Interpolation

MATH-151: Mathematical Algorithms in Matlab

September 18, 2023

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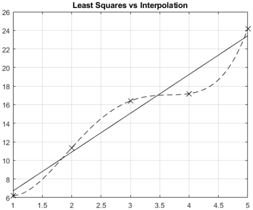

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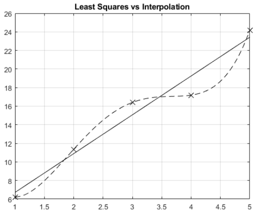

- Most often we use polynomials as our **interpolants**, but different sets of functions can also be used!

When is Interpolation Used?

- Interpolation is useful in many purposes when we are interested in representing data with a smooth function that captures changes in slope. For example
  - Plotting a smooth curve through data
  - Reading between entries of a data table
  - Differentiation or integration of function data
  - Approximating a complicated function with a simpler one
  - Using trigonometric functions to interpolate uniformly-spaced data points is called the Discrete Fourier Transform (or DFT) and commonly used in signal processing.

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- How do we make it correct when $x = x_i$? We take our product above and divide by $(x_i - x_j)$ for each $j \neq i$ so we get 1, then multiply by $f(x_i)$

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- We repeat this for each of our points and add them together

$$p(x) = \sum_{i=1}^{N} f(x_i) \frac{\prod_{j=1, j\neq i}^{N} (x - x_j)}{\prod_{j=1, j\neq i}^{N} (x_i - x_j)}$$

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- We start with $x_1 = 2, f(x_1) = 5$. Our term to make sure this works is

$$f(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = 5\frac{(x - 7)(x - 8)}{(2 - 7)(2 - 8)}$$

- Repeating this process for $x_2 = 7$ and $x_3 = 8$ we can find our interpolating polynomial

$$p(x) = 5\frac{(x - 7)(x - 8)}{(2 - 7)(2 - 8)} + 0\frac{(x - 2)(x - 8)}{(7 - 2)(7 - 8)} + 11\frac{(x - 2)(x - 7)}{(8 - 2)(8 - 7)}$$

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- We can simplify this, but we don't have to. The computer loves to grind through arithmetic!

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- And we use these to build our polynomial as follows

$$p(x) = f[x_1] + f[x_1, x_2](x - x_1) + \cdots + f[x_1, \ldots, x_n](x - x_1) \times \cdots \times (x - x_{n-1})$$

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- Just like Lagrange's method, this looks much more complicated when writing it abstractly than when performing it in practice, so lets see an example again..

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- Putting it all together we can find our interpolating polynomial is

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or, if we want to be fancy, $f(x) = 5 + (x-2)\Big(-1 + (x-7)(2)\Big)$

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- If we want to do the work, we can see this is the same answer that Lagrange's method gave us!

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  - Lagrange's method is easier to understand and more straightfoward. But it is less flexible to new data, and often less computationally efficient
  - Newton's divided differences are easier to add new data points to and can be written to be very fast, however it is harder to understand and implement