Volatility Surface Construction and Parametric Fitting

From Option Prices to SVI/SSVI Calibration

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Abstract

This short note develops a complete self-contained pipeline for constructing and parameterizing equity index volatility surfaces from market option prices. Starting with 'first principles' – defining options, volatility, and the Black-Scholes framework – then progress to the extraction of implied volatilities from market data, and finally to the calibration of the Stochastic Volatility Inspired (SVI) and Surface SVI (SSVI) parameterizations. The implementation uses SPY options data from the public Yahoo Finance API. We can see that per-expiry SVI achieves near-perfect fits (RMSE $< 10^{-4}$ in variance space) but requires 105 parameters across 21 expiries, while the joint SSVI parameterization captures the entire surface with only 3 parameters at the cost of systematic misfit in short-dated options. All replication code is provided.

1 Introduction

The volatility surface is one of the most important objects in more *quantitative finance*. It encodes the market's collective view on the distribution of future asset returns across different time horizons and price levels. Understanding how to construct, parameterize, and interpret volatility surfaces is essential for option pricing, risk management, and trading applications..

My aim is for this note is to be 'self-contained': a reader with some basic calculus and probability under their belt, should be able to follow the development from options principles to a working implementation. The 'order of exploration' is as follows:

1. **Options and Payoffs**: What is an option? What determines its value?

2. **The Black-Scholes Framework**: How do we price options under idealized assumptions?

3. **Implied Volatility**: How do we invert market prices to extract the market's view?

4. **The Volatility Surface**: How do implied volatilities vary across strikes and expiries?

5. **SVI Parameterization**: How do we fit a smooth curve to each expiry slice?

6. **SSVI Parameterization**: How do we fit the entire surface jointly?

7. **Implementation**: How can we build this in practice?

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†This work used AI assistants (Claude, Gemini) for coding support – debugging, boilerplate, and the occasional "what is happening here" moment. I'm an economics student, so the code is functional first and elegant second.. All mistakes and questionable decisions remain my own.

2 Options: Definitions and Payoffs

2.1 What is an Option?

An *option* is a financial contract that gives its holder the *right*, but *not* the obligation, to buy or sell an underlying asset at a predetermined price on or before a specified date.

Definition 2.1 (European Call and Put Options). Let S_T denote the price of an underlying asset at time T. A **European call option** with strike K and expiry T pays

$$C_T = \max(S_T - K, 0) = (S_T - K)^+ \tag{1}$$

at time T. A **European put option** pays

$$P_T = \max(K - S_T, 0) = (K - S_T)^+ \tag{2}$$

at time T. European options can only be exercised at expiry while American options can be exercised before as long as the contract is in the money ($S_T > K$ for calls and $S_T < K$ for puts).

The call payoff is positive when $S_T > K$ (the asset is worth more than the strike), and zero otherwise. The holder "calls away" the asset at a favorable price. Conversely, the put payoff is positive when $S_T < K$ – the holder "puts" the asset to the counterparty at a price above market value.

Example 2.1. Consider a call option on SPY with strike $K = 600$ and one month to expiry. If SPY is trading at $S_T = 620$ at expiry, the payoff is $620 - 600 = \$20$. If SPY is at 580, the payoff is $0.

2.2 "Moneyness"

Options are categorized by their relationship to the current spot price:

Definition 2.2 (Moneyness). An option is:

- **At-the-money (ATM)**: $K \approx S$ (strike near current price)
- **In-the-money (ITM)**: Positive intrinsic value ($K < S$ for calls, $K > S$ for puts)
- **Out-of-the-money (OTM)**: Zero intrinsic value ($K > S$ for calls, $K < S$ for puts)

For volatility surface construction, we prefer OTM options because they are more liquid and their prices are more sensitive to volatility (ITM options are dominated by intrinsic value as they are, well, in the money)..

2.3 Put-Call Parity

A fundamental relationship links call and put prices:

Theorem 2.1 (Put-Call Parity). *For European options on a non-dividend-paying asset,*

$$C - P = S - Ke^{-rT} \tag{3}$$

where C and P are call and put prices, S is spot, K is strike, r is the risk-free rate, and T is time to expiry.

Proof. Consider two portfolios at time 0:

- Portfolio A: Long call + cash Ke^{-rT}
- Portfolio B: Long put + long stock

At expiry T: Portfolio A is worth $\max(S_T, K)$ (exercise call if $S_T > K$, otherwise keep cash K). Portfolio B is also worth $\max(S_T, K)$ (exercise put if $S_T < K$, otherwise keep stock S_T). By no-arbitrage, they must have equal value today. □

For assets with dividends (like SPY, which is simply a proxy to the S&P 500), the common approach is to work with the *forward price* which I'll denote as F:

$$C - P = e^{-rT}(F - K) \tag{4}$$

Rearranging gives us a way to extract F from market prices..

$$F = K + e^{rT}(C - P) \tag{5}$$

This is how the implementation extracts the forward price *without* needing to know the dividend yield explicitly..

3 The Black-Scholes Framework

3.1 Geometric Brownian Motion

The Black-Scholes model assumes the underlying asset follows *geometric Brownian motion*:

Definition 3.1 (GBM Dynamics). Under the risk-neutral measure $\mathbb{Q}$, the asset price satisfies

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{6}$$

where r is the risk-free rate, $\sigma > 0$ is the **volatility**, and W_t is a standard Brownian motion.

The solution is:

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] \tag{7}$$

Since $W_T \sim \mathcal{N}(0, T)$, we have $\log(S_T/S_0) \sim \mathcal{N}\left((r - \sigma^2/2)T, \sigma^2 T\right)$.

3.2 Black-Scholes Framework

The expected discounted payoff under $\mathbb{Q}$ gives the option price:

Theorem 3.1 (Black-Scholes Formula, Black and Scholes (1973)). *The price of a European call option is given by..*

$$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{8}$$

where $\Phi(\cdot)$ is the standard normal CDF and

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \tag{9}$$

The put price follows from put-call parity: $P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$.

3.3 Black's Model (Forward Pricing)

When working with forward prices, it's cleaner to use **Black's model** (Black, 1976):

$$C = e^{-rT}\left[F\Phi(d_1) - K\Phi(d_2)\right], \quad P = e^{-rT}\left[K\Phi(-d_2) - F\Phi(-d_1)\right] \tag{10}$$

where now

$$d_1 = \frac{\log(F/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \tag{11}$$

This formulation is *internally consistent*: the forward F already incorporates discounting and dividends, so the pricing formula is cleaner. The implementation uses Black's model throughout.

Remark 3.1 (Why Black's Model?). Using Black's model avoids a common pitfall: mixing discounted market prices with undiscounted theoretical prices. Since market option prices are present values, and Black's formula outputs present values (with the e^{-rT} factor), the comparison is consistent.

"The Trillion Dollar Equation"

Before solving for option price, Black and Scholes (1973) derived the partial differential equation that *any* derivative on a non-dividend-paying stock must satisfy:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0} \tag{12}$$

This is the Nobel Prize-winning **Black-Scholes PDE**.. The closed-form solution in Theorem 3.1 is what you get when you solve this equation with the boundary condition $V(S,T) = (S-K)^+$ for a call option.

The equation says: "In a no-arbitrage world, the instantaneous change in any derivative's value (first term), plus the convexity effect from the asset's randomness (second term), plus the drift from the asset's expected growth (third term), must exactly equal the risk-free return on that derivative (fourth term)." If it didn't, you could construct a riskless portfolio that beats the bank – free money!! (or free lunch as you'll see in many finance textbooks)!

This single beautiful equation underpins the $500+ trillion derivatives market..

4 Implied Volatility

4.1 Definition and Interpretation

The Black-Scholes model assumes constant volatility σ. In reality, *volatility is stochastic,* and the log-return distribution has fatter tails than normal. Rather than abandoning the model entirely, practitioners use it as a *quoting convention*:

Definition 4.1 (Implied Volatility). The **implied volatility** σ_{IV} of an option is the unique $\sigma > 0$ such that

$$\text{BS}(S, K, T, r, \sigma) = \text{Market Price} \tag{13}$$

where $\text{BS}(\cdot)$ denotes the Black-Scholes (or Black) pricing formula.

Implied volatility is a *model-free* transformation of option prices into volatility units. It allows comparison across strikes, expiries, and underlying securities.

4.2 Numerical Inversion

The BS formula is monotonic in σ, so inversion is well-posed.. Brent's method is used:

Algorithm 1 Implied Volatility via Brent's Method

Require: Market price V_{mkt}, forward F, strike K, expiry T, rate r, option type
Ensure: Implied volatility σ_{IV}
 1: Define $f(\sigma) = \text{Black}(F, K, T, \sigma, r) - V_{\text{mkt}}$
 2: Check arbitrage bounds: $V_{\text{mkt}} \geq \max(\text{intrinsic}, 0)$
 3: $\sigma_{\text{IV}} \leftarrow \texttt{brentq}(f, 0.001, 5.0)$ ▷ Root-finding on $[\sigma_{\min}, \sigma_{\max}]$
 4: **return** σ_{IV}

> **Implementation Reference**
>
> `data_pipeline.py: implied_vol(price, F, K, T, r, opt)` implements Black's model inversion using `scipy.optimize.brentq`. Arbitrage bounds are checked before root-finding.

5 The Volatility Surface

5.1 From Smile to Surface

If the Black-Scholes model were correct, implied volatility would be constant across all strikes and expiries. In practice, it varies systematically:

Definition 5.1 (Volatility Smile and Surface). The **volatility smile** is the function $K \mapsto \sigma_{\text{IV}}(K)$ for a fixed expiry T. The **volatility surface** is the function $(K, T) \mapsto \sigma_{\text{IV}}(K, T)$.

For equity indices like SPY, the smile is typically a *skew*: IV decreases as strike increases.. This reflects:

- **Leverage effect**: Stock declines increase firm leverage, hence volatility

- **Jump risk**: Markets crash more than they rally; OTM puts are expensive

- **Supply/demand**: Portfolio insurance creates demand for downside protection

5.2 Coordinates: Log-Moneyness and Total Variance

For parameterization, transform to standardized coordinates:

Definition 5.2 (Log-Moneyness and Total Variance). The **log-moneyness** is

$$k = \log(K/F) \tag{14}$$

Recall F is the forward price. The **total implied variance** is

$$w = \sigma_{\text{IV}}^2 \cdot T \tag{15}$$

These coordinates have nice properties:

- $k = 0$ corresponds to ATM ($K = F$)

- $k < 0$ is OTM puts, $k > 0$ is OTM calls

- w accumulates variance over time; for constant σ_{IV}, w is linear in T

5.3 No-Arbitrage Constraints

Not every surface $w(k, T)$ is valid. Arbitrage-free surfaces must satisfy:

Theorem 5.1 (Arbitrage Constraints). *A variance surface $w(k, T)$ is arbitrage-free if and only if:*

1. ***Calendar spread**: $\partial w / \partial T \geq 0$ (variance increases with time)*

2. ***Butterfly spread***: *The local volatility is non-negative, equivalent to*

$$g(k) = \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{(w'(k))^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2} \geq 0 \qquad (16)$$

The butterfly condition ensures you cannot construct a portfolio that profits from any price move (a "free lunch").

6 SVI Parameterization

6.1 The Raw SVI Formula

Gatheral (2004) introduced the **Stochastic Volatility Inspired (SVI)** parameterization:

Definition 6.1 (Raw SVI). The SVI parameterization of total variance is

$$w(k) = a + b\left[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\right] \qquad (17)$$

with five parameters:

- $a \in \mathbb{R}$: Overall variance level
- $b \geq 0$: Slope of the wings (controls how fast variance grows for large $|k|$)
- $\rho \in (-1, 1)$: Skew (correlation-like; $\rho < 0$ gives equity-style downward skew)
- $m \in \mathbb{R}$: Horizontal translation (shifts the smile left/right)
- $\sigma > 0$: Smoothness at the vertex (controls curvature at $k = m$)

6.2 Geometric Interpretation

The SVI formula can be understood as a *hyperbola.*. For large $|k - m|$:

$$w(k) \approx a + b(\rho + 1)(k - m) \quad \text{as } k \to +\infty \qquad (18)$$

$$w(k) \approx a + b(\rho - 1)(k - m) \quad \text{as } k \to -\infty \qquad (19)$$

So the left wing has slope $b(\rho - 1)$ and the right wing has slope $b(\rho + 1)$. For equity skew ($\rho < 0$), the left wing is steeper.

The parameter σ controls the "roundness" near $k = m$: as $\sigma \to 0$, the smile becomes a V-shape (piecewise linear)..

6.3 Arbitrage-Free Constraints

For SVI to satisfy the butterfly condition (16), we need:

Proposition 6.1 (SVI Butterfly Condition). *Define*

$$w'(k) = b \left[\rho + \frac{k - m}{\sqrt{(k - m)^2 + \sigma^2}} \right] \tag{20}$$

$$w"(k) = \frac{b\sigma^2}{((k - m)^2 + \sigma^2)^{3/2}} \tag{21}$$

Then $g(k) \geq 0$ for all k if and only if the parameters satisfy certain constraints.

> **Implementation Reference**
>
> `svi_calibration.py`: `check_butterfly(k, params)` evaluates (16) on a grid of k values and returns whether all $g(k) \geq 0$.

6.4 Calibration

We can fit SVI to each expiry slice by minimizing the sum of squared errors:

$$\min_{a,b,\rho,m,\sigma} \sum_{i=1}^{n} \left[w_{\text{SVI}}(k_i; a, b, \rho, m, \sigma) - w_i^{\text{mkt}} \right]^2 \tag{22}$$

This is a non-convex optimization problem. We can use differential evolution (a global optimizer) with bounds:

Parameter	Lower	Upper
a	-0.5	0.5
b	0.001	2.0
ρ	-0.999	0.999
m	-0.5	0.5
σ	0.001	1.0

> **Implementation Reference**
>
> `svi_calibration.py`: `calibrate_surface(data)` fits SVI to all expiries using `scipy.optimize.differential_evolution`. Outputs: `svi_params_*.csv`, `svi_fitted_*.csv`.

7 SSVI Parameterization

7.1 Motivation

Per-expiry SVI has a problem: nothing connects different expiries.. With 5 parameters per slice and 20+ expiries, we have 100+ parameters and no guarantee of calendar arbitrage freedom.

Surface SVI (SSVI) addresses this by parameterizing the entire surface jointly with only 3 global parameters.

7.2 The SSVI Formula

Definition 7.1 (SSVI, Gatheral and Jacquier (2014))**.** The SSVI parameterization is

$$w(k, \theta) = \frac{\theta}{2} \left[1 + \rho \varphi(\theta) k + \sqrt{(\varphi(\theta)k + \rho)^2 + 1 - \rho^2} \right] \quad (23)$$

where:

- $\theta = \theta(T)$ is the ATM total variance at expiry T
- $\rho \in (-1, 0]$ controls skew (typically negative for equities)
- $\varphi(\theta)$ is a function controlling how the smile shape varies with θ

The key insight: the smile *shape* is controlled by $\varphi(\theta)$, which depends only on the ATM variance level, *not* on time directly. This encodes the empirical observation that smile dynamics are driven by variance levels.

7.3 The Power-Law 'φ'

Gatheral and Jacquier (2014) propose:

$$\varphi(\theta) = \frac{\eta}{\theta^\gamma (1 + \theta)^{1-\gamma}} \quad (24)$$

with $\eta > 0$ (vol-of-vol) and $\gamma \in (0, 1)$ (interpolation between power laws).
For small θ: $\varphi(\theta) \approx \eta/\theta^\gamma$ (smile steepens as variance decreases).
For large θ: $\varphi(\theta) \approx \eta/\theta$ (smile flattens).

7.4 Two-Stage Calibration

SSVI calibration proceeds in two stages:

Stage 1: ATM Term Structure. Extract $\theta(T)$ from market data by finding the ATM variance at each expiry. Interpolate to get a continuous function, ensuring monotonicity (for calendar arbitrage freedom).

Stage 2: Global Fit. With $\theta(T)$ fixed, optimize (ρ, η, γ) to minimize:

$$\min_{\rho, \eta, \gamma} \sum_{\text{all } (k_i, T_i)} \left[w_{\text{SSVI}}(k_i, \theta(T_i)) - w_i^{\text{mkt}} \right]^2 \quad (25)$$

This is a 3-parameter optimization over thousands of data points..

Implementation Reference

`ssvi_calibration.py`: `SSVICalibrator.fit(data)` implements two-stage calibration. Stage 1 uses linear interpolation with monotonicity enforcement. Stage 2 uses differential evolution. Outputs: `ssvi_params_*.csv`, `ssvi_atm_term_structure_*.csv`, `ssvi_fitted_*.csv`.

7.5 No-Arbitrage Guarantees

SSVI has remarkable arbitrage properties:

Theorem 7.1 (SSVI No-Arbitrage, Gatheral and Jacquier (2014))**.** *If $\theta(T)$ is increasing in T and $|\rho| < 1$, then SSVI is free of calendar arbitrage. Under additional parameter constraints, it is also free of butterfly arbitrage.*

This is why practitioners often prefer SSVI for risk management and scenario analysis: it produces a *guaranteed arb-free* surface.

8 Implementation Details

8.1 Data Pipeline

The implementation fetches SPY option chains from Yahoo Finance and processes them:

1. **Fetch**: Download call and put chains for all available expiries
2. **Filter**: Keep expiries in $[3, 365]$ DTE range
3. **Forward Extraction**: Use put-call parity (5) on liquid ATM strikes
4. **IV Calculation**: Invert Black's formula (10) for each option
5. **Transform**: Compute $k = \log(K/F)$ and $w = \sigma_{\text{IV}}^2 T$
6. **Quality Filter**: Remove illiquid options (wide spreads, low open interest)

> **Implementation Reference**
>
> `data_pipeline.py`: Key functions:
> - `black_price(F, K, T, sigma, r, opt)`: Black's model pricing
> - `implied_vol(price, F, K, T, r, opt)`: IV inversion via Brent's method
> - `extract_forward(calls, puts, S, T, r)`: Forward via put-call parity
> - `SPYDataPipeline.fetch()`: Main entry point
>
> Output: `spy_vol_surface_TIMESTAMP.csv`

8.2 Filtering Criteria

Use OTM options only (puts for $K < F$, calls for $K \geq F$) and filter by:

- Bid-ask spread $\leq 50\%$ of mid price
- Open interest ≥ 10 contracts
- Log-moneyness $k \in [-0.5, 0.5]$
- Valid IV solution exists

8.3 Workflow Sketch

```
1  # Step 1: Fetch data
2  python data_pipeline.py
3  # -> spy_vol_surface_TIMESTAMP.csv
4
5  # Step 2: Fit SVI (per-expiry)
6  python svi_calibration.py
7  # -> svi_params_TIMESTAMP.csv, svi_fitted_TIMESTAMP.csv
8
9  # Step 3: Fit SSVI (joint)
10 python ssvi_calibration.py
11 # -> ssvi_params_TIMESTAMP.csv, ssvi_atm_term_structure_TIMESTAMP.csv,
       ssvi_fitted_TIMESTAMP.csv
12
13 # Step 4: Generate plots
14 python visualization.py
15 # -> 10 plot_*.png files
```

Listing 1: Running the pipeline

Each script will auto-detect the most recent output from the previous step.

9 Results and Analysis

9.1 Data Summary

From SPY options (December 30, 2025):

- Spot price: $687.01
- Expiries: 21 (from 3 DTE to 365 DTE)
- Total data points: 2,333
- Strikes per expiry: 15 to 219 (monthlies have more liquidity)

9.2 SVI Results

'Per-expiry' SVI achieves excellent fits:

- All 21 expiries fit successfully
- All 21 fits are arbitrage-free (butterfly condition satisfied)
- RMSE range: 1.4×10^{-5} to 7.9×10^{-4} (variance space)
- Average RMSE: 2.1×10^{-4}

However, the fitted ρ values vary wildly across expiries (from -0.37 to $+0.65$), indicating that per-expiry SVI does not enforce consistency. Some slices produce economically odd parameters despite excellent RMSE; this illustrates non-identifiability of raw SVI without regularization/cross-expiry structure..

9.3 SSVI Results

Joint SSVI calibration yields:

Parameter	Value	Interpretation
ρ	-0.62	Strong negative skew (typical for equity options)
η	0.77	Vol-of-vol level
γ	0.61	Power-law interpolation

Fit quality:

- RMSE: 0.00132 (variance) $\sim 1\%$ (IV points)
- Max error: 0.00862 (variance) $\sim 5\%$ (IV points)

9.4 SVI vs SSVI: The Tradeoff

The key finding is a clear *bias-variance tradeoff*:

	SVI (per-expiry)	SSVI (joint)
Parameters	$5 \times 21 = 105$	3
Short-dated fit ($T < 30$ DTE)	Excellent	Poor (3-5% RMSE)
Long-dated fit ($T > 90$ DTE)	Excellent	Excellent ($< 1\%$ RMSE)
Calendar arbitrage	Not guaranteed	Guaranteed
Interpolation to unlisted expiries	Not possible	Smooth

SSVI's error concentrates in short-dated options because:

1. Near-expiry smiles are kinked and change rapidly day-to-day
2. The SSVI $\varphi(\theta)$ function assumes smooth dependence on θ
3. Microstructure effects (tick size, bid-ask) dominate for small T

9.5 Practical Recommendations

For production use:

- **Pricing short-dated exotics**: Use per-expiry SVI
- **Risk management**: Use SSVI (guarantees no-arbitrage)
- **Scenario analysis**: Use SSVI (smooth interpolation)
- **Hybrid approach**: SVI for $T < 30$ DTE, SSVI for longer tenors

10 Closing Remarks

I hope this note was a nice self-contained introduction to volatility surface construction – accessible to anyone with basic calc, probability, and finance background. From defining what an option is, through Black-Scholes, to calibrating SVI and SSVI on real SPY data, all in one place!

The main takeaways are that SSVI is remarkably parsimonious (3 parameters for the entire surface) and guarantees no-arbitrage, *but* struggles with short-dated options where smile dynamics are too erratic. Per-expiry SVI fits beautifully but offers no cross-expiry consistency. In practice, a hybrid approach (SVI for $T < 30$ DTE, SSVI beyond) often works best. For anyone wishing to go deeper: Gatheral's *The Volatility Surface* remains the definitive reference.

Implementation Reference

Complete source code is available in the four Python scripts!

- `data_pipeline.py`
- `svi_calibration.py`
- `ssvi_calibration.py`
- `visualization.py`

Dependencies: `numpy`, `pandas`, `scipy`, `matplotlib`, `yfinance`

References

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Appendix A: Black-Scholes Derivation

Setup: Geometric Brownian Motion

Under the risk-neutral measure $\mathbb{Q}$:

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{26}$$

Equivalently:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \tag{27}$$

Applying Itô's Lemma: Itô (1951)

Let $f(S) = \log S$. Compute the derivatives:

$$f'(S) = \frac{1}{S} \tag{28}$$

$$f"(S) = -\frac{1}{S^2} \tag{29}$$

Itô's lemma states:

$$\mathrm{d}f(S_t) = f'(S_t)\,\mathrm{d}S_t + \frac{1}{2}f"(S_t)\,(\mathrm{d}S_t)^2 \tag{30}$$

Compute $(\mathrm{d}S_t)^2$:

$$(\mathrm{d}S_t)^2 = (rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t)^2 \tag{31}$$

$$= r^2 S_t^2\,(\mathrm{d}t)^2 + 2r\sigma S_t^2\,\mathrm{d}t\,\mathrm{d}W_t + \sigma^2 S_t^2\,(\mathrm{d}W_t)^2 \tag{32}$$

$$= \sigma^2 S_t^2\,\mathrm{d}t \tag{33}$$

where: $(\mathrm{d}t)^2 = 0$, $\mathrm{d}t\,\mathrm{d}W_t = 0$, $(\mathrm{d}W_t)^2 = \mathrm{d}t$.
Substitute into Itô's lemma..

$$\mathrm{d}(\log S_t) = \frac{1}{S_t}\,\mathrm{d}S_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2\,\mathrm{d}t \tag{34}$$

$$= \frac{1}{S_t}\left(rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t\right) - \frac{\sigma^2}{2}\,\mathrm{d}t \tag{35}$$

$$= r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \frac{\sigma^2}{2}\,\mathrm{d}t \tag{36}$$

$$= \left(r - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{37}$$

Integrating the SDE !

Integrate from 0 to T:

$$\int_0^T \mathrm{d}(\log S_t) = \int_0^T \left(r - \frac{\sigma^2}{2}\right)\mathrm{d}t + \int_0^T \sigma\,\mathrm{d}W_t \tag{38}$$

Left side:

$$\int_0^T \mathrm{d}(\log S_t) = \log S_T - \log S_0 = \log\frac{S_T}{S_0} \tag{39}$$

Right side:

$$\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \tag{40}$$

Sooo:

$$\log \frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \tag{41}$$

Exponentiate both sides:

$$\frac{S_T}{S_0} = \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] \tag{42}$$

$$\boxed{S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right]} \tag{43}$$

Distribution of S_T

Since $W_T \sim \mathcal{N}(0, T)$, we have $\sigma W_T \sim \mathcal{N}(0, \sigma^2 T)$.

Define:

$$X = \log \frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \tag{44}$$

Then:

$$\mathbb{E}[X] = \left(r - \frac{\sigma^2}{2}\right)T + \sigma \cdot \mathbb{E}[W_T] = \left(r - \frac{\sigma^2}{2}\right)T \tag{45}$$

$$\text{Var}(X) = \sigma^2 \text{Var}(W_T) = \sigma^2 T \tag{46}$$

Therefore:

$$\log S_T \sim \mathcal{N}\left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \, \sigma^2 T\right) \tag{47}$$

Equivalently, S_T is **lognormally distributed**.

Risk-Neutral Pricing Formula

The call option price is the discounted expected payoff:

$$C = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] \tag{48}$$

Expand the expectation:

$$C = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\mathbf{1}_{\{S_T > K\}}\right] \tag{49}$$

Split into two terms:

$$C = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbf{1}_{\{S_T > K\}}\right] - e^{-rT} K \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T > K\}}\right] \tag{50}$$

Define:

$$I_1 = \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbf{1}_{\{S_T > K\}}\right] \tag{51}$$

$$I_2 = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T > K\}}\right] = \mathbb{Q}(S_T > K) \tag{52}$$

So:

$$C = e^{-rT} I_1 - e^{-rT} K I_2 \tag{53}$$

Computing $I_2 = \mathbb{Q}(S_T > K)$

The condition $S_T > K$ is equivalent to:

$$S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] > K \tag{54}$$

$$\exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] > \frac{K}{S_0} \tag{55}$$

$$\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T > \log\frac{K}{S_0} \tag{56}$$

Rearrange:

$$\sigma W_T > \log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T \tag{57}$$

$$W_T > \frac{1}{\sigma}\left[\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T\right] \tag{58}$$

Standardize: let $Z = W_T/\sqrt{T} \sim \mathcal{N}(0,1)$, so $W_T = Z\sqrt{T}$.

$$Z\sqrt{T} > \frac{1}{\sigma}\left[\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T\right] \tag{59}$$

$$Z > \frac{\log(K/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{60}$$

Multiply numerator and denominator by -1 (and flip inequality):

$$Z < \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{61}$$

Define:

$$\boxed{d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}} \tag{62}$$

Therefore:

$$I_2 = \mathbb{Q}(Z < d_2) = \Phi(d_2) \tag{63}$$

Computing $I_1 = \mathbb{E}^{\mathbb{Q}}[S_T\,\mathbf{1}_{\{S_T > K\}}]$

Write explicitly using the PDF of W_T:

$$I_1 = \int_{-\infty}^{\infty} S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma w\right] \mathbf{1}_{\{w > w^*\}} \cdot \frac{1}{\sqrt{2\pi T}} e^{-w^2/(2T)}\,\mathrm{d}w \tag{64}$$

where $w^* = \frac{1}{\sigma}\left[\log(K/S_0) - (r - \sigma^2/2)T\right]$.
Substitute $z = w/\sqrt{T}$, so $w = z\sqrt{T}$, $\mathrm{d}w = \sqrt{T}\,\mathrm{d}z$:

$$I_1 = S_0 \int_{z^*}^{\infty} \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z\right] \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z \tag{65}$$

where $z^* = -d_2$.
Combine the exponentials:

$$I_1 = S_0 \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z - \frac{z^2}{2}\right]\,\mathrm{d}z \tag{66}$$

Focus on the exponent:

$$\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z - \frac{z^2}{2} \tag{67}$$

$$= rT - \frac{\sigma^2 T}{2} + \sigma\sqrt{T}z - \frac{z^2}{2} \tag{68}$$

$$= rT - \frac{1}{2}\left(z^2 - 2\sigma\sqrt{T}z + \sigma^2 T\right) \tag{69}$$

$$= rT - \frac{1}{2}\left(z - \sigma\sqrt{T}\right)^2 \tag{70}$$

Substitute back:

$$I_1 = S_0 e^{rT} \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z - \sigma\sqrt{T})^2}{2}\right] \mathrm{d}z \tag{71}$$

Change variables: let $u = z - \sigma\sqrt{T}$, so $\mathrm{d}u = \mathrm{d}z$.
When $z = z^* = -d_2$:

$$u^* = -d_2 - \sigma\sqrt{T} = -\frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \tag{72}$$

Simplify u^*:

$$u^* = -\frac{\log(S_0/K) + (r - \sigma^2/2)T + \sigma^2 T}{\sigma\sqrt{T}} \tag{73}$$

$$= -\frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{74}$$

$$= -d_1 \tag{75}$$

where we define:

$$\boxed{d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}} \tag{76}$$

Therefore:

$$I_1 = S_0 e^{rT} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, \mathrm{d}u = S_0 e^{rT} \, \mathbb{Q}(U > -d_1) \tag{77}$$

Since $U \sim \mathcal{N}(0,1)$:

$$\mathbb{Q}(U > -d_1) = \mathbb{Q}(U < d_1) = \Phi(d_1) \tag{78}$$

Thus:

$$I_1 = S_0 e^{rT} \Phi(d_1) \tag{79}$$

Assembly

Recall:

$$C = e^{-rT} I_1 - e^{-rT} K I_2 \tag{80}$$

Substitute $I_1 = S_0 e^{rT} \Phi(d_1)$ and $I_2 = \Phi(d_2)$:

$$C = e^{-rT} \cdot S_0 e^{rT} \Phi(d_1) - e^{-rT} K \Phi(d_2) \tag{81}$$

$$= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) \tag{82}$$

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$$C = S_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2) \qquad (83)$$

with:

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad (84)$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad (85)$$

Put Price via Put-Call Parity

From put-call parity: $P = C - S_0 + Ke^{-rT}$.

$$P = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) - S_0 + Ke^{-rT} \qquad (86)$$

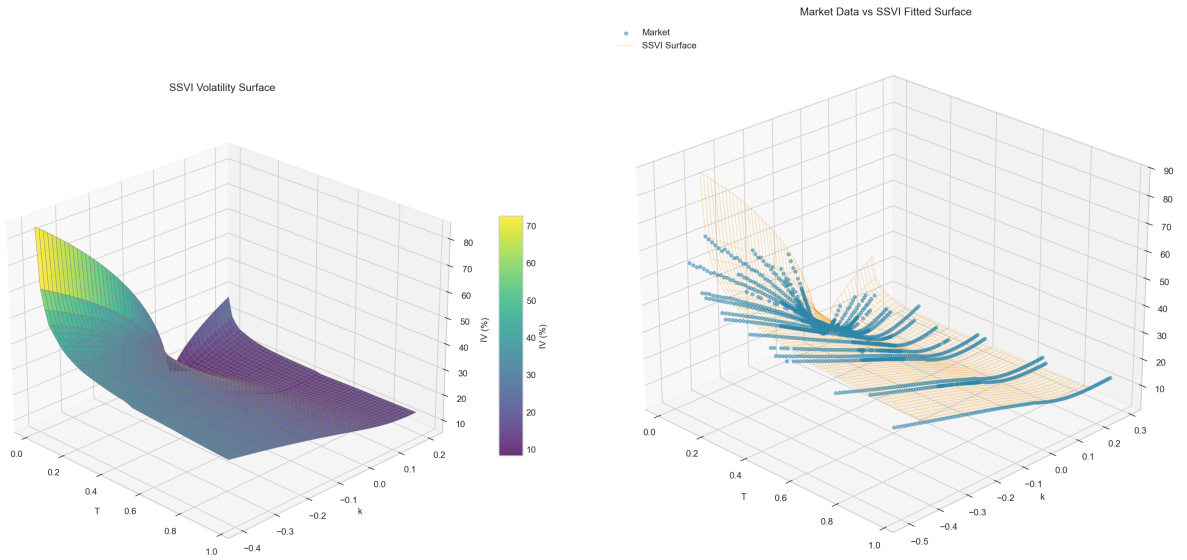$$= S_0(\Phi(d_1) - 1) + Ke^{-rT}(1 - \Phi(d_2)) \qquad (87)$$

$$= -S_0(1 - \Phi(d_1)) + Ke^{-rT}(1 - \Phi(d_2)) \qquad (88)$$

$$= Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \qquad (89)$$

using $1 - \Phi(x) = \Phi(-x)$.

$$P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \qquad (90)$$

Appendix B: Results and Diagnostics

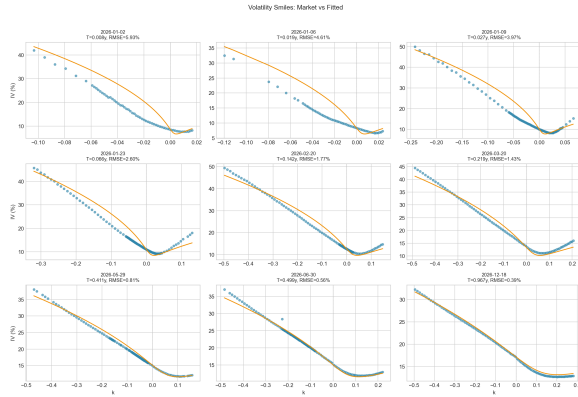(a) SSVI fitted surface (joint calibration).

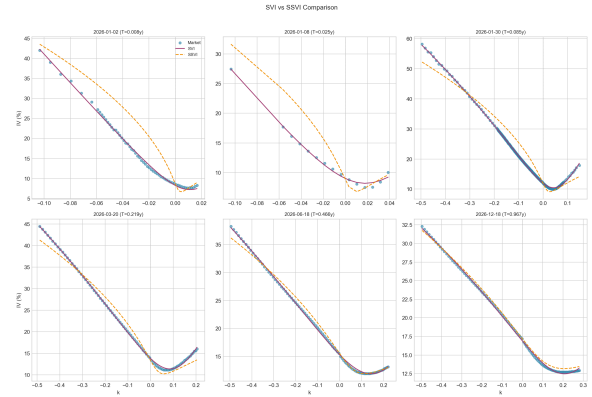(b) Market vs SSVI wireframe overlay.
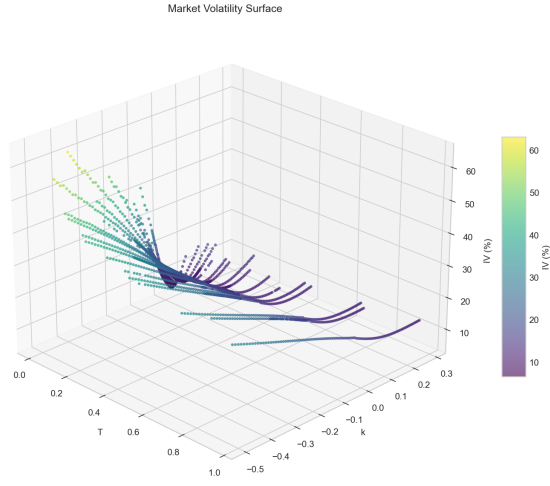
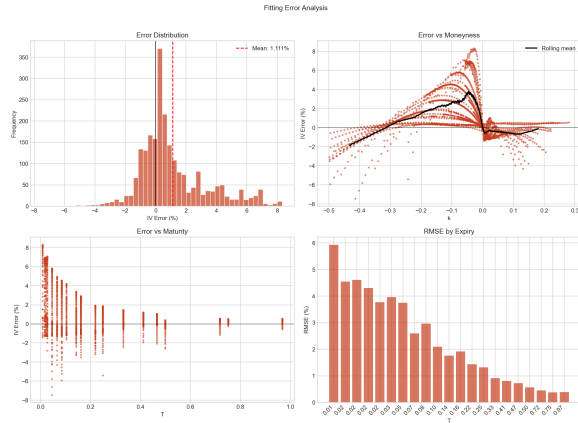(a) Raw market implied volatility surface.
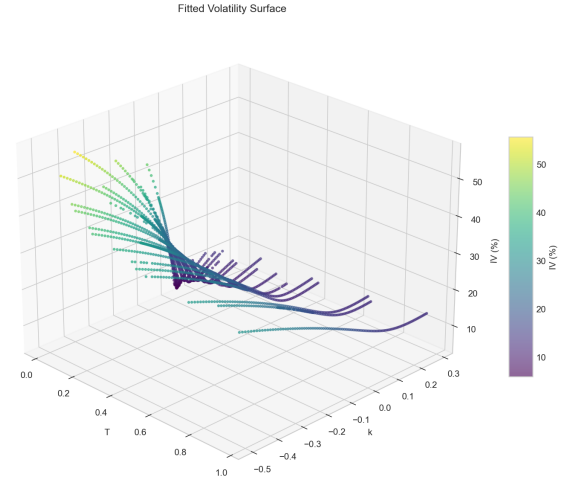
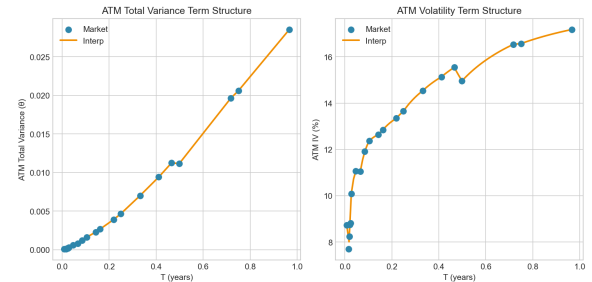(b) SVI fitted surface (per-expiry calibration).

(c) Volatility smile cross-sections by expiry.

(d) SVI vs SSVI residual comparison.

(e) SSVI fit error analysis by expiry.

(f) ATM term structure of implied volatility.