Nonlinear Solvers

MATH-151: Mathematical Algorithms in Matlab

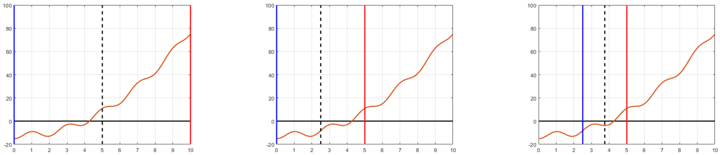
October 9, 2023

Why Solve Nonlinear Equations?

- We are going to focus now on finding x for a function such that $f(x) = 0$
- Lines are great, we know how to solve for where a line crosses zero quite easily, but many things in real life are nonlinear so we need to develop techniques for finding zeros for those equations as well
- Why do we want to find zeros in the first place?
  - If we know how to find where $f(x) = 0$, we know how to find where it is any other c since we can instead solve $f(x) - c = g(x) = 0$
  - If we want to find where two curves $f(x)$ and $g(x)$ intersect, we can do $f(x) - g(x) = h(x) = 0$
  - Most engineering problems are interested in optimization. We remember from Calc I, if we want to find a minimizer or maxmizer, we start by finding a critical point where $f'(x) = 0$
- Basically any time we want to know where anything occurs, we either run into a line or a case to solve for the root of a nonlinear equation!
  - We usually like to know when/where things occur

BISECTION METHOD

- We've already seen one of these a few weeks ago! The Bisection method!
- Suppose we know $f(a)$ is negative and $f(b)$ is positive. If f is a continuous function, it must pass through 0 between a and b.
- So we cut it in half and try $f(m)$, where $m = \frac{a+b}{2}$
  - If it has the same sign as $f(a)$, m becomes our new a
  - If it has the same sign as $f(b)$, m becomes our new b



- This gives us a new, smaller possible range, so we do Bisection Method on that range
- We continue until $f(m)$ is sufficiently small.

Fixed Point Equations

- How do we tell the computer to know when it's found a zero? For that we introduce the concept of a **fixed point**.
- If $g(x)$ is a function that tells us our next guess for the zero, a fixed point satisfies the following

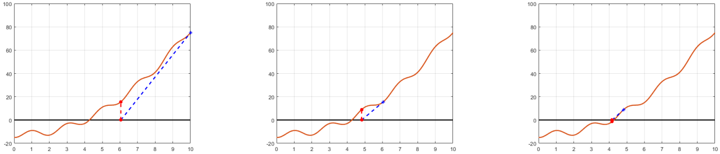
$$x_n = g(x_n) = x_n + f(x_n)h(x_n)$$

- When this occurs, our guess for the next point is the same as our current guess
  - This will occur when $f(x_n) = 0$
- That's what we are looking for! We found an x_n such that $f(x_n) = 0$.
  - Remember, we will almost never get exactly 0 so we repeat our process until $f(x_n)$ or $x_n - x_{n-1}$ become small enough
  - Or perhaps, we repeat *while* $f(x_n)$ is too big
- Great! So what can we do to set these up? What would this $g(x_n)$ look like?

Newton's Method

- This is a widely used method that is usually much faster than bisection method!
- We pretend our function is the tangent line at our current guess, so that means our next guess for x is where the tangent line crosses 0

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

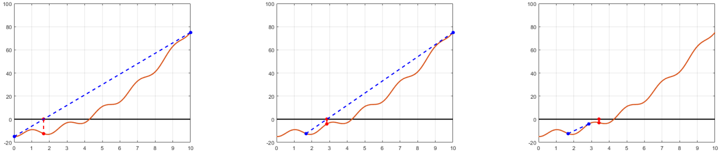


- We are not only using the values of our function, but also information about what the curve looks like. This will help us get closer to our 0 faster!
  - This does have some risk though, we could encounter a point where $f'(x_n) = 0$...

Secant Method

- Sometimes we don't have a derivative function, or it would be very difficult to implement
- In those cases, we can estimate our derivative using finite difference method with our last 2 points!
    - But this means we need to make 2 initial guesses to start
- Doing this we have a slightly different method called **Secant Method**

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$



- The code I use in my research is actually using Secant method because our gradients are too complicated to find analytically!