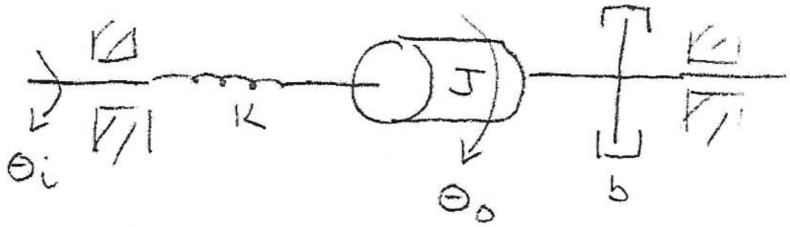
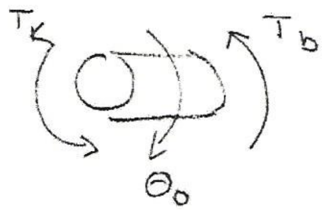
PROBLEM SET 3

Problem 1

(a)





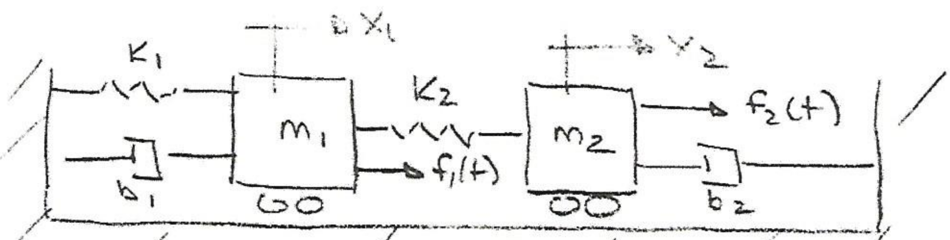
$$\circlearrowleft \Sigma M = J\ddot{\theta}_o = -T_b - T_K$$
$$= -b\dot{\theta}_o - K(\theta_o - \theta_i)$$
$$\Rightarrow \underline{J\ddot{\theta}_o + b\dot{\theta}_o + K\theta_o = K\theta_i}$$

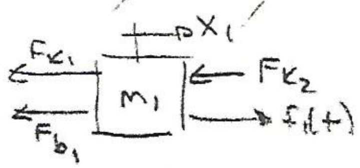
For transfer functions : IC's = 0

$$\Rightarrow [Js^2 + bs + K]\,\Theta_o(s) = K\,\Theta_i(s)$$

$$\therefore \boxed{G(s) = \frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K}{Js^2 + bs + K}}$$

(b)

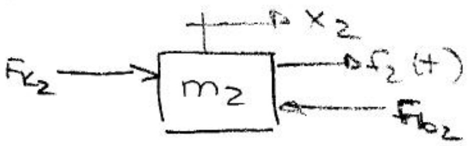




$$\xrightarrow{+} \Sigma F = m_1\ddot{x}_1$$
$$= -F_{k_2} + f_1(t) - F_{k_1} - F_{b_1}$$
$$= -K_2(x_1 - x_2) + f_1(t) - K_1x_1 - b_1\dot{x}_1$$
$$\Rightarrow \underline{m_1\ddot{x}_1 + b_1\dot{x}_1 + (K_1 + K_2)x_1 = f_1(t) + K_2x_2} \quad (1)$$

Problem 1 :(cont'd)

(b)(cont'd)



$$\overset{+}{\rightarrow}\Sigma F = m_2\ddot{x}_2$$
$$= F_{k2} + f_2(t) - F_{b2}$$
$$= k_2(x_1 - x_2) + f_2(t) - b_2\dot{x}_2$$
$$\Rightarrow \underline{m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = k_2x_1 + f_2(t)} \quad (2)$$

$$(1)\ \left[m_1s^2 + b_1s + (k_1+k_2)\right]X_1(s) + \left[-k_2\right]X_2(s) = F_1(s)$$

$$(2)\ \left[-k_2\right]X_1(s) + \left[m_2s^2 + b_2s + k_2\right]X_2(s) = F_2(s)$$

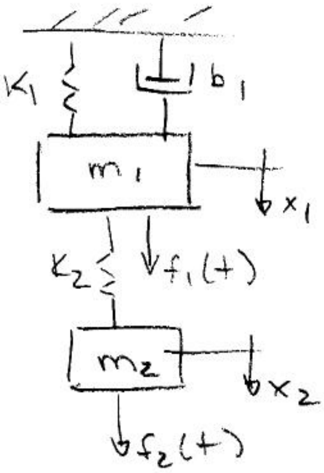
$$G(s) = \frac{sX_1(s)}{F_2(s)} \Rightarrow \text{set } F_1(s) = 0$$

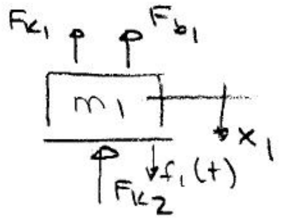
$$X_1(s) = \frac{\begin{vmatrix} 0 & -k_2 \\ F_2(s) & m_2s^2+b_2s+k_2 \end{vmatrix}}{\left[m_1s^2+b_1s+(k_1+k_2)\right]\left[m_2s^2+b_2s+k_2\right] - k_2^2}$$

$$\frac{X_1(s)}{F_2(s)} = \frac{k_2}{\left[m_1s^2+b_1s+(k_1+k_2)\right]\left[m_2s^2+b_2s+k_2\right] - k_2^2}$$

$$\therefore \boxed{G(s) = \frac{sX_1(s)}{F_1(s)} = \frac{sk_2}{\left[m_1s^2+b_1s+(k_1+k_2)\right]\left[m_2s^2+b_2s+k_2\right] - k_2^2}}$$

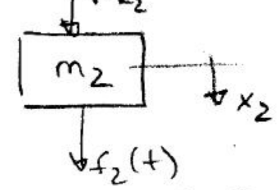
Problem 1 (cont'd)





$$\downarrow \Sigma F = m_1 \ddot{x}_1$$
$$= -F_{K_1} - F_{b_1} - F_{K_2} + f_1(t)$$
$$= -K_1 x_1 - b_1 \dot{x}_1 - K_2(x_1 - x_2) + f_1(t)$$

$$(1) \Rightarrow m_1 \ddot{x}_1 + b_1 \dot{x}_1 + (K_1 + K_2) x_1 = K_2 x_2 + f_1(t)$$



$$\downarrow \Sigma F = m_2 \ddot{x}_2$$
$$= F_{K_2} + f_2(t)$$
$$= K_2(x_1 - x_2) + f_2(t)$$

$$(2) \Rightarrow m_2 \ddot{x}_2 + K_2 x_2 = K_2 x_1 + f_2(t)$$

$$(1)\ [m_1 s^2 + b_1 s + (K_1 + K_2)] X_1(s) + [-K_2] X_2(s) = F_1(s)$$

$$(2)\ [-K_2] X_1(s) + [m_2 s^2 + K_2] X_2(s) = F_2(s)$$

$$\boxed{\Delta(s) = [m_1 s^2 + b_1 s + (K_1 + K_2)][m_2 s^2 + K_2] - K_2^2}$$

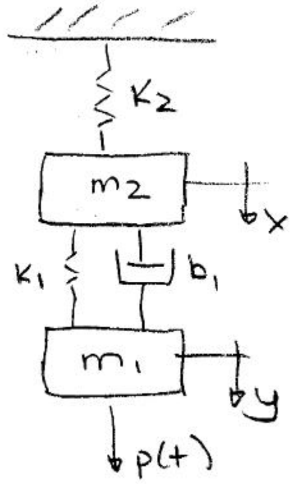
$$X_1(s) = \frac{\begin{vmatrix} F_1(s) & -K_2 \\ F_2(s) & m_2 s^2 + K_2 \end{vmatrix}}{\Delta(s)} = \frac{[m_2 s^2 + K_2] F_1(s) + K_2 F_2(s)}{\Delta(s)}$$

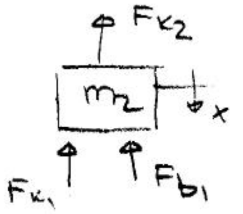
$$\therefore \boxed{\begin{aligned} G_{11}(s) &= \frac{X_1(s)}{F_1(s)} = \frac{m_2 s^2 + K_2}{\Delta(s)} \\ G_{12}(s) &= \frac{X_1(s)}{F_2(s)} = \frac{K_2}{\Delta(s)} \end{aligned}}$$

$$X_2(s) = \frac{\begin{vmatrix} m_1 s^2 + b_1 s + (K_1 + K_2) & F_1(s) \\ -K_2 & F_2(s) \end{vmatrix}}{\Delta(s)} = \frac{K_2 F_1(s) + [m_1 s^2 + b_1 s + (K_1 + K_2)] F_2(s)}{\Delta(s)}$$

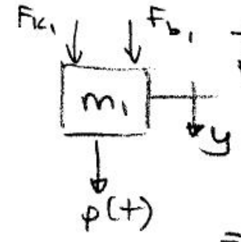
$$\boxed{G_{21}(s) = \frac{X_2(s)}{F_1(s)} = \frac{K_2}{\Delta(s)} \quad ; \quad G_{22}(s) = \frac{X_2(s)}{F_2(s)} = \frac{[m_1 s^2 + b_1 s + (K_1 + K_2)]}{\Delta(s)}}$$

Problem 2





$$\downarrow \Sigma F = m_2 \ddot{x}$$
$$= -F_{k_2} - F_{k_1} - F_{b_1}$$
$$= -k_2 x - k_1(x-y) - b_1(\dot{x}-\dot{y})$$
$$\Rightarrow \underline{m_2\ddot{x} + b_1\dot{x} + (k_1+k_2)x = b\dot{y} + k_1 y}$$



$$\downarrow \Sigma F = m_1 \ddot{y}$$
$$= F_{k_1} + F_{b_1} + p(t)$$
$$= k_1(x-y) + b_1(\dot{x}-\dot{y}) + p(t)$$
$$\Rightarrow \underline{m_1\ddot{y} + b_1\dot{y} + k_1 y = b_1\dot{x} + k_1 x + p(t)}$$

$$[m_2 s^2 + b_1 s + (k_1+k_2)]X(s) + [-b_1 s - k_1]Y(s) = 0$$
$$[-b_1 s - k_1]X(s) + [m_1 s^2 + b_1 s + k_1]Y(s) = P(s)$$

$$X(s) = \frac{\begin{vmatrix} 0 & -b_1 s - k_1 \\ P(s) & m_1 s^2 + b_1 s + k_1 \end{vmatrix}}{\Delta(s)} \Rightarrow \boxed{\Delta(s) = [m_2 s^2 + b_1 s + (k_1+k_2)][m_1 s^2 + b_1 s + k_1] - [b_1 s + k_1]^2}$$

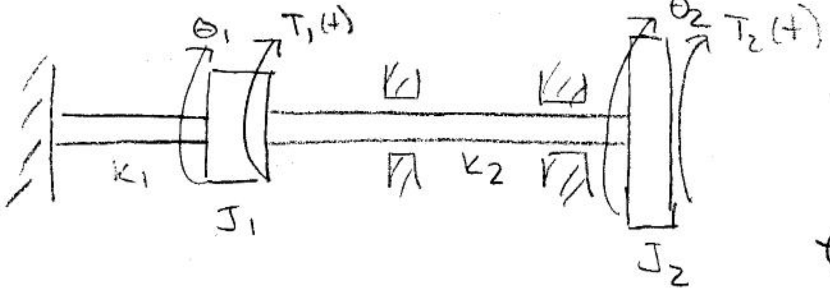
$$= \frac{[b_1 s + k_1]\,P(s)}{\Delta(s)}$$

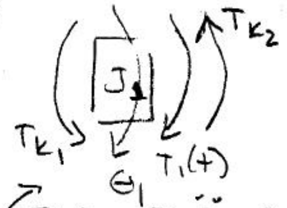
$$Y(s) = \frac{\begin{vmatrix} m_2 s^2 + b_1 s + (k_1+k_2) & 0 \\ b_1 s - k_1 & P(s) \end{vmatrix}}{\Delta(s)} = \frac{[m_2 s^2 + b_1 s + (k_1+k_2)]P(s)}{\Delta(s)}$$

(a) $G_{ax}(s) = \dfrac{s\,X(s)}{P(s)} = \dfrac{s[b_1 s + k_1]}{\Delta(s)}$; $G_{ay}(s) = \dfrac{s\,Y(s)}{P(s)} = \dfrac{s[m_2 s^2 + b_1 s + (k_1+k_2)]}{\Delta(s)}$

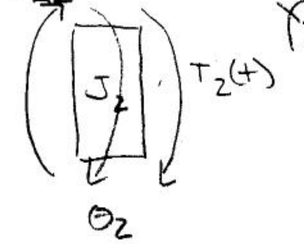
(b) $G_{bx}(s) = \dfrac{s^2 X(s)}{P(s)} = \dfrac{s^2[b_1 s + k_1]}{\Delta(s)}$; $G_{by}(s) = \dfrac{s^2 Y(s)}{P(s)} = \dfrac{s^2[m_2 s^2 + b_1 s + (k_1+k_2)]}{\Delta(s)}$

Problem 3





$$\sum M = J_1\ddot{\Theta}_1 = -T_{K_1} + T_1(t) - T_{K_2}$$
$$= -K_1\Theta_1 + T_1(t) - K_2(\Theta_1 - \Theta_2)$$
$$\Rightarrow \underline{J_1\ddot{\Theta}_1 + (K_1+K_2)\Theta_1 = K_2\Theta_2 + T_1(t)}$$



$$\sum M = J_2\ddot{\Theta}_2$$
$$= T_{K_1} + T_2(t)$$
$$= K_2(\Theta_1 - \Theta_2) + T_2(t) \Rightarrow \underline{J_2\ddot{\Theta}_2 + K_2\Theta_2 = K_2\Theta_1 + T_2(t)}$$

$$[J_1 s^2 + (K_1+K_2)]\,\Theta_1(s) + [-K_2]\,\Theta_2(s) = T_1(s)$$
$$[-K_2]\,\Theta_1(s) + [J_2 s^2 + K_2]\,\Theta_2(s) = T_2(s)$$

(a)

$$\Theta_1(s) = \frac{\begin{vmatrix} T_1(s) & -K_2 \\ T_2(s) & J_2 s^2 + K_2 \end{vmatrix}}{\boxed{\Delta(s) = [J_1 s^2 + (K_1+K_2)][J_2 s^2 + K_2] - K_2^2}}$$

$$= \frac{[J_2 s^2 + K_2]\,T_1(s) + K_2\,T_2(s)}{\Delta(s)}$$

$$\boxed{G_{11}(s) = \frac{\Theta_1(s)}{T_1(s)} = \frac{J_2 s^2 + K_2}{\Delta(s)} \qquad G_{12} = \frac{\Theta_1(s)}{T_2(s)} = \frac{K_2}{\Delta(s)}}$$

$$\Theta_2(s) = \frac{\begin{vmatrix} J_1 s^2 + (K_1+K_2) & T_1(s) \\ -K_2 & T_2(s) \end{vmatrix}}{\Delta(s)} = \frac{[J_1 s^2 + (K_1+K_2)]\,T_2(s) + K_2\,T_1(s)}{\Delta(s)}$$

$$\boxed{G_{21}(s) = \frac{\Theta_2(s)}{T_1(s)} = \frac{K_2}{\Delta(s)}\ ; \quad G_{22}(s) = \frac{\Theta_2(s)}{T_2(s)} = \frac{J_1 s^2 + (K_1+K_2)}{\Delta(s)}}$$

Problem 3 (cont'd)

(b)

$$\Theta_1(s) = G_{11}(s)\cdot T_1(s) + G_{12}(s)\cdot T_2(s)$$

$$\boxed{\Theta_1(s) = \frac{J_2 s^2 + K_2}{\Delta(s)}\cdot T_1(s) + \frac{K_2}{\Delta(s)}\cdot T_2(s) \qquad \to \Delta(s) = [J_1 s^2 + (K_1+K_2)][J_2 s^2 + K_2] - K_2^2}$$

$$\Theta_2(s) = G_{21}(s)\cdot T_1(s) + G_{22}(s)\cdot T_2(s)$$

$$\boxed{\Theta_2(s) = \frac{K_2}{\Delta(s)}\cdot T_1(s) + \frac{J_1 s^2 + (K_1+K_2)}{\Delta(s)}\cdot T_2(s)}$$

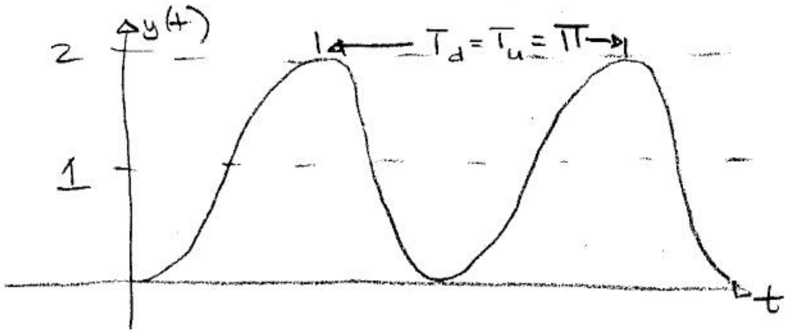
Problem 4

$$\frac{Y(s)}{U(s)} = G(s) = \frac{K}{Ms^2 + Bs + K}$$; unit step input $(A = 1)$

(a) $M = 1,\ B = 0,\ K = 4$: $G(s) = \frac{4}{s^2+4} = \frac{1}{\frac{s^2}{4}+1} \Rightarrow \underline{\bar{K} = 1}$

$\hookrightarrow \underline{\omega_n = 2}$

$\zeta = 0 \Rightarrow$ undamped system



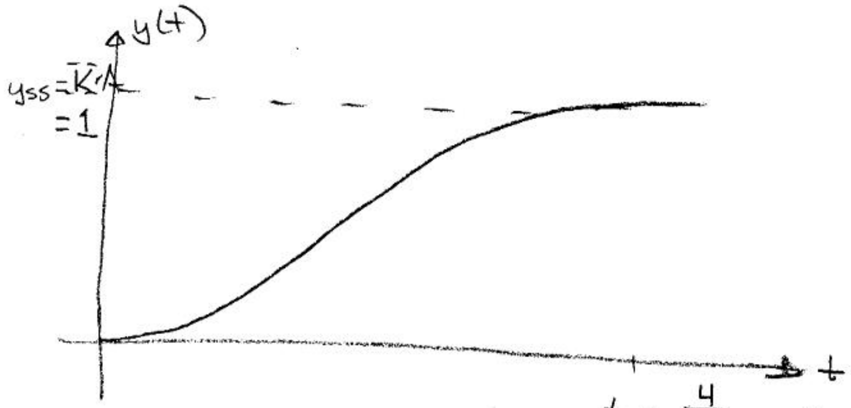
(b) $M = 1,\ B = 4,\ K = 4$: $G(s) = \frac{4}{s^2+4s+4} = \frac{1}{\frac{s^2}{4} + s + 1}$

$\underline{\bar{K} = 1}$

$\underline{\omega_n = 2}$

$\frac{2\zeta}{\omega_n} = 1$

$\zeta = \frac{\omega_n}{2} \Rightarrow \underline{\zeta = 1}$ critically damped system



$t_s = \frac{4}{\zeta\omega_n} = 2$

Check w/ Fig 8-24

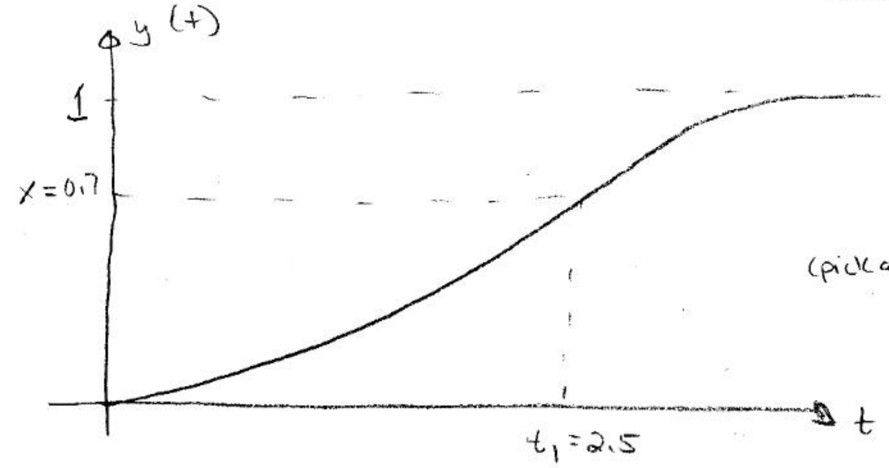
Problem 4 (cont'd)

c) $M = 1$, $B = 8$, $K = 4$: $\quad G(s) = \dfrac{4}{s^2 + 8s + 4} = \dfrac{1}{\frac{s^2}{4} + 2s + 1}$

$\omega_n = 2$ $\qquad \dfrac{2\zeta}{\omega_n} = 2$

$K = 1$

$\zeta = \omega_n \Rightarrow \zeta = 2$ overdamped system



$x_{ss} = K \cdot A = 1$

Check w/ Fig 8-24 (pick any pt): @ $x = 0.7$, $\omega_n t = 5$
$t = 2.5$

$t_s \to \approx 7.5$

d) $M = 1$, $B = 1$, $K = 4$: $\quad G(s) = \dfrac{4}{s^2 + s + 4} = \dfrac{1}{\frac{s^2}{4} + \frac{s}{4} + 1}$

$K = 1$ $\qquad \dfrac{2\zeta}{\omega_n} = \dfrac{1}{4}$

$\omega_n = 2$

$\zeta = \dfrac{\omega_n}{8} = \dfrac{2}{8} \Rightarrow \zeta = 0.25$

$x_{ss} = K \cdot A = 1$

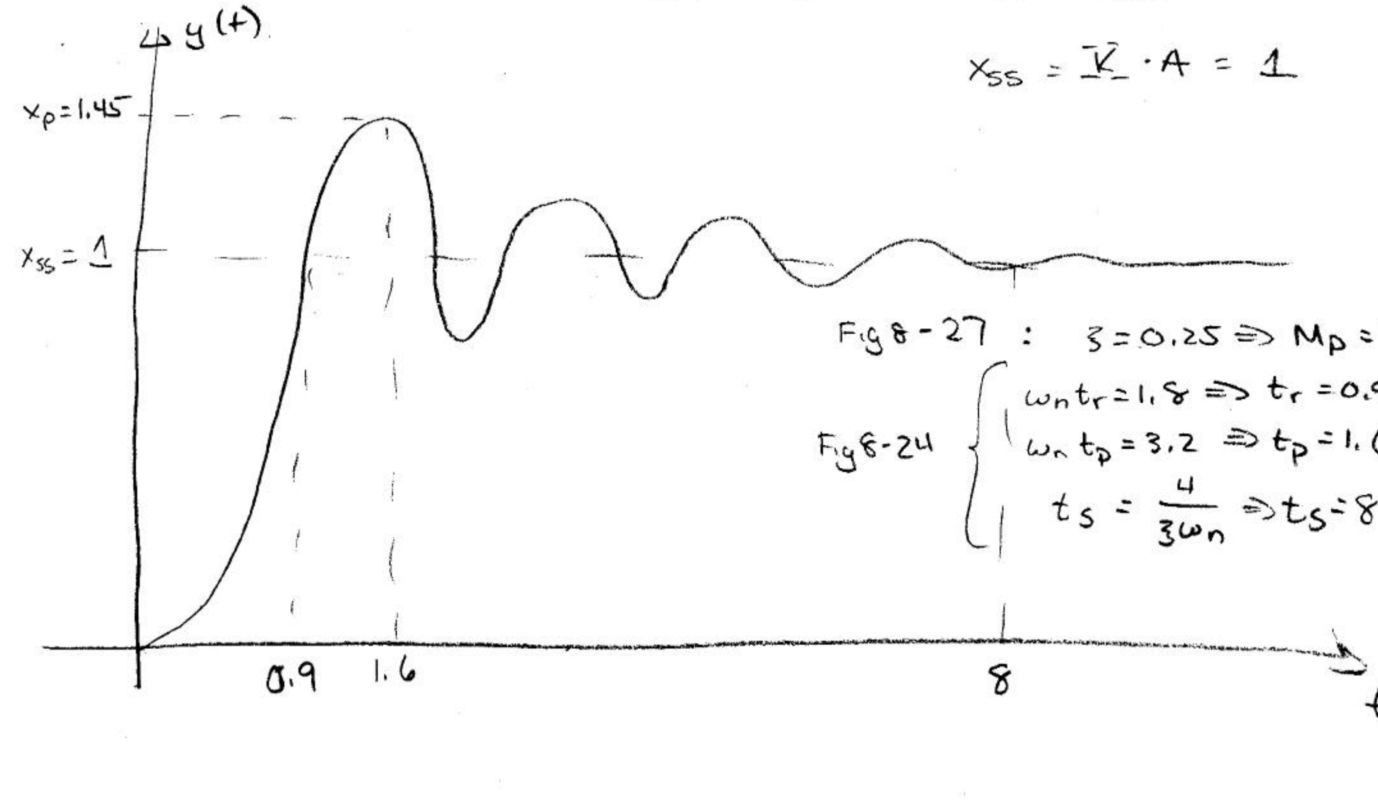


Fig 8-27 : $\zeta = 0.25 \Rightarrow M_p = 45\%$

Fig 8-24
$\omega_n t_r = 1.8 \Rightarrow t_r = 0.9$
$\omega_n t_p = 3.2 \Rightarrow t_p = 1.6$
$t_s = \dfrac{4}{\zeta \omega_n} \Rightarrow t_s = 8$

1^{st} order: $G(s) = \dfrac{K}{\tau s + 1}$

step input magnitude 2 $\Rightarrow A = 2$

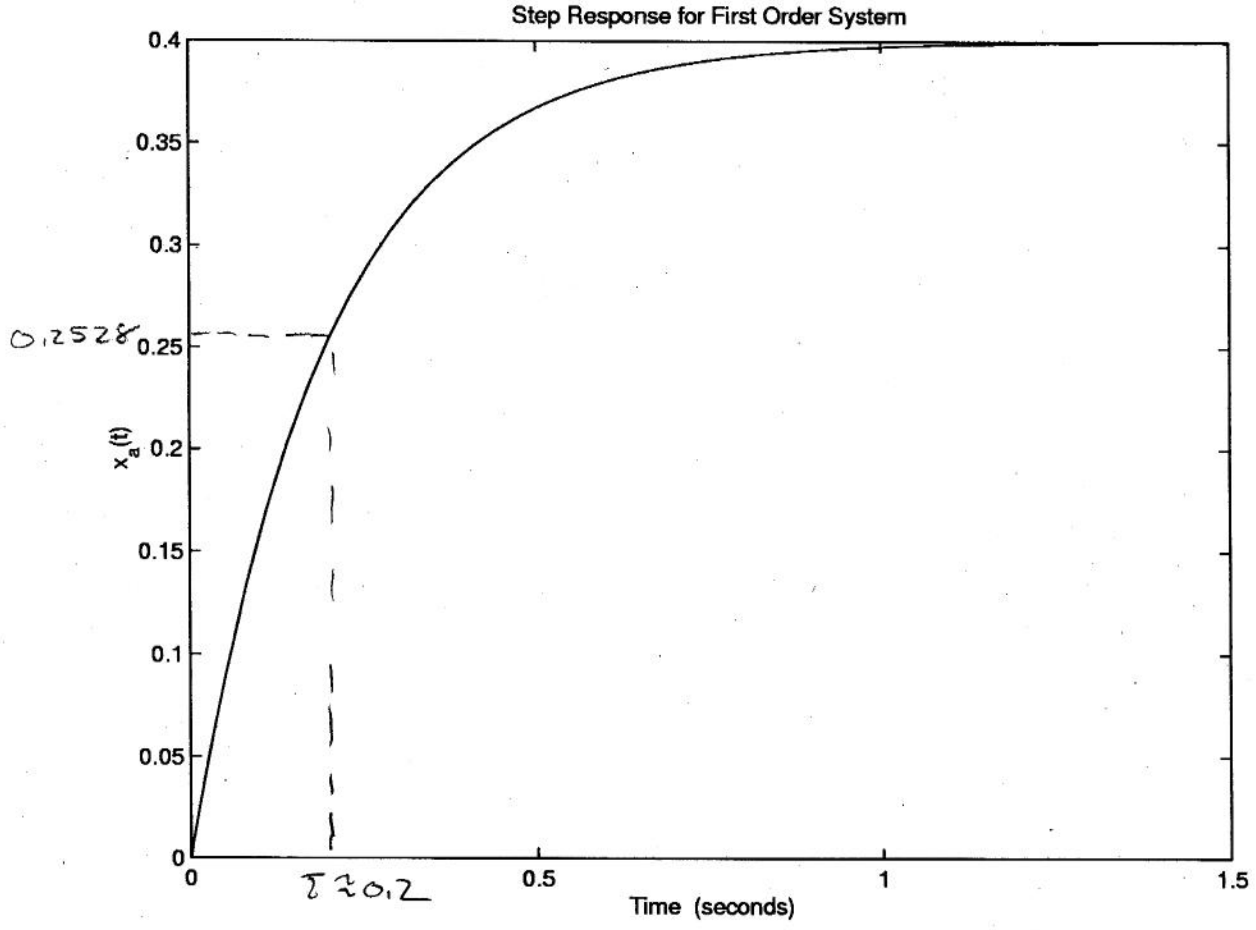


Figure 1: Figure 5(a)

$X_{ss} = K \cdot A = 0.4 \Rightarrow K = \dfrac{0.4}{A}$

$\underline{K = 0.2}$

$x(@t=\tau) = 0.632\, X_{ss}$

$= 0.632(0.4)$

$= 0.2528 \Rightarrow \therefore \underline{\tau \approx 0.2}$

$G(s) = \dfrac{0.2}{0.2s+1}$

$\Rightarrow \boxed{G(s) = \dfrac{1}{s+5}}$

2nd order $G(s) = \dfrac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$ $\underline{A = 2}$

$x_{ss} = K \cdot A \Rightarrow K = \frac{x_{ss}}{A} = \frac{0.5}{2} \Rightarrow \underline{K = 0.25}$

$M_p = \frac{x_p - x_{ss}}{x_{ss}} \times 100\% = \frac{0.73-0.5}{0.5} \times 100\%$; $M_p \approx 45\%$

Fig 8-27: $\underline{\zeta = 0.25}$

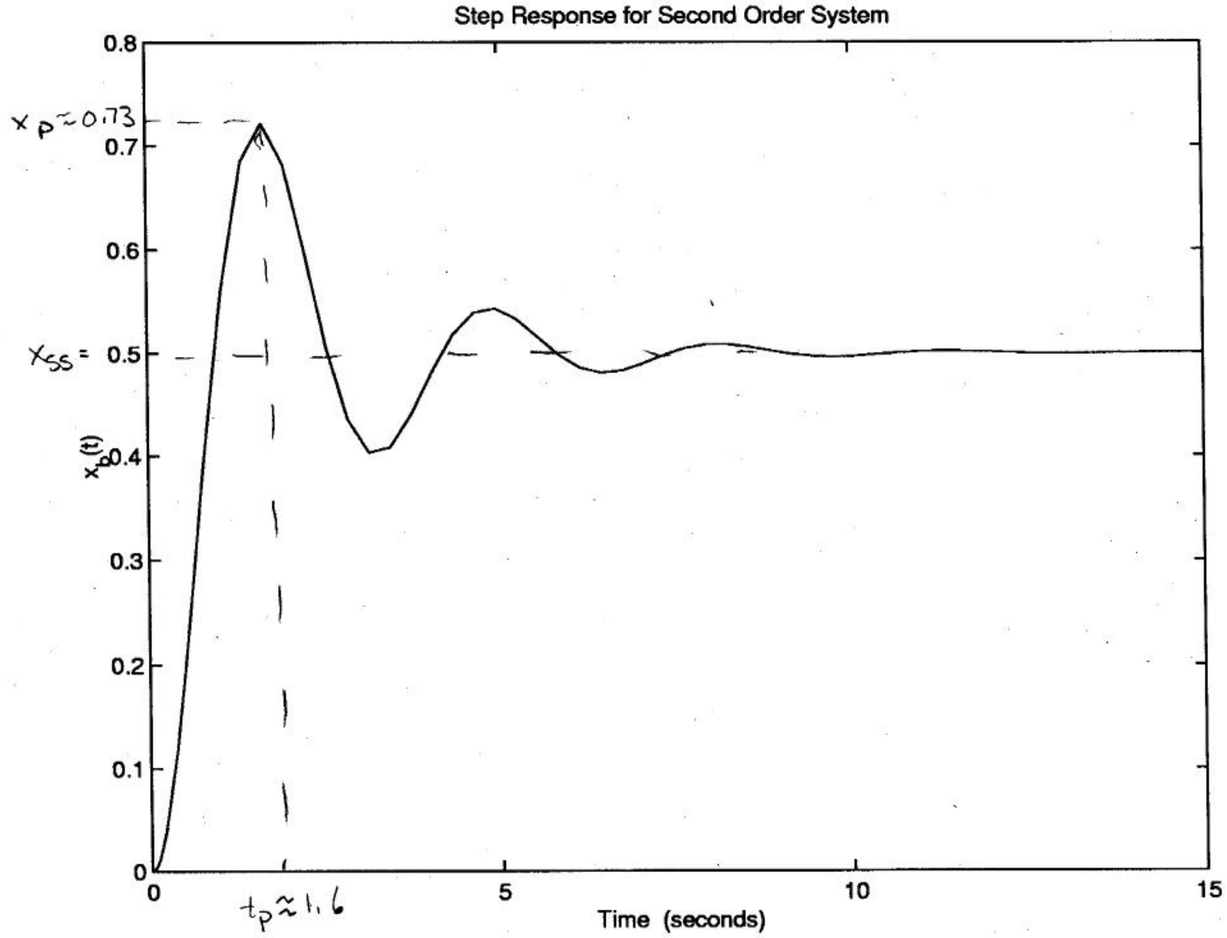


Figure 2: Figure 5(b)

From Fig 8-24.
for $\zeta = 0.25$: $t_{p,\text{chart}} = 3.3 = \omega_n t_p$

$\omega_n = \frac{3.3}{t_p} = \frac{3.3}{1.6} \Rightarrow \underline{\omega_n = 2}$

$$G(s) = \frac{0.25}{\frac{s^2}{4} + \frac{2(0.25)s}{2} + 1} \Rightarrow \boxed{G(s) = \frac{1}{s^2 + s + 4}}$$