9.1

Wednesday, April 18, 2018 2:41 AM

9.1.

$$\vec{V} \cdot \nabla \vec{V}$$

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\vec{V} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\vec{V} \cdot \nabla \vec{V} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\vec{i}$$

$$+ \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\vec{j}$$

$$+ \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\vec{k}$$

9.2

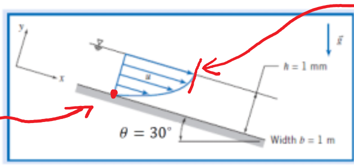
Tuesday, April 5, 2016 10:45 PM

Look for key words

Problem 9.2

A liquid flows down on an inclined plane surface in a steady, fully developed laminar film of thickness h. Simplify the continuity and Navier-Stokes equations to model this flow field. Obtain expressions for the liquid velocity profile, the shear stress distribution, the volume flow rate, and the average velocity. Relate the liquid film thickness to the volume flow rate per unit depth of surface normal to the flow. Calculate the volume flow rate in a film of water $h = 1mm$ thick, flowing on a surface $1m$ wide, inclined at $\theta = 30°$ to the horizontal.

"no shear" at free surface: $\left.\frac{\partial u}{\partial y}\right|_{y=h} = \emptyset$



"no slip" at wall: $u(y=0) = \emptyset$

Figure 2: Sketch of fully developed laminar flow on an inclined plane.

It is a liquid flow, so we can assume it is incompressible: $\frac{D\rho}{Dt} = 0 \Leftrightarrow \nabla \cdot \vec{v} = 0$

Governing equations

Continuity: $\cancel{\frac{\partial u}{\partial x}}^{(2)} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{(3)} = 0$

x-mom.: $\rho\left(\cancel{\frac{\partial u}{\partial t}}^{(1)} + u\cancel{\frac{\partial u}{\partial x}}^{(2)} + \cancel{v}^{(4)}\frac{\partial u}{\partial y} + \cancel{w}^{(3)}\frac{\partial u}{\partial z}\right) = \rho g_x - \cancel{\frac{\partial p}{\partial x}}^{(2)} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{(2)} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{(3)}\right)$

y-mom.: $\rho\left(\cancel{\frac{\partial v}{\partial t}}^{(1)} + u\cancel{\frac{\partial v}{\partial x}}^{(2)} + \cancel{v}^{(4)}\cancel{\frac{\partial v}{\partial y}}^{(4)} + \cancel{w}^{(3)}\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{(2)(4)} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{(4)} + \cancel{\frac{\partial^2 v}{\partial z^2}}^{(4)}\right)$

z-mom.: $\rho\left(\cancel{\frac{\partial w}{\partial t}}^{(1)} + u\cancel{\frac{\partial w}{\partial x}}^{(2)(3)} + \cancel{v}^{(4)}\cancel{\frac{\partial w}{\partial y}}^{(3)} + \cancel{w}^{(3)}\cancel{\frac{\partial w}{\partial z}}^{(3)}\right) = \rho \cancel{g_z}^{(3)} - \cancel{\frac{\partial p}{\partial z}}^{(3)} + \mu\left(\cancel{\frac{\partial^2 w}{\partial x^2}}^{(2)(3)} + \cancel{\frac{\partial^2 w}{\partial y^2}}^{(3)} + \cancel{\frac{\partial^2 w}{\partial z^2}}^{(3)}\right)$

Let's simplify: (1) steady flow: $\frac{\partial}{\partial t} = 0$

(2) fully developed flow: $\frac{\partial}{\partial x} = 0$ (In this case $\frac{\partial p}{\partial x} = 0$ as well, since $p = p_{atm}$ at free surface)

(3) No flow ($w = \emptyset$) or variation of flow properties in z-direction ($\frac{\partial}{\partial z} = 0$)

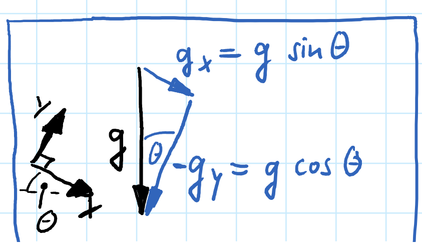
With assumptions (2) and (3), the continuity equation reduces to:

$$\frac{\partial v}{\partial y} = \emptyset \Leftrightarrow \int dv = v(y) = \text{const.}$$

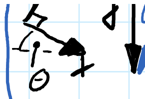
w/ kinematic B.C.: $v(y=0) = 0 \quad \therefore \underline{v = 0}$ (4)

x-momentum: $\boxed{0 = \rho g_x + \mu \frac{\partial^2 u}{\partial y^2}}$

y-momentum: $\boxed{0 = \rho g_y - \frac{\partial p}{\partial y}}$



y-momentum: $\boxed{0 = \rho g_y - \frac{\partial p}{\partial y}}$



$-g_y = g \cos\theta$

For fully developed flow, u can at most be a function of y:

$$\frac{d^2u}{dy^2} = -\frac{\rho g_x}{\mu} = -\rho g \frac{\sin\theta}{\mu}$$

$$\frac{du}{dy} = -\rho g \frac{\sin\theta}{\mu} y + C_1$$

$$\boxed{u(y) = -\rho g \frac{\sin\theta}{2\mu} y^2 + C_1 y + C_2}$$

evaluate boundary conditions:

"no-slip" at solid wall, $u(y=0)=0$: $\quad 0 + 0 + C_2 = 0$

"no-shear" at free surface, $\left.\frac{du}{dy}\right|_{y=h} = 0$: $\quad -\rho g \frac{\sin\theta}{\mu} h + C_1 = 0$

$\therefore \quad C_1 = \rho g \frac{\sin\theta}{\mu} h$

- velocity profile $\boxed{u(y) = \rho g \frac{\sin\theta}{\mu}\left(hy - \frac{1}{2}y^2\right)}$

- shear stress distribution $\quad \tau_{yx} = \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) = \rho g \sin\theta (h-y)$ (2)

- volume flow rate $\quad \dot{V} = \iint_A u\, dA = \int_0^b \int_0^h u\, dy\, dz = b\int_0^h u\, dy =$

$$= b\int_0^h \rho g \frac{\sin\theta}{\mu}\left(hy - \frac{1}{2}y^2\right)dy = b \rho g \frac{\sin\theta}{\mu}\left[\frac{1}{2}hy^2 - \frac{1}{6}y^3\right]_0^h =$$

$$= b \rho g \frac{\sin\theta}{\mu}\left[\frac{1}{2}h^3 - \frac{1}{6}h^3\right] = \frac{1}{3\mu}\rho g \sin\theta\, b\, h^3$$

- average velocity

$$u_{avg} = \frac{\dot{V}}{A} = \frac{\dot{V}}{b \cdot h} = \frac{1}{3\mu}\rho g \sin\theta\, h^2$$

- relate liquid film thickness to volume flow rate per unit depth

- relate liquid film thickness to volume flow rate per unit depth

$$h^3 = \left(\frac{\dot{V}}{b}\right)\frac{3\mu}{\rho g \sin\theta\, h^3} \iff h = \left[\left(\frac{\dot{V}}{b}\right)\frac{3\mu}{\rho g \sin\theta\, h^3}\right]^{\frac{1}{3}}$$

- Calculate volume flow rate for:

$h = 1\ mm$
$b = 1\ m$
$\theta = 30°C$

Appendix A, properties at 20°C:

$\rho = 998\ kg/m^3$
$\mu = 1.01 \times 10^{-3}\ \frac{Ns}{m^2}$

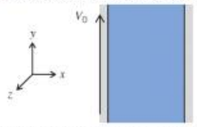
$$\dot{V} = \frac{1}{3\mu}\rho g \sin\theta\, b\, h^3 = \frac{1}{3 \cdot 1.01\times10^{-3}\ \frac{\cancel{kg}}{\cancel{m\,s}}} \cdot 998\frac{\cancel{kg}}{\cancel{m^3}} \cdot 9.81\frac{\cancel{m}}{\cancel{s^2}} \cdot \frac{1}{2} \cdot 1\cancel{m} \cdot (0.001m)^3 =$$

$$= 0.0016\frac{m^3}{s} = 1.6\ l/s$$

9.3

Tuesday, April 5, 2016 10:45 PM

Problem 9.3
Consider a steady, laminar, fully developed, incompressible flow between two infinite plates, as shown below. The flow is due to the upward motion of the left plate as well a pressure gradient that is applied in the y direction. Given the conditions that $\vec{v} \neq \vec{v}(z); w = 0$, and that gravity points in the negative y direction, prove that $u = 0$ and that the pressure gradient in the y direction must be constant.



$\downarrow g$ or: $\vec{g} = \begin{pmatrix} \phi \\ g \\ \phi \end{pmatrix}$

Figure 1: Sketch of flow between two vertical plates.

Look for key words that will allow you to simplify the problem!

- steady (state): $\Rightarrow \frac{\partial}{\partial t} = 0$
- laminar flow: $\Rightarrow$ not turbulent flow: o.k. to use equations as we have derived them to calculate (average) velocities.
- fully developed: $\Rightarrow$ "Fully developed" flow means that if we go some distance downstream (in flow direction), the velocity field will not change.
  Here, the flow will occur in the y-direction: $\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} = \phi$
  Careful: We cannot simply say $\frac{\partial}{\partial y} = 0$, because there can be a pressure gradient $\frac{\partial p}{\partial y}$, which would be balance by viscous stresses at the walls.
- incompressible: $\Rightarrow \frac{D\rho}{Dt} = \phi \Leftrightarrow \underline{\nabla \cdot \vec{v} = 0}$
  this simplifies the viscous terms in the momentum equation
- "between infinite plates" $\Rightarrow$ Typically this means the flow is of large extent in one or more dimensions and implies that flow properties are homogeneous in this direction. Here, it implies $\frac{\partial}{\partial z} = 0$. "Infinite plates" usually also implies 2-D.
- $\vec{v} \neq \vec{v}(z)$: $\Rightarrow \vec{v}$ is at most a function of x and y, i.e. 2-D, $w = \phi$

(a) Prove that $u = 0$:

Continuity equation: $\nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{=0} + \cancel{\frac{\partial w}{\partial z}}^{=0} = 0$

$\therefore \quad \frac{\partial u}{\partial x} = 0 \iff \int du = u(x) = c_1$

with kinematic boundary condition $u(x)\big|_{\text{at wall}} = 0$

$\therefore \quad c_1 = 0 \iff u = 0$

(b) Prove that the pressure gradient in the y-direction must be constant:

Momentum (Navier-Stokes) equation for incompressible flow:

$$x\text{-dir.:} \quad \rho \frac{Du}{Dt} = \rho\left(\cancel{\frac{\partial u}{\partial t}}^{=0} + \cancel{u}\frac{\partial u}{\partial x} + v\cancel{\frac{\partial u}{\partial y}} + \cancel{w}\frac{\partial u}{\partial z}\right) = \rho \cancel{g_x} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \cancel{\frac{\partial^2 u}{\partial y^2}} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$y\text{-dir.:} \quad \rho \frac{Dv}{Dt} = \rho\left(\cancel{\frac{\partial v}{\partial t}}^{=0} + \cancel{u}\frac{\partial v}{\partial x} + v\cancel{\frac{\partial v}{\partial y}} + \cancel{w}\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}\right) + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}$$

$$z\text{-dir.:} \quad \rho \frac{Dw}{Dt} = \rho\left(\cancel{\frac{\partial w}{\partial t}}^{=0} + \cancel{u}\frac{\partial w}{\partial x} + v\cancel{\frac{\partial w}{\partial y}} + \cancel{w}\frac{\partial w}{\partial z}\right) = \rho \cancel{g_z} - \cancel{\frac{\partial p}{\partial z}} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

steady state | $u=0$ | fully developed | 2-D, $w=0$, $\frac{\partial}{\partial z}=0$ | gravity in y-dir. only | fully developed

$\Rightarrow$ All terms in the z-component equation cancel, and the x- and y-components reduce to:

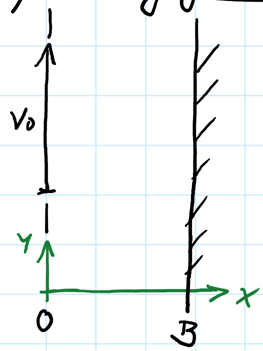
x: $\quad \frac{\partial p}{\partial x} = 0 \iff \int dp = p(x) = \text{const.} = c_2$ (pressure is const. in x-direction)

y: $\quad \underbrace{\frac{\partial p}{\partial y} - \rho g_y}_{f(y)} = \underbrace{\mu \frac{\partial^2 v}{\partial x^2}}_{f(x)} = \text{const.} = c_3$

The left-hand side of the equation is a function of y only.
The right-hand side of the equation is a function of x only.
$\Rightarrow$ If each side of an equation is a function of a different independent variable, then the equation can only be satisfied for all values if each side is equal to a constant.

$$\frac{\partial p}{\partial y} - \rho g_y = \text{const} \quad \therefore \quad \underline{\underline{\frac{\partial p}{\partial y} = \text{const.}}}$$

Let's gain some additional insight by solving for the velocity field (not asked for in problem statement)



Boundary Conditions:

$v(x=0) = V_0$ (left plate moves w/ V_0)

$v(x=B) = \emptyset$ (right plate does not move)

$$\frac{d^2v}{dx^2} = \frac{C_3}{\mu}$$

$$d\left(\frac{dv}{dx}\right) = \frac{C_3}{\mu}\,dx$$

$$\int d\left(\frac{dv}{dx}\right) = \int \frac{C_3}{\mu}\,dx + C_4$$

$$\frac{dv}{dx} = \frac{C_3}{\mu}x + C_4$$

$$dv = \frac{C_3}{\mu}x\,dx + C_4\,dx$$

$$\int dv = \int \frac{C_3}{\mu}x\,dx + \int C_4\,dx + C_5$$

$$\boxed{v(x) = \frac{1}{2\mu}C_3x^2 + C_4x + C_5}$$

Apply B.C.s: $v(x=0) = V_0 \quad \therefore \quad 0 + 0 + \underline{\underline{C_5}} = \underline{\underline{V_0}}$

$v(x=B) = 0 \quad \therefore \quad \frac{1}{2\mu}C_3B^2 + C_4B + V_0 = 0$

$\therefore \quad \underline{\underline{C_4}} = \underline{\underline{-\frac{1}{2\mu}C_3B - \frac{V_0}{B}}}$

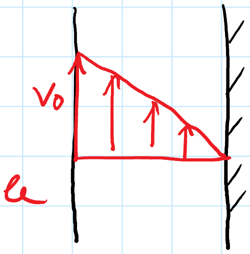
$$\therefore \quad \boxed{v(x) = \frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)x^2 - \frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)Bx - \frac{V_0}{B}x + V_0}$$

Case 1: If $\boxed{\frac{dp}{dy} - \rho g_y = 0}$ (or: $p(y) = p_0 + \rho g_y y$) (with: $g_y = -g$)

$\Rightarrow$ hydrostatic pressure distribution

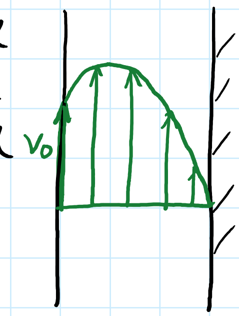
then $v(x) = v_0 \left(1 - \frac{x}{B}\right)$

linear velocity profile



Case 2: If $\boxed{\frac{dp}{dy} - \rho g_y \neq 0}$, or $\frac{dp}{dy} \neq \rho g_y$

$\Rightarrow$ this means there is an additional pressure gradient aside from the hydrostatic pressure gradient and we get the full parabolic velocity profile.

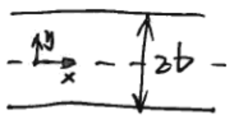


9.4

Wednesday, April 18, 2018 2:41 AM

9.4.

Given: $u = u_m \left[1 - \left(\frac{y}{b}\right)^2\right]$



(a) rate of linear deformation

$$\frac{\partial u}{\partial x} = 0.$$

volume dilation is zero

rate of angular deformation in xy plane

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} = -\frac{u_m}{b^2} \cdot 2y$$

(b) Vorticity vector

$$\vec{\xi} = \nabla \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= -\frac{\partial u}{\partial y}\vec{k} = \frac{u_m}{b^2} \cdot 2y\,\vec{k}$$

(c) $\vec{\xi} = \frac{u_m}{b^2} 2y \vec{k}$

$$\xi_z = \frac{u_m}{b^2} \cdot 2y$$

$$\frac{d\vec{\xi}_z}{dy} = \frac{u_m}{b^2} \cdot 2 \rightarrow \text{constant}$$

At $y = -b$. $\vec{\xi}_z = \frac{u_m}{b^2} \cdot 2 \cdot -b = -\frac{2u_m}{b}$ min

At $y = b$, $\vec{\xi}_z = \frac{u_m}{b^2} \cdot 2 \cdot b = \frac{2u_m}{b}$ max