Showing that $\mathbb{E}[\mathbb{E}[V \mid U, W] \mid W] = \mathbb{E}[V \mid W]$.

Consider the specific case of $W = w$:

$$\mathbb{E}[\underbrace{\mathbb{E}[V \mid U, W]} \mid W = w] = \sum_u \underbrace{\mathbb{E}[V \mid U = u, W = w]}_{f(u,w)} \Pr(U = u \mid W = w)$$

Recall that this is a R.V. that is a function $f(U,W)$

$$= \sum_u \left(\sum_v v \cdot \Pr(V = v \mid U = u, W = w) \right) \Pr(U = u, W = w)$$

$$= \sum_u \sum_v v \cdot \Pr(V = v, U = u \mid W = w)$$

↓ Bayes

$$= \sum_u \sum_v v \cdot \frac{\Pr(V = v, U = u, W = w)}{\Pr(W = w)}$$

$$= \sum_v \frac{v}{\Pr(W = w)} \left(\sum_u \Pr(V = v, U = u, W = w) \right)$$

↓ total probability

$$= \sum_v \frac{v}{\Pr(W = w)} \left(\Pr(V = v, W = w) \right)$$

$$= \sum_v v \cdot \frac{\Pr(V = v, W = w)}{\Pr(W = w)}$$

$$= \sum_v v \cdot \Pr(V = v \mid W = w)$$

↓ by definition

$$= \mathbb{E}[V \mid W = w]$$

Since this derivation holds for arbitrary w, we have

$$\mathbb{E}[\mathbb{E}[V \mid U, W] \mid W] = \mathbb{E}[V \mid W].$$ ←

Important "nugget"...
whatever appears in the inner conditioning but not in the outer conditioning is "averaged out" (i.e. disappears)