Nonlinear Solvers

MATH-151: Mathematical Algorithms in Matlab

October 9, 2023

Why Solve Nonlinear Equations?

- We are going to focus now on finding x for a function such that $f(x) = 0$
- Lines are great, we know how to solve for where a line crosses zero quite easily, but many things in real life are nonlinear so we need to develop techniques for finding zeros for those equations as well

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- Why do we want to find zeros in the first place?
  - If we know how to find where $f(x) = 0$, we know how to find where it is any other c since we can instead solve $f(x) - c = g(x) = 0$
  - If we want to find where two curves $f(x)$ and $g(x)$ intersect, we can do $f(x) - g(x) = h(x) = 0$
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- Basically any time we want to know where anything occurs, we either run into a line or a case to solve for the root of a nonlinear equation!
  - We usually like to know when/where things occur

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- So we cut it in half and try $f(m)$, where $m = \frac{a+b}{2}$
  - If it has the same sign as $f(a)$, m becomes our new a
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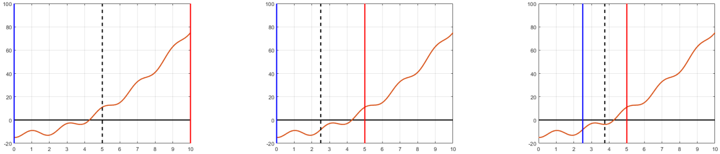

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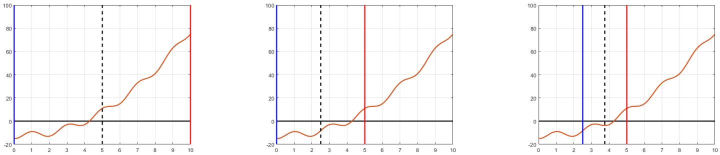

- This gives us a new, smaller possible range, so we do Bisection Method on that range
- We continue until $f(m)$ is sufficiently small.

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  - Or perhaps, we repeat *while* $f(x_n)$ is too big
- Great! So what can we do to set these up? What would this $g(x_n)$ look like?

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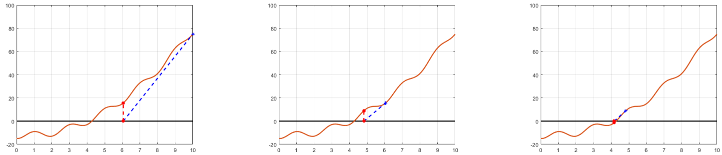

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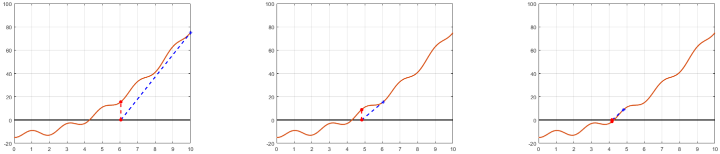

- We are not only using the values of our function, but also information about what the curve looks like. This will help us get closer to our 0 faster!
  - This does have some risk though, we could encounter a point where $f'(x_n) = 0$...

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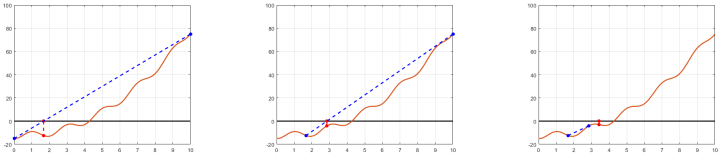

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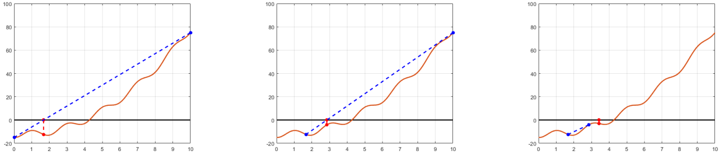

- The code I use in my research is actually using Secant method because our gradients are too complicated to find analytically!