ME 709/809 Final [20 points]

You will have 72 hours to complete this assignment. You will be simulating a 2D, fully developed channel flow using the $k-\epsilon$ model.

- TASK 1 [2 points]: Without using index notation, show the governing equations for a 2D, time averaged $k-\epsilon$ model.
  - Show the time averaged continuity equation.
  - Show each of the components of the averaged Naveir-Stokes equations. Use the Boussinesq assumption as listed in section 6 of the notes. You can ignore the term in blue from the assumption.
  - Start from the modeled k and ϵ equations (equations 11.97 and 11.98 in `http://www.tfd.chalmers.se/~lada/postscript_files/solids-and-fluids_turbulent-flow_turbulence-modelling.pdf`) and show the modeled k and ω equations for a 2D, time-averaged flow.

- TASK 2 [2 points]: Simplify the equations from TASK1 for a 2D fully developed channel flow. Clearly indicate what terms you are setting to 0 and why. Prove that the governing equations reduce to:

  $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \nu_t)\frac{\partial U}{\partial y}\right];$$

  $$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \epsilon;$$

  $$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial y}\right] + \frac{\epsilon}{k}\left(c_1 P_k - c_2 \epsilon\right);$$

  where

  $$P_k = \nu_t\left(\frac{\partial U}{\partial y}\right)^2; \text{ and}$$

  $$\nu_t = c_\mu \frac{k^2}{\epsilon}.$$

- TASK 3 [5 points]: Discretize the equations using central difference and a non-equidistant mesh. Note that you need to keep k and ϵ positive to avoid divergence of the solution. To achieve that, put all negative source terms in S_P where $S = S_P\Phi_P + S_U$. For example, the dissipation term in the k equations should be in the S_P and look similar to: $S_P = -\frac{\epsilon}{k}\Delta y$ or $S_P = -\frac{\epsilon}{k}\Delta V$ (depending on how you integrate the equations).

- TASK 4 [3 points]: Write down the steps of the algorithm for solving this problem in Matlab. For the boundary conditions use $U = k = \frac{\partial \epsilon}{\partial y} = 0$.
  Use channel height of 2 m, $\rho = 1$ kg/m^3, $u_\tau = 1$ m/s and use $-\frac{\partial P}{\partial x} = \tau_w$ $= \mu\left(\frac{du}{dy}\right)_{y=0}$
  Use a Gauss-Seidel solver.

- TASK 5 [8 points] Simulate this flow using Matlab.
  Plot the velocity profile.

$-\epsilon = \frac{S_P k}{\Delta y}$

Please submit the following on Canvas under FINAL exam submission:

- A pdf file containing the work for TASK 1, 2, 3, and 4; The plot from TASK 5, and a printout of the code from TASK 5. The work from TASKS 1 – 4 can be scanned papers, an exported one-note document or have any other form, as long as everything is legible and the submission is a single pdf file.
- A zipped folder with all your work on the assignment, including the final .m file.

ME 709 Final

Charlie Nitschelm, 1/20/19

Task 1:

a) Show the time-averaged continuity equation for a 2D, fully developed flow.

$$\rho \frac{\partial u_i}{\partial x_j} = 0 \Rightarrow \rho \neq 0 \Rightarrow \frac{\partial u_i}{\partial x_j} = 0 \longrightarrow$$ Now account for flow in the x and y direction and getting rid of the indic

$$\rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$ where u is the average velocity in the x-direction and v is the average velocity in the y-direction.

b) Show each component (x and y) of the averaged NS equations. Ignore terms in blue.

$$\bigstar\ \rho\left[\overline{u_j' \frac{\partial u_i}{\partial x_j}} + \overline{u_i \frac{\partial u_j'}{\partial x_j}}\right] = \frac{\partial}{\partial x_j'}\left(-\rho\gamma_t\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

Continuity Equation

Calculating as a nested loop with $(i,j) = (1,1), (2,1)$ and $(1,2)$ and $(2,2)$ then adding all terms together. This brings us a 2D

$$\rightarrow \cancel{\rho\left[\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right]} \rightarrow$$

$$\cancel{\rightarrow = \frac{\partial}{\partial x}}$$

Full Equation: 2

$$\rho\left[\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\rho \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right]$$

↓ imparting i and j indicies

Final Answer

$$\rho\left[\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right] = -\left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\right) + \rightarrow$$

$$\rightarrow \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) + \frac{\partial}{\partial x}\left[\rho \nu_t \left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] - \rightarrow$$

$$-\frac{\partial}{\partial y}\left[\rho \nu_t \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial y}\right)\right]$$ ← Answer (3 lines)

$i=1$ and $i=2$ equations combined

* $i=2$ components all zero

c) Show a Modeled k and w equation for a 2D, time averaged flow:

Using equations 11.97 and 11.98 ($k-\varepsilon$ model equations)

$\omega \propto \frac{\varepsilon}{k} \longrightarrow$ Change ε equation where $\varepsilon = \omega k$

$\varepsilon \rightarrow \omega$ equation:

$$\frac{\partial(k\omega)}{\partial t} + \bar{V}_j \frac{\partial(k\omega)}{\partial x_j} = \frac{k\omega}{k} C_{k\omega 1} \nu_t \left(\frac{\partial \bar{V}_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\frac{\partial V_i}{\partial x_j} + \longrightarrow$$

$$\rightarrow C_{k\omega 1}\, g_i \frac{k\omega}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i} - C_{k\omega 2}\frac{(k\omega)^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\overset{= C_\mu \frac{k^2}{(k\omega)}}{\nu_t}}{\sigma_\varepsilon}\right)\frac{\partial k\omega}{\partial x_j}\right]$$

↓ simplify : ANSWER → ω equation

$$k\frac{\partial\omega}{\partial t} + \bar{V}_j k \frac{\partial \omega}{\partial x_j} = \omega C_{\omega 1}\left(C_\mu \frac{k}{\omega}\right)\left(\frac{\partial \bar{V}_i}{\partial x_j} + \frac{\partial \bar{V}_j}{\partial x_i}\right)\frac{\partial \bar{V}_i}{\partial x_j} + \frac{C_{\omega 1} g_i k C_\mu}{\sigma_\theta}\left(\frac{\partial\bar{\theta}}{\partial x_i}\right)$$

$$- C_{\omega 2}\frac{\omega^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{C_\mu k}{\sigma_\omega \omega}\right)\cdot k\frac{\partial\omega}{\partial x_j}\right]$$

Equation for a k-ω model for a 2D, time averaged flow

↓ k-equation

$$\frac{\partial k}{\partial t} + \bar{V}_j \frac{\partial k}{\partial x_j} = \overset{\nu_t \omega}{\downarrow} C_\mu \frac{k}{\omega}\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)\frac{\partial V_i}{\partial x_j} + g_i\beta\frac{C_\mu k}{\sigma_\theta \omega}\frac{\partial\bar{\theta}}{\partial x_i} - k\omega + \rightarrow$$

$$\rightarrow + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{C_\mu k}{\sigma_k \omega}\right)\frac{\partial k}{\partial x_j}\right]$$

Task 2:

Averaged NS equation simplified for a 2D, fully developed Channel flow. → Assumptions: $\frac{\partial}{\partial t} = 0$: steady state

$\frac{\partial \vec{V}}{\partial x} = 0$: Fully developed

$V = 0$: 2D flow in X direction

$\frac{\partial P}{\partial y} = 0$: Pressure distributions is hydro static

$\frac{\partial \rho}{\partial x} = 0$ → Incompressible

left side:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{\,S.S} + \cancel{\frac{\partial v}{\partial t}}^{\,S.S} + u\cancel{\frac{\partial u}{\partial x}}^{\,\frac{\partial}{\partial x}} + V\cancel{\frac{\partial u}{\partial y}}^{\,V=0} + V\cancel{\frac{\partial u}{\partial y}}\right) = \text{STUFF} \rightarrow 0 = \frac{1}{\rho}(\text{STUFF})$$

Divide over

$$0 = \frac{1}{\rho}\left(-\frac{\partial P}{\partial x} - \cancel{\frac{\partial P}{\partial y}}^{\,\frac{\partial P}{\partial y}=0} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \cancel{\frac{\partial \tau_{xx}}{\partial y}} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \cancel{\frac{\partial \tau_{xx}}{\partial y}} + \cancel{\frac{\partial}{\partial x}}^{\,\frac{\partial}{\partial x}=0}\left(\cancel{V_t\left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)}^{\,\frac{\partial}{\partial x}=0}\right)\right.$$

$$\left.-\frac{\partial}{\partial x}\left(V_t\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}^{\,\frac{\partial}{\partial x}=0} + 2\frac{\partial u}{\partial y}\right)\right)\right)$$

No shear, Channel flow

↓ So! $\uparrow \frac{\partial u}{\partial y}$ ✓

$$\boxed{0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial y}\left(\nu + \nu_t\right)\right]}$$ ✓

K-ω equations: Same assumption as before → $i=1,\ j=2$

$$K:\ \overset{S.S}{\cancel{\frac{\partial k}{\partial t}}} + \bar{V}_j\, \overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial k}{\partial x_j}}} = \nu_t \left(\overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial V_i}{\partial x_j}}} + \overset{=\frac{\partial u}{\partial y}}{\frac{\partial V_j}{\partial x_i}}\right)\overset{\frac{\partial u}{\partial y}}{\frac{\partial V_i}{\partial x_j}} + g_i \beta \frac{\nu_t C}{\sigma_\theta}\overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial \theta}{\partial x_i}}} - \underset{\varepsilon}{k\omega}$$

$$+ \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$0 = \underbrace{\nu_t\left(\frac{\partial U}{\partial y}\right)^2}_{P_k} - \varepsilon + \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] \checkmark$$

$$\omega:\ k\,\overset{\frac{\partial}{\partial t}=0}{\cancel{\frac{\partial \omega}{\partial t}}} + \overset{V=0}{\cancel{V_j}}\, k\, \overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial \omega}{\partial x_j}}} = \omega C_{\omega 1}\left(\nu_t\right)\left(\overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial V_i}{\partial x_j}}} + \overset{\frac{\partial u}{\partial y}}{\frac{\partial V_j}{\partial x_i}}\right)\overset{\frac{\partial u}{\partial y}}{\frac{\partial V_i}{\partial x_j}} + \frac{C_{\omega 1} g_i k C_\mu}{\sigma_\theta}\overset{\frac{\partial}{\partial x}=0}{\cancel{\frac{\partial \theta}{\partial x_i}}}$$

$$- C_{\omega 2}\frac{\omega^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right) k \frac{\partial \omega}{\partial x_j}\right]$$

↓ simplify again

$$0 = \omega\left(C_1 \nu_t \left(\frac{\partial u}{\partial y}\right)^2 - C_2 \underset{\varepsilon}{\omega k}\right) + \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x}\right] \qquad k\frac{\partial \omega}{\partial x} \rightarrow \frac{\partial \varepsilon}{\partial x}$$

↓ Sub in

$$0 = \omega\left(C_1 P_k - C_2 \varepsilon\right) + \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x}\right] \checkmark$$

Task 3:

Equation 1:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[(\nu+\nu_t)\frac{\partial U}{\partial y}\right] = 0$$

$$\int_S^N \left(-\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[(\nu+\nu_t)\frac{\partial U}{\partial x}\right] + S\right) dy = 0$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}\Delta x \Delta y + \int_S^N \frac{\partial}{\partial y}\left[(\nu+\nu_t)\frac{\partial U}{\partial y}\right] + S \Delta y = \cancel{0}$$

$$-\frac{1}{\rho}\frac{dP}{\partial x}\Delta y + \left[\left((\nu+\nu_t)\frac{\partial U}{\partial x}\right)_N - \left((\nu+\nu_t)\frac{\partial U}{\partial y}\right)_S\right] + S \Delta y = 0$$

$$\frac{P_E - P_P}{\delta x_e} - \frac{P_P - P_W}{\delta x_w}$$

Discretiz

$$\boxed{-\frac{1}{\rho}\left(\frac{P_E - P_P}{\delta x_e} - \frac{P_P - P_W}{\delta x_w}\right)\Delta y + (\nu+\nu_t)\left(\frac{U_N - U_P}{\delta y_N} - \frac{U_P - U_S}{\delta y_S}\right) + S \Delta y = 0}$$

Equation 2:

$$\frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial K}{\partial y}\right] + P_k - \varepsilon = 0$$

$$\int_S^N \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial K}{\partial y}\right] dx\,dy + \int_S^N P_k\, dx\,dy - \int_S^N \varepsilon\, dx\,dy + \int_S^N S\, dx\,dy = 0$$

$$\frac{\partial K}{\partial y_N} - \frac{\partial K}{\partial y_S}$$

$$\nu_t\left(\frac{\partial U}{\partial y_N}^2 - \frac{\partial U}{\partial y_S}^2\right)$$

$$\boxed{\left(\nu+\frac{\nu_t}{\sigma_k}\right)\left[\frac{k_N-k_P}{\delta y_N}-\frac{k_P-k_S}{\delta y_S}\right]\Delta x+\nu_t\left[\left(\frac{U_N-U_P}{\delta y_N}\right)^2-\left(\frac{U_P-U_S}{\delta y_S}\right)^2\right]\Delta y + KS_p\Delta x+S\Delta x\Delta y}$$

Equation 3:

$$\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial\varepsilon}{\partial y}\right]+\frac{\varepsilon}{K}\left(C_1P_k-C_2\varepsilon\right)=0$$

$$\int_S^N\int_W^E\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial\varepsilon}{\partial y}\right]+\int_S^N\int_W^E\frac{\varepsilon}{K}\left(C_1P_k-C_2\varepsilon\right)+\int_S^N\int_W^E S\,dx=0$$

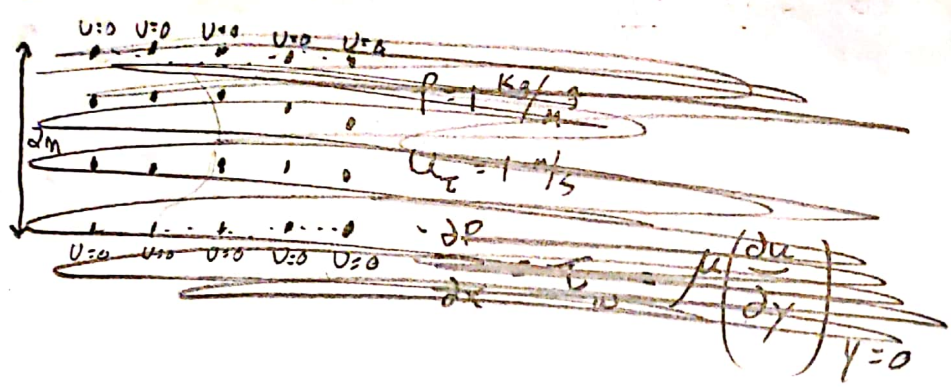
$$\downarrow$$

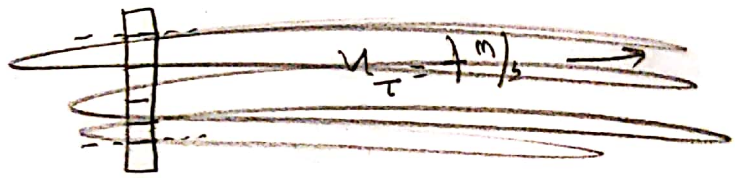
$$\left(\nu+\frac{\nu_t}{\sigma_k}\right)\left[\frac{\partial\varepsilon}{\partial y_N}-\frac{\partial\varepsilon}{\partial x_S}\right]\Delta x+\frac{\varepsilon}{K}C_1\nu_t\left[\frac{\partial U}{\partial y_N}^2-\frac{\partial U}{\partial y_S}^2\right]\Delta x\Delta y$$
$$-C_2\frac{\varepsilon^2}{K}\Delta x\Delta y+S\Delta x\Delta y$$

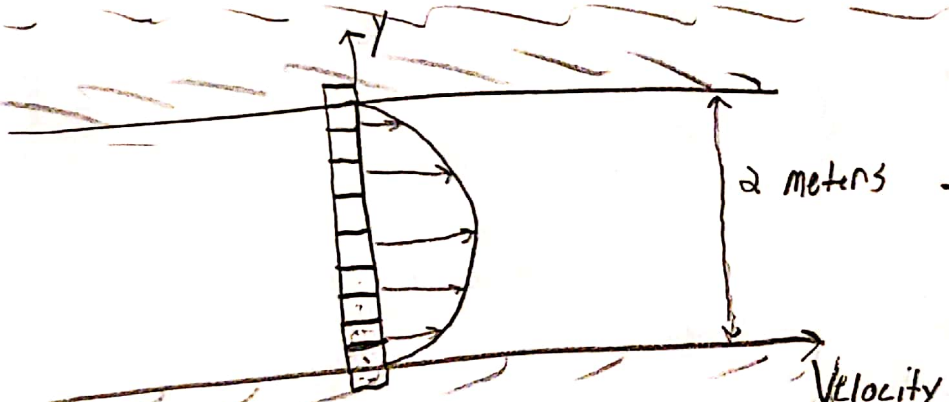
$$\downarrow$$

$$\left(\nu+\frac{\nu_t}{\sigma_k}\right)\left[\frac{\varepsilon_N-\varepsilon_P}{\delta y_N}-\frac{\varepsilon_P-\varepsilon_S}{\delta y_S}\right]\Delta x+C_1C_\mu K\left[\frac{U_N-U_P}{\delta y_N}-\frac{U_P-U_S}{\delta y_S}\right]\Delta y$$
$$+\Delta x\Delta y\left(S_pK\Delta x+S\right)$$

Task 4:







$H = 2m$, $\rho = 1 \frac{kg}{m^3}$, $u_\tau = 1$ m/s

$$-\frac{\partial P}{\partial x} = \tau_w = \mu\left(\frac{\partial u}{\partial x}\right)_{y=0}$$

$$u_\tau = u^* = \sqrt{\frac{\tau_w}{\rho}}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

Steps to solve problem with MATLAB:

1. Set up column of initial values (BC.'s)
2. Starting from the bottom, above wall BC, guess a u, k and ε values
3. Solve for $\nu_t = C_\mu \frac{k^2}{\varepsilon}$ where $C_\mu = 0.09$, and k and ε are the guesses
4. Solve for $P_k = \nu_t \left(\frac{\partial u}{\partial y}\right)^2$ where ν_t was solved above and $\frac{\partial u}{\partial y} = \rightarrow$

$\rightarrow \frac{U_N - 2U_P + U_S}{\Delta y}$ where $U_N = (i+1)$ velocity and $U_S = (i-1)$ velocity and $U_P =$ (i) velocity and $\Delta y = \frac{H}{\text{Number of cells}}$

5. From the NS equation in task 2, solve for $\frac{\partial U}{\partial y}$.
(When $-\frac{\partial P}{\partial x} = \tau_w = \frac{U_\tau^2}{\rho}$

6. When $\frac{\partial U}{\partial y} = const$, break apart $\frac{\partial U}{\partial y}$ and solve for new U_p

7. With equations (2 + 3) from task 2, solve for k and ε respectively with all constants defined.

8. Save all new values of U, k and ε and repeat over again

Task 5:

$\rho = 1 \frac{kg}{m^3}$, $u_\tau = 1\, m/s = \sqrt{\frac{\tau_w}{\rho}}$ (u^*), $\tau_w = -\frac{\partial p}{\partial x} = \mu \left(\frac{\partial U}{\partial y}\right)_{y=0} = \rho$

$\nu_t = C_\mu \frac{k^2}{\varepsilon}$, $P_k = \nu_t \left(\frac{\partial u}{\partial y}\right)^2$

For U: $-\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[(\nu+\nu_t)\frac{\partial u}{\partial x}\right] = 0$

$$1 + (\nu+\nu_t)\left(\frac{U_N + U_S - 2U_P}{\Delta y}\right) = 0$$

$$\frac{-\Delta y}{(\nu+\nu_t)} = U_N + U_S - 2U_P$$

$$\boxed{U_P = \frac{\Delta y}{2(\nu+\nu_t)} + \frac{U_N}{2} + \frac{U_S}{2}}$$

For k: $\frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \varepsilon = 0$

Integrate

$$(\nu+\nu_t)\left(\frac{k_N + k_S - 2k_P}{\Delta y}\right) + (\nu_t P_k - \varepsilon) = 0$$

$$k_N + k_S - 2k_P = \frac{(\varepsilon - \nu_t P_k)\Delta y}{\nu + \nu_t}$$

$$\boxed{k_P = \frac{\overset{S}{(\nu_t P_k} - \varepsilon)\Delta y}{2\left(\nu + \frac{\nu_t}{\sigma_k}\right)} + \frac{k_N}{2} + \frac{k_S}{2}}$$

For ε: Same as k but S contains

$$S = \left(\frac{\varepsilon}{k} C_1 P_k - C_2 \frac{\varepsilon^2}{k}\right)$$

So, $$\boxed{\varepsilon_P = \frac{\left(\frac{\varepsilon}{k} C_1 P_k - C_2 \frac{\varepsilon^2}{k}\right)\Delta y^2}{2\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)} + \frac{\varepsilon_N}{2} + \frac{\varepsilon_S}{2}}$$