Advanced Algorithms

南京大学

尹一通

Course Info

- Instructor: 尹一通、郑朝栋
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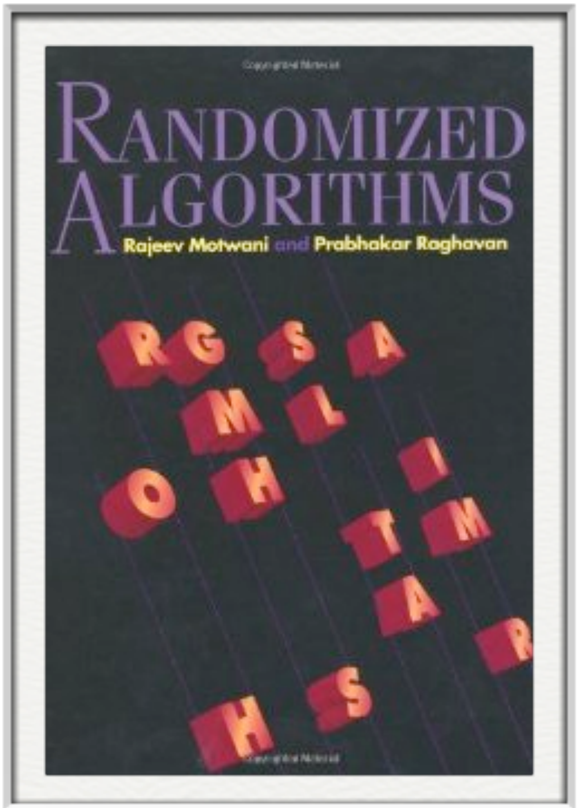
  - http://tcs.nju.edu.cn/wiki/

Textbooks



Rajeev Motwani and Prabhakar Raghavan.
Randomized Algorithms.
Cambridge University Press, 1995.

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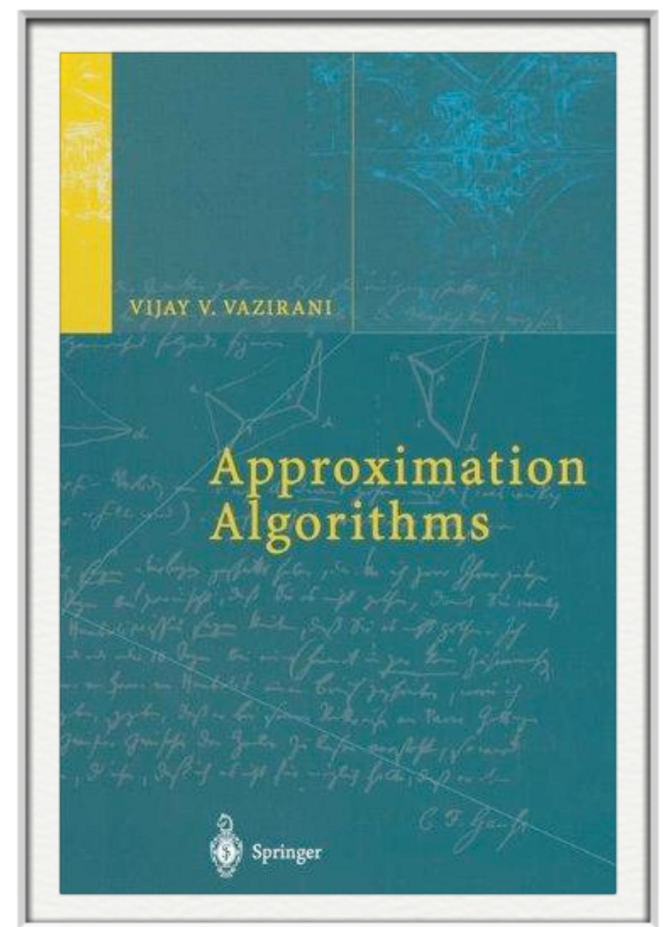
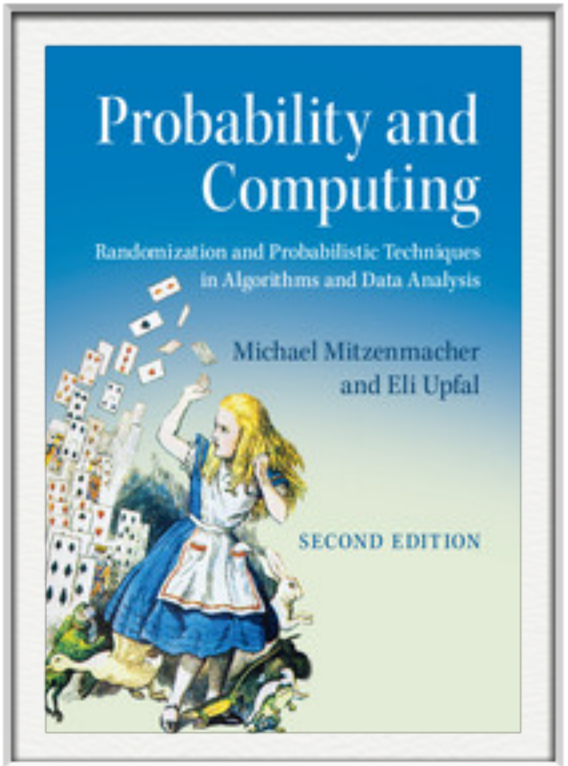
Approximation Algorithms.
Spinger-Verlag, 2001.



References



Mitzenmacher and Upfal.
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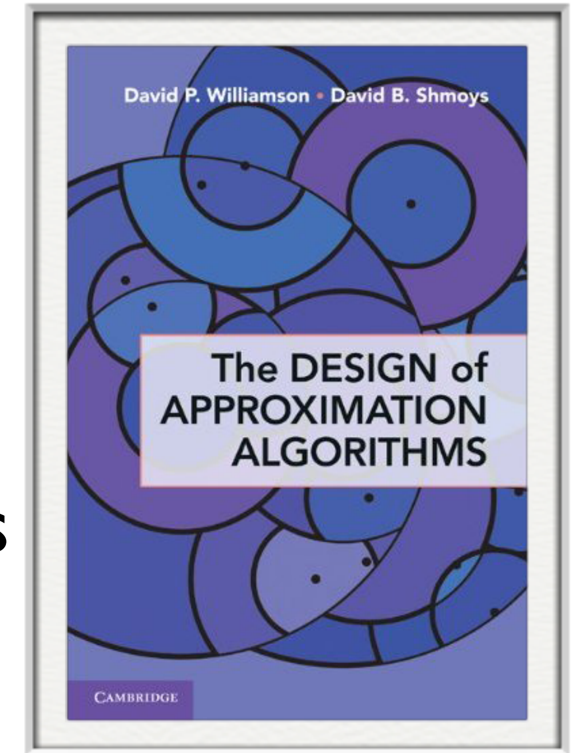
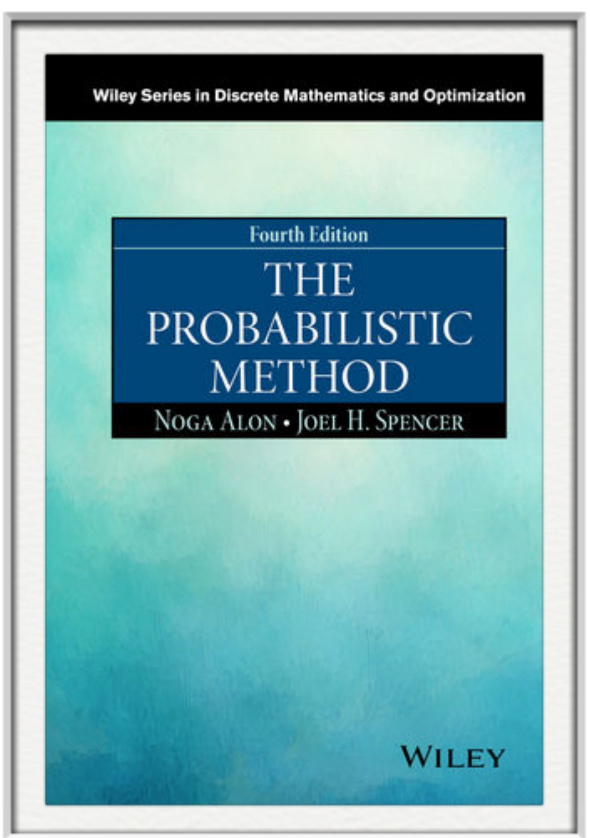
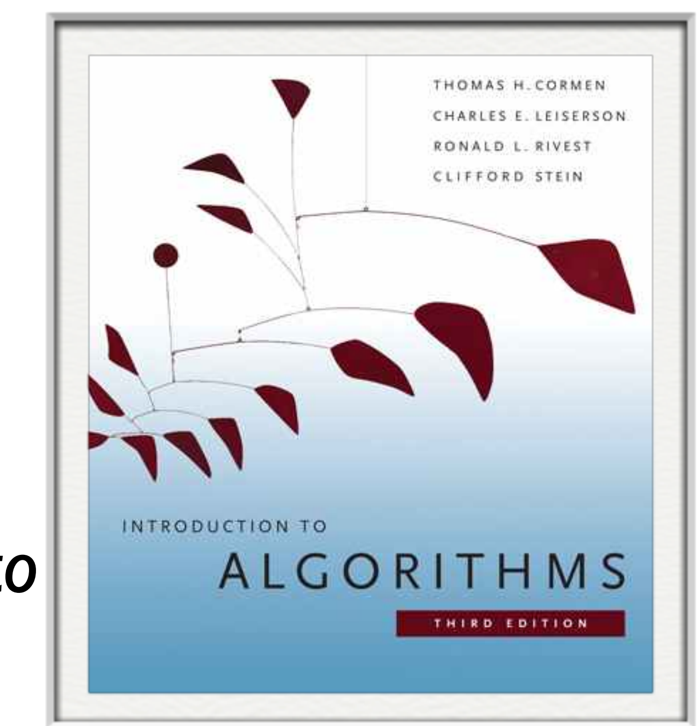
Williamson and Shmoys
The Design of Approximation Algorithms





Alon and Spencer
The Probabilistic Method, 4th Ed.

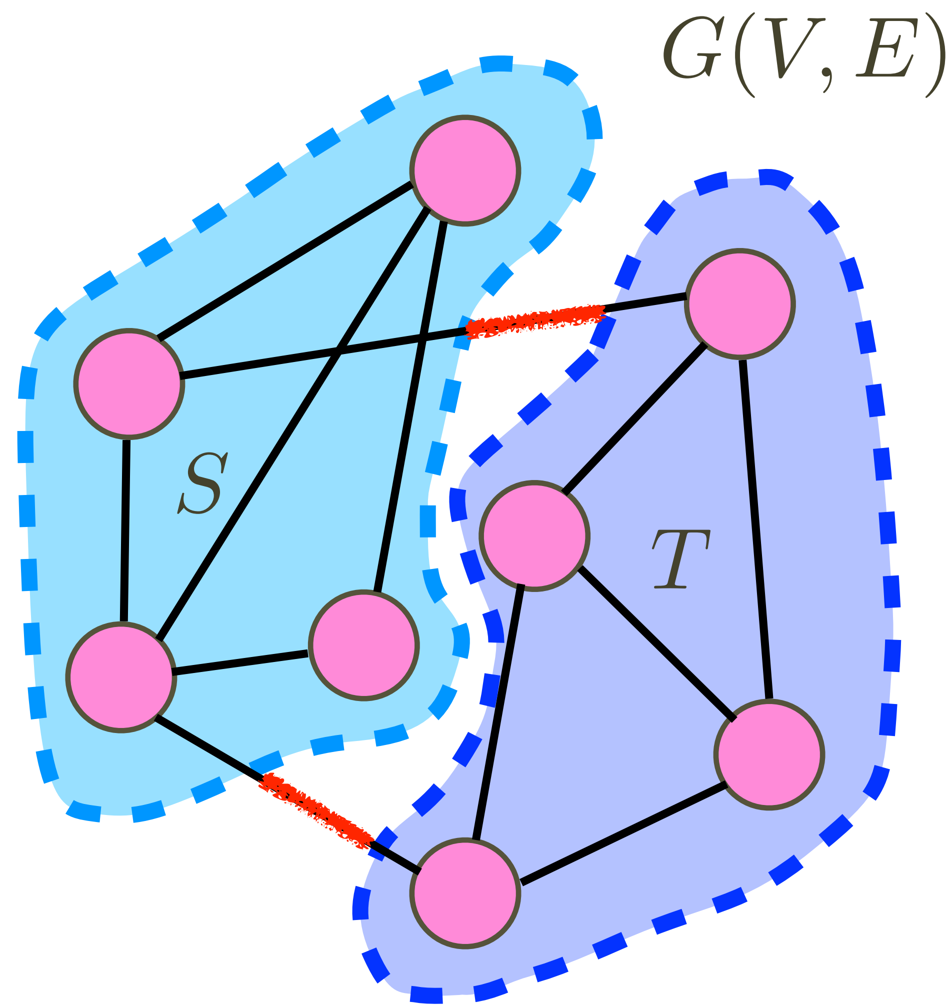
CLRS
Introduction to Algorithms



“Advanced” Algorithms

Min-Cut

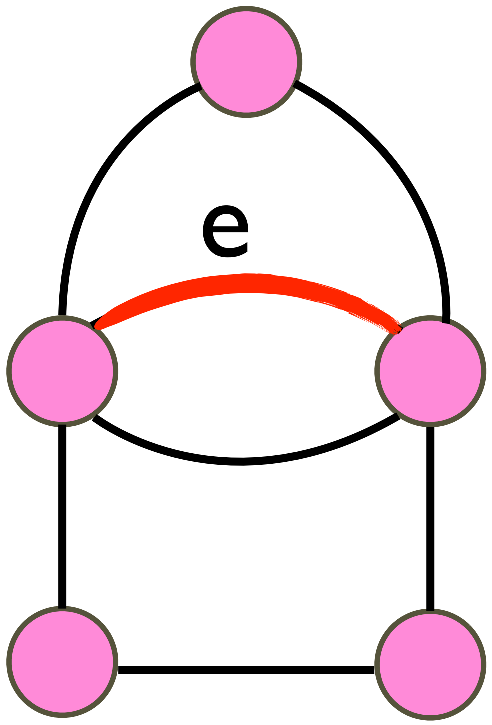
- Partition V into two parts: S and T
- Minimize the cut $E(S,T)$
- deterministic algorithm:
  - max-flow min-cut
  - best known upper bound: $O(mn + n^2 \log n)$



$$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$$

Contraction

- multigraph $G(V, E)$
- multigraph: allow parallel edges
- for an edge e, $\text{contract}(e)$ merges the two endpoints.



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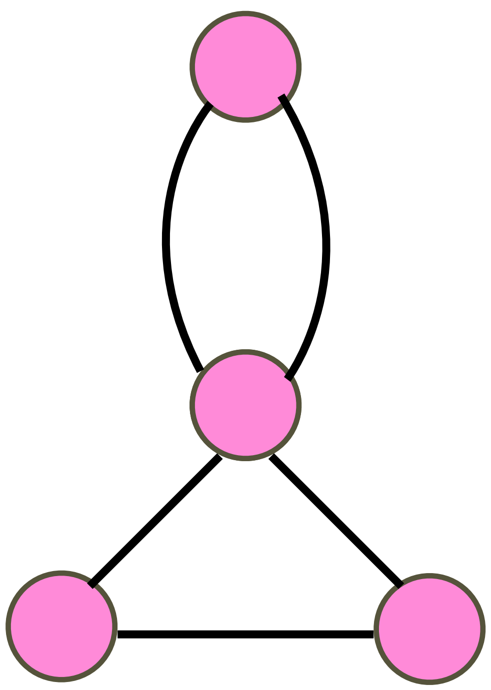
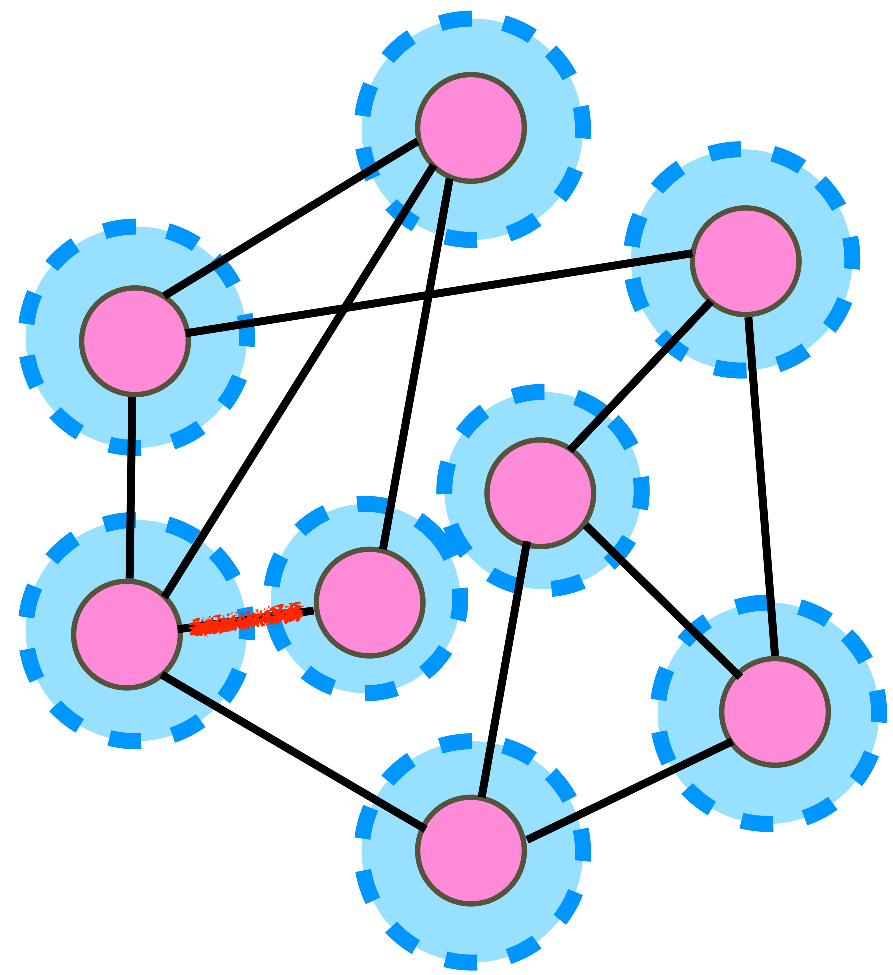


Karger's min-cut Algorithm

MinCut (multigraph $G(V,E)$)

while $\lvert V\rvert>2$ do
choose a uniform $e \in E$;
contract(e);
return remaining edges;



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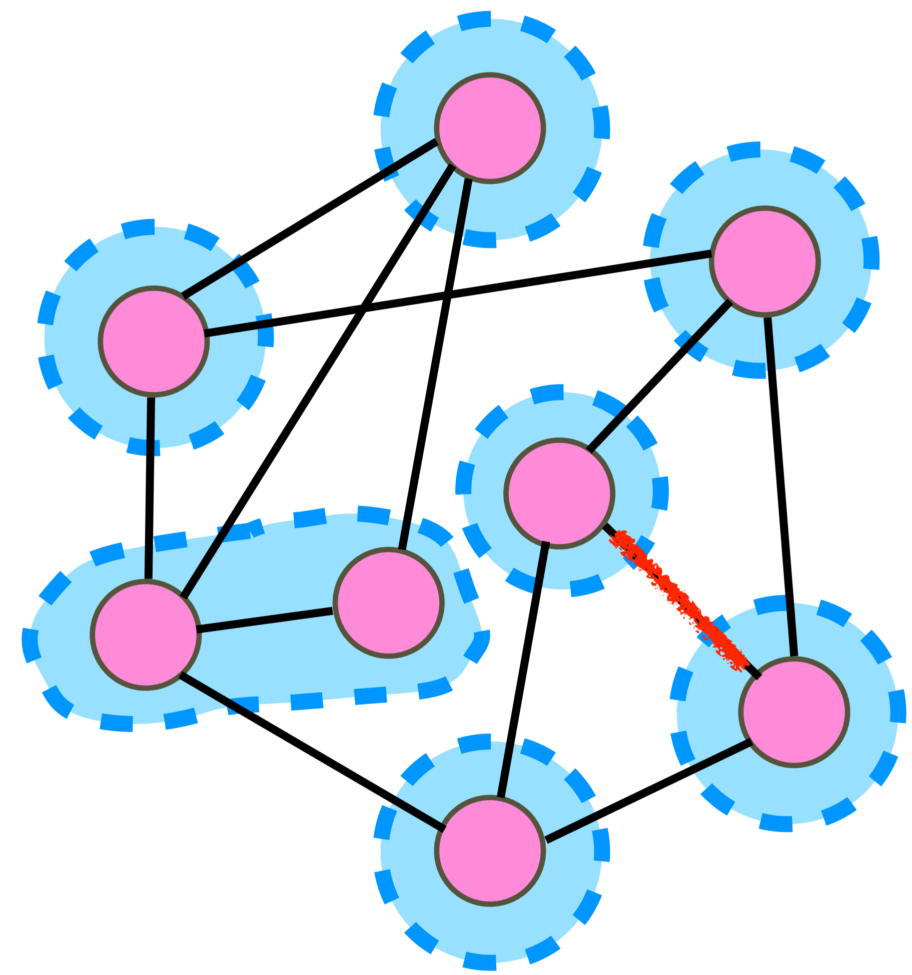

Karger's min-cut Algorithm

MinCut (multigraph $G(V,E)$)

while $|V|>2$ do

  choose a uniform $e \in E$;

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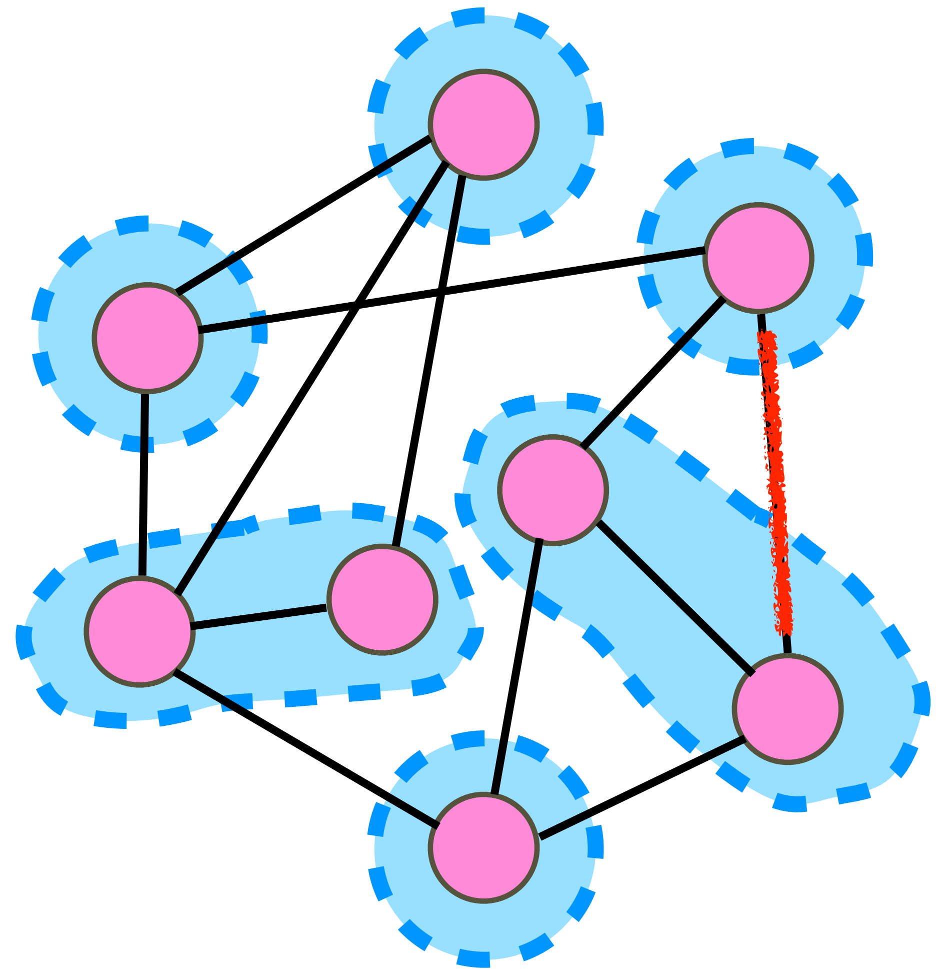

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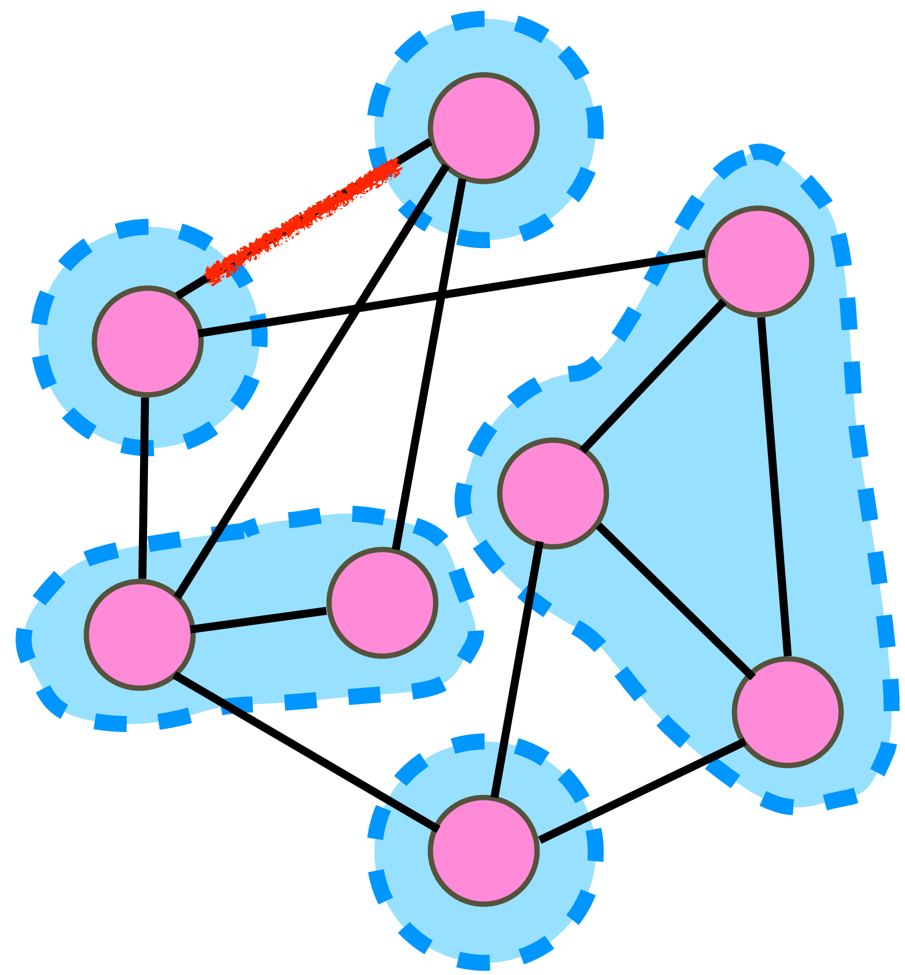

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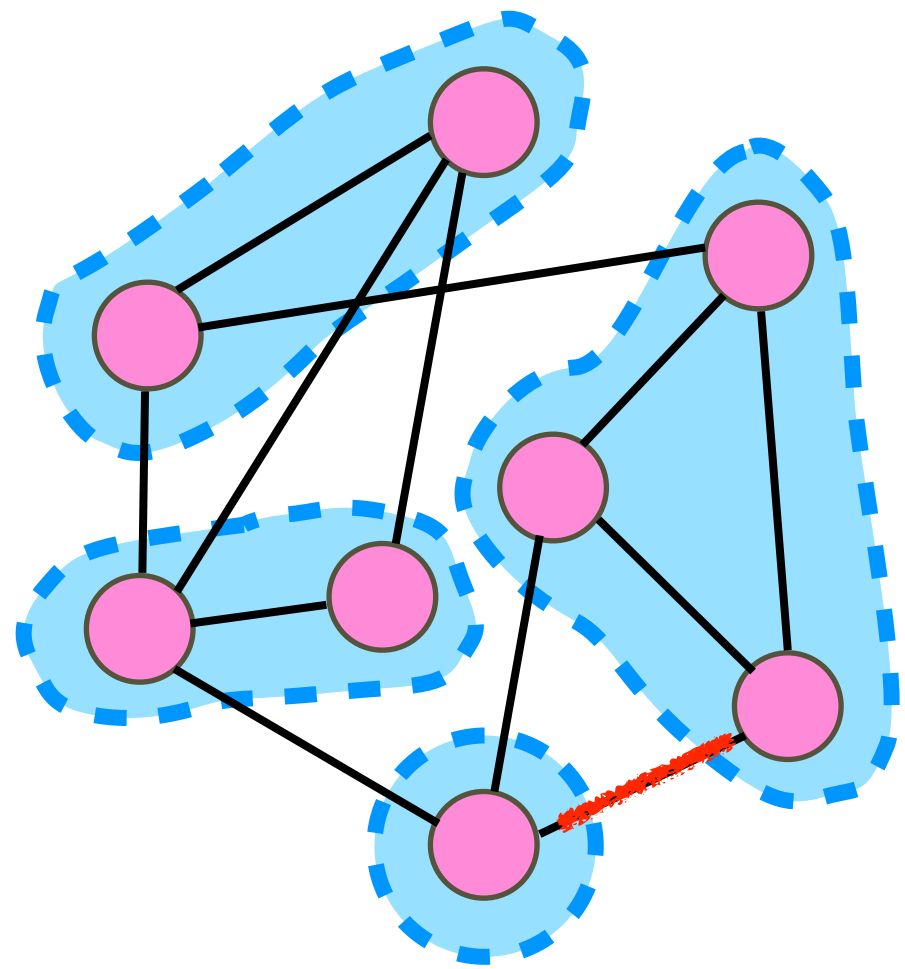

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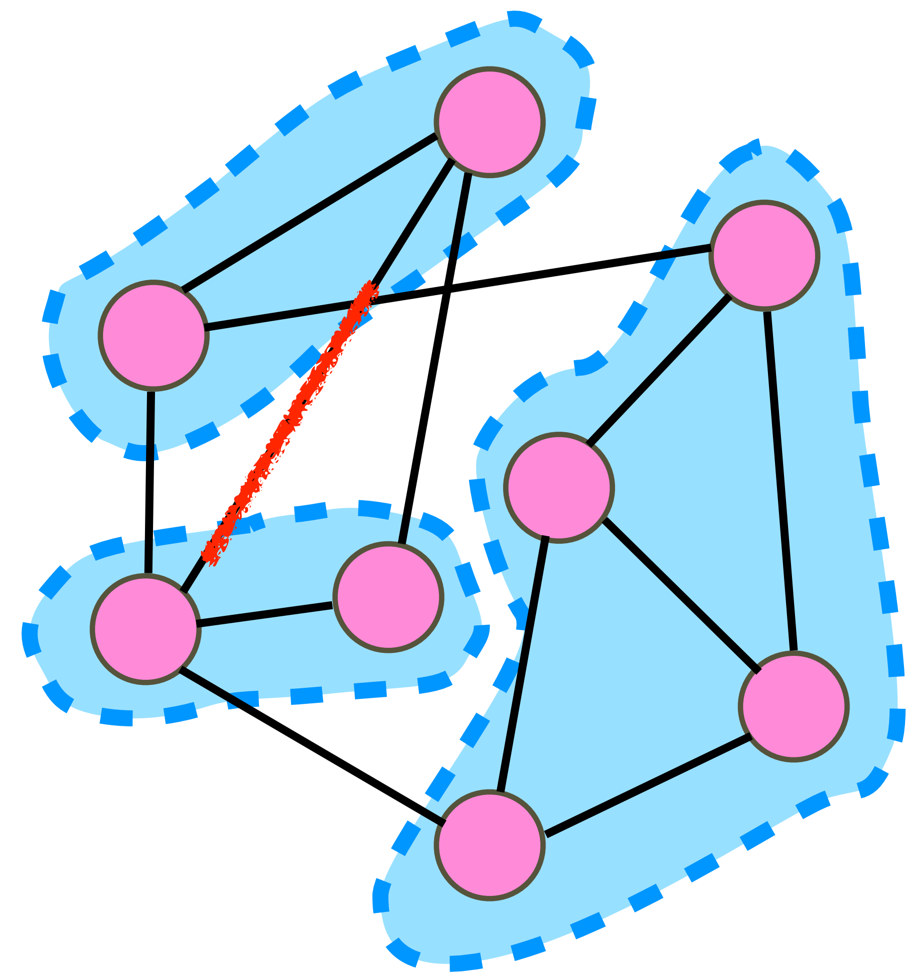

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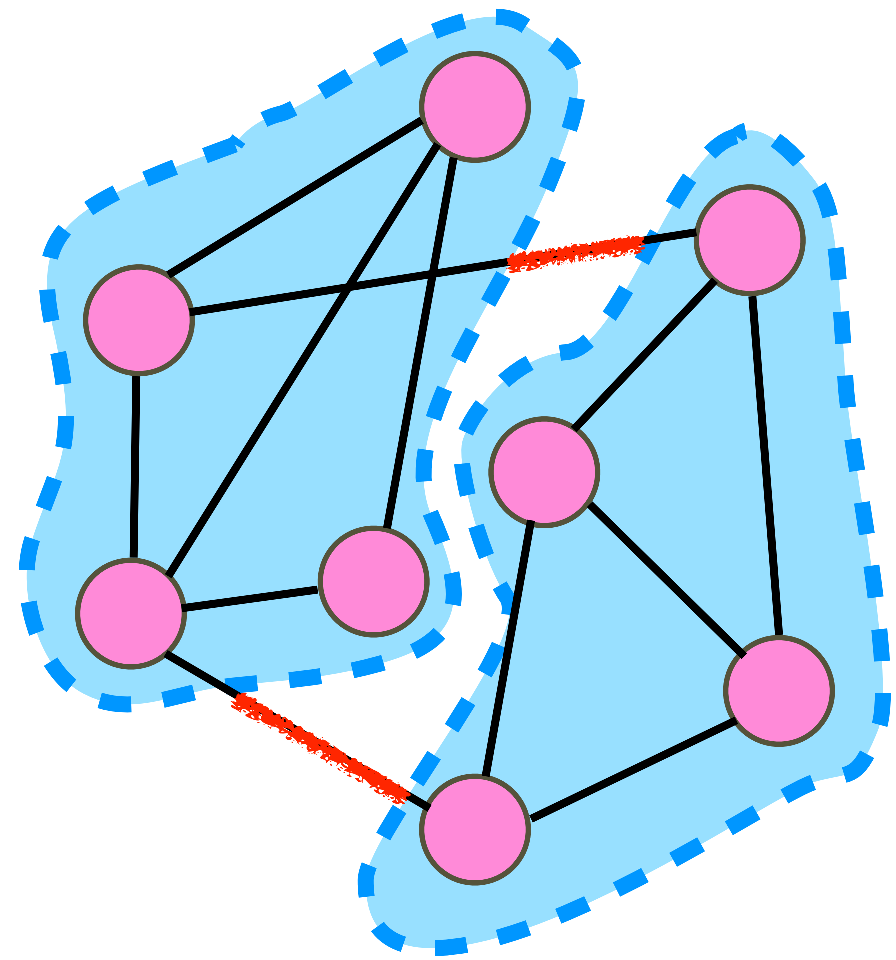


edges returned

MinCut (multigraph $G(V,E)$)

while $|V|>2$ do

choose a uniform $e \in E$;

contract(e);

return remaining edges;

Theorem (Karger 1993)**:**

Pr[a min-cut is returned] $\geq \frac{2}{n(n-1)}$

repeat independently
for $n(n\text{-}1)/2$ times
and return the smallest cut

$$\Pr[\text{ fail to finally return a min-cut }]$$

$$= \Pr[\text{ fail to construct a min-cut in one trial }]^{n(n-1)/2}$$

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)/2} < \frac{1}{\mathrm{e}}$$

MinCut (multigraph $G(V,E)$)

while $\|V\|>2$ do
  choose a uniform $e \in E$;
  contract(e);
return remaining edges;

suppose $e_1, e_2, \ldots, e_{n-2}$ are contracted edges

initially: $G_1 = G$

i-th round:

$$G_i = \text{contract}(G_{i-1}, e_{i-1})$$

$$\left.\begin{array}{c} C \text{ is a min-cut in } G_{i-1} \\ e_{i-1} \notin C \end{array}\right\} \Rightarrow C \text{ is a min-cut in } G_i$$

C: a min-cut of G

$$\Pr[C \text{ is returned}] \geq \Pr[e_1, e_2, \ldots, e_{n-2} \notin C]$$

chain rule:

$$= \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

suppose $e_1, e_2, \ldots, e_{n-2}$ are contracted edges

initially: $G_1 = G$ i-th round: $G_i = \mathrm{contract}(G_{i-1}, e_{i-1})$

$$\left.\begin{array}{c} C \text{ is a min-cut in } G_{i-1} \\ e_{i-1} \notin C \end{array}\right\} \Rightarrow C \text{ is a min-cut in } G_i$$

C: a min-cut of G

$$\Pr[C \text{ is returned}] \geq \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

$\Downarrow$ C is a min-cut in G_i

C is a min-cut in $G(V, E)$

$\Rightarrow |E| \geq \frac{1}{2}|C||V|$

Proof:

min-degree of $G \geq |C|$

$$\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{(n-i+1)}\right)$$

$$= \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{2}{n(n-1)}$$

MinCut (multigraph $G(V,E)$)

while $|V|>2$ do

  choose a uniform $e \in E$;

  contract(e);

return remaining edges;

Theorem (Karger 1993)**:**

For any min-cut C,

$$\Pr[C \text{ is returned}] \geq \frac{2}{n(n-1)}$$

running time: $O(n^2)$

repeat *independently* for $O(n^2 \log n)$ times

returns a min-cut with probability $1\text{-}O(1/n)$

total running time: $O(n^4 \log n)$

Number of Min-Cuts

Theorem (Karger 1993)**:**

For any min-cut C,

$$\Pr[C \text{ is returned}] \geq \frac{2}{n(n-1)}$$

Corollary

The number of distinct min-cuts in a graph of n vertices is at most $n(n\text{-}1)/2$.

An Observation

MinCut (multigraph $G(V,E)$)

while $\|V\|> t$ do
  choose a uniform $e \in E$;
  contract(e);
return remaining edges;

C: a min-cut of G

$$\Pr[e_1, \ldots, e_{n-t} \not\in C] = \prod_{i=1}^{n-t} \Pr[e_i \not\in C \mid e_1, \ldots, e_{i-1} \not\in C]$$

$$\geq \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)}$$

only getting bad when t is small

Fast Min-Cut

MinCut (multigraph $G(V,E)$)

while $\lvert V\rvert > t$ do
  choose a uniform $e \in E$;
  contract(e);
return remaining edges;

FastCut (G)

if $\lvert V\rvert \leq 6$ then return a min-cut by brute force;
else: (t to be fixed later)
  G_1= Contract(G,t);
  G_2= Contract(G,t); (independently)
  return min{FastCut(G_1), FastCut(G_2)};

FastCut (G)

if $|V| \le 6$ then return a min-cut by brute force;

else: (t to be fixed later)

G_1= Contract(G,t);

G_2= Contract(G,t);

(independently)

return min{FastCut(G_1), FastCut(G_2)};

C: a min-cut in G

A: no edge in C is contracted during Contract(G,t)

$$\Pr[A] = \prod_{i=1}^{n-t} \Pr[e_i \not\in C \mid e_1, \ldots, e_{i-1} \not\in C]$$

$$\ge \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} \ge \frac{t(t-1)}{n(n-1)} \ge \left(\frac{t-1}{n-1}\right)^2$$

FastCut (G)
if $\|V\|\leq 6$ then return a min-cut by brute force;
else: (t to be fixed later)
G_1= Contract(G,t);
G_2= Contract(G,t); (independently)
return min{FastCut(G_1), FastCut(G_2)};

C: a min-cut in G set $t = \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$

A: no edge in C is contracted during Contract(G,t)

$$\Pr[A] \geq \left(\frac{t-1}{n-1}\right)^2 \geq \frac{1}{2}$$

$$p(n) = \min_{G:|V|=n} \Pr[\,\mathrm{FastCut}(G)\ \underbrace{\text{returns a mincut in } G}_{\text{succeeds}}\,]$$

$$\geq 1 - (1 - \Pr[A]\Pr[\mathrm{FastCut}(G_1) \text{ succeeds} \mid A])^2$$

$$\geq 1 - \left(1 - \left(\frac{t-1}{n-1}\right)^2 p(t)\right)^2 \geq p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) - \frac{1}{4} p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)^2$$

FastCut (G)

if $|V|\le 6$ then return a min-cut by brute force;

else: set $t = \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$

G_1= Contract(G,t);

G_2= Contract(G,t);

(independently)

return min{FastCut(G_1), FastCut(G_2)};

$$p(n) = \min_{G:|V|=n} \Pr[\,\text{FastCut}(G) \text{ returns a mincut in } G\,]$$

$$\ge p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) - \frac{1}{4} p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)^2$$

by induction: $p(n) = \Omega\left(\frac{1}{\log n}\right)$

running time: $T(n) = 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) + O(n^2)$

by induction: $T(n) = O(n^2 \log n)$

FastCut (G)
if $\lvert V\rvert \leq 6$ then return a min-cut by brute force;
else: set $t = \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$
G_1= Contract(G,t);
G_2= Contract(G,t); (independently)
return min{FastCut(G_1), FastCut(G_2)};

Theorem (Karger-Stein 1996)**:**

FastCut runs in time $O(n^2 \log n)$ and returns a min-cut with probability $\Omega(1/\log n)$.

repeat *independently* for $O((\log n)^2)$ times

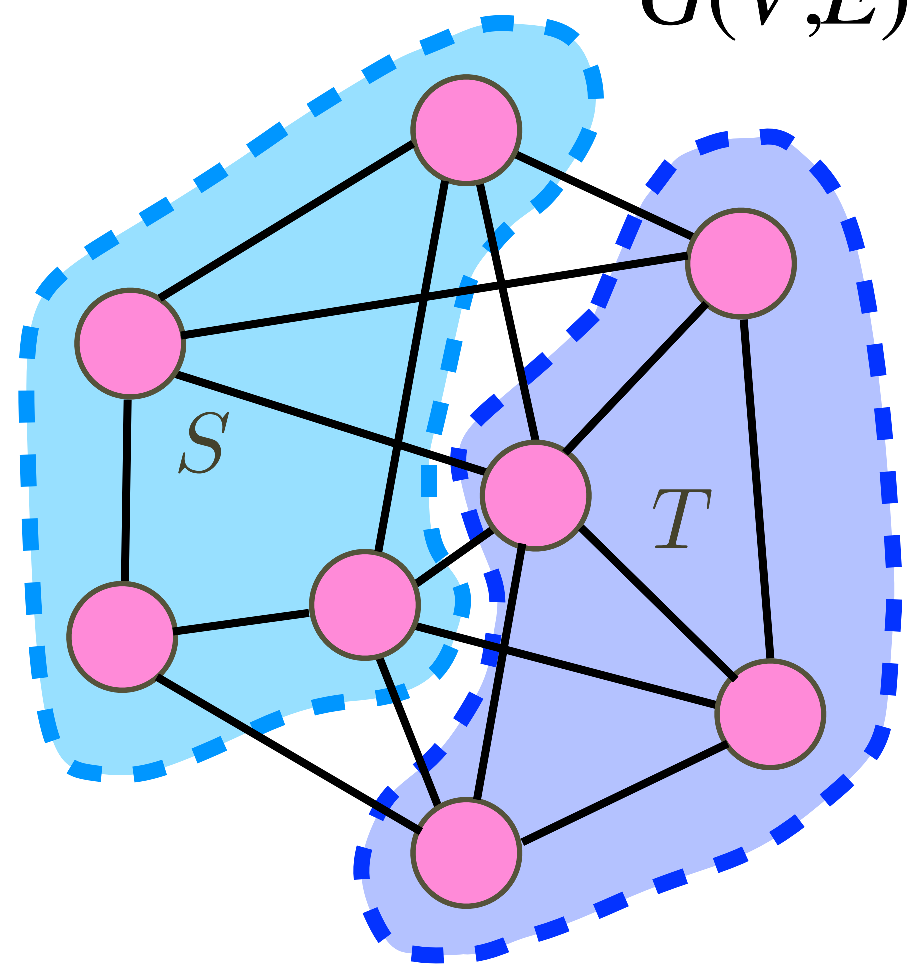
total running time: $O(n^2 \log^3 n)$

returns a min-cut with probability $1\text{-}O(1/n)$

Max-Cut

- Partition V into two parts: S and T
- Maximize the cut $E(S,T)$
- **NP-hard**
  - one of Karp's 21 NP-complete problems
- Approximation algorithms?

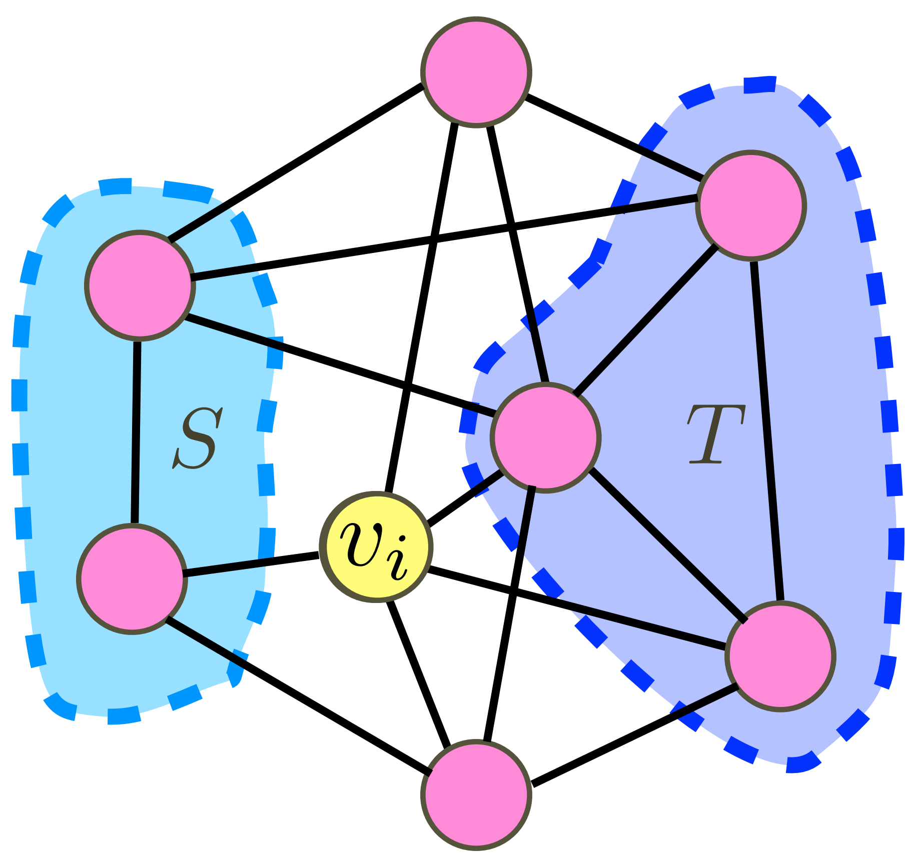
$G(V,E)$



$E(S,T) = \{uv \in E \mid u \in S,\ v \in T\}$

Greedy *Heuristics*

initially, $S{=}T{=}\varnothing$
for $i = 1,2, \ldots, n$
  v_i joins one of S, T
  to maximize *current* $E(S,T)$



$E(S,T) = \{uv \in E \mid u{\in}S,\ v{\in}T\}$

Greedy *Heuristics*

initially, S=T=$\varnothing$
for $i = 1,2, ..., n$
  v_i joins one of S, T
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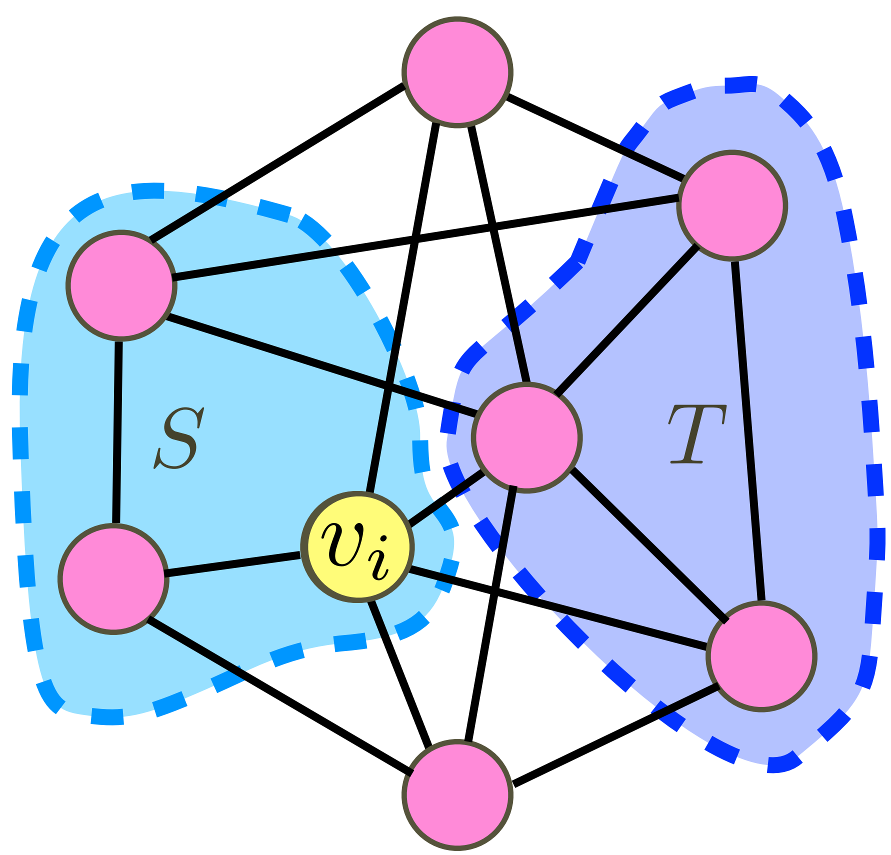
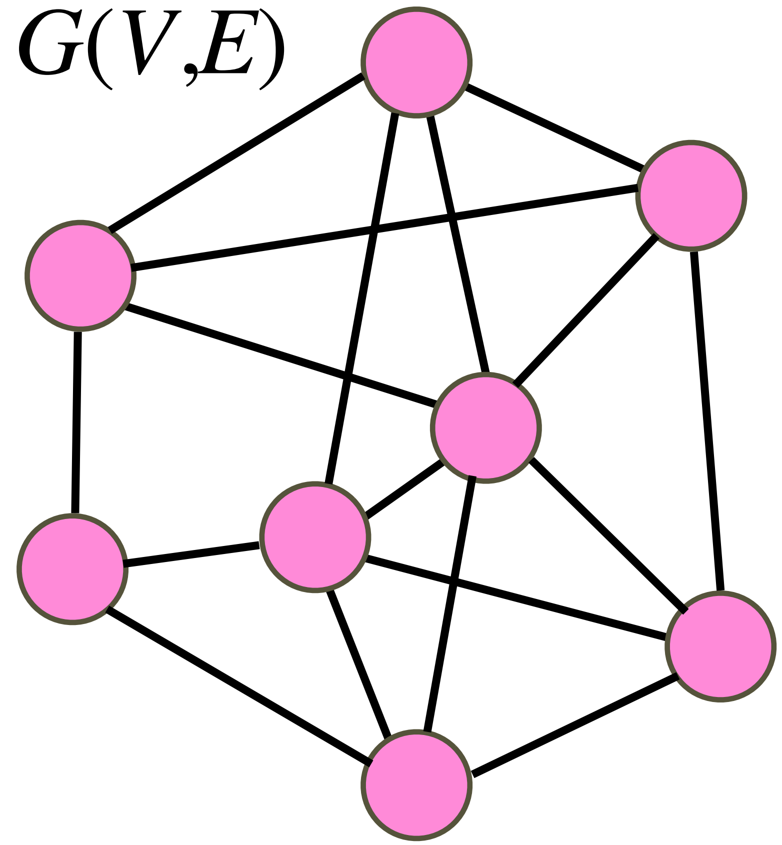


$$E(S,T) = \{uv \in E \mid u \in S,\ v \in T\}$$

Approximation Ratio

instance $G(V,E)$



algorithm A:

initially, $S=T=\emptyset$
for $i = 1,2, \ldots, n$
v_i joins one of S, T
to maximize *current* $E(S,T)$

OPT_G: value of maximum cut of G

SOL_G: value of the cut returned by algorithm A on G

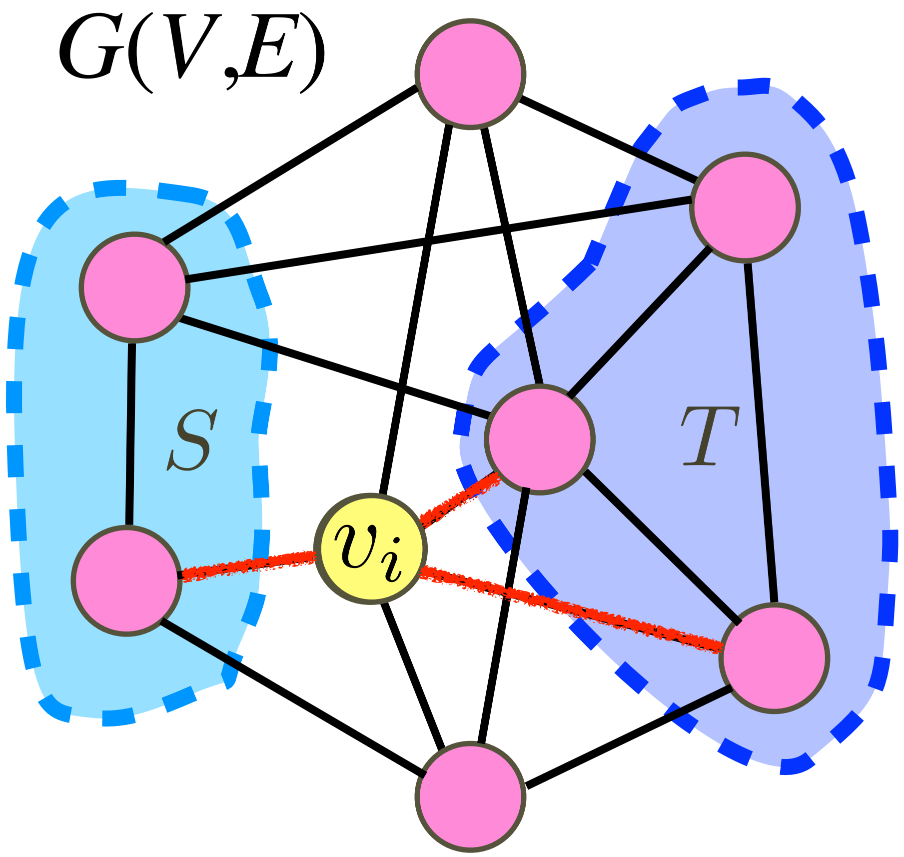
algorithm A has *approximation ratio* α if

$$\forall \text{ input } G, \quad \frac{\mathrm{SOL}_G}{\mathrm{OPT}_G} \geq \alpha$$

Approximation Algorithm

initially, $S=T=\emptyset$
for $i = 1,2, \ldots, n$
  v_i joins one of S, T
  to maximize *current* $E(S,T)$

$G(V,E)$



$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$

$$\frac{\mathrm{SOL}_G}{\mathrm{OPT}_G} \geq \frac{\mathrm{SOL}_G}{|E|} \geq \frac{1}{2}$$

$\forall v_i$, $\geq 1/2$ of $|E(S_i,v_i)| + |E(T_i,v_i)|$ contributes to SOL_G

$$|E| = \sum_{i=1}^{n} \left(|E(S_i, v_i)| + |E(T_i, v_i)| \right)$$

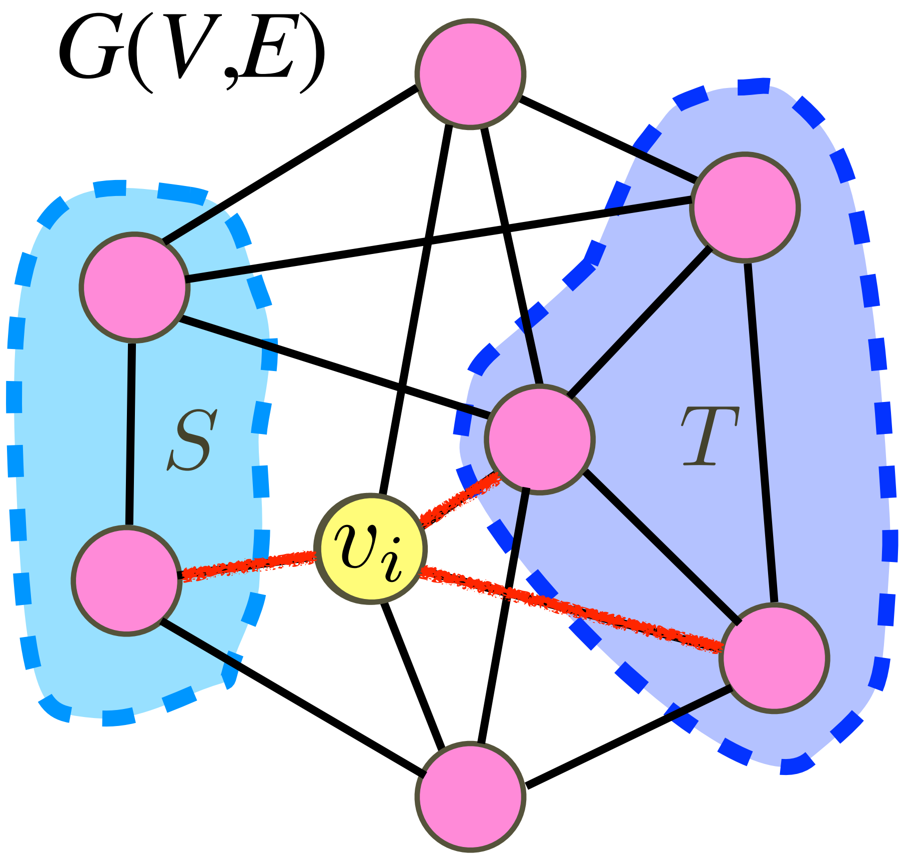
Approximation Algorithm

initially, $S=T=\emptyset$

for $i = 1,2, \ldots, n$

v_i joins one of S, T

to maximize *current* $E(S,T)$



$E(S,T) = \{uv \in E \mid u\in S, v\in T\}$

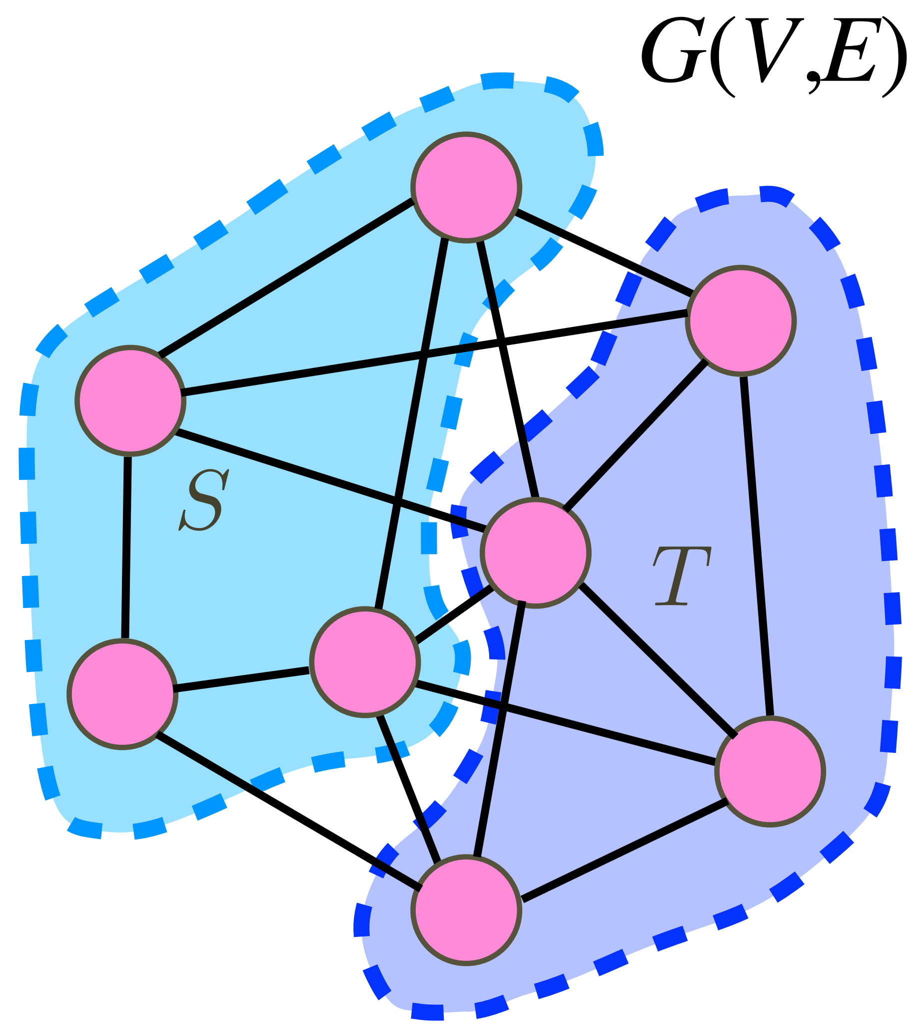
$$\frac{\mathrm{SOL}_G}{\mathrm{OPT}_G} \geq \frac{\mathrm{SOL}_G}{|E|} \geq \frac{1}{2}$$

approximation ratio: $1/2$

running time: $\mathrm{O}(m)$

Max-Cut

- Partition V into two parts: S and T
- Maximize the cut $E(S,T)$
- **NP-hard**
  - one of Karp's 21 NP-complete problems
- greedy algorithm: 0.5-approximation



$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$

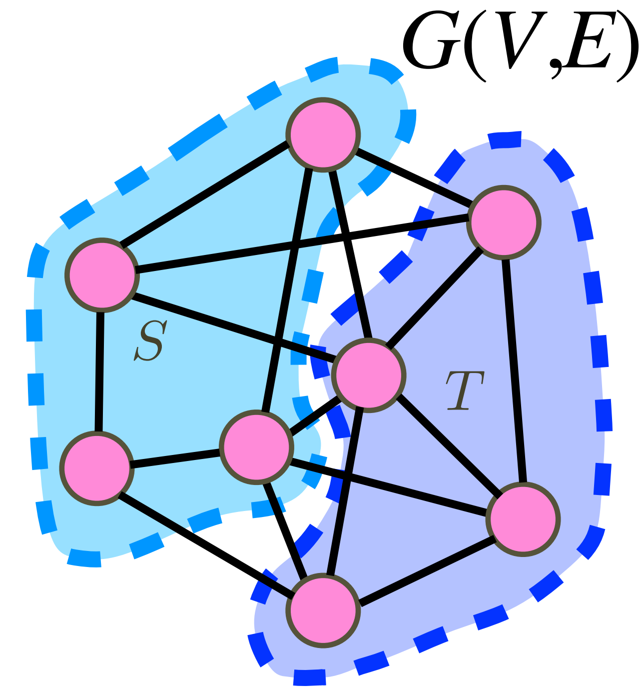
Random Cut

for each vertex $v \in V$

uniform & independent $Y_v \in \{0,1\}$

$$Y_v = 1 \Rightarrow v \in S$$

$$Y_v = 0 \Rightarrow v \in T$$



for each edge $uv \in E$

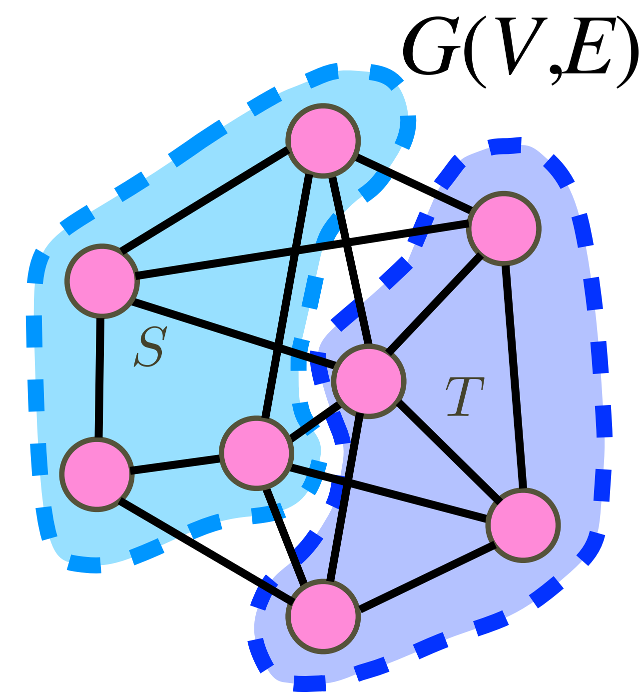
$$Y_{uv} = \begin{cases} 1 & Y_u \neq Y_v \\ 0 & Y_u = Y_v \end{cases} \qquad |E(S,T)| = \sum_{uv \in E} Y_{uv}$$

$$\mathbf{E}[|E(S,T)|] = \sum_{uv \in E} \Pr[Y_u \neq Y_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

Random Cut

for each vertex $v \in V$

uniform & 2-wise independent $Y_v \in \{0,1\}$



$$Y_v = 1 \Rightarrow v \in S$$
$$Y_v = 0 \Rightarrow v \in T$$

for each edge $uv \in E$

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$$\mathbf{E}[|E(S,T)|] = \sum_{uv \in E} \Pr[Y_u \neq Y_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

Definition:

Events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are **mutually independent** if for any subset $I \subseteq \{1, 2, \ldots, n\}$,

$$\Pr\left[\bigwedge_{i \in I} \mathcal{E}_i\right] = \prod_{i \in I} \Pr[\mathcal{E}_i].$$

Definition:

Random variables $X_1, X_2, \ldots, X_n$ are **mutually independent** if for any subset $I \subset [n]$ and any values x_i, where $i \in I$,

$$\Pr\left[\bigwedge_{i \in I} (X_i = x_i)\right] = \prod_{i \in I} \Pr[X_i = x_i].$$

k-wise Independence

Definition:
Events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are k-**wise independent** if for any subset $I \subseteq \{1, 2, \ldots, n\}$, with $|I| \leq k$

$$\Pr\left[\bigwedge_{i \in I} \mathcal{E}_i\right] = \prod_{i \in I} \Pr[\mathcal{E}_i].$$

Definition:
Random variables $X_1, X_2, \ldots, X_n$ are k-**wise independent** if for any subset $I \subset [n]$ and any values x_i, where $i \in I$, with $|I| \leq k$

$$\Pr\left[\bigwedge_{i \in I} (X_i = x_i)\right] = \prod_{i \in I} \Pr[X_i = x_i].$$

pairwise: 2-wise

2-wise Independent Bits

uniform & independent bits: (random source)

$$X_1, X_2, \ldots, X_m \in \{0,1\}$$

Goal: 2-wise independent uniform bits:

$$Y_1, Y_2, \ldots, Y_n \in \{0,1\} \qquad n \gg m$$

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

nonempty subsets:

$$\emptyset \neq S_1, S_2, \ldots, S_{2^m-1} \subseteq \{1, 2, \ldots, m\}$$

$$\boxed{Y_j = \bigoplus_{i \in S_j} X_i}$$

uniform & independent bits: $X_1, X_2, \ldots, X_m \in \{0,1\}$

nonempty subsets: $S_1, S_2, \ldots, S_{2^m-1} \subseteq \{1,2,\ldots,m\}$

$$\boxed{Y_j = \bigoplus_{i \in S_j} X_i}$$

2-wise independent uniform bits:

$$Y_1, Y_2, \ldots, Y_{2^m-1} \in \{0,1\}$$

$\log_2 n$ total random bits

⇒ n-1 pairwise independent bits

Derandomization

for each vertex $v \in V$

uniform & 2-wise independent $Y_v \in \{0,1\}$

$$Y_v = 1 \Rightarrow v \in S$$
$$Y_v = 0 \Rightarrow v \in T$$

for each edge $uv \in E$

$$\mathbf{E}[|E(S,T)|] = \sum_{uv \in E} \Pr[Y_u \neq Y_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

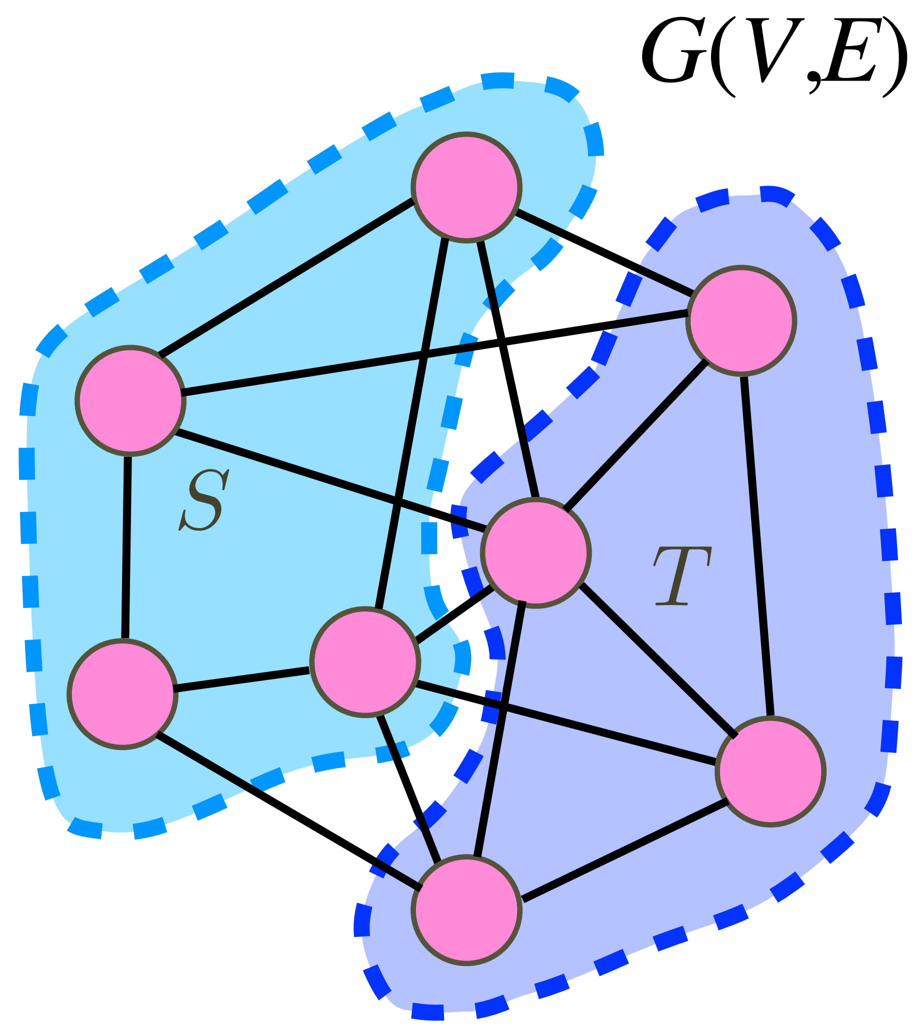
$$V = \{v_1, v_2, \ldots, v_n\}$$

$Y_{v_1}, Y_{v_2}, \ldots, Y_{v_n}$ constructed from $\lceil \log_2(n+1) \rceil$ bits

try all $2^{\lceil \log_2(n+1) \rceil} = O(n^2)$ possibilities!

Max-Cut

- Partition V into two parts: S and T
- Maximize the cut $E(S,T)$
- **NP-hard**
- greedy algorithm: 0.5-approx.
- best known approx. ratio for poly-time algorithms: 0.878~
- unique game conjecture: no poly-time algorithm with approx. ratio >0.878~



$$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$$