Numerical Integration

MATH-151: Mathematical Algorithms in Matlab

September 25, 2023

Integration in Practice

- Reminder: The definite integral of a function $f(x)$ between points a and b outputs the area beneath that curve between $x = a$ and $x = b$ and is represented
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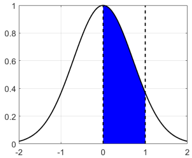

- Integrals effectively "accumulate" the effect of f, for a few examples
  - Integrating velocity over some time tells us how far an object traveled
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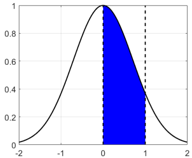

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- There are many different rules for performing these integrals, and many of them are very complicated for a computer to perform, so we find approximations!

Rectangular Approximations

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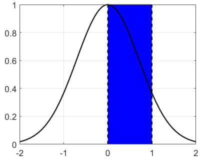
  - **Left-hand rule**, $f(x) \approx f(a) \Rightarrow \int_a^b f(x)dx \approx (b-a)f(a)$



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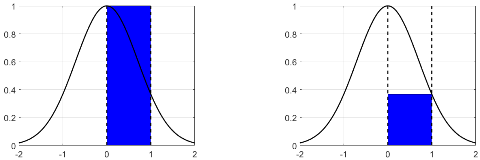

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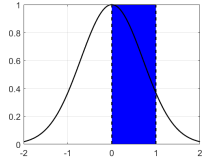
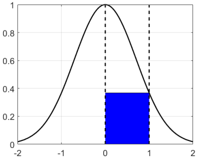
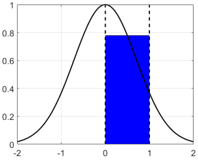
  - **Midpoint rule**, $f(x) \approx f(\frac{a+b}{2}) \Rightarrow \int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2})$







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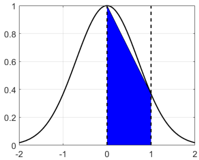

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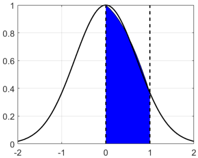

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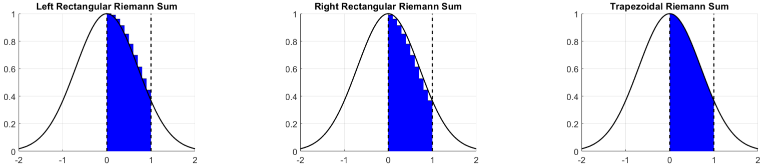
- Here are our Riemann sums using 10 equal sized "slices"



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- This sequence can allow us to see an estimate of the antiderivative (plus or minus some constant C)