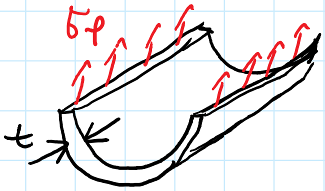
(= what the answer should be!)

(d) H is not needed in the calculation of the net force, since the net force only depends on the relative change of pressure with depth.

The net force will always be equal to the weight of the volume of fluid displaced by the object.

(e) structural integrity of the submarine hull is what limits its diving depth.

From Mechanics:



tangential stresses in pressurized thin-walled hull:

$$\boxed{\sigma_\varphi = (p - p_{inside})\frac{R}{t}}$$

(f) In theory, increasing the pressure inside the hull can increase the maximum dive depth. In practice, there are (low) limits of what pressure increase the crew can withstand.

Note: A double-hulled submarine with an intermediate pressure between the two hulls would be another way to increase dive limits.