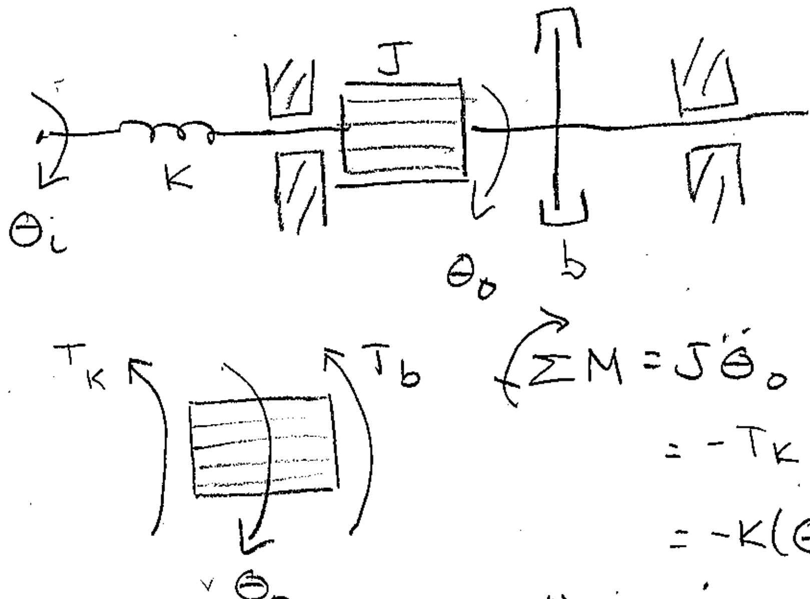
PROBLEM SET 5 SOLUTIONS

Problem 1



$$\sum M = J\ddot{\theta}_o$$

$$= -T_K - T_b$$

$$= -K(\theta_o - \theta_i) - b\dot{\theta}_o$$

$$\underline{J\ddot{\theta}_o + b\dot{\theta}_o + K\theta_o = K\theta_i}$$

Standard 2nd order form: $\dfrac{\ddot{q}}{\omega_n^2} + \dfrac{2\zeta\dot{q}}{\omega_n} + q = \bar{K} q_i(t)$

$$\frac{J\ddot{\theta}_o}{K} + \frac{b}{K}\dot{\theta}_o + \theta_o = \theta_i$$

$$\Rightarrow \omega_n = \sqrt{\frac{K}{J}} \qquad \bar{K} = 1 \qquad \underline{\zeta = \frac{b}{2\sqrt{KJ}}} \qquad \Rightarrow b = 2\sqrt{KJ}\,\zeta$$

Choose $K = J = 5$

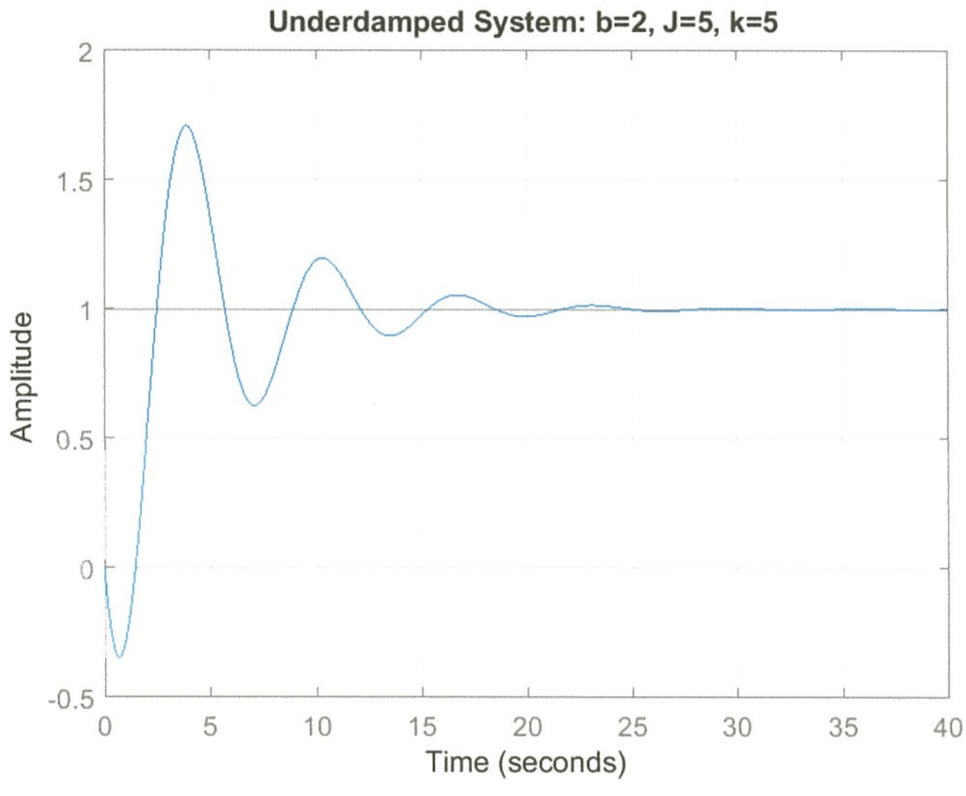
1) underdamped system ($\zeta \overset{e.g.}{=} 0.2$) $\Rightarrow$ $\underline{b = 2}$

2) critically damped system ($\zeta = 1$) $\Rightarrow$ $\underline{b = 10}$

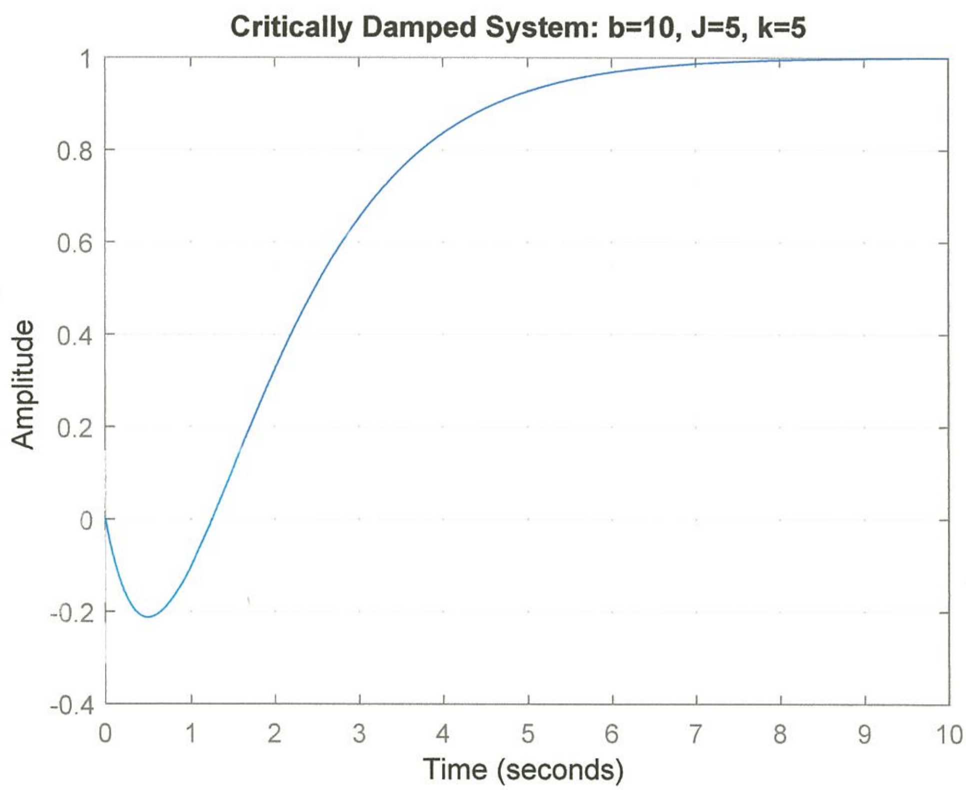
3) overdamped system (e.g. $\zeta = 2$) $\Rightarrow$ $\underline{b = 20}$

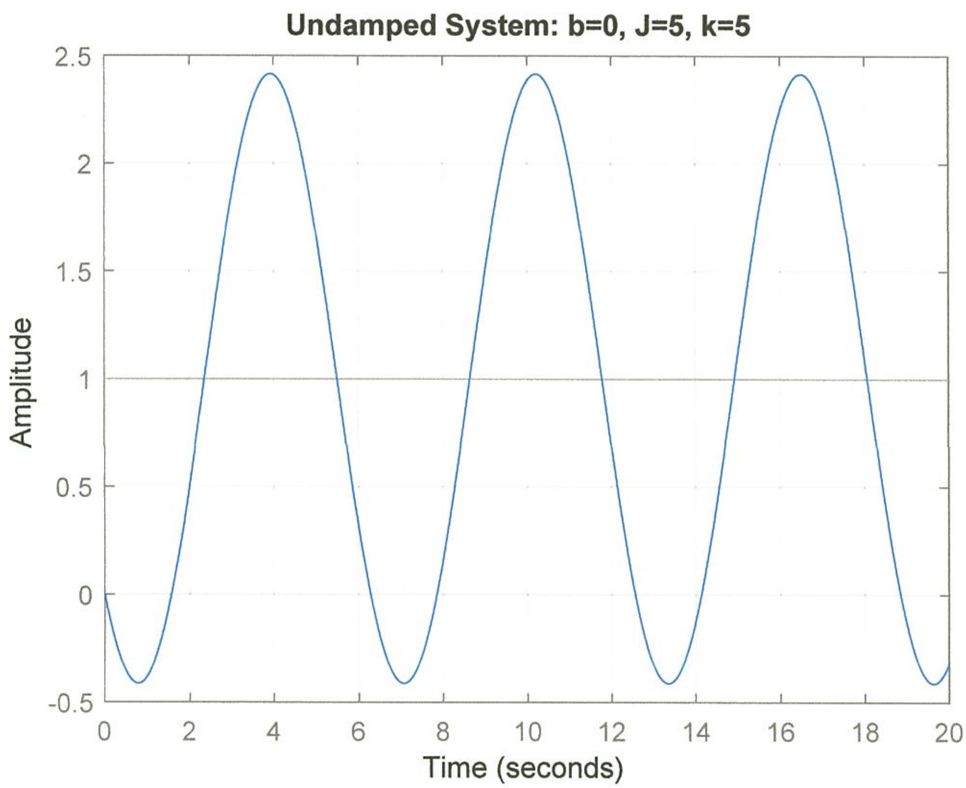
4) undamped system ($\zeta = 0$) $\Rightarrow$ $\underline{b = 0}$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Problem Set 5 Solutions
% Problem 1 - From dynamic system of Ogata B-3-13
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all

% Define system parameters
k = 5;
J = 5;
b = 2;        % Underdamped System: zeta=0.2 --> b=2
              % Criticially Damped System: zeta=1.0 --> b=10
              % Overdamped System: zeta=2.0 --> b=20
              % Undamped System: zeta=0 --> b=0

% Calculate dynamic parameters
K = 1;
w_n = sqrt(k/J);
zeta = b/(2*sqrt(k*J));

% Define dynamic system as a state space model (necessary to use
% non-zero initial conditions for LSIM command
num = K;
den = [1/w_n 2*zeta/w_n 1];
sys_tf = tf(num,den);         % system as a transfer function
sys_ss = ss(sys_tf);          % system as a state space model


figure(1)
%step(sys)    % The step command does not take into account initial
              % conditions

t = 0:0.01:40;
u = ones(1,4001);
lsim(sys_ss,u,t,[-1 0]);
grid;
title('Underdamped System: b=2, J=5, k=5')
% xlabel('Time (s)');
ylabel('Amplitude');
```



Underdamped System: b=2, J=5, k=5
Amplitude
Time (seconds)
2
1.5
1
0.5
0
-0.5
0
5
10
15
20
25
30
35
40

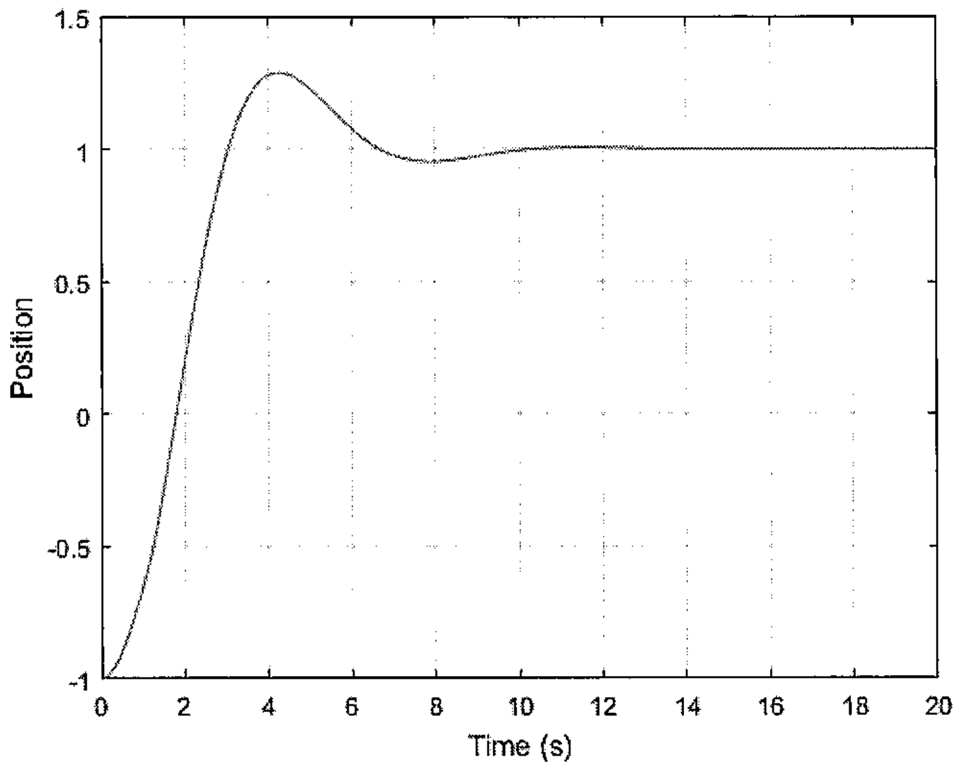

Critically Damped System: b=10, J=5, k=5
Amplitude
Time (seconds)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
2
3
4
5
6
7
8
9
10


Undamped System: b=0, J=5, k=5
Amplitude
Time (seconds)
2.5
2
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
12
14
16
18
20

```
clc
close all
clear all
```

Problem 2

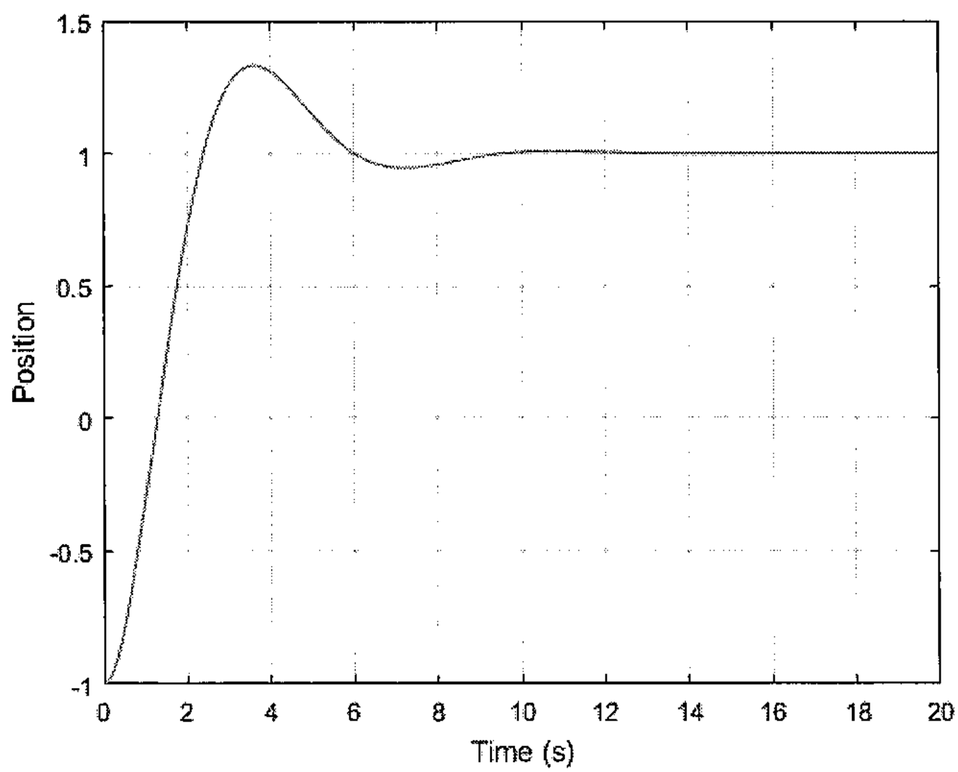
```
% System Parameters (Modified to change system response)
J = 1;
K = 1;
B = 1;

% Undamped: J,B,K = 1,0,1
% Underdamped: J,B,K = 1,1,1
% Critically Damped: J,B,K = 1,2,1
% Overdamped: J,B,K = 1,4,1

% Run simulation
sim('Problem2')

% Plot Figure
figure;
plot(tout,theta)
grid on
xlabel('Time (s)')
ylabel('Position')
```



Problem 3

```
clc
close all
clear all

 % System Parameters (Modified to change system response)
 J = 1;
 K = 1;
 B = 1;

 % Undamped: J,B,K = 1,0,1
 % Underdamped: J,B,K = 1,1,1
 % Critically Damped: J,B,K = 1,2,1
 % Overdamped: J,B,K = 1,4,1

% Time settings
start_time = 0;
final_time = 20;
dt = 0.01;

% Number of data points
steps = final_time/dt;

% State State Representation
```

```
A = [0 1; -K/J -B/J];
B = [0;K/J];
C = [1 0; 0 1];
D = [0;0];
x = [-1;0]; % Inital Condition
u = 1;
x_dot = (A*x)+(B*u);

% Integration with For Loop
for t = 1:steps
    x = x_dot*dt + x; % Integrate State Matrix
    x_dot = A*x+B*u; % Determine State Values
    y(t) = x(1); % Assign Output Values
end

time = linspace(0,steps*dt,steps); % Extract time data from data
 points

% Plot Response
figure;
plot(time,y);
grid on
xlabel('Time (s)')
ylabel('Position')
```

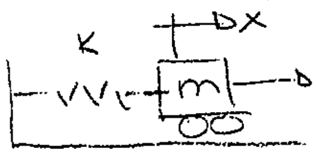


Published with MATLAB® R2017a

PROBLEM SET 5

Problem 4 : B-5-1

(a)



$$\Sigma F = m\ddot{x} = -F_K + u(t)$$
$$= -Kx + u(t)$$

$$\Rightarrow \underline{m\ddot{x} + Kx = u(t)}$$

$$\underline{x} = \begin{Bmatrix} x_1 = x \\ x_2 = \dot{x} \end{Bmatrix} \qquad u = u(t) \qquad y = x$$

$$\dot{x}_1 = \dot{x} = x_2$$
$$\dot{x}_2 = \ddot{x} = \frac{-K}{m}x + \frac{1}{m}u = \frac{-K}{m}x_1 + \frac{1}{m}u$$
$$y = x = x_1$$

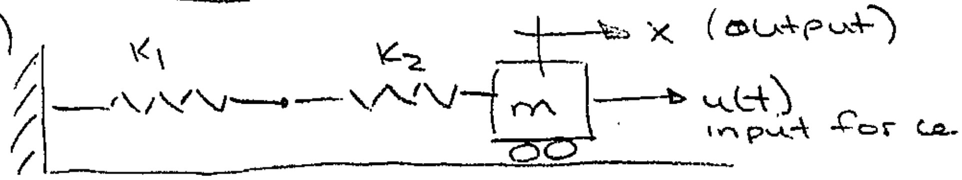
$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{m} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

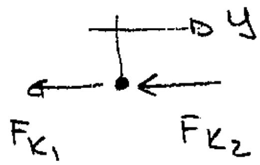
$$\underline{x} = \begin{Bmatrix} x_1 = x \\ x_2 = \dot{x} \end{Bmatrix} \qquad u = u(t)$$

Problem 4 (cont'd)

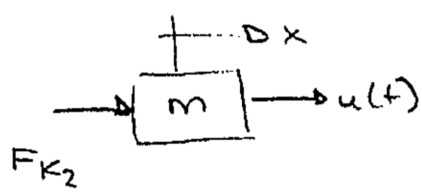
(b)



$\Rightarrow$ define y as displacement of node between springs



$$\overrightarrow{\sum F} = 0$$
$$= -F_{k_1} - F_{k_2}$$
$$= -k_1 y - k_2(y - x)$$
$$(k_1 + k_2) y = k_2 x \quad : \underline{\text{not}} \text{ ODE} \quad y = \frac{k_2 x}{k_1 + k_2}$$



$$\overrightarrow{\sum F} = m\ddot{x}$$
$$= F_{k_2} + u(t)$$
$$= k_2(y - x) + u(t)$$

$$\Rightarrow m\ddot{x} + k_2 x = k_2 y + u(t)$$
$$\therefore m\ddot{x} + k_2 x = \frac{k_2^2 x}{k_1 + k_2} + u(t)$$
$$\underline{m\ddot{x} + \frac{k_1 k_2}{k_1 + k_2} x = u(t)}$$

$$\underline{x} = \begin{Bmatrix} x_1 = x \\ x_2 = \dot{x} \end{Bmatrix} \qquad u = u(t) \qquad y = x$$

$$\dot{x}_1 = \dot{x} = x_2$$
$$\dot{x}_2 = \ddot{x} = \frac{-k_1 k_2}{m(k_1 + k_2)} x + \frac{1}{m} u(t) = -\frac{k_1 k_2}{m(k_1 + k_2)} x_1 + \frac{1}{m} u$$
$$y = x = x_1$$

Problem 4 (cont'd)

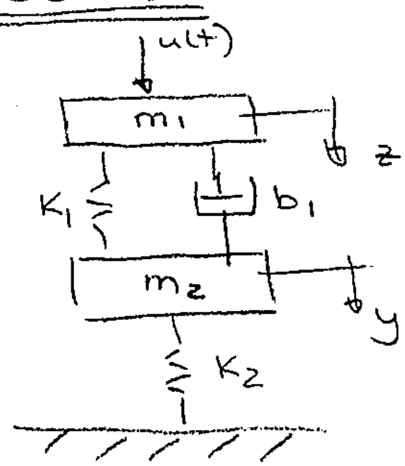
(b) cont'd

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1 k_2}{m(k_1+k_2)} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

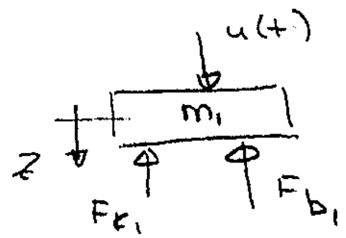
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$\underline{x} = \begin{Bmatrix} x_1 = x \\ x_2 = \dot{x} \end{Bmatrix} \qquad u = u(t)$$

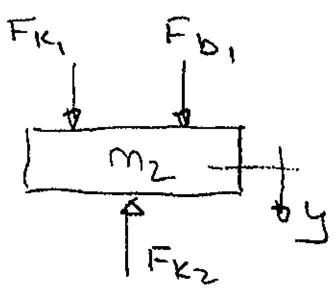
Problem 5 : B-5-3



output : y, z



$$\downarrow \Sigma F = m_1 \ddot{z}$$
$$= -F_{k_1} - F_{b_1} + u(t)$$
$$= K_1(z-y) - b_1(\dot{z}-\dot{y}) + u(t)$$
$$\Rightarrow \underline{m_1 \ddot{z} + b_1 \dot{z} + K_1 z = b_1 \dot{y} + K_1 y + u(t)}$$



$$\downarrow \Sigma F = m_2 \ddot{y}$$
$$= F_{k_1} + F_{b_1} - F_{k_2}$$
$$= K_1(z-y) + b_1(\dot{z}-\dot{y}) - K_2 y$$
$$\Rightarrow \underline{m_2 \ddot{y} + b_1 \dot{y} + (K_1+K_2) y = b_1 \dot{z} + K_1 z}$$

$$\underline{x} = \begin{Bmatrix} x_1 = z \\ x_2 = \dot{z} \\ x_3 = y \\ x_4 = \dot{y} \end{Bmatrix} \qquad u = u(t) \qquad y = \begin{bmatrix} y_1 = y \\ y_2 = z \end{bmatrix}$$

$$\dot{x}_1 = \dot{z} = x_2$$
$$\dot{x}_2 = \ddot{z} = \frac{1}{m_1}\left[-b_1\dot{z} - K_1 z + b_1\dot{y} + K_1 y + u\right]$$
$$= \frac{1}{m_1}\left[-b_1 x_2 - K_1 x_1 + b_1 x_4 + K_1 x_3 + u\right]$$

$$\dot{x}_3 = \dot{y} = x_4$$
$$\dot{x}_4 = \ddot{y} = \frac{1}{m_2}\left[-b_1\dot{y} - (K_1+K_2)y + b_1\dot{z} + K_1 z\right]$$
$$= \frac{1}{m_2}\left[-b_1 x_4 - (K_1+K_2)x_3 + b_1 x_2 + K_1 x_1\right]$$
$$y_1 = y = x_3$$

Problem 5. (Cont'd)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & \frac{-b_1}{m_1} & \frac{k_1}{m_1} & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & \frac{-(k_1+k_2)}{m_2} & \frac{-b_1}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\underline{x} = \begin{Bmatrix} x_1 = z \\ x_2 = \dot{z} \\ x_3 = y \\ x_4 = \dot{y} \end{Bmatrix} \qquad u = u(t)$$

Problem 6 : B-5-5

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Method I

$\Rightarrow y = x_1$

① $\dot{x}_1 = x_2 + u$

② $\dot{x}_2 = x_3 + u$

③ $\dot{x}_3 = x_1 - 3x_2 + 3x_3 + u$

$\Rightarrow$ ① $\dot{y} = x_2 + u \therefore x_2 = \dot{y} - u$

$\therefore \underline{\dot{x}_2 = \ddot{y} - \dot{u}}$

② $\ddot{y} - \dot{u} = x_3 + u \therefore x_3 = \ddot{y} - \dot{u} - u$

$\underline{\dot{x}_3 = \dddot{y} - \ddot{u} - \dot{u}}$

③ $\dddot{y} - \ddot{u} - \dot{u} = y - 3\left[\dot{y} - u\right] + 3\left[\ddot{y} - \dot{u} - u\right] + u$

$\Rightarrow \boxed{\dddot{y} - 3\ddot{y} + 3\dot{y} - y = \ddot{u} - 2\dot{u} + 2u}$

Method II

$$G(s) = C\left[sI - A\right]^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 3 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0$$

Problem 6 (cont'd)

Method II (cont'd)

$$G(s) = [1 \quad 0 \quad 0] \frac{\text{adj}[sI-A]}{\det[sI-A]} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\det[sI-A] = [s^2(s-3)-1] - [-3s]$$
$$= s^3 - 3s^2 + 3s - 1$$

$$\text{adj}[sI-A] = C^T_{[sI-A]}$$

$$C_{11} = (-1)^{1+1} \det \begin{vmatrix} s & -1 \\ 3 & s-3 \end{vmatrix} = s^2 - 3s + 3$$

$$C_{12} = (-1)^{1+2} \det \begin{vmatrix} 0 & -1 \\ -1 & s-3 \end{vmatrix} = 1$$

$$C_{13} = (-1)^{1+3} \det \begin{vmatrix} 0 & s \\ -1 & 3 \end{vmatrix} = s$$

$$C_{21} = (-1)^{2+1} \det \begin{vmatrix} -1 & 0 \\ 3 & s-3 \end{vmatrix} = s-3$$

$$C_{22} = (-1)^{2+2} \det \begin{vmatrix} s & 0 \\ -1 & s-3 \end{vmatrix} = s^2 - 3s$$

$$C_{23} = (-1)^{2+3} \det \begin{vmatrix} s & -1 \\ -1 & 3 \end{vmatrix} = -3s + 1$$

$$C_{31} = (-1)^{3+1} \det \begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} = 1$$

$$C_{32} = (-1)^{3+2} \det \begin{vmatrix} s & 0 \\ 0 & -1 \end{vmatrix} = s$$

$$C_{33} = (-1)^{3+3} \det \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} = s^2$$

Problem 6 (cont'd)

Method II (cont'd)

$$C^T_{[sI-A]} = \begin{bmatrix} s^2-3s+3 & s-3 & 1 \\ 1 & s^2-3s & s \\ s & -3s+1 & s^2 \end{bmatrix}$$

$$\therefore\ G(s) = \frac{[1 \quad 0 \quad 0]}{s^3-3s^2+3s-1}\begin{bmatrix} s^2-3s+3 & s-3 & 1 \\ 1 & s^2-3s & s \\ s & -3s+1 & s^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{[s^2-3s+3 \quad s-3 \quad 1]}{s^3-3s^2+3s-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{[s^2-3s+3]+[s-3]+1}{s^3-3s^2+3s-1}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^2-2s+1}{s^3-3s^2+3s-1}$$

$$[s^3-3s^2+3s-1]Y(s) = [s^2-2s+1]U(s)$$

$$\boxed{\dddot{y} - 3\ddot{y} + 3\dot{y} - y = \ddot{u} - 3\dot{u} + u}$$

✓

Problem 7   B-5-6

$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$, output $= y$

$$\underline{x} = \begin{Bmatrix} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \end{Bmatrix} \qquad u = u \qquad y = y$$

$$\dot{x}_1 = \dot{y} = x_2$$
$$\dot{x}_2 = \ddot{y} = x_3$$
$$\dot{x}_3 = \dddot{y} = -6\ddot{y} - 11\dot{y} - 6y + 6u$$
$$= -6x_3 - 11x_2 - 6x_1 + 6u$$

$$y = y = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \underline{u}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \underline{u}$$

$$\underline{x} = \begin{Bmatrix} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \end{Bmatrix} \qquad \underline{u} = u$$

Problem 8 : B-5-7

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$G(s) = \underline{C} \left[s\underline{I} - \underline{A} \right]^{-1} \underline{B} + \underline{D}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}}{(s+4)(s+1)+3} \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s+1 & -1 \end{bmatrix}}{s^2 + 5s + 7} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{(s+1) + (-1)}{s^2 + 5s + 7}$$

$$\boxed{G(s) = \frac{s}{s^2 + 5s + 7}}$$