PROBLEM SET 6 SOLUTIONS

Problem 1

$$G_1(s) = \frac{s(s+100)}{(s^2+s+4)(s+10)(s+1000)}$$

$$f(t) = 5\sin(100t) \Rightarrow y_{ss}(t) = A_o \sin(100t + \phi_o)$$

$$A_o = \overset{5}{A_i} \cdot |G_1(j\omega)|_{\omega=100} \qquad \phi_o = \cancel{\phi_i}^{\,0} + \measuredangle G(j\omega)\big|_{\omega=100}$$

From Bode plot: $20|G_1(100j)| = -100\,dB$

$$\therefore \underline{|G_1(100j)| = 10^{-5}} \Rightarrow \underline{A_o = 5\times10^{-5}}$$

$$\measuredangle G(100j) = -130^\circ \Rightarrow \underline{\phi_o = -2.27\text{ rad}}$$

$$\therefore \boxed{y_{ss}(t) = 5\times10^{-5}\sin(100t - 2.27)}$$

$$G_1(j\omega) = \frac{j\omega(j\omega+100)}{(-\omega^2+j\omega+4)(j\omega+10)(j\omega+1000)}$$

$$|G(j\omega)| = \frac{\prod_{i=1}^{m}\sqrt{(Re)^2+(Im)^2}}{\prod_{j=1}^{n}\sqrt{(Re)^2+(Im)^2}}$$

$$|G_1(j\omega)|_{10^3} = \left.\frac{\sqrt{(\omega)^2}\sqrt{(100)^2+(\omega)^2}}{\sqrt{(4-\omega^2)^2+(\omega)^2}\sqrt{(10)^2+\omega^2}\sqrt{(10^3)^2+\omega^2}}\right|_{10^3}$$

$$\boxed{|G_1(j\omega)|_{10^3} = 7.1\times10^{-7}} \Rightarrow \text{matches Bode plot}$$

Problem 1 (Cont'd)

$$\measuredangle G(j\omega) = \left[\sum_{num} \tan^{-1}\left(\frac{Im}{Re}\right)\right] - \left[\sum_{den} \tan^{-1}\left(\frac{Im}{Re}\right)\right]$$

$$\measuredangle G_1(j\omega)\Big|_{10^3} = \left[\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{\omega}{100}\right)\right]$$

$$- \left[\tan^{-1}\left(\frac{\omega}{4-\omega^2}\right) + \tan^{-1}\left(\frac{\omega}{10}\right) + \tan^{-1}\left(\frac{\omega}{10^3}\right)\right]\Bigg|_{10^3}$$

$$= \left[\frac{\pi}{2} + 1.47\right] - \left[(\pi - 0.001) + 1.56 + \frac{\pi}{4}\right]$$

$$\boxed{\measuredangle G_1(j\omega)\Big|_{10^3} = -2.45 \text{ rad}}$$

Problem 2

$$G_2(s) = \frac{(s+10)(s^2+s+10^4)}{s(s+100)(s+10^{-4})}$$

$$f(t) = 10\sin t \implies y_{ss}(t) = A_0 \sin(t + \phi_0)$$

$$A_0 = 10 \times |G(j\omega)|\Big|_{\omega=1} \qquad \phi_0 = 0^\circ + \measuredangle G(j\omega)\Big|_{\omega=1}$$

From Bode plot:

$$20\log|G(j)| \approx 55\,dB \rightarrow |G(j)| = 560$$

$$\measuredangle G(j) \approx -220^\circ \rightarrow \measuredangle G(j) = -3.8 \text{ rad}$$

$$\therefore \boxed{y_{ss} = 5600\sin(t - 3.8)}$$

$$G_2(j\omega) = \frac{(j\omega+10)(-\omega^2+j\omega+10^4)}{j\omega(j\omega+100)(j\omega+10^{-4})}$$

$$G_2(j\omega)\Big|_{\omega=10^4} = \frac{\sqrt{(10)^2+(10^4)^2}\;\overbrace{\sqrt{(10^4-10^8)^2+(10^4)^2}}^{\approx 10^{12}}}{\underbrace{\sqrt{(10^4)^2}\sqrt{(100)^2+(10^4)^2}\;\sqrt{(10^{-4})^2+(10^4)^2}}_{\approx 10^{12}}} \approx \boxed{1.0 = G_2(10^4 j)}$$

$$\measuredangle G_2(j\omega)\Big|_{\omega=10^4} = \left[\tan^{-1}\left(\frac{10^4}{10}\right) + \tan^{-1}\left(\frac{10^4}{10^4-(10^4)^2}\right)\right]$$
$$- \left[\tan^{-1}\left(\frac{10^4}{0}\right) + \tan^{-1}\left(\frac{10^4}{100}\right) + \tan^{-1}\left(\frac{10^4}{10^{-4}}\right)\right]$$
$$= \left[\cancel{1.57} + (\approx\pi)\right] - \left[\cancel{\tfrac{\pi}{2}} + 1.56 + 1.57\right]$$

$$\boxed{\measuredangle G_2(10^4 j) = 0.01 \text{ rad}}$$

Problem 3

$$G_3(s) = \frac{s(s+1000)}{(s+10)(s^2+10^5 s+10^{10})}$$

(a) $$G_3(s) = \frac{s\left[10^3\cdot\left(\frac{s}{10^3}+1\right)\right]}{\left[10\left(\frac{s}{10}+1\right)\right]\left[10^{10}\left(\frac{s^2}{10^{10}}+\frac{s}{10^5}+1\right)\right]} = \frac{\overset{a}{10^{-8}}\ \overset{b}{s}\ \overset{c}{\left(\frac{s}{10^3}+1\right)}}{\underset{d}{\left(\frac{s}{10}+1\right)}\underset{e}{\left(\frac{s^2}{10^{10}}+\frac{s}{10^5}+1\right)}}$$

Determine break frequencies for each factor:

a $\omega_{b,a} = 0$

b $\omega_{b,b} = 0$

c $\omega_{b,c} = \frac{1}{\tau} = \frac{1}{10^{-3}} = 10^3$

d $\omega_{b,d} = \frac{1}{\tau} = \frac{1}{10^{-1}} = 10$

$\frac{2\zeta}{\omega_n} = 10^{-5}$

e $\omega_{b,e} = \omega_n = \sqrt{10^{10}} = 10^5 \Rightarrow \zeta \approx 0.5$, "hump" = 1dB

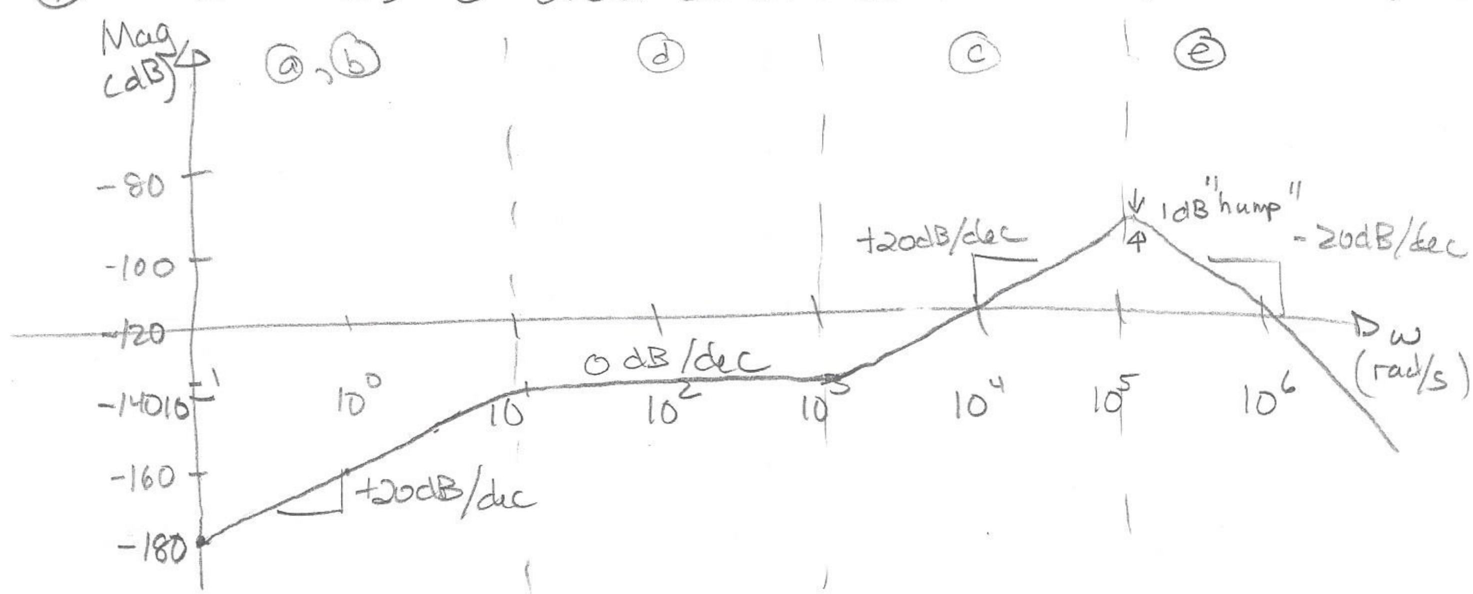
Identify factors in order of ω_b

1 $\Rightarrow$ a, b $\Rightarrow$ +20dB/dec slope

2 $\Rightarrow$ d $\Rightarrow$ 1st order denominator : Δslope = -20dB/dec

3 $\Rightarrow$ c $\Rightarrow$ 1st order numerator : Δslope = +20dB/dec

4 $\Rightarrow$ e $\Rightarrow$ 2nd order denominator : Δslope = -40dB/dec



Problem 3 (cont'd)

(a) (cont'd)

$$G_3(j\omega) = \frac{j\omega(j\omega + 1000)}{(j\omega+10)(-\omega^2 + 10^5 j\omega + 10^{10})}$$

$$|G_3(j\omega)|_{10^5} = \frac{\sqrt{(10^5)^2}\sqrt{1000^2+(10^5)^2}}{\sqrt{10^2+(10^5)^2}\sqrt{10^{10}-(10^5)^2+(10^5\cdot 10^5)^2}}$$

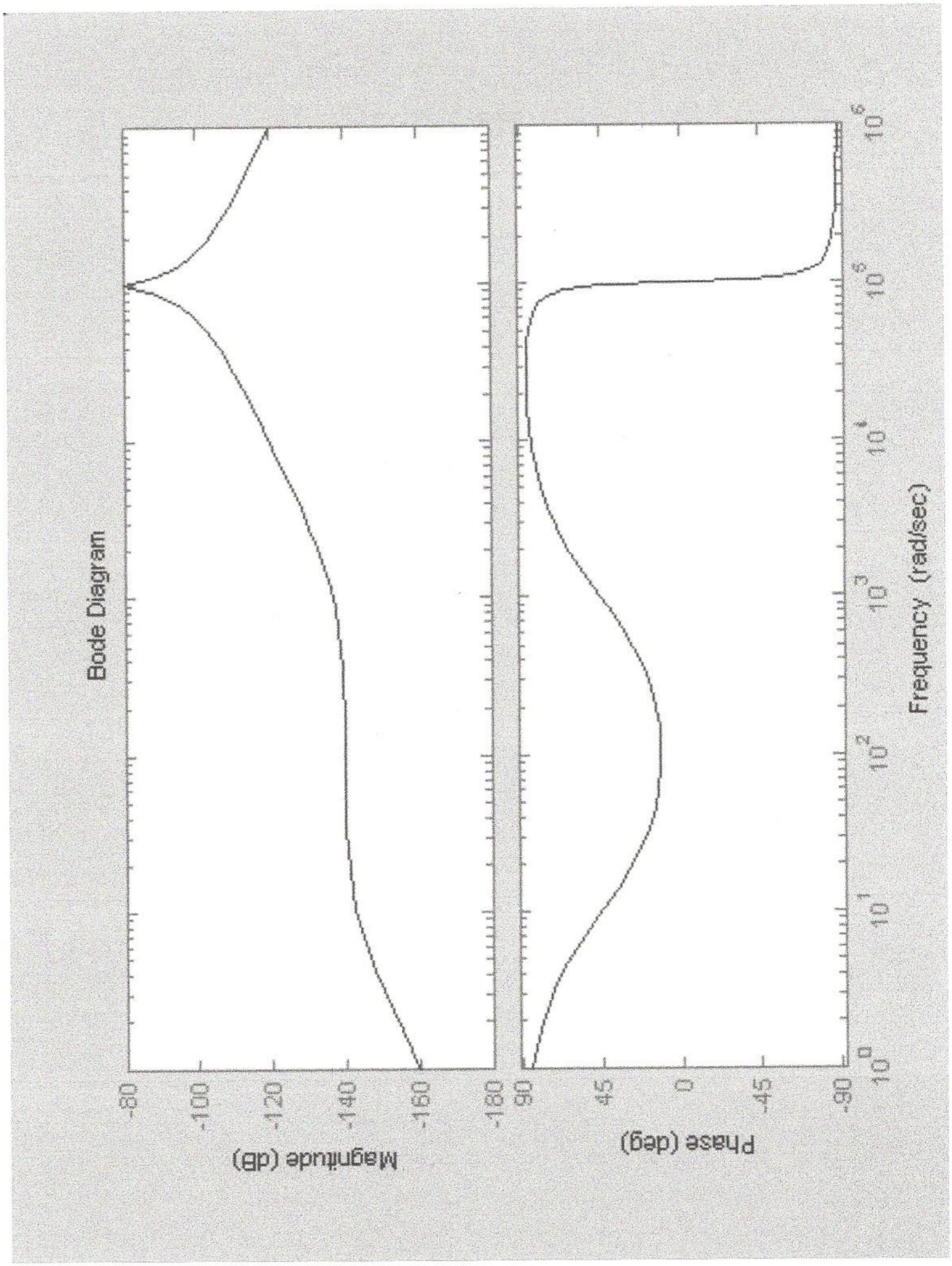
$$\boxed{|G(10^5 j)| \simeq 10^{-5}}$$

$$\measuredangle G_3(j\omega)\Big|_{10^5} = \left[\tan^{-1}\left(\frac{10^5}{0}\right) + \tan^{-1}\left(\frac{10^5}{10^3}\right)\right] - \left[\tan^{-1}\left(\frac{10^5}{10}\right) + \tan^{-1}\left(\frac{10^5\cdot 10^5}{10^{10}-(10^5)^2}\right)\right]$$

$$\boxed{\measuredangle G_3(10^5 j) \simeq 0 \text{ rad}}$$

Problem 3

(b)



Bode Diagram
Magnitude (dB)
-80
-100
-120
-140
-160
-180
Phase (deg)
90
45
0
-45
-90
Frequency (rad/sec)
10^0
10^1
10^2
10^3
10^4
10^5
10^6

Problem 4

$$G_4(s) = \frac{(s+1)(s^2+10^3 s+10^4)}{s(s+10^{-2})(s+10^8)}$$

$$(a)\ G_4(s) = \frac{(s+1)\left[10^4\left(\frac{s^2}{10^4}+10^{-1}s+1\right)\right]}{s\left[10^{-2}(10^2 s+1)\right]\left[10^8\left(\frac{s}{10^8}+1\right)\right]} = \frac{\overset{a}{10^{-2}}\ \overset{b}{(s+1)}\ \overset{c}{\left(\frac{s^2}{10^4}+10^{-1}s+1\right)}}{\underset{d}{s}\ \underset{e}{(10^2 s+1)}\ \underset{f}{\left(\frac{s}{10^8}+1\right)}}$$

break frequencies: a $\nexists\ \omega_b$

b $\omega_b = \frac{1}{\tau} = 1$

c $\omega_b = \omega_n = 10^2$

d $\nexists\ \omega_b$

e $\omega_b = \frac{1}{\tau} = 10^{-2}$

f $\omega_b = \frac{1}{\tau} = 10^8$

Ascending break frequencies:

a, d = -20 dB/dec slope

e 1^{st} order denominator $\Rightarrow \Delta$ slope $= -20$ dB/dec

b 1^{st} order numerator $\Rightarrow \Delta$ slope $= +20$ dB/dec

c 2^{nd} order numerator $\Rightarrow \Delta$ slope $= +40$ dB/dec

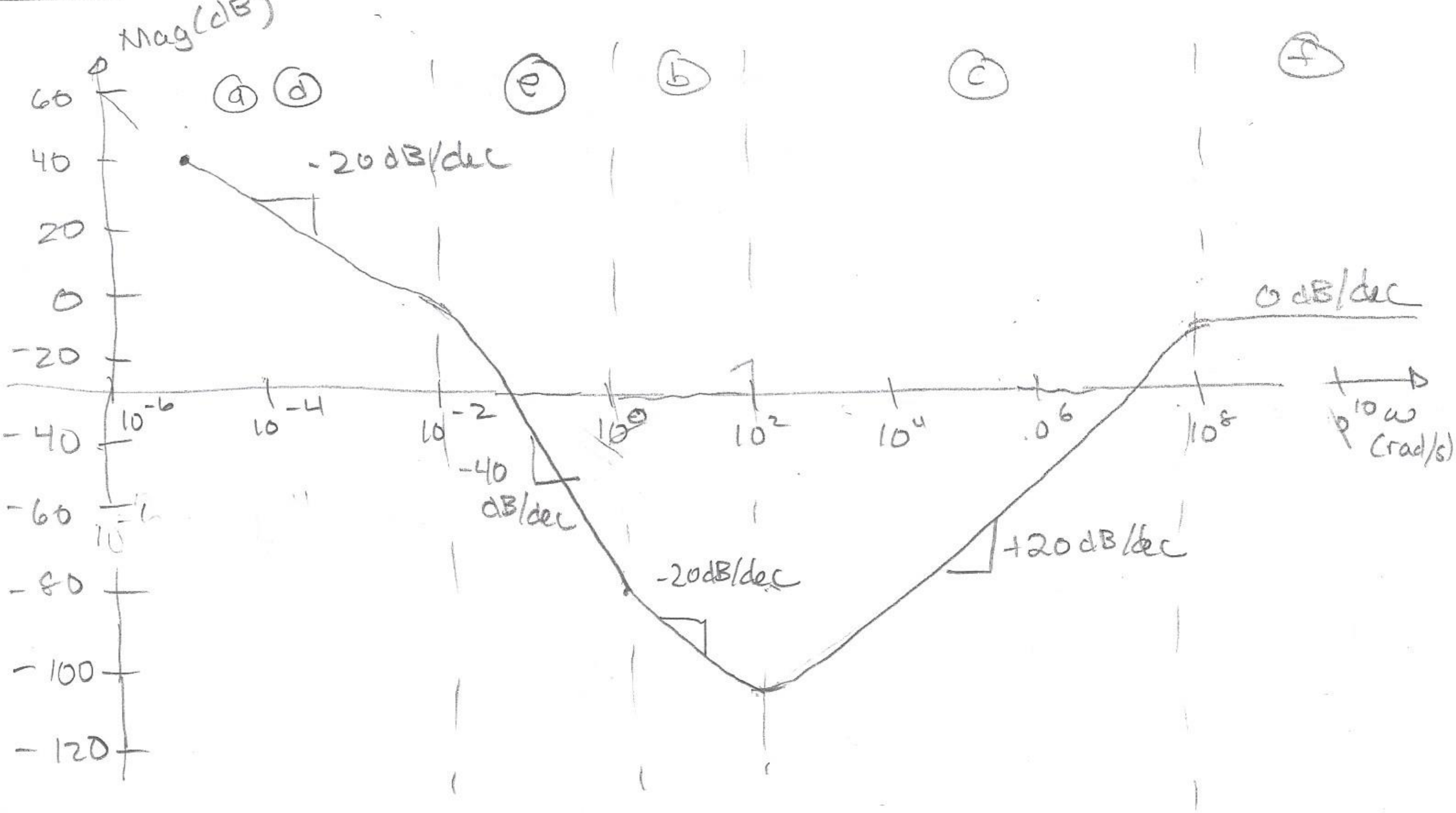
f 1^{st} order denominator $\Rightarrow \Delta$ slope $= -20$ dB/dec

$\Rightarrow$ for a, d intercept, choose $\omega < 1^{st}\ \omega_{break}$

$$G_{ad}(j\omega) = \frac{10^{-2}}{j\omega} \Rightarrow 20\log\left|G_{ad}(j\omega)\right|_{10^{-4}} = 20\log\left|\frac{10^{-2}}{10^{-4}}\right|$$

$$@\ \omega = 10^{-4} \Rightarrow 40\text{ dB}$$

Problem 4 (cont'd)



$$G_4(j\omega) = \frac{(j\omega+1)(10^3 j\omega + 10^4 - \omega^2)}{j\omega(j\omega+10^{-2})(j\omega+10^8)}$$

$$\left|G_4(j\omega)\right|_{10^3} = \left.\frac{\sqrt{1^2+\omega^2}\,\sqrt{(10^4-\omega^2)^2+(10^3\omega)^2}}{\sqrt{0^2+\omega^2}\,\sqrt{(10^{-2})^2+\omega^2}\,\sqrt{(10^8)^2+(\omega)^2}}\right|_{10^3}$$

3+6

3 3 8

$$\boxed{\left|G_4(10^3 j)\right| \approx 10^5}$$

$$\measuredangle G_4(j\omega)\Big|_{10^3} = \left[\tan^{-1}\left(\frac{\omega}{1}\right) + \tan^{-1}\left(\frac{10^3\omega}{10^4-\omega^2}\right)\right] - \left[\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{\omega}{10^{-2}}\right) + \tan^{-1}\left(\frac{\omega}{10^8}\right)\right]$$

$$= \left[1.57 + (\pi - 0.79)\right] - \left[\frac{\pi}{2} + 1.57 + 0\right]$$

$$\boxed{\measuredangle G(10^3 j) = 0.78\,\text{rad}}$$

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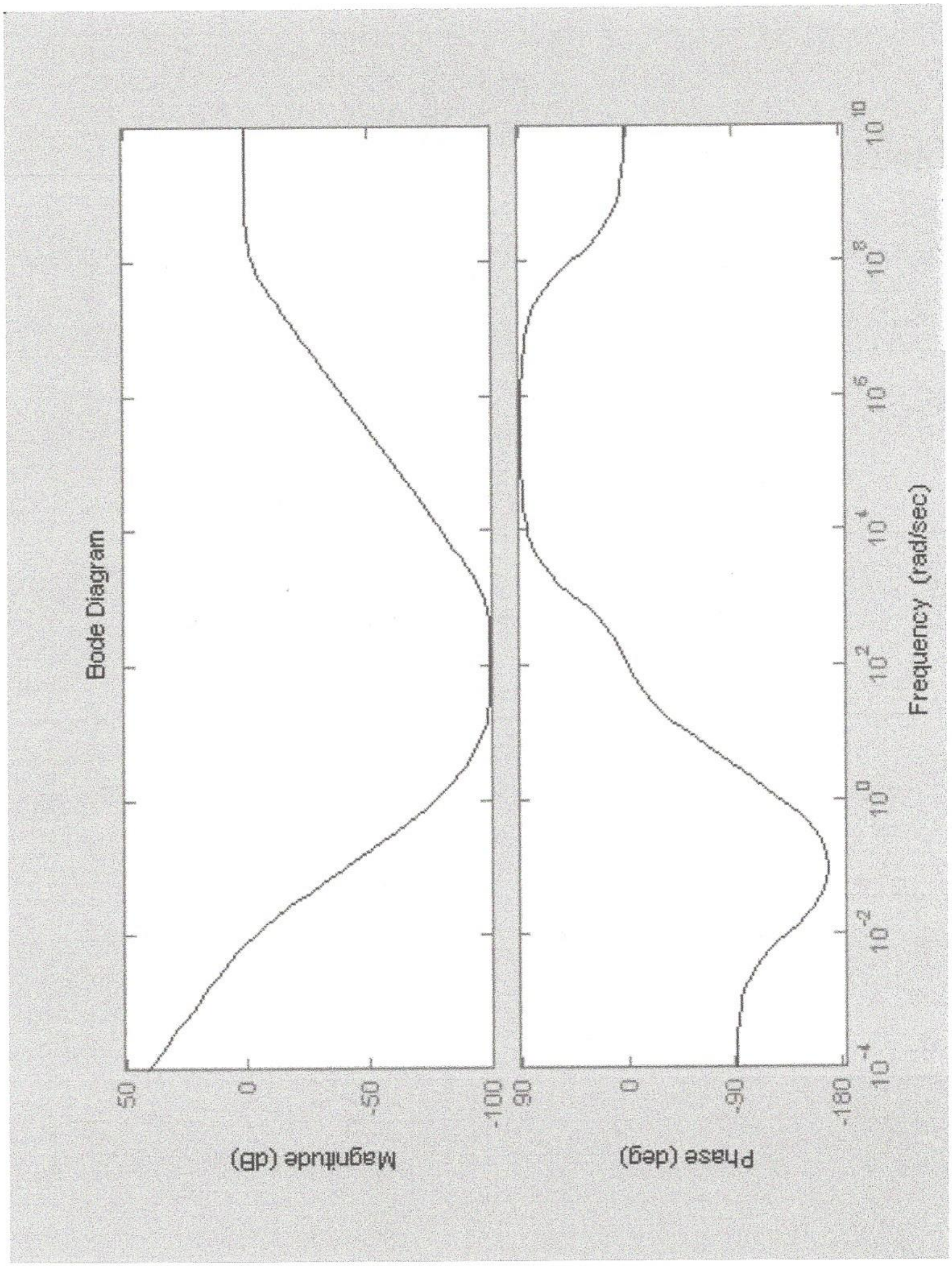
Problem 4

(b)


Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
90
0
-90
-180
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
Frequency (rad/sec)

Problem 5

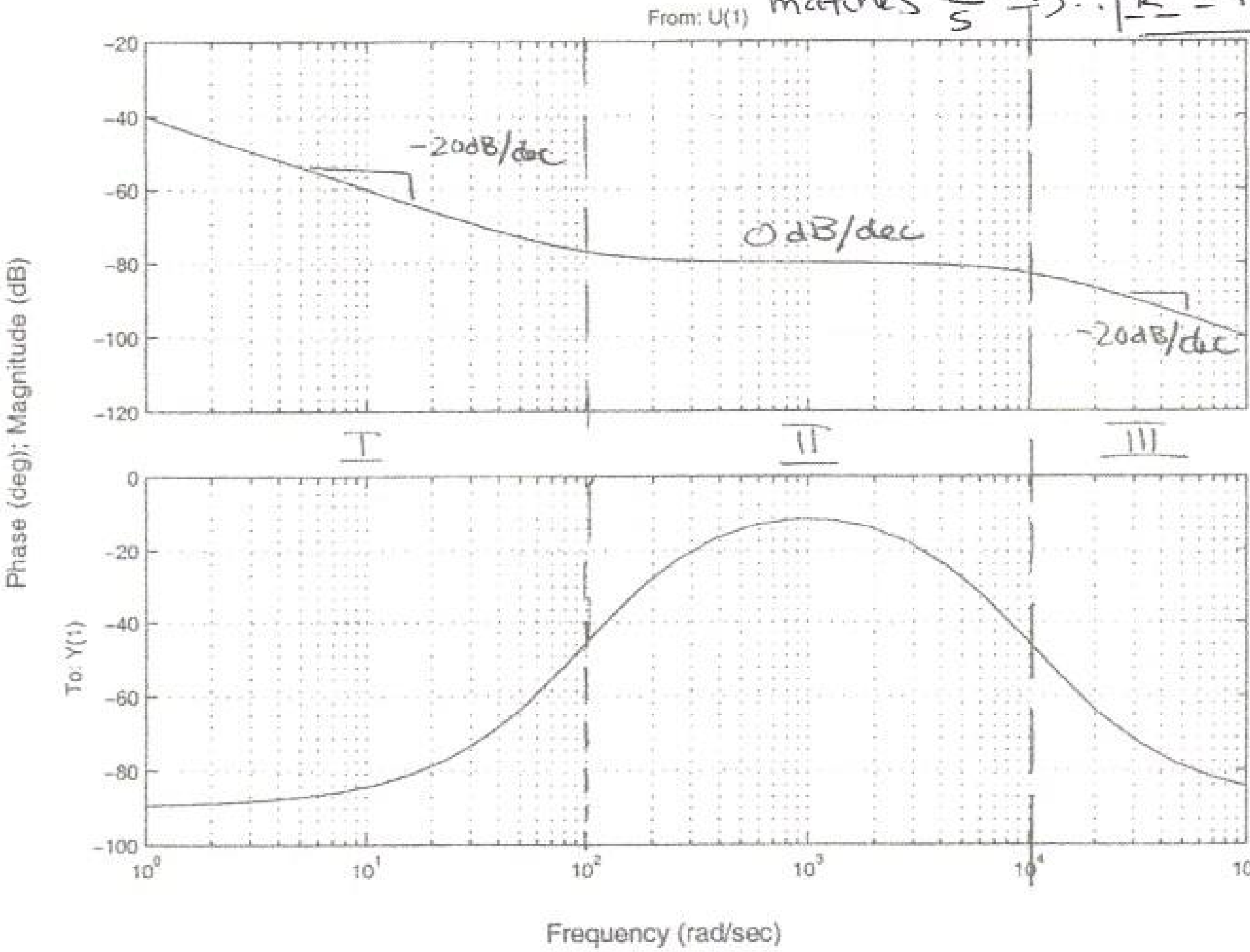
I : slope of Mag = -20dB/dec and no break frequency
$\therefore$ have a $\boxed{\frac{1}{s}}$ term

$\frac{1}{s}$: $\left|\frac{1}{j\omega}\right|_{\omega=1} = 1$ or 0dB $\Rightarrow \therefore$ should cross 0dB at $\omega=1$

But @ $\omega=1$, $|G(j\omega)| = -40$dB $\Rightarrow \therefore$ have gain K

$20\log\left|K\frac{1}{j\omega}\right|_{\omega=1} = -40\text{dB} \quad |K| = 10^{-2}$

Initial phase is $-90°$, which matches $\frac{1}{s} \Rightarrow \therefore \boxed{K = +10^{-2}}$



Bode Diagram for Problem 5

II @ $\omega = 100$, -20dB/dec $\to$ 0dB/dec
change of +20dB/dec $\Rightarrow$ 1st order numerator

$\tau s + 1 \to \tau = \frac{1}{\omega_{break}} = \frac{1}{100} \Rightarrow \boxed{\frac{s}{100} + 1}$

III @ $\omega = 10^4$, 0dB/dec $\to$ -20dB/dec : change of -20dB/dec
$\therefore$ 1st order denominator

$\frac{1}{\tau s + 1} \to \tau = \frac{1}{\omega_{break}} = \frac{1}{10^4} \Rightarrow \boxed{\frac{1}{10^{-4}s + 1}}$

Problem 5 : (Cont'd)

$$\therefore\ G(s) = \left[\frac{1}{s}\right]\left[10^{-2}\right]\left[\frac{s}{100}+1\right]\left[\frac{1}{s/10^4+1}\right]$$

$$\boxed{G(s) = \frac{10^{-2}\left[\frac{s}{100}+1\right]}{s\left[\frac{s}{10^4}+1\right]}} \quad \text{or} \quad \frac{s+100}{s\left[s+10^4\right]}$$

confirmed w/ phase plot

$f(t) = 10\sin(100t)$

$x_{ss}(t) = A_0 \sin(100t+\phi)$

$A_0 = 10 \cdot |G(100j)| \Rightarrow$ from Bode diagram, $20\log|G(100j)| \approx -80\text{dB}$

$= 10 \cdot \left[10^{\frac{-80}{20}}\right]$

$\underline{A_0 = 10^{-3}}$

$\phi : \angle G(100j) \approx -45^\circ = \frac{-\pi}{4}$

$$\therefore\ \boxed{x_{ss}(t) = 10^{-3}\sin\left(100t - \frac{\pi}{4}\right)}$$

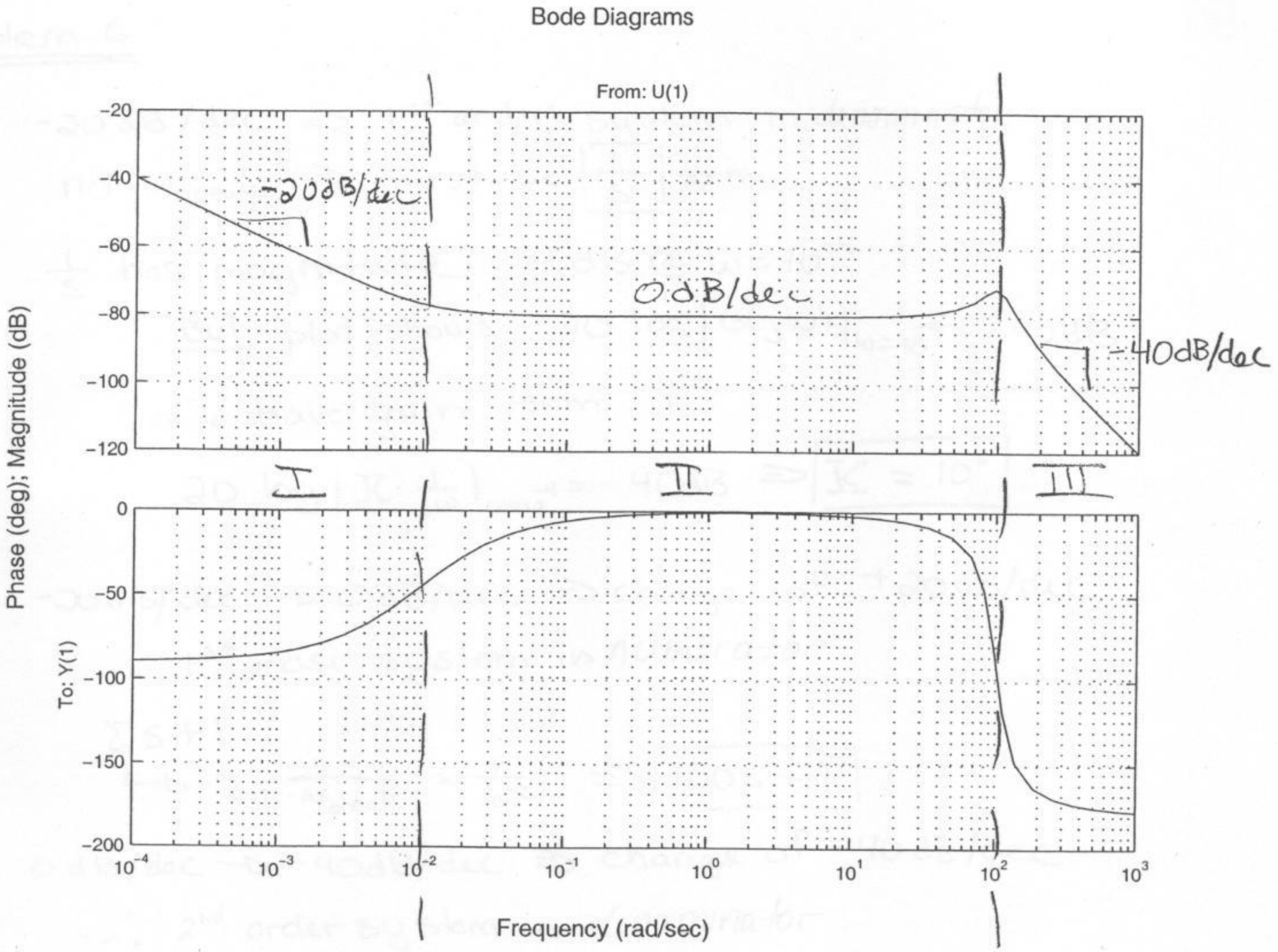


Figure 4: Bode Diagram for Problem 6

Problem 5 From the Bode diagram in Figure 3, determine the transfer function of the system. In addition, determine the steady-state equation for the system response, given that the input is $f(t) = 10\sin(1000t)$.

Problem 6 From the Bode diagram in Figure 4, determine the transfer function of the system. In addition, determine the steady-state equation for the system response, given that the input is $f(t) = 10\sin(0.1t)$.

Problem 6

I: $-20\,dB/dec \Rightarrow 1^{st}$ order system in denominator

no $\omega_{break} \Rightarrow$ must be $\boxed{\frac{1}{s}}$ term

$\frac{1}{s}$ has magnitude $-80dB$ @ $\omega = 10^{-4}$

But plot shows $20\log|G(j\omega)|_{\omega=10^{-4}} - 40\,dB$

$\therefore$ have gain term K

$20\log\left|K \cdot \frac{1}{j\omega}\right|_{\omega=10^{-4}} = -40dB \Rightarrow |K| = 10^{-6}$

$\rightarrow$ Initial phase $= -90^\circ$ which matches $\frac{1}{s} \Rightarrow \therefore \boxed{K = +10^{-6}}$

II: $-20dB/dec \rightarrow 0dB/dec \Rightarrow$ change of $+20dB/dec$

$\therefore 1^{st}$ order system in numerator

$\tau s + 1$

$\rightarrow \tau = \frac{1}{\omega_{break}} = \frac{1}{10^{-2}} \Rightarrow \boxed{100s + 1}$

III: $0\,dB/dec \rightarrow -40dB/dec \Rightarrow$ change of $-40\,dB/dec$

$\therefore 2^{nd}$ order system in denominator

$$\frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$$

ζ from phase plot + "hump" @ $\omega = 100\,rad/s$ and chart

$\zeta = 0.2$

$\rightarrow \omega_n = \omega_{break} = 10^2$

$$\therefore \frac{1}{\frac{s^2}{10^4} + \frac{2(.2)}{100}s + 1} = \boxed{\frac{1}{\frac{s^2}{10^4} + 4\times10^{-3}s + 1}}$$

$$\boxed{G(s) = \frac{10^{-6}\left[100s+1\right]}{s\left[\frac{s^2}{10^4} + \frac{4s}{10^3} + 1\right]}} \quad \text{or} \quad \frac{s + 10^{-2}}{s\left[s^2 + 40s + 10^4\right]}$$

Problem 6 : Cont'd

$f(t) = 10 \sin(0.1t)$

$\Rightarrow x_{ss}(t) = A_o \sin(0.1t + \phi)$

From Bode diagram, $20 \log |G(0.1j)| = -80 dB$

$\Rightarrow \therefore |G(0.1j)| = 10^{-4}$

$A_o = 10 \cdot 10^{-4} \Rightarrow \underline{A = 10^{-3}}$

ϕ from Bode diagram is $\approx -5° = \underline{-0.087 rad}$

$\therefore \boxed{x_{ss}(t) = 10^{-3} \sin(0.1t - 0.087)}$