ME 608 Fluid Dynamics, M. Wosnik

Problem 5.1

Consider flow of water in a long circular pipe. The inlet velocity is uniform, while the exit velocity distribution is given by $u_2(r) = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$. Given are length $L = 8m$, pipe radius $R = 0.005m$, and $u_{max} = 0.2m/s$. Using conservation of mass in integral formulation, determine the maximum inlet velocity, u_1.

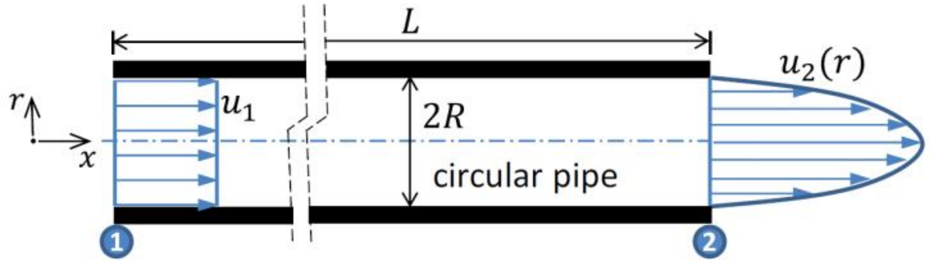
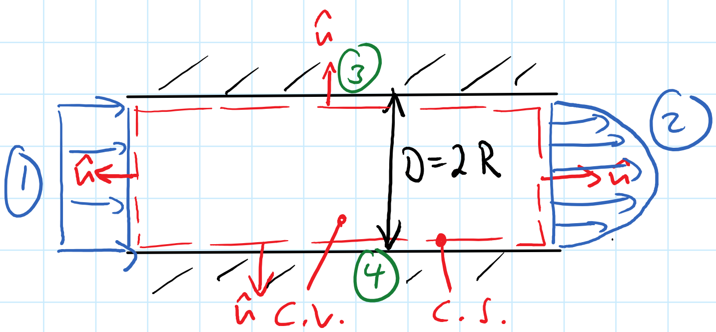


Figure 1: Sketch of flow in long pipe with uniform inflow and exit velocity distribution $u_2(r)$.



for a circular cross-section:

$$dA_1 = dA_2 = r\,dr\,d\theta$$

$$\iint dA_1 = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = 2\pi\int_0^R r\,dr = 2\pi\left[\frac{r^2}{2}\right]_0^R = \underline{\pi R^2}$$

conservation of mass

$$\cancel{\frac{\partial}{\partial t}\iiint_{C.V.} \rho\, d\forall}^{\,0} + \iint_{C.S.} \rho \vec{V}\cdot d\vec{A} = 0$$

4 sides to C.S.:

$$\iint_{CS(1)} \rho \underbrace{\vec{V}\cdot\hat{n}}_{=-V_1} dA_1 + \iint_{CS(2)} \rho \underbrace{\vec{V}\cdot\hat{n}}_{=V_2(r)} dA_2 + \iint_{CS(3)} \rho \underbrace{\vec{V}\cdot\hat{n}}_{=0,\ \vec{V}\perp\hat{n}} dA_3 + \iint_{CS(4)} \rho \underbrace{\vec{V}\cdot\hat{n}}_{=0,\ \vec{V}\perp\hat{n}} dA_4 = 0$$

$$-\rho V_1 \underbrace{\int_0^{2\pi}\int_0^R r\,dr\,d\theta}_{=\pi R^2} + \rho\, u_{max}\int_0^{2\pi}\int_0^R \left(1-\left(\frac{r}{R}\right)^2\right) r\,dr\,d\theta = 0$$

$$\underbrace{\qquad\qquad}_{=\pi R^2}$$

$$\rho\, u_{max}\, 2\pi \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$\parallel \qquad \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

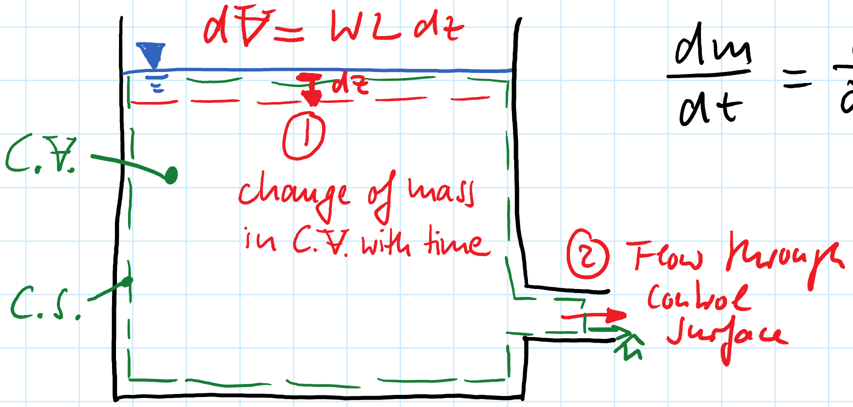
$$\parallel \qquad \left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$\therefore \quad V_1 = u_{max} \frac{\cancel{\rho}\, \cancel{2}\cancel{\pi}\, \frac{\cancel{R^2}}{4}}{\cancel{\rho}\, \cancel{\pi} \cancel{R^2}} = \frac{1}{2} u_{max} = 0.1 \, \frac{m}{s}$$

$\Rightarrow$ For a parabolic velocity distribution, as it occurs in fully developed laminar pipe flow, the average velocity is $\frac{1}{2}$ of the maximum velocity.

Problem 5.2

Consider flow of water out of a spout with a circular cross section with $D = 10mm$, attached near the bottom of a tank 0.3m wide, 0.5m long, and filled to a depth of 0.4m. At the moment the valve is opened, water flows through the valve tube at a Reynolds number of $Re_D = 2{,}000$. What is the rate of change of the water level in the tank at this moment? Use conservation of mass in integral form! (Hint: define a convenient control volume.)



$$\frac{dm}{dt} = \underbrace{\frac{\partial}{\partial t}\iiint_{C.V.} \rho \, d\forall}_{1} + \underbrace{\iint_{C.S.} \rho \vec{V} \cdot d\vec{A}}_{2} = 0$$

1 $$\frac{\partial}{\partial t}\iiint_{C.V.} \rho \, d\forall = \frac{\partial}{\partial t} W L \int \rho \, dz = \rho W L \frac{\partial}{\partial t}(z) = \rho W L \frac{dz}{dt}$$

2 $$\iint_{C.S.} \rho \vec{V} \cdot d\vec{A} = \iint_{C.S.} \rho \underbrace{\vec{V} \cdot \hat{n}}_{V_{avg}} \, dA =$$

($\vec{V}$ and $\hat{n}$ are parallel and point in same direction)

$$dA = r \, dr \, d\theta$$

w/ $Re_D = \frac{V_{avg} D}{\nu}$ ∴ $V_{avg} = Re_D \frac{\nu}{D} = 2000 \frac{1 \times 10^{-6}}{1 \times 10^{-2}} = 0.2 \ \frac{m}{s}$

$$\Rightarrow = \int_0^{2\pi}\int_0^{R} \rho V_{avg} \, r \, dr \, d\theta = 2\pi \int_0^R \rho V_{avg} \, r \, dr = 2\pi \rho V_{avg} \left[\tfrac{1}{2} r^2\right]_0^R =$$

$$= \rho V_{avg} \pi R^2 = \rho V_{avg} A_{circular\ tube}$$

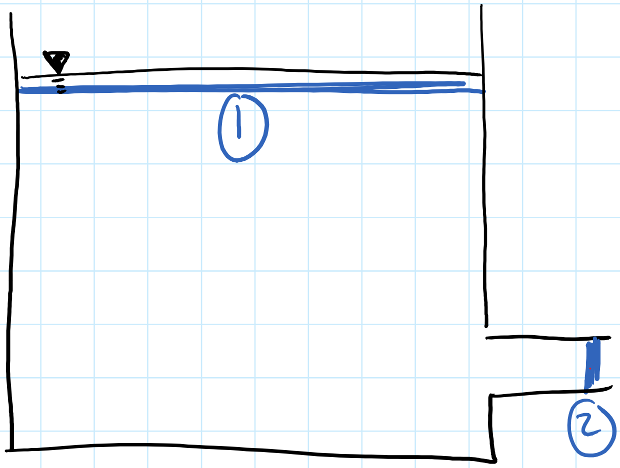
$$① + ② = 0: \therefore \boxed{\frac{dz}{dt} = \frac{\rho V_{avg} \pi R^2}{\rho W L} = V_{avg} \frac{\pi R^2}{W L}}$$

w/ numerical values:

$$\frac{dz}{dt} = 0.2 \frac{m}{s} \frac{\pi \left(\frac{10mm}{2}\right)^2}{0.3m \; 0.5m} = 1.0472 \times 10^{-4} \frac{m}{s}$$

$$= 0.10 \frac{mm}{s}$$

Notes & alternative way to solve it:



The result essentially says:

$$(\rho A V)_{①} = (\rho A V)_{②} \quad \left(= 1.57 \times 10^{-2} \frac{kg}{s}\right)$$

→ For the moment when the valve is first opened, the flow situation is equivalent to the steady state situation where water would be replaced @ ① at the same rate as it leaves the C.V. @ ②.

Check: ①

$$\iint_{C.S._①} \rho \vec{V} \cdot \hat{n} \, dA = \iint_{C.S._①} \rho \left(-\frac{dz}{dt}\right) dy\, dz = -\rho W L \frac{dz}{dt} \checkmark$$

same answer as from unsteady term (volume integral)

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Problem 5.3
A piston inside a cylinder filled with oil moves downward with velocity v_P (cf. Figure 2). The velocity distribution of the oil exiting at the top surface of the cylinder is measured *relative* to the piston and is given by $w(r) = W_o\left[1-\left(\frac{r}{R_1}\right)^2\right]$.
For known R_1, R_2, v_P and $\rho = const.$, determine the maximum oil velocity W_o

(a) using a coordinate system moving with the piston (piston-fixed).

(b) using a coordinate system fixed in space.

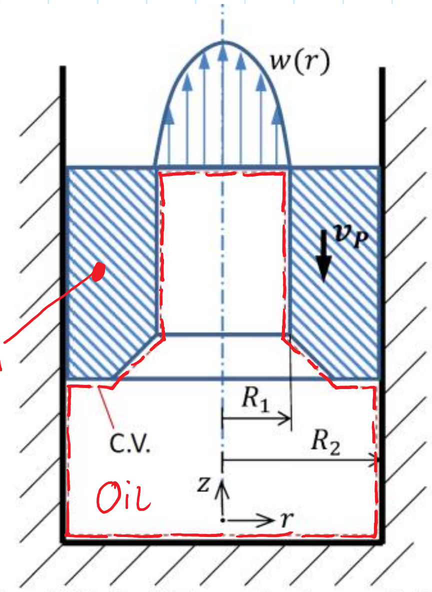
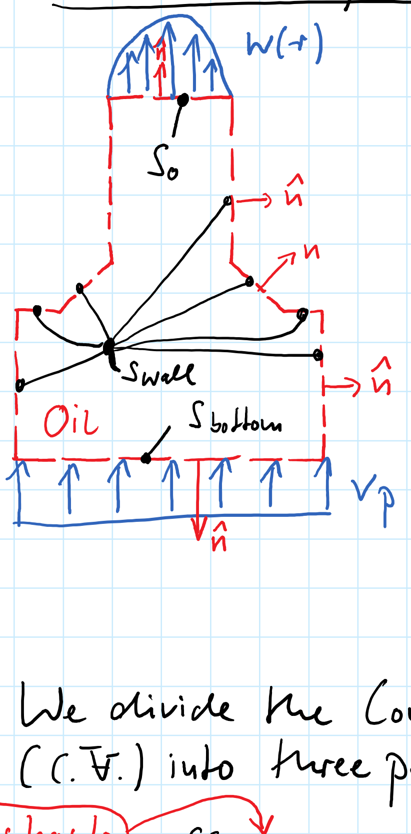


Figure 2: Sketch of piston moving downward inside a cylinder filled with oil.

(a) For coordinate system moving w/ piston:



For a coordinate system moving with the piston at v_P, the lower cylinder wall is moving towards the piston with v_P. This means that the relative velocity at the bottom boundary of the C.V. is v_P.

This can be treated as a steady-state problem $\frac{\partial}{\partial t} = 0$

Governing equation conservation of mass:

$$\cancel{\frac{\partial}{\partial t}}\underbrace{\iiint_{C.V.} \rho \, d\forall}_{=0} + \iint_{C.S.} \rho \vec{v}\cdot\hat{n}\, dS = 0$$

We divide the Control Surface (C.S.) bounding the Control Volume (C.∀.) into three parts:

this has to be the relative velocity

$$\iint_{S_0} \cancel{\rho} \underbrace{\vec{v}\cdot\hat{n}}_{=w(r)} dS_0 + \iint_{S_{wall}} \cancel{\rho} \underbrace{\vec{v}\cdot\hat{n}}_{=0} dS_{wall} + \iint_{S_{bottom}} \cancel{\rho} \underbrace{\vec{v}\cdot\hat{n}}_{=-v_P} dS_{bottom} = \emptyset$$

$=0$, since $\vec{v} \perp \hat{n}$ everywhere on stationary walls

$=-v_P$ ||, but opposite direction

velocity $S_0 \quad = w(r) \quad$ and S_{bottom}

$\vec{v} \perp \hat{n}$ everywhere on stationary walls

$= -V_p$

$\parallel$, but opposite direction ($\cos\pi = -1$)

$$\therefore \quad \iint_{S_0} w(r)\, dS_0 - V_p \iint_{S_{bottom}} dS_{bottom} = 0$$

<u>cylindrical coordinates:</u>

$dS = r\,dr\,d\varphi$

$$V_p \int_0^{2\pi}\int_0^{R_2} r\,dr\,d\varphi = \int_0^{2\pi}\int_0^{R_1} \bar{W}_0 \left(1-\left(\frac{r}{R_1}\right)^2\right) r\,dr\,d\varphi$$

$$V_p \cancel{2}\pi \cancel{\tfrac{1}{2}} R_2^2 = 2\pi \int_0^{R_1} \bar{W}_0 \left(1-\left(\frac{r}{R_1}\right)^2\right) r\,dr$$

Integrate w/ substitution: $a = \left(\frac{r}{R_1}\right)^2 \Leftrightarrow \frac{da}{dr} = 2\frac{r}{R_1^2} \Leftrightarrow dr = \frac{R_1^2}{2r}da$

$$\therefore \quad \int_0^{R_1} \bar{W}_0 \left(1-\left(\frac{r}{R_1}\right)^2\right) r\,dr = \bar{W}_0 \int_0^1 (1-a)\,\cancel{r}\,\frac{R_1^2}{2\cancel{r}}\,da =$$

$$= \bar{W}_0 \frac{R_1^2}{2} \int_0^1 (1-a)\,da =$$

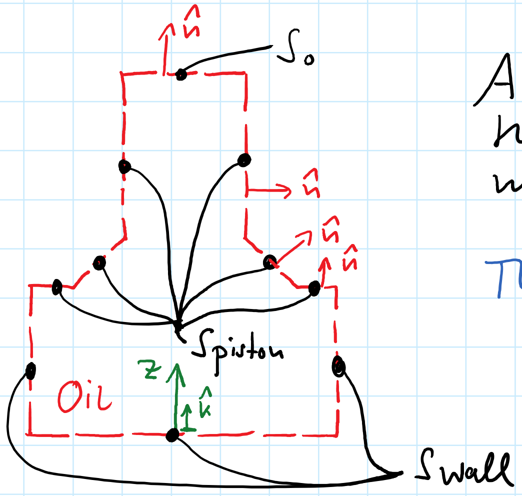
$$= \bar{W}_0 \frac{R_1^2}{2}\left[a - \frac{a^2}{2}\right]_0^1 = \frac{1}{4}\bar{W}_0 R_1^2$$

$$\therefore \quad V_p \cancel{\pi} R_2^2 = \cancel{2\pi}\,\frac{1}{\cancel{4}\,2}\bar{W}_0 R_1^2$$

$$\boxed{\bar{W}_0 = 2\bar{V}_p \left(\frac{R_2}{R_1}\right)^2}$$

(twice the piston velocity times the area ratio. This makes sense, since we have a parabolic profile at the top and different diameters.)

(b) <u>For a fixed coordinate system:</u>



Again, we can divide our C.V. into three parts, but we now have S_{piston} moving downward with V_p.

The relative velocity @ S_0 is

$$\vec{v} = \vec{W}(r) - \vec{V}_p \quad \text{(has only z-component)}$$

has to be <u>relative velocity</u>:

$\iint \ldots$

has to be relative velocity:

$$\iint_{S_0} \cancel{\rho}\, \underbrace{\vec{v}}_{=\left\{W_0\left[1-\left(\frac{r}{R_1}\right)^2\right]-v_p\right\}} \cdot \hat{n}\, dS_0 + \underbrace{\iint_{S_{piston}} \cancel{\rho}\, \vec{v}\cdot\hat{n}\, dS_{piston}}_{\text{(see below)}} + \iint_{S_{wall}} \cancel{\rho}\, \vec{v}\cdot\hat{n}\, dS_{wall} = \emptyset$$

(see below)

$\vec{v}\cdot\hat{n} = \emptyset$, since $\vec{v} \perp \hat{n}$

On piston surfaces:

$$\vec{v}_{rel}\cdot\hat{n} = \underbrace{\vec{v}_{wall}}\cdot\hat{n} = \underbrace{-v_p\,\hat{k}}\cdot\hat{n}$$

piston wall move in z-dir. with $-v_p$

$$\therefore \iint_{S_{piston}} \vec{v}\cdot\hat{n}\, dS_p = \iint_{S_{piston}} -v_p\, \underbrace{\hat{k}\cdot\hat{n}\, dS_p} = \ldots$$

Note that this is the projection of the piston in the z-direction, i.e.

$$\hat{k}\cdot\hat{n}\, dS_{piston} = \pm \underbrace{dA_z}$$

area with $\hat{k}$ as surface normal

On S_{piston}: $\hat{k}$ and $\hat{n}$ always have an angle $\leq \frac{\pi}{2}$

$\therefore$ sign $(\hat{k}\cdot\hat{n}) \geq 0$ (positive)

$$\therefore \ldots = -v_p \int_0^{2\pi}\int_{R_1}^{R_2} r\, dr\, d\varphi = -v_p \cancel{2}\pi \frac{1}{\cancel{2}}\left(R_2^2 - R_1^2\right)$$

On outflow surface S_0:

$$\iint_{S_0} \vec{v}\cdot\hat{n}\, dS_0 = \int_0^{2\pi}\int_0^{R_1} \left\{W_0\left(1-\left(\frac{r}{R_1}\right)^2\right) - v_p\right\} r\, dr\, d\varphi$$

$$= 2\pi\left(\frac{1}{4}W_0 R_1^2 - \frac{1}{2}R_1^2 v_p\right) = \pi R_1^2\left(\frac{1}{2}W_0 - v_p\right)$$

Putting it together:

$$-v_p\pi\left(R_2^2 - R_1^2\right) = -\pi R_1^2\left(\frac{1}{2}W_0 - v_p\right)$$

$$-v_p\pi R_2^2 + \cancel{v_p\pi R_1^2} = -\frac{1}{2}\pi R_1^2 W_0 + \cancel{\pi R_1^2 v_p}$$

$$\therefore \boxed{W_0 = 2 v_p\left(\frac{R_2}{R_1}\right)^2}$$

(same result as for (a), as it should be)
(This is a good check whether we did it correctly.)

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Problem 5.4

A short circular pipe with diameter D_1 has water (ρ, μ) flowing through it from left to right at a volumetric flow rate Q. At the exit a plug with diameter D_2 that partially blocks the water flow is inserted (to provide upstream pressure p_{1g} (gage) to the pipe and/or to reduce flow rate). Both upstream pressure and flow rate can be measured.

(a) Using the conservation equations (mass, momentum) in control volume formulation, derive an expression for the force required to hold the plug in place in terms of known quantities. State all the assumptions you are making.

(b) For $D_1 = 0.1m$, $D_2 = 0.075m$, $\rho = 1000\ kg/m^3$, upstream $p_{1g} = 500\ kPa$ and $Q = 0.1\ m^3/s$, calculate the magnitude of this force.

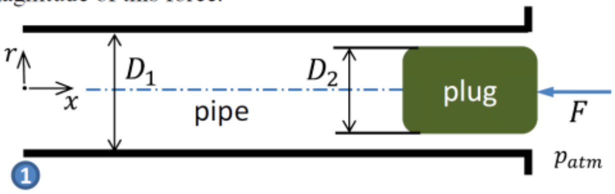
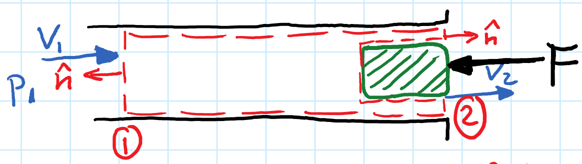


Figure 3: Sketch of pipe with plug of diameter D_2 at outlet on right.



$A_1 = D_1^2 \frac{\pi}{4}$; $V_1 = \frac{Q}{A_1}$

$A_2 = (D_1^2 - D_2^2)\frac{\pi}{4}$

$p_1 = p_{1g} + p_{atm}$
(p_{1g} is gage pressure)

$p_2 = p_{atm}$

assumptions:
- steady state, $\frac{\partial}{\partial t} = 0$
- upstream pressure p_{1g} and flow rate Q are known.
- uniform flow @ 1 and 2
- neglect friction on walls ("short" pipe)
- const. density $\rho = const.$

(a) conservation of mass

$$\cancel{\frac{\partial}{\partial t}\iiint_{C.V.} \rho\, dV}^{=0} + \iint_{C.S.} \rho \vec{v}\cdot\hat{n}\, dS = 0$$

$$\iint_{C.S.1} \rho \underbrace{\vec{v}\cdot\hat{n}}_{=-V_1} dS_1 + \iint_{C.S.2} \rho \underbrace{\vec{v}\cdot\hat{n}}_{=V_2} dS_2 = 0$$

$$-\cancel{\rho} V_1 A_1 + \cancel{\rho} V_2 A_2 = 0$$

$$\therefore\ V_2 = V_1 \frac{A_1}{A_2} = \frac{Q}{\cancel{A_1}}\frac{\cancel{A_1}}{A_2} = \frac{Q}{(D_1^2 - D_2^2)\frac{\pi}{4}}$$

conservation of momentum

$$A_2 \quad A_1 \quad A_2 \quad (D_1^2 - D_2^2)\frac{\pi}{4}$$

Conservation of momentum

$$\cancel{\frac{\partial}{\partial t}}^{=0} \iiint_{V} \rho \vec{v}\, dV + \iint_{C.S.} \rho \vec{v} (\vec{v}\cdot\hat{n}) dS = \cancel{\vec{F}_B}^{=0,\text{ no body force in x-direction}} + \vec{F}_S$$

x-component:

$$\iint_{C.S.①} \rho v_1 \underbrace{(\vec{v}\cdot\hat{n})}_{=-v_1} dS_① + \iint_{C.S.②} \rho v_2 \underbrace{(\vec{v}\cdot\hat{n})}_{=v_2} dS_② = \iint_{C.S.①} \underbrace{p_1}_{(p_{1g}+p_{atm})} dS_① - \iint_{C.S.②} \underbrace{p_2}_{(p_{atm})} dS_② + \vec{F}\cdot\hat{n}$$

force exerted on fluid by plug

$$\therefore F = \rho v_1^2 A_1 - \rho v_2^2 A_2 + p_{1g} A_1$$

$$= \rho\left(\frac{Q}{A_1}\right)^2 A_1 - \rho\left(\frac{Q}{A_2}\right)^2 A_2 + p_{1g} A_1$$

$$\left[\frac{N}{m^2} m^2\right] = [N] (✓)$$

$$= \frac{4\rho Q^2}{D_1^2 \pi} - \frac{4\rho Q^2}{(D_1^2 - D_2^2)\pi} + p_{1g} D_1^2 \frac{\pi}{4} = \frac{4\rho Q^2}{\pi}\left(\frac{1}{D_1^2} - \frac{1}{D_1^2 - D_2^2}\right) + p_{1g} D_1^2 \frac{\pi}{4}$$

unit check: $\left[\frac{kg}{\cancel{m^3}} \frac{m^3}{s} \frac{\cancel{m^3}}{s} \frac{1}{\cancel{m^2}}\right] = \left[\frac{kg\, m}{s^2}\right] = [N]$ (✓)

(b) with: $D_1 = 0.1\ m$, $D_2 = 0.075\ m$, $\rho = 1000 \frac{kg}{m^3}$, $p_1 = 500\ kPa$, $Q = 0.1 \frac{m^3}{s}$

$$v_1 = \frac{Q}{A_1} = 12.7 \frac{m}{s} \quad ; \quad v_2 = \frac{Q}{A_2} = 29.1 \frac{m}{s}$$

$$\left(Re_{D_1} = \frac{D_1 v_1}{\nu} = \frac{0.1 m \cdot 12.7 \frac{m}{s}}{1\times10^{-6} \frac{m^2}{s}} = 1.27\times10^6 \rightarrow \text{high Re\# turbulent flow}\right)$$

$$\therefore F = \underbrace{\rho v_1^2 A_1}_{\text{I}} - \underbrace{\rho v_2^2 A_2}_{\text{II}} + \underbrace{p_1 A_1}_{\text{III}}$$

$$= 1273\,N - 2910\,N + 3927\,N = 2290\,N$$

Discussion:

$$\boxed{F = \frac{4\rho Q^2}{\pi}\left(\frac{1}{D_1^2} - \frac{1}{D_1^2 - D_2^2}\right) + p_{1g} D_1^2 \frac{\pi}{4}}$$

As $D_2 \rightarrow D_1$: $\frac{1}{D_1^2 - D_2^2} \rightarrow \infty$, hence $F \rightarrow \infty$

Why is that? Well, we are prescribing a fixed flow rate Q, hence as the area through which Q has to flow decreases, the difference in (rate of) momentum required to do so increases. As the area through which Q has to flow goes to $\emptyset$, the Force required to maintain the flow-rate, and hence the force to hold the piston in place, goes to infinity.

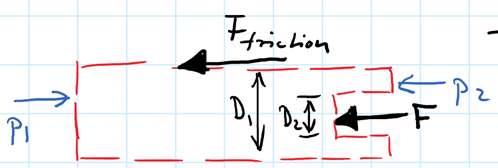
As $D_2 \to 0$: This means we shrink the piston until it vanishes.

$$F = \frac{4\rho Q^2}{\pi}\underbrace{\left(\frac{1}{D_1^2} - \underbrace{\frac{1}{D_1^2 - D_2^2}}_{\to 1/D_1^2}\right)}_{\to \emptyset} + p_{1g} D_1^2 \frac{\pi}{4}$$

so that $F \to p_{1g} D_1^2 \frac{\pi}{4}$ (?)

This is a peculiar result. With the piston gone ($D_2 \to 0$), there should be no force.

Ask yourself: With (inflow momentum) = (outflow momentum), what is there that could cause $p_1 > p_2$? (or: $p_{1g} > 0$)?



$\to$ Friction on the walls!
(which we neglected. $\to$ Draw a free-body diagram, and include the frictional force on the walls. This is the result we are getting!)