Derivation of the Newton - Raphson method

The one-dimensional case

let $f(x)$ be an analytic function for which we want to find the root.

Assume x_{k+1} is a root and x_k is an approximation such that $f'(x_k) \neq 0$ and $|x_{k+1} - x_k|$ is small.

Consider the first-order Taylor series for $f(x)$ about x_k evaluated at x_{k+1} :

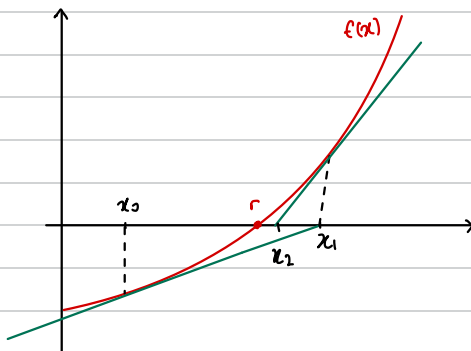$$f(x_{k+1}) \simeq f(x_k) + f'(x_k)(x_{k+1} - x_k) \quad (*)$$

Since we have assumed $|x_{k+1} - x_k|$ is small, the other terms in the Taylor series will be negligible and can be ignored.

Since we assume x_{k+1} is a root we have $f(x_{k+1}) = 0$ and can rearrange $(*)$ to obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

which is the Newton - Raphson formula for a function in one variable.

Geometric Interpretation

Two-dimensional case

We now want to find solutions (x_1, x_2) of
$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$
where f_1, f_2 are analytic functions.

We can write this in vector notation as $F(\underline{x}) = 0$ where $F = (f_1, f_2)^T$, $\underline{x} = (x_1, x_2)^T$

let h, k be small increments.

We consider the Taylor series of the functions

$$f_1(x_1+h, x_2+k) \simeq f_1(x_1, x_2) + h\frac{\partial f_1}{\partial x_1}(x_1, x_2) + k\frac{\partial f_1}{\partial x_2}(x_1, x_2)$$

$$f_2(x_1+h, x_2+k) \simeq f_2(x_1, x_2) + h\frac{\partial f_2}{\partial x_1}(x_1, x_2) + k\frac{\partial f_2}{\partial x_2}(x_1, x_2)$$

where we have assumed that h, k are small and so the higher order terms can be neglected.

We assume that (x_1+h, x_2+k) is a root of both functions and so $f_1(x_1+h, x_2+k) = 0$ and $f_2(x_1+h, x_2+k) = 0$.

We can write this as
$$0 = F(\underline{x}) + \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\[2mm] \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} \Delta\underline{x}$$

where $\Delta\underline{x} = (h, k)^T$.

let $J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\[2mm] \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_1} \end{pmatrix}$

This is called the Jacobian of F and can be written as $J_F(x)$

Thus we can write the equation as

$$0 = F(x) + J_F(x)\,\Delta x$$

$$\Rightarrow \quad \Delta x = -J_F(x)^{-1}\,(F(x))$$

Thus we use this formula to compute Δx given our initial guess x and then add Δx to find the next guess.

We continue this process until Δx is small enough.