8.1

Wednesday, April 18, 2018 2:18 AM

c.f. problem 4.212 from text book (solution posted separately)

8.2

Monday, April 2, 2018 10:44 PM

ME 608 Fluid Dynamics

Problem 8.2
Using the continuity equation in differential form, evaluate which of the following velocity distributions (sets of equations) represent possible **three-dimensional incompressible flow** cases.

(a) $u = 2y^2 + 2xz$; $v = -2yz + 6x^2yz$; $w = 3x^2z^2 + x^3y^4$
(b) $u = xyzt$; $v = -xyzt^2$; $w = z^2(xt^2 - yt)$
(c) $u = x^2 + 2y + z^2$; $v = x - 2y + z$; $w = -2xz + y^2 + 2z$

Continuity equation: $\boxed{\dfrac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0}$

incompressible flow: $\dfrac{D\rho}{Dt} = \dfrac{\partial \rho}{\partial t} + \vec{V}\cdot\nabla\rho = \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + v\dfrac{\partial \rho}{\partial y} + w\dfrac{\partial \rho}{\partial z} = \phi$

$$\Longleftrightarrow \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

∴ With velocity fields given, check whether they satisfy $\nabla \cdot \vec{V} = 0$

(a)
$$u = 2y^2 + 2xz \Rightarrow \frac{\partial u}{\partial x} = 2z$$
$$v = -2yz + 6x^2yz \Rightarrow \frac{\partial v}{\partial y} = -2z + 6x^2z$$
$$w = 3x^2z^2 + x^3y^4 \Rightarrow \frac{\partial w}{\partial z} = 6x^2z$$

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 12x^2z \neq \phi$$

Not incompressible

(b)
$$u = xyzt \Rightarrow \frac{\partial u}{\partial x} = yzt$$
$$v = -xyzt^2 \Rightarrow \frac{\partial v}{\partial y} = -xzt^2$$
$$w = z^2(xt^2 - yt) \Rightarrow \frac{\partial w}{\partial z} = 2xzt^2 - 2yzt$$

$$\nabla\cdot\vec{V} = -yzt + xzt^2 \neq 0$$

Not incompressible

(c)
$$u = x^2 + 2y + z^2 \Rightarrow \frac{\partial u}{\partial x} = 2x$$
$$v = x - 2y + z \Rightarrow \frac{\partial v}{\partial y} = -2$$
$$w = -2xz + y^2 + 2z \Rightarrow \frac{\partial w}{\partial z} = -2x + 2$$

$$\nabla\cdot\vec{V} = 2x - 2 - 2x + 2 = \phi$$

incompressible

⇒ divergence-free velocity field

8.3

Monday, April 2, 2018 10:45 PM

Problem 8.3
A useful approximation for the x component of velocity in an incompressible, two-dimensional laminar boundary layer is a parabolic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U_∞, at the edge of the boundary layer ($y = \delta$). The equation for the profile is ← look for key words!

u-profile is given: $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

where $\delta = cx^{0.5}$ and c is a constant. Using the continuity equation in differential form, show that the simplest expression for the y component of velocity is

$\delta \propto x^{\frac{1}{2}}$

Show that v-profile is: $\frac{v}{U_\infty} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$

For $\delta = 5mm$ and $x = 0.5m$, plot v/U_∞ versus y/δ to find the location of the max. value of ratio v/U_∞.

Continuity equation: $\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = 0}$

or: $= \frac{\partial\rho}{\partial t} + \vec{V}\cdot\nabla\rho + \rho\nabla\cdot\vec{V} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) =$

$$= \frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho w)}{\partial t} =$$

$$= \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} + v\frac{\partial\rho}{\partial y} + \rho\frac{\partial v}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\frac{\partial w}{\partial z} = 0$$

$= \vec{V}\cdot\nabla\rho$ $\quad$ $= \rho\nabla\cdot\vec{V}$

Assumptions (from problem description):

- incompressible: $\frac{D\rho}{Dt} = \emptyset \iff \nabla\cdot\vec{V} = \phi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$
- Two-dimensional: $w = 0,\ \frac{\partial}{\partial z} = 0 \quad \therefore \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$ governing equation

Given: velocity profile $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ $\quad$ w/ $\delta = c\,x^{\frac{1}{2}}$ $\quad \therefore x^{-\frac{1}{2}} = \frac{c}{\delta}$ (*) $\quad$ and: $c^2 = \frac{\delta^2}{x}$ (**)

from continuity eqn: $\boxed{\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}}$

$$\therefore \quad v = -\int\frac{\partial u}{\partial x}\,dy + C_2(x)$$

additive constant can at most be a function of x (and still satisfy continuity)

$$\text{w/}\ \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[U_\infty\left(2\frac{y}{cx^{\frac{1}{2}}} - \frac{y^2}{c^2x}\right)\right] = \frac{\partial}{\partial x}\left[U_\infty\left(\frac{2y}{c}x^{-\frac{1}{2}} - \frac{y^2}{c^2}x^{-1}\right)\right]$$

$$= U_\infty\left(-\frac{1}{\cancel{2}}\frac{\cancel{2}y}{c}x^{-\frac{3}{2}} + \frac{y^2}{c^2}x^{-2}\right) \overset{(*)}{=} U_\infty\left(-\frac{y}{c}\left(\frac{c}{\delta}\right)^3 + \frac{y^2}{c^2}\left(\frac{c}{\delta}\right)^4\right)$$

$$= U_\infty\frac{c^2}{\delta^2}\left(-\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right)$$

$$\therefore \quad v = -\int U_\infty\frac{c^2}{\delta^2}\left(-\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right)dy + C_2(x) =$$

$$= -U_\infty\frac{c^2}{\delta^2}\left(-\frac{1}{2}\frac{y^2}{\ldots} + \frac{1}{\ldots}\frac{y^3}{\ldots}\right) + C_2(x) =$$

$$= -U_\infty \frac{c^2}{\delta^2}\left(-\frac{1}{2}\frac{y^2}{\delta} + \frac{1}{3}\frac{y^3}{\delta^2}\right) + C_2(x) =$$

$$= U_\infty \frac{c^2}{\delta}\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right) + C_2(x) \underset{(**)}{\overset{\text{w/ } c^2 = \frac{\delta^2}{x}}{=}}$$

$$= U_\infty \frac{\delta^{\cancel{2}}}{\cancel{\delta} x}\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right) + C_2(x)$$

- no-slip condition @ wall: $u(y=0) = 0$
- kinematic boundary condition @ wall: $\boxed{v(y=0) = 0}$ (there is no flow through the wall)

$$v(y=0) = 0 + C_2(x) = 0 \iff C_2 = 0$$

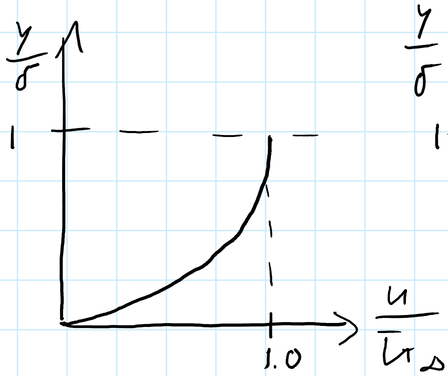
Thus: $\dfrac{v}{U_\infty} = \dfrac{\delta}{x}\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right)$ (q.e.d.)

Plotting: $\dfrac{v}{U_\infty}$ vs. $\dfrac{y}{\delta}$

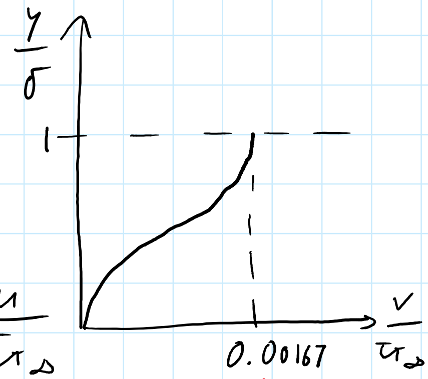
u-profile is 2nd order polynomial
v-profile is 3rd order polynomial

Plot profiles non-dimensionally, i.e., $\dfrac{v}{U_\infty}$ vs. $\dfrac{y}{\delta}$, (or: $\dfrac{u}{U_\infty}$ vs. $\dfrac{y}{\delta}$) not asked for

Use: $\delta = 5\text{ mm}$, $x = 500\text{ mm} \Rightarrow \dfrac{\delta}{x} = 0.01$



u velocity increases all the way to U_∞ through boundary layer (this is, in fact, the definition of the "boundary layer")

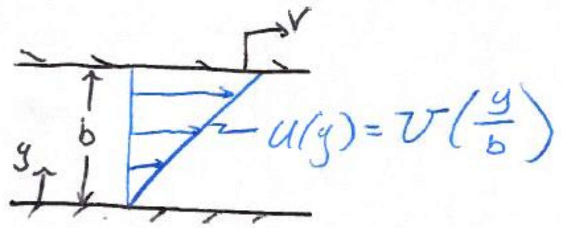


v is a small fraction of U_∞ — but not zero, since the boundary layer is a developing flow.

8.4

Wednesday, April 18, 2018 2:30 AM

8.4) Given:



find: a) Volumetric dilation Rate

b) angular velocity Vector

c) Vorticity vector

d) Rate of Angular deformation

Analysis:

a) $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{0} + \cancel{\frac{\partial w}{\partial z}}^{0}$

$\hookrightarrow$ u Not a fcn of x

$\boxed{\text{Volumetric dilation} = 0}$

b) $\frac{1}{2}(\nabla \times \vec{V}) \Rightarrow -\frac{1}{2}\frac{\partial u}{\partial y}\hat{k} = \boxed{-\frac{1}{2}\frac{V}{b}\hat{k}}$

c) $\omega = \nabla \times \vec{V} \Rightarrow -\frac{\partial u}{\partial y}\hat{k} = \boxed{-\frac{V}{b}\hat{k}}$

d) rate of Angular def $= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \boxed{\frac{V}{b}}$