Numerical Integration

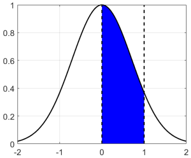
MATH-151: Mathematical Algorithms in Matlab

September 25, 2023

Integration in Practice

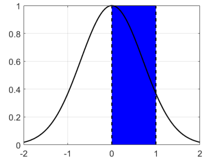
- Reminder: The definite integral of a function $f(x)$ between points a and b outputs the area beneath that curve between $x = a$ and $x = b$ and is represented
$$\int_a^b f(x)dx$$
- Integrals effectively "accumulate" the effect of f, for a few examples
  - Integrating velocity over some time tells us how far an object traveled
  - The area under a probability distribution gives us a probability
  - Computing oddly shaped areas and volumes
  - Often used for solutions of differential equations
- There are many different rules for performing these integrals, and many of them are very complicated for a computer to perform, so we find approximations!

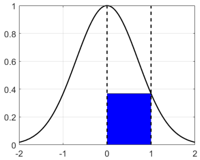
Ex: $\int_0^1 e^{-x^2} dx$

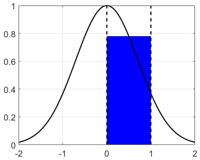


RECTANGULAR APPROXIMATIONS

- A very basic way to quickly approximate an integral is to assume that the function is constant, $f(x) \approx c$
- This is pretty good because our area will become a rectangle, and we know how to calculate the area of a rectangle! $A = wh$
- We have many choices for our c, lets look at some popular options
  - **Left-hand rule**, $f(x) \approx f(a) \Rightarrow \displaystyle\int_a^b f(x)dx \approx (b-a)f(a)$
  - **Right-hand rule**, $f(x) \approx f(b) \Rightarrow \displaystyle\int_a^b f(x)dx \approx (b-a)f(b)$
  - **Midpoint rule**, $f(x) \approx f(\frac{a+b}{2}) \Rightarrow \displaystyle\int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2})$



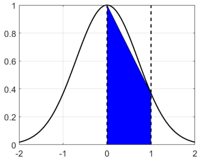




TRAPEZOIDAL RULE

- If we want to allow out function to actually change with varying x values, the next easiest thing we can do is assume the function is a line between $(a, f(a))$ and $(b, f(b))$
- We could write this out as a function, but we can think of this as forming a trapezoid with our area. We know how to calculate the area of a trapezoid! $A = w\frac{h_1+h_2}{2}$
- This doesn't have any choices so we write it as follows
  Trapezoidal Rule: $\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$

Ex: $$\int_0^1 e^{-x^2}dx \approx (1-0)\frac{e^{-(0^2)}+e^{-(1^2)}}{2}$$



Simpson's Rule

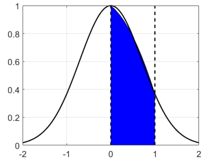
- Now that we have seen interpolation, we can go another step further and find a parabola that goes through our edges and our midpoint $m = \frac{a+b}{2}$. We then find a surprisingly nice formula comes out of that!

$$f(x) \approx f(a) + \frac{f(m) - f(a)}{\frac{b-a}{2}}(x-a) + \frac{f(b) - 2f(m) + f(a)}{\frac{(b-a)^2}{2}}(x-a)(x-m)$$

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b \left(f(a) + \frac{f(m) - f(a)}{\frac{b-a}{2}}(x-a) + \frac{f(b) - 2f(m) + f(a)}{\frac{(b-a)^2}{2}}(x-a)(x-m)\right)dx \\ &= \frac{b-a}{6}(f(a) + 4f(m) + f(b))\end{aligned}$$

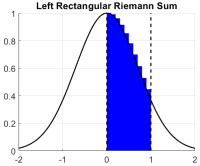
- This gives us **Simpson's Rule:** $\int_a^b f(x)dx \approx \frac{b-a}{6}(f(a) + 4f(m) + f(b))$

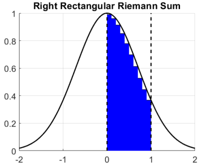
Ex: $\int_0^1 e^{-x^2}dx \approx \frac{(1-0)}{6}\left(e^{-(0^2)} + 4e^{-(\frac{1}{2}^2)} + e^{-(1^2)}\right)$

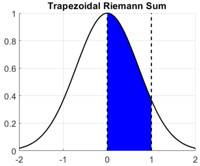


Riemann Sums

- If we can get data for many points, we can also make our approximation more accurate by cutting our range into tinier slices!
- This is normally referred to as a **Riemann Sum**
- We can use each of the methods above to calculate the areas of each our our slices.
    - As we saw from Simpson's rule, we get much better estimates at the cost of needing to know more points.
- Here are our Riemann sums using 10 equal sized "slices"







Estimating the Antiderivative

- Since when we are performing a Riemann sum we are just making approximation of the integral of smaller sections and then adding them together, we can get a sequence of sums.
- For example, if we consider our Riemann sum earlier
  $\int_0^1 e^{-x^2} dx \approx \sum_{n=1}^{10} \frac{1}{10} e^{-\frac{n-1}{10}^2}$
  If we only take the first half half of that sum, we get the integral along half of that range!
  $\sum_{n=1}^{5} \frac{1}{10} e^{-\frac{-1}{10}^2} \approx \int_0^{\frac{1}{2}} e^{-x^2} dx$
- This sequence can allow us to see an estimate of the antiderivative (plus or minus some constant C)