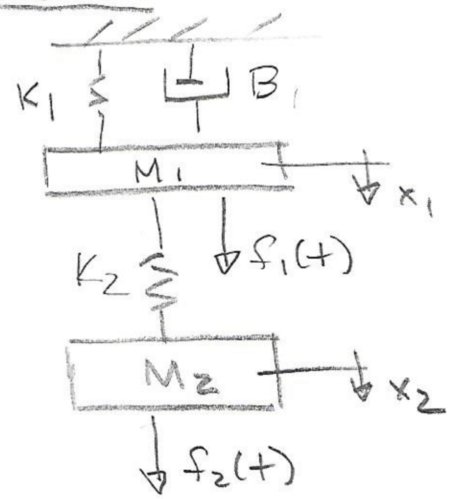
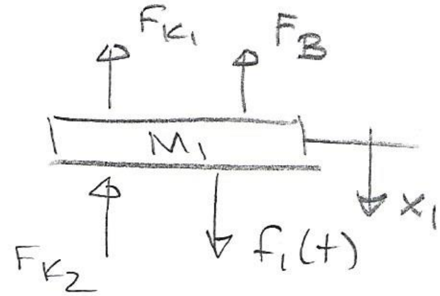
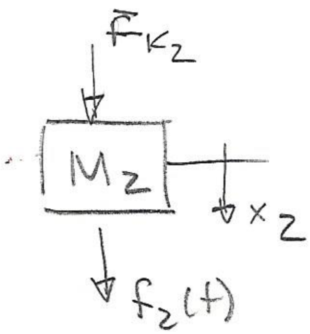
PROBLEM SET 2
SOLUTIONS

Problem 1



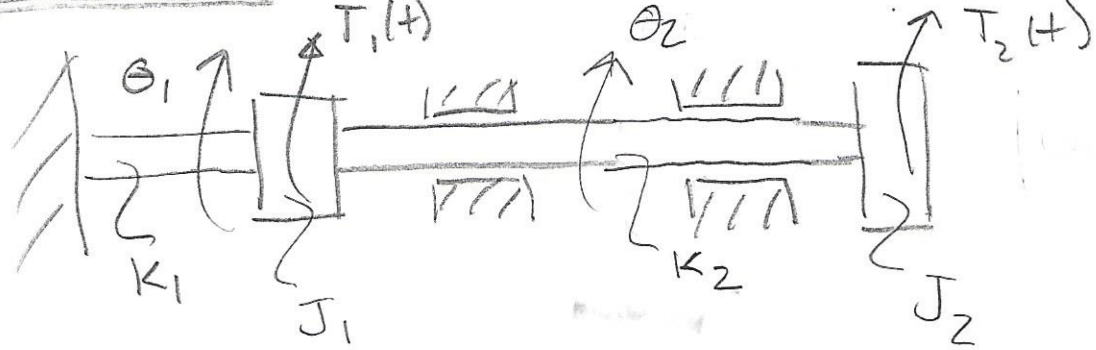


$$\sum F = M_1\ddot{x}_1$$
$$= -F_{K_1} - F_B - F_{K_2} + f_1(t)$$
$$= -K_1x_1 - B\dot{x}_1 - K_2(x_1 - x_2) + f_1(t)$$
$$\Rightarrow \boxed{M_1\ddot{x}_1 + B\dot{x}_1 + (K_1+K_2)x_1 = K_2x_2 + f_1(t)}$$



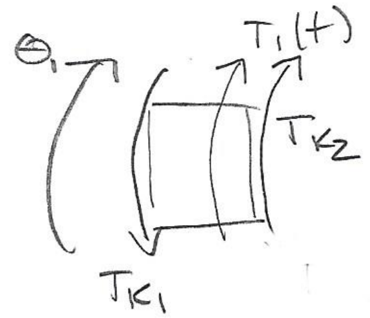
$$\sum F = M_2\ddot{x}_2$$
$$= F_{K_2} + f_2(t)$$
$$= K_2(x_1 - x_2) + f_2(t)$$
$$\Rightarrow \boxed{M_2\ddot{x}_2 + K_2x_2 = K_2x_1 + f_2(t)}$$

Problem 2

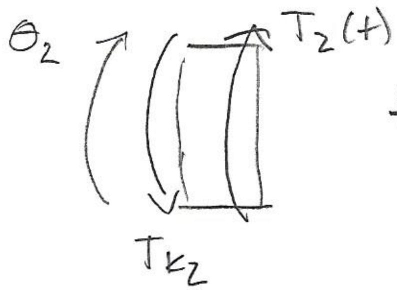


As the problem is given, θ_2 should be considered the relative angular displacement between J_2 and J_1. That is $\theta_2 = \theta_{J_2} - \theta_{J_1}$

However, full points will be awarded if you interpretted θ_2 as the absolute angular displacement of J_2

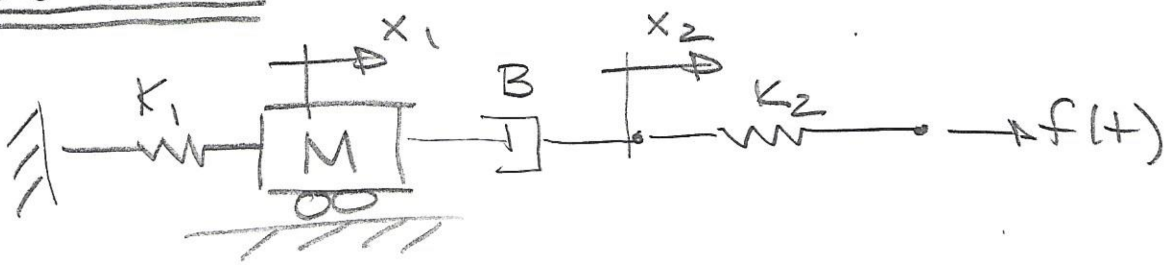


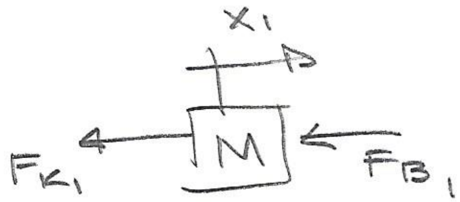
$$\sum M = J_1 \ddot{\theta}_1$$
$$= -T_{K_1} + T_1(t) + T_{K_2}$$
$$= -K_1\theta_1 + T_1(t) + K_2\theta_2$$
$$\boxed{J_1 \ddot{\theta} + K_1\theta_1 = K_2\theta_2 + T_1(t)}$$



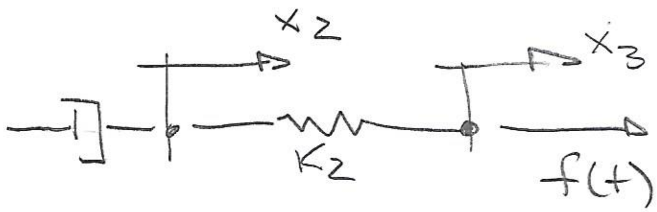
$$\sum M = J_2 \ddot{\theta}_2$$
$$= -T_{K_2} + T_2(t)$$
$$= -K_2\theta_2 + T_2(t)$$
$$\Rightarrow \boxed{J_2 \ddot{\theta}_2 + K_2\theta_2 = T_2(t)}$$

Problem 3

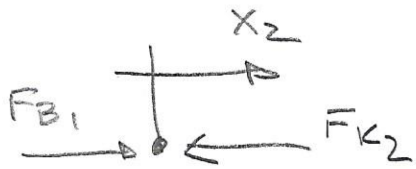




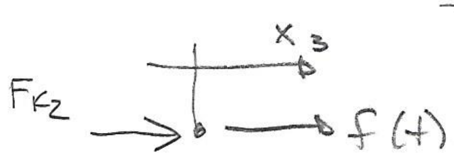
$$\sum \vec{F} = M\ddot{x}_1$$
$$= -F_{K_1} - F_{B_1}$$
$$= -K_1 x_1 - B_1(\dot{x}_1 - \dot{x}_2)$$
$$\Rightarrow \boxed{M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1 x_1 = B_2\dot{x}_2}$$



(introduce this variable to completely define deflection in spring)



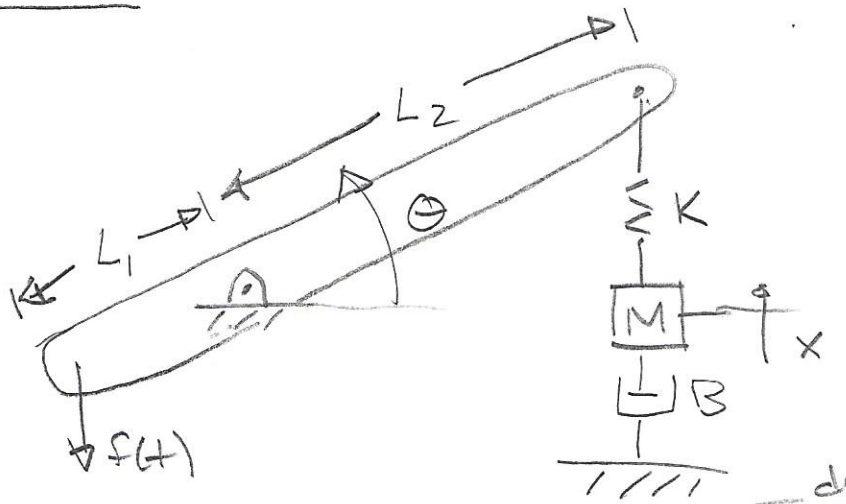
$$\sum \vec{F} = \cancel{M_2}\ddot{x}_2 \quad 0$$
$$= F_{B_1} - F_{K_2}$$
$$= B_1(\dot{x}_1 - \dot{x}_2) - K_2(x_2 - x_3)$$
$$\underline{B\dot{x}_2 + K_2 x_2 = B_1\dot{x}_1 + K_2 x_3}$$



$$\sum \vec{F} = \cancel{M_3}x_3 \quad 0$$
$$= F_{K_2} + f(t)$$
$$= K_2(x_2 - x_3) + f(t)$$
$$\therefore\ x_3 = \frac{1}{K_2}\left[K_2 x_2 + f(t)\right]$$

$$\Rightarrow \boxed{B\dot{x}_2 + K_2 x_2 = B_1\dot{x}_1 + \left[K_2 x_2 + f(t)\right]}$$

Problem 4

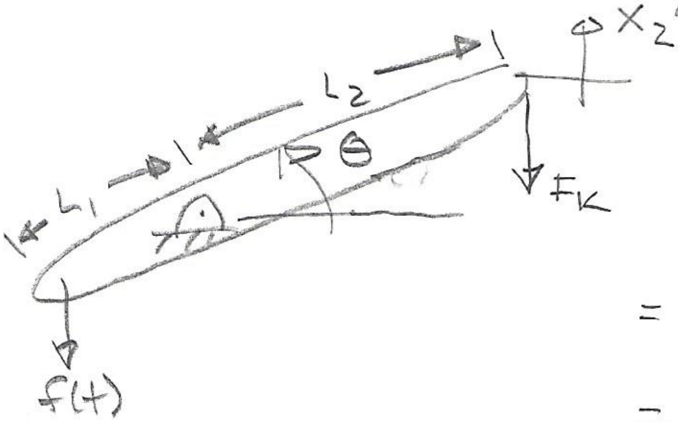


$\theta << 1;$

$\sin\theta \approx \theta$

$\cos\theta \approx 1$

define variable x_2



$x_2 = L_2 \sin\theta \approx L_2 \theta$

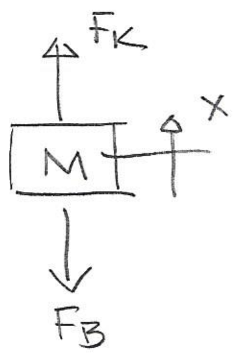
$$\sum M = \cancel{J\ddot{\theta}}$$

$$= L_1 \cos\theta\, f(t) - L_2 \cos\theta\, F_K$$

$$= L_1 f(t) - L_2 K (x_2 - x)$$

$$= L_1 f(t) - L_2 K (L_2 \theta - x)$$

$$\Rightarrow \boxed{L_2^2 K \theta = L_2 K x + L_1 f(t)}$$



$$\sum F = M\ddot{x}$$

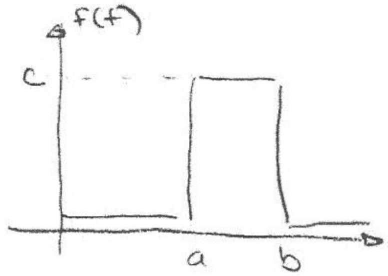
$$= F_K - F_B$$

$$= K(x_2 - x) - B\dot{x}$$

$$= K(L_2\theta - x) - B\dot{x}$$

$$\Rightarrow \boxed{M\ddot{x} + B\dot{x} + Kx = K L_2 \theta}$$

Problem 5 : B-2-6

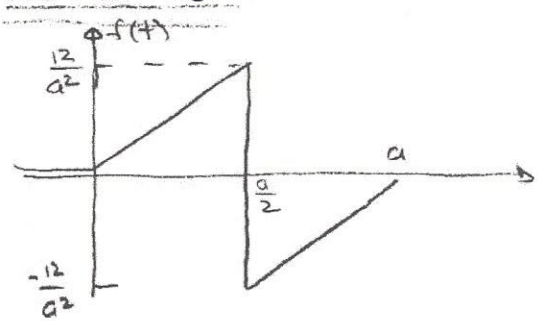


$$f(t) = c \cdot 1(t-a) - c \cdot 1(t-b)$$

$$\mathcal{L}[f(t)] = c \cdot e^{-as} \cdot \frac{1}{s} - c \cdot e^{-bs} \cdot \frac{1}{s}$$

$$\boxed{F(s) = \frac{c}{s}\left[e^{-as} - e^{-bs}\right]}$$

Problem 6 : B-2-8



$$f(t) = \frac{12/a^2}{a/2}\, t \; - \frac{24}{a^2} \cdot 1\left(t-\frac{a}{2}\right)$$

$$- \frac{12/a^2}{a/2}\, t \cdot 1(t-a)$$

$$\mathcal{L}[f(t)] = \frac{24}{a^3} \cdot \frac{1}{s^2} - \frac{24}{a^2} \cdot e^{-a/2 s} \frac{1}{s} - \frac{24}{a^3} e^{-as} \frac{1}{s^2}$$

$$\boxed{F(s) = \frac{24}{a^3} \cdot \frac{1}{s^2}\left[1 - as e^{-\frac{a}{2}s} - e^{-as}\right]}$$

$$\lim_{a \to 0} \mathcal{L}[f(t)] = \lim_{a \to 0} F(s)$$

$$= \lim_{a \to 0} \left[\frac{24}{a^3 s^2}\left(1 - as e^{-\frac{a}{2}s} - e^{-as}\right)\right]$$

$$= \lim_{a \to 0} \left[\frac{\cancel{24}\,4\left(+as\left(\frac{s}{2}\right)e^{-\frac{a}{2}s} - \cancel{s}e^{-\frac{a}{2}s} + \cancel{s}e^{-as}\right)}{3a^2 s^{\cancel{2}}}\right]$$

$$= \lim_{a \to 0} \left[\frac{8\left(-\frac{as}{2} \cdot \frac{s}{2} e^{-\frac{a}{2}s} + \cancel{s}e^{-\frac{a}{2}s} - \cancel{s}e^{-as}\right)}{2a\cancel{s}}\right]$$

$$= \lim_{a \to 0} 4\left[\frac{-\frac{as}{4}\left(-\frac{s}{2}\right)e^{-\frac{a}{2}s} - \frac{s}{4}e^{-\frac{a}{2}s} - \frac{s}{2}e^{-\frac{a}{2}s} + se^{-as}}{1}\right]$$

$$= 4\left(-\frac{3s}{4} + s\right) \Rightarrow \boxed{\lim_{a \to 0} \mathcal{L}[f(t)] = s}$$

Problem 7   B-2-10

$$F(s) = \frac{2(s+2)}{s(s+1)(s+3)}$$

(a) $f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \cancel{s}\left[\frac{2(s+2)}{\cancel{s}(s+1)(s+3)}\right]$

$$\boxed{f(0^+) = 0}$$

(b) $f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \cancel{s}\left[\frac{2(s+2)}{\cancel{s}(s+1)(s+3)}\right]$

$$\boxed{f(\infty) = \frac{4}{3}}$$

Problem 8 : B-2-12

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0)$$

$$\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

Let $h(t) = \ddot{f}(t) \Rightarrow \mathcal{L}[h(t)] = \mathcal{L}[\ddot{f}(t)]$

$$\mathcal{L}[\dot{h}(t)] = sH(s) - h(0)$$

$$= s\left[s^2F(s) - sf(0) - \dot{f}(0)\right] - \ddot{f}(0)$$

$$\boxed{\mathcal{L}[\dddot{f}(t)] = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)}$$

Problem 9 : B-2-17

$$F(s) = \frac{s}{s^2+2s+10}$$

$\Rightarrow$ 2nd order denominator :

In order, check

① $\frac{s+a}{(s+a)^2+\omega^2}$, $\frac{\omega}{(s+a)^2+\omega^2}$

② $\frac{s}{(s+a)(s+b)}$, $\frac{1}{(s+a)(s+b)}$

③ $\frac{s}{\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1}$, $\frac{\omega_n^2}{\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1\right)}$

check ① :

$$F(s) = \frac{s}{(s+1)^2+3^2}$$ ① works ✓

$$= \frac{s+1}{(s+1)^2+3^2} - \frac{1}{(s+1)^2+3^2}$$

$$\boxed{f(t) = e^{-t}\cos 3t - \tfrac{1}{3}e^{-t}\sin 3t}$$

Problem 10 : B-2-19

$$F(s) = \frac{2s+10}{(s+1)^2(s+4)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$$

$$A = \frac{d}{dt}\left(\frac{2s+10}{s+4}\right)\bigg|_{s=-1} = \frac{(s+4)(2)-(2s+10)(1)}{(s+4)^2}\bigg|_{s=-1} = \frac{-2}{9}$$

$$B = \frac{2s+10}{s+4}\bigg|_{s=-1} = \frac{8}{3}$$

$$C = \frac{2s+10}{(s+1)^2}\bigg|_{s=-4} = \frac{2}{9}$$

$$F(s) = \frac{-2}{9}\frac{1}{s+1} + \frac{8}{3}\frac{1}{(s+1)^2} + \frac{2}{9}\frac{1}{(s+4)}$$

$$\Rightarrow \boxed{f(t) = \frac{-2}{9}e^{-t} + \frac{8}{3}te^{-t} + \frac{2}{9}e^{-4t}}$$

Problem 11 : B-2-22

$\ddot{x} + 4x = 0, \quad x(0) = 5, \quad \dot{x}(0) = 0$

(a) $x(t) = x_c(t) + x_p(t)$

$x_c(t): D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore x_c(t) = C_1 \cos 2t + C_2 \sin 2t$

$x_p(t): f(t) = 0 \Rightarrow x_p(t) = 0$

$x(t) = x_c(t) = C_1 \cos 2t + C_2 \sin 2t$

$$\left.\begin{array}{l} (1)\ x(0) = 5 = C_1 \\ (2)\ \dot{x}(0) = 0 = 2C_2 \end{array}\right\} \quad \boxed{x(t) = 5\cos 2t}$$

(b)
$$\left.\begin{array}{l} \mathcal{L}[x(t)] = X(s) \\ \mathcal{L}[\dot{x}(t)] = sX(s) - x_0 \\ \mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx_0 - \cancel{\dot{x}_0}^{\,0} \end{array}\right\} \quad \begin{array}{l} [s^2X(s) - s(5)] + 4X(s) = 0 \\ X(s)[s^2+4] = 5s \\ X(s) = \dfrac{5s}{s^2+4} \end{array}$$

$\# 11 \Rightarrow \omega = 2 \Rightarrow \mathcal{L}^{-1}[X(s)] = 5\cos 2t$

$\Rightarrow \boxed{x(t) = 5\cos 2t}$

Problem 12 : B-2-24

$2\ddot{x} + 2\dot{x} + x = 1$; $x(0) = 0$, $\dot{x}(0) = 2$

(a) $x(t) = x_c(t) + x_p(t)$

$x_c(t)$: $2D^2 + 2D + 1 = 0$

$$D_{1,2} = \frac{-2 \pm \sqrt{4-8}}{4}$$

$$= -\frac{1}{2} \pm \frac{1}{2} i$$

$$\therefore x(t) = e^{-\frac{1}{2}t}\left[C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2}\right]$$

$x_p(t)$: $f(t) = 1 \Rightarrow \therefore x_p(t) = A$

$$2[0] + 2[0] + [A] = 1$$

$$A = 1 \Rightarrow x_p(t) = 1$$

$$x(t) = e^{-\frac{t}{2}}\left[C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2}\right] + 1$$

(1) $x(0) = 0 = C_1 + 1 \rightarrow C_1 = -1$

(2) $\dot{x}(0) = 2 = \frac{C_2}{2} - \frac{1}{2}(C_1) \rightarrow C_2 = 3$

$$\therefore \boxed{x(t) = e^{-\frac{t}{2}}\left[-\cos\frac{t}{2} + 3\sin\frac{t}{2}\right] + 1}$$

(b) $2[s^2 X(s) - 2] + 2[sX(s)] + [X(s)] = \frac{1}{s}$

$$[2s^2 + 2s + 1]X(s) = \frac{1}{s} + 4$$

$$X(s) = \frac{1 + 4s}{s(2s^2 + 2s + 1)}$$

Use 20, 21 : $X(s) = \frac{1}{2}\frac{1}{s\left[(s+\frac{1}{2})^2 + (\frac{1}{2})^2\right]} + \frac{2}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}$

$$= \frac{1}{2}\left[\frac{A}{s} + \frac{Bs + C}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}\right] + \frac{2}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}$$

$$As^2 + As + \frac{1}{2}A + Bs^2 + Cs = 1$$

$A = 2$

$B = -2$

$C = -2$

Problem 8 (cont'd)

(b) (cont'd)

$$X(s) = \frac{1}{s} - \frac{s+1}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2} + \frac{2}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}$$

$$= \frac{1}{s} - \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2} + 3\frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}$$

$$\boxed{x(t) = 1 - e^{-\frac{t}{2}}\cos\frac{t}{2} + 3e^{-\frac{t}{2}}\sin\frac{t}{2}}$$