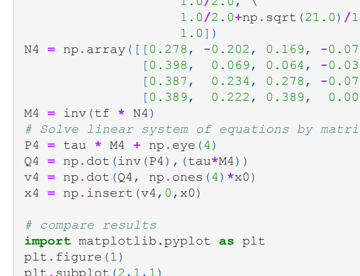
Orthogonal Collocation on Finite Elements

Discretization of a continuous time representation allow large-scale nonlinear programming (NLP) solvers to find solutions at specified intervals in a time horizon. There are many names and related techniques for obtaining mathematical relationships between derivatives and non-derivative values. Some of the terms that are relevant to this discussion include orthogonal collocation on finite elements, direct transcription, Gauss pseudospectral method, Gaussian quadrature, Lobatto quadrature, Radau collocation, Legendre polynomials, Chebyshev polynomials, Jacobi polynomials, Laguerre polynomials, any many more. There are many papers that discuss the details of the derivation and theory behind these methods1-5. The purpose of this section is to give a practical introduction to orthogonal collocation on finite elements with Lobatto quadrature for the numerical solution of differential algebraic equations.

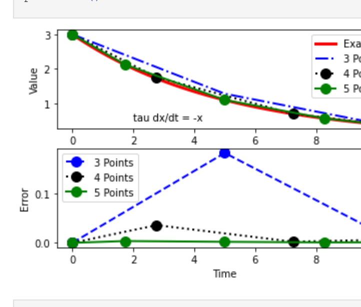
In [2]:

```python
#@markdown ###Orthogonal Collocation 1

# Solve tau dx/dt = -x
import numpy as np
from numpy.linalg import inv

## problem statement
x0 = 3.0  # initial condition
tf = 10.0 # final time
tau = 5.0 # model parameter (time constant)

## analytic solution
time = np.linspace(0,tf,100)
x = x0*np.exp(-time/tau)

## numeric solutions
# 2nd order polynomial (3 points)
t2 = tf * np.array([0.0,0.5,1.0])
N2 = np.array([[0.75,-0.25], \
               [1.00, 0.00]])
M2 = inv(tf * N2)
# Solve linear system of equations by matrix manipulation
P2 = tau*M2 + np.eye(2)
Q2 = np.dot(inv(P2),(tau*M2))
v2 = np.dot(Q2, np.ones(2)*x0)
x2 = np.insert(v2,0,x0)

# 3rd order polynomial (4 points)
t3 = tf * np.array([0.0, \
                    1.0/2.0-np.sqrt(5.0)/10.0, \
                    1.0/2.0+np.sqrt(5.0)/10.0, \
                    1.0])
N3 = np.array([[0.436,-0.281, 0.121], \
               [0.614, 0.064, 0.046], \
               [0.603, 0.230, 0.167]])
M3 = inv(tf * N3)
# Solve linear system of equations by matrix manipulation
P3 = tau * M3 + np.eye(3)
Q3 = np.dot(inv(P3),(tau*M3))
v3 = np.dot(Q3, np.ones(3)*x0)
x3 = np.insert(v3,0,x0)

# 4th order polynomial (5 points)
t4 = tf * np.array([0.0, \
                    1.0/2.0-np.sqrt(21.0)/14.0, \
                    1.0/2.0, \
                    1.0/2.0+np.sqrt(21.0)/14.0, \
                    1.0])
N4 = np.array([[0.278, -0.202, 0.169, -0.071], \
               [0.398,  0.069, 0.064, -0.031], \
               [0.387,  0.234, 0.278, -0.071], \
               [0.389,  0.222, 0.389,  0.000]])
M4 = inv(tf * N4)
# Solve linear system of equations by matrix manipulation
P4 = tau * M4 + np.eye(4)
Q4 = np.dot(inv(P4),(tau*M4))
v4 = np.dot(Q4, np.ones(4)*x0)
x4 = np.insert(v4,0,x0)

# compare results
import matplotlib.pyplot as plt
plt.figure(1)
plt.subplot(2,1,1)
plt.plot(time,x,'r-',linewidth=3)
plt.plot(t2,x2,'b-.',linewidth=2,markersize=20)
plt.plot(t3,x3,'k:.',linewidth=2,markersize=20)
plt.plot(t4,x4,'g.-',linewidth=2,markersize=20)
plt.xlabel('Time')
plt.ylabel('Value')
plt.legend(['Exact','3 Points','4 Points','5 Points'])
plt.text(2,0.5,'tau dx/dt = -x')

plt.subplot(2,1,2)
# exact solutions at collocation points
y2 = x0*np.exp(-t2/tau)
y3 = x0*np.exp(-t3/tau)
y4 = x0*np.exp(-t4/tau)
plt.plot(t2,x2-y2,'b--.',linewidth=2,markersize=20)
plt.plot(t3,x3-y3,'k:.',linewidth=2,markersize=20)
plt.plot(t4,x4-y4,'g.-',linewidth=2,markersize=20)
plt.xlabel('Time')
plt.ylabel('Error')
plt.legend(['3 Points','4 Points','5 Points'])
plt.show()
```



In [3]:

```python
#@markdown ###Orthogonal Collocation 2

# Solve tau dx/dt = -x + k u
import numpy as np
from numpy.linalg import inv

## problem statement
x0 = 0.0  # initial condition
tf = 10.0 # final time
tau = 5.0 # model parameter (time constant)
k = 2.0   # gain

## analytic solution
time = np.linspace(0,tf)
x = k*(1.0-np.exp(-time/tau))

## numeric solutions
# 2nd order polynomial (3 points)
t2 = tf * np.array([0.0,0.5,1.0])
N2 = np.array([[0.75,-0.25], \
               [1.00, 0.00]])
M2 = inv(tf * N2)
# Solve linear system of equations by matrix manipulation
P2 = inv(tau*M2 + np.eye(2))
Q2 = np.dot(tau*M2,np.ones(2)*x0) + k*np.ones(2)
v2 = np.dot(P2, Q2)
x2 = np.insert(v2,0,x0)

# 3rd order polynomial (4 points)
t3 = tf * np.array([0.0, \
                    1.0/2.0-np.sqrt(5.0)/10.0, \
                    1.0/2.0+np.sqrt(5.0)/10.0, \
                    1.0])
N3 = np.array([[0.436,-0.281, 0.121], \
               [0.614, 0.064, 0.046], \
               [0.603, 0.230, 0.167]])
M3 = inv(tf * N3)
# Solve linear system of equations by matrix manipulation
P3 = inv(tau*M3 + np.eye(3))
Q3 = np.dot(tau*M3,np.ones(3)*x0) + k*np.ones(3)
v3 = np.dot(P3, Q3)
x3 = np.insert(v3,0,x0)

# 4th order polynomial (5 points)
t4 = tf * np.array([0.0, \
                    1.0/2.0-np.sqrt(21.0)/14.0, \
                    1.0/2.0, \
                    1.0/2.0+np.sqrt(21.0)/14.0, \
                    1.0])
N4 = np.array([[0.278, -0.202, 0.169, -0.071], \
               [0.398,  0.069, 0.064, -0.031], \
               [0.387,  0.234, 0.278, -0.071], \
               [0.389,  0.222, 0.389,  0.000]])
M4 = inv(tf * N4)
# Solve linear system of equations by matrix manipulation
P4 = inv(tau*M4 + np.eye(4))
Q4 = np.dot(tau*M4,np.ones(4)*x0) + k*np.ones(4)
v4 = np.dot(P4, Q4)
x4 = np.insert(v4,0,x0)

# compare results
import matplotlib.pyplot as plt
plt.figure(1)
plt.subplot(2,1,1)
plt.plot(time,x,'r-',linewidth=3)
plt.plot(t2,x2,'b-.',linewidth=2,markersize=20)
plt.plot(t3,x3,'k:.',linewidth=2,markersize=20)
plt.plot(t4,x4,'g.-',linewidth=2,markersize=20)
plt.xlabel('Time')
plt.ylabel('Value')
plt.legend(['Exact','3 Points','4 Points','5 Points'])
plt.text(4,0.5,'tau dx/dt = -x + k u')

plt.subplot(2,1,2)
# exact solutions at collocation points
y2 = k*(1-np.exp(-t2/tau))
y3 = k*(1-np.exp(-t3/tau))
y4 = k*(1-np.exp(-t4/tau))
plt.plot(t2,x2-y2,'b--.',linewidth=2,markersize=20)
plt.plot(t3,x3-y3,'k:.',linewidth=2,markersize=20)
plt.plot(t4,x4-y4,'g.-',linewidth=2,markersize=20)
plt.xlabel('Time')
plt.ylabel('Error')
plt.legend(['3 Points','4 Points','5 Points'])
plt.show()
```



Exercise 1

Objective: Solve a differential equation with orthogonal collocation on finite elements. Create a MATLAB or Python script to simulate and display the results. Estimated Time: 2-3 hours

Solve the following differential equation from time 0 to 1 with orthogonal collocation on finite elements with 4 nodes for discretization in time.

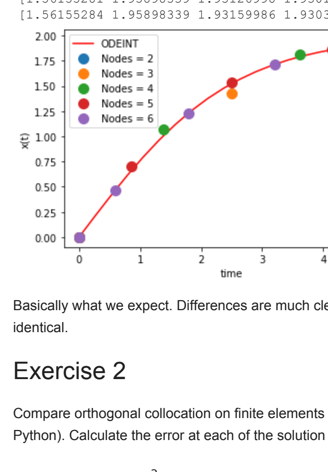
$$5dx/dt = -x2 + u$$

Specify the initial condition for x as 0 and the value of the input, u, as 4. Compare the solution result with 2-6 time points (nodes). Report the solution at the final time for each and comment on how the solution changes with an increase in the number of nodes.

In [31]:

```python
import numpy as np
from scipy.optimize import fsolve
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# final time
tf = 5.0

# solve with ODEINT (for comparison)
def model(x,t):
    u = 4.0
    return (-x**2 + u)/5.0
t = np.linspace(0,tf,20)
y0 = 0
y = odeint(model,y0,t)
plt.figure(1)
plt.plot(t,y,'r-',label='ODEINT')

# ----------------------------------------------------
# Approach #1 - Write the model equations in Python
# ----------------------------------------------------
# define collocation matrices
def colloc(n):
    if (n==2):
        NC = np.array([[1.0]])
    if (n==3):
        NC = np.array([[0.75,-0.25], \
                       [1.00, 0.00]])
    if (n==4):
        NC = np.array([[0.436,-0.281, 0.121], \
                       [0.614, 0.064, 0.0461], \
                       [0.603, 0.230, 0.167]])
    if (n==5):
        NC = np.array([[0.278, -0.202, 0.169, -0.071], \
                       [0.398,  0.069, 0.064, -0.031], \
                       [0.387,  0.234, 0.278, -0.071], \
                       [0.389,  0.222, 0.389,  0.000]])
    if (n==6):
        NC = np.array([[0.191, -0.147, 0.139, -0.113, 0.047],
                       [0.276,  0.059, 0.051, -0.050, 0.022],
                       [0.267,  0.193, 0.252, -0.114, 0.045],
                       [0.269,  0.178, 0.384,  0.032, 0.019],
                       [0.269,  0.181, 0.374,  0.110, 0.067]])
    return NC

# define collocation points from Lobatto quadrature
def tc(n):
    if (n==2):
        time = np.array([0.0,1.0])
    if (n==3):
        time = np.array([0.0,0.5,1.0])
    if (n==4):
        time = np.array([0.0, \
                         0.5-np.sqrt(5)/10.0, \
                         0.5+np.sqrt(5)/10.0, \
                         1.0])
    if (n==5):
        time = np.array([0.0,0.5-np.sqrt(21)/14.0, \
                         0.5,0.5+np.sqrt(21)/14.0, 1])
    if (n==6):
        time = np.array([0.0, \
                         0.5-np.sqrt((7.0+2.0*np.sqrt(7.0))/21.0)/2.0, \
                         0.5-np.sqrt((7.0-2.0*np.sqrt(7.0))/21.0)/2.0, \
                         0.5+np.sqrt((7.0-2.0*np.sqrt(7.0))/21.0)/2.0, \
                         0.5+np.sqrt((7.0+2.0*np.sqrt(7.0))/21.0)/2.0, \
                         1.0])
    return time*tf

# solve with SciPy fsolve
def myFunction(z,*param):
    n = param[0]
    m = param[1]
    # rename z as x and xdot variables
    x = np.empty(n-1)
    xdot = np.empty(n-1)
    x[0:n-1] = z[0:n-1]
    xdot[0:n-1] = z[n-1:m]

    # initial condition (x0)
    x0 = 0.0
    # input parameter (u)
    u = 4.0
    # final time
    tn = tf

    # function evaluation residuals
    F = np.empty(m)
    # nonlinear differential equations at each node
    # 5 dx/dt = -x^2 + u
    F[0:n-1] = 5.0 * xdot[0:n-1] + x[0:n-1]**2 - u
    # collocation equations
    # tn * NC * xdot = x - x0
    NC = colloc(n)
    F[n-1:m] = tn * np.dot(NC,xdot) - x + x0 * np.ones(n-1)
    return F

sol_py = np.empty(5) # store 5 results
for i in range(2,7):
    n = i
    m = (i-1)*2
    zGuess = np.ones(m)
    z = fsolve(myFunction,zGuess,args=(n,m))
    # add to plot
    yc = np.insert(z[0:n-1],0,0)
    plt.plot(tc(n),yc,'o',markersize=10,label='Nodes = '+str(i))
    # store just the last x[n] value
    sol_py[i-2] = z[n-2]
plt.legend(loc='best')

# ----------------------------------------------------
# Approach #2 - Write model in APMonitor and let
#   modeling language create the collocation equations
# ----------------------------------------------------
# load GEKKO
from gekko import GEKKO

sol_apm = np.empty(5) # store 5 results
i = 0
for nodes in range(2,7):
    m = GEKKO()

    u = m.Param(value=4)
    x = m.Var(value=0)
    m.Equation(5*x.dt() == -x**2 + u)

    m.time = [0,tf]

    m.options.imode = 4
    m.options.time_shift = 0
    m.options.nodes = nodes

    m.solve() # solve problem
    sol_apm[i] = x.value[-1] # store solution (last point)
    i += 1

# print the solutions
print(sol_py)
print(sol_apm)

# show plot
plt.ylabel('x(t)')
plt.xlabel('time')
plt.show()
```

```
apm 10.32.114.116_gk_model20 <br><pre> ----------------------------------------------------------------
 APMonitor, Version 1.0.1
 APMonitor Optimization Suite
 ----------------------------------------------------------------


 --------- APM Model Size ------------
 Each time step contains
   Objects      :            0
   Constants    :            0
   Variables    :            2
   Intermediates:            0
   Connections  :            0
   Equations    :            1
   Residuals    :            1

 Variable time shift OFF
 Number of state variables:              2
 Number of total equations: -            2
 Number of slack variables: -            0
 ---------------------------------------
 Degrees of freedom       :              0

 **********************************************
 Dynamic Simulation with Interior Point Solver
 **********************************************


 Info: Exact Hessian

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver ma57.

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        1

Total number of variables............................:        2
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        2
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 4.00e+00 0.00e+00   0.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  0.0000000e+00 2.00e+00 0.00e+00 -11.0 4.00e+00    -  1.00e+00 5.00e-01h  2
   2  0.0000000e+00 1.60e-01 0.00e+00 -11.0 4.00e-01    -  1.00e+00 1.00e+00h  1
   3  0.0000000e+00 1.45e-03 0.00e+00 -11.0 3.81e-02    -  1.00e+00 1.00e+00h  1
   4  0.0000000e+00 1.24e-07 0.00e+00 -11.0 3.52e-04    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 4

                                   (scaled)                 (unscaled)
Objective...............:   0.0000000000000000e+00    0.0000000000000000e+00
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   1.2384702396950331e-07    1.2384702396950331e-07
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   1.2384702396950331e-07    1.2384702396950331e-07


Number of objective function evaluations             = 7
Number of objective gradient evaluations             = 5
Number of equality constraint evaluations            = 7
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 5
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 4
Total CPU secs in IPOPT (w/o function evaluations)   =      0.003
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.

 The solution was found.

 The final value of the objective function is   0.000000000000000E+000

 ---------------------------------------------------
 Solver         :  IPOPT (v3.12)
 Solution time  :   6.099999998696148E-003 sec
 Objective      :   0.000000000000000E+000
 Successful solution
 ---------------------------------------------------

apm 10.32.114.116_gk_model21 <br><pre> ----------------------------------------------------------------
 APMonitor, Version 1.0.1
 APMonitor Optimization Suite
 ----------------------------------------------------------------


 --------- APM Model Size ------------
 Each time step contains
   Objects      :            0
   Constants    :            0
   Variables    :            2
   Intermediates:            0
   Connections  :            0
   Equations    :            1
   Residuals    :            1

 Variable time shift OFF
 Number of state variables:              4
 Number of total equations: -            4
 Number of slack variables: -            0
 ---------------------------------------
 Degrees of freedom       :              0

 **********************************************
 Dynamic Simulation with Interior Point Solver
 **********************************************


 Info: Exact Hessian

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver ma57.

Number of nonzeros in equality constraint Jacobian...:        9
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        2

Total number of variables............................:        4
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 4.00e+00 0.00e+00   0.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  0.0000000e+00 2.00e+00 0.00e+00 -11.0 4.00e+00    -  1.00e+00 5.00e-01h  2
   2  0.0000000e+00 2.50e-01 0.00e+00 -11.0 5.00e-01    -  1.00e+00 1.00e+00h  1
   3  0.0000000e+00 4.90e-03 0.00e+00 -11.0 7.00e-02    -  1.00e+00 1.00e+00h  1
   4  0.0000000e+00 1.84e-06 0.00e+00 -11.0 1.36e-03    -  1.00e+00 1.00e+00h  1
   5  0.0000000e+00 2.44e-13 0.00e+00 -11.0 4.93e-07    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:   0.0000000000000000e+00    0.0000000000000000e+00
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   2.4358293160275934e-13    2.4358293160275934e-13
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   2.4358293160275934e-13    2.4358293160275934e-13


Number of objective function evaluations             = 8
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 8
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 6
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.002
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.

 The solution was found.

 The final value of the objective function is   0.000000000000000E+000

 ---------------------------------------------------
 Solver         :  IPOPT (v3.12)
 Solution time  :   6.600000000617001E-003 sec
 Objective      :   0.000000000000000E+000
 Successful solution
 ---------------------------------------------------

apm 10.32.114.116_gk_model22 <br><pre> ----------------------------------------------------------------
 APMonitor, Version 1.0.1
 APMonitor Optimization Suite
 ----------------------------------------------------------------


 --------- APM Model Size ------------
 Each time step contains
   Objects      :            0
   Constants    :            0
   Variables    :            2
   Intermediates:            0
   Connections  :            0
   Equations    :            1
   Residuals    :            1

 Variable time shift OFF
 Number of state variables:              6
 Number of total equations: -            6
 Number of slack variables: -            0
 ---------------------------------------
 Degrees of freedom       :              0

 **********************************************
 Dynamic Simulation with Interior Point Solver
 **********************************************


 Info: Exact Hessian

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver ma57.

Number of nonzeros in equality constraint Jacobian...:       18
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        3

Total number of variables............................:        6
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        6
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 4.00e+00 0.00e+00   0.0 0.00e+00    -  0.00e+00 0.00e+00   0
Reallocating memory for MA57: lfact (277)
   1  0.0000000e+00 6.83e+00 0.00e+00 -11.0 4.00e+00    -  1.00e+00 1.00e+00H  1
   2  0.0000000e+00 8.32e-01 0.00e+00 -11.0 9.12e-01    -  1.00e+00 1.00e+00h  1
   3  0.0000000e+00 7.77e-03 0.00e+00 -11.0 9.52e-02    -  1.00e+00 1.00e+00h  1
   4  0.0000000e+00 5.79e-06 0.00e+00 -11.0 2.41e-03    -  1.00e+00 1.00e+00h  1
   5  0.0000000e+00 2.56e-12 0.00e+00 -11.0 1.60e-06    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:   0.0000000000000000e+00    0.0000000000000000e+00
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   2.5641710976742615e-12    2.5641710976742615e-12
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   2.5641710976742615e-12    2.5641710976742615e-12


Number of objective function evaluations             = 8
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 8
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 6
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.003
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.

 The solution was found.

 The final value of the objective function is   0.000000000000000E+000

 ---------------------------------------------------
 Solver         :  IPOPT (v3.12)
 Solution time  :   6.900000000314321E-003 sec
 Objective      :   0.000000000000000E+000
 Successful solution
 ---------------------------------------------------

apm 10.32.114.116_gk_model23 <br><pre> ----------------------------------------------------------------
 APMonitor, Version 1.0.1
 APMonitor Optimization Suite
 ----------------------------------------------------------------


 --------- APM Model Size ------------
 Each time step contains
   Objects      :            0
   Constants    :            0
   Variables    :            2
   Intermediates:            0
   Connections  :            0
   Equations    :            1
   Residuals    :            1

 Variable time shift OFF
 Number of state variables:              8
 Number of total equations: -            8
 Number of slack variables: -            0
 ---------------------------------------
 Degrees of freedom       :              0

 **********************************************
 Dynamic Simulation with Interior Point Solver
 **********************************************


 Info: Exact Hessian

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver ma57.

Number of nonzeros in equality constraint Jacobian...:       27
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        4

Total number of variables............................:        8
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        8
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 4.00e+00 0.00e+00   0.0 0.00e+00    -  0.00e+00 0.00e+00   0
Reallocating memory for MA57: lfact (474)
   1  0.0000000e+00 2.00e+00 0.00e+00 -11.0 4.00e+00    -  1.00e+00 5.00e-01h  2
   2  0.0000000e+00 3.52e-01 0.00e+00 -11.0 5.93e-01    -  1.00e+00 1.00e+00h  1
   3  0.0000000e+00 3.61e-03 0.00e+00 -11.0 6.01e-02    -  1.00e+00 1.00e+00h  1
   4  0.0000000e+00 7.67e-07 0.00e+00 -11.0 8.76e-04    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 4

                                   (scaled)                 (unscaled)
Objective...............:   0.0000000000000000e+00    0.0000000000000000e+00
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   7.6675927906943997e-07    7.6675927906943997e-07
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   7.6675927906943997e-07    7.6675927906943997e-07


Number of objective function evaluations             = 9
Number of objective gradient evaluations             = 5
Number of equality constraint evaluations            = 9
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 5
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 4
Total CPU secs in IPOPT (w/o function evaluations)   =      0.003
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.

 The solution was found.

 The final value of the objective function is   0.000000000000000E+000

 ---------------------------------------------------
 Solver         :  IPOPT (v3.12)
 Solution time  :   6.799999999202555E-003 sec
 Objective      :   0.000000000000000E+000
 Successful solution
 ---------------------------------------------------

apm 10.32.114.116_gk_model24 <br><pre> ----------------------------------------------------------------
 APMonitor, Version 1.0.1
 APMonitor Optimization Suite
 ----------------------------------------------------------------


 --------- APM Model Size ------------
 Each time step contains
   Objects      :            0
   Constants    :            0
   Variables    :            2
   Intermediates:            0
   Connections  :            0
   Equations    :            1
   Residuals    :            1

 Variable time shift OFF
 Number of state variables:             10
 Number of total equations: -           10
 Number of slack variables: -            0
 ---------------------------------------
 Degrees of freedom       :              0

 **********************************************
 Dynamic Simulation with Interior Point Solver
 **********************************************


 Info: Exact Hessian

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver ma57.

Number of nonzeros in equality constraint Jacobian...:       40
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        5

Total number of variables............................:       10
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:       10
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 4.00e+00 0.00e+00   0.0 0.00e+00    -  0.00e+00 0.00e+00   0
Reallocating memory for MA57: lfact (726)
   1  0.0000000e+00 7.64e+00 0.00e+00 -11.0 4.00e+00    -  1.00e+00 1.00e+00H  1
   2  0.0000000e+00 1.08e+00 0.00e+00 -11.0 1.04e+00    -  1.00e+00 1.00e+00h  1
   3  0.0000000e+00 1.06e-02 0.00e+00 -11.0 1.31e-01    -  1.00e+00 1.00e+00h  1
   4  0.0000000e+00 3.24e-07 0.00e+00 -11.0 1.69e-03    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 4

                                   (scaled)                 (unscaled)
Objective...............:   0.0000000000000000e+00    0.0000000000000000e+00
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   3.2350532930269793e-07    3.2350532930269793e-07
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   3.2350532930269793e-07    3.2350532930269793e-07


Number of objective function evaluations             = 7
Number of objective gradient evaluations             = 5
Number of equality constraint evaluations            = 7
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 5
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 4
Total CPU secs in IPOPT (w/o function evaluations)   =      0.003
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.

 The solution was found.

 The final value of the objective function is   0.000000000000000E+000

 ---------------------------------------------------
 Solver         :  IPOPT (v3.12)
 Solution time  :   6.600000000617001E-003 sec
 Objective      :   0.000000000000000E+000
 Successful solution
 ---------------------------------------------------

[1.56155281 1.95898339 1.93126998 1.9301535  1.92970515]
[1.56155284 1.95898339 1.93159986 1.93034749 1.9282544 ]
```



Basically what we expect. Differences are much clearer with a longer integration time. Ill try 5 seconds. Fsolve and Gekko are basically identical.

Exercise 2

Compare orthogonal collocation on finite elements with 3 nodes with a numerical integrator (e.g. ODE15s in MATLAB or ODEINT in Python). Calculate the error at each of the solution points for the equation (same as for Exercise 1):

$$5 dx/dt = -x^2 + u$$

In [36]:

```python
#@markdown Solution ODEINT Python

from scipy.integrate import odeint
import numpy as np


def collocation(x,t):
    u = 4
    xdot = -x**2 / 5.0 + u / 5.0

    return xdot

y3 = odeint(collocation,[0],tf*np.array([0, 0.5, 1]))

print(y3[-2:].flatten())

#gekko doesn't do internal points, and the equation is nonlinear, so I need to use a nonlinear optmiizer.
i = 3
n = i
m = (i-1)*2
zGuess = np.ones(m)
z = fsolve(myFunction,zGuess,args=(n,m))

#recall my function returns [x,xdot] solution to the collocation problem so I only need the first 2 points.
print(z[:2])
```

```
[1.52318835 1.92805517]
[1.42864153 1.95898339]
```

pretty close. Again, gets worse with longer final time.