Numerical Differentiation

MATH-151: Mathematical Algorithms in Matlab

October 2, 2023

A Refresher on the Derivative

- Since we learned about numerical integration last week, lets now look at the other primary focus of calculus, the derivative. Which at some point x is defined as

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- Rates of change are very useful, as we are almost always interested in how things change
  - The rate of change of the position of an object over time gives us its velocity. The rate of change of that is its acceleration
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- Derivatives are very informative, but very hard to do so we need to come up with approximations

Finite Difference Methods

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- These sure are straight-forward to do, but how good are they?

Finite Difference Error

- We can use the Taylor series approximations to analyze our error for the forward difference equation

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \dots$$

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- Doing the same thing with the backward finite difference equation will give similar results.

Central Difference Method

- Since we know how to find error now, lets see if we can use $f(x)$, $f(x+h)$, and $f(x-h)$ to get a more accurate estimate that is $\mathcal{O}(h^2)$

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&+ c(f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \dots)
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$\Rightarrow$ We Want $a + b + c = 0$ *(This makes sure $f(x)$ terms cancel out)*

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- Which has error $\mathcal{O}(h^2)$ like we wanted. So now making h 10 times smaller will make your error 100 times smaller!

Second Order Derivatives

- Lets also see if we can use those same points to get an estimate of the second derivative

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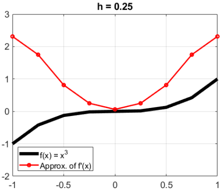
- If we look further into the error, we see this is also a $\mathcal{O}(h^2)$ method

Derivative Functions

- If we have a set of data points from a function, we can estimate its derivative as a function by computing our finite difference estimate at each point.



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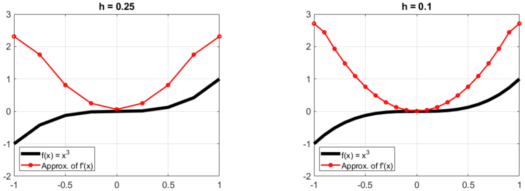

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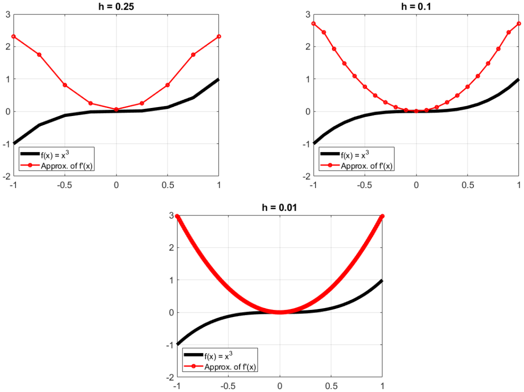

- We can see clearly that more points gives us a more accurate estimate

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- I was actually rollerblading, so good guess!


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