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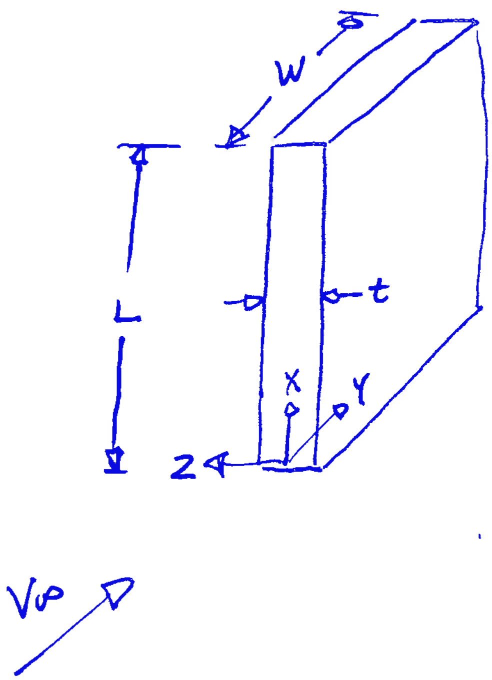
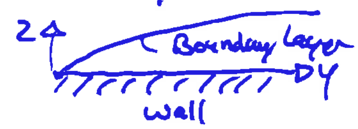


$\ast$ Convective Cooling on Sides - Assume laminar flow

Reynolds Analogy: $St\,Pr^{2/3} = \frac{C_f}{2}$ $St = \frac{Nu}{Re\,Pr}$

$$\therefore \quad Nu = Re\,Pr^{1/3}\,\frac{C_f}{2}$$

Blasius solution for flow over a flat plate: $C_f(y) = 0.664\,Re_y^{-1/2}$
(in y-direction)



$$\therefore \quad Nu(y) = 0.332\,Pr^{1/3}\left[Re(y)\right]^{1/2}$$

$$Re(y) = \frac{\rho_{air} V_\infty y}{\mu_{air}} \; ; \quad Pr = \frac{\mu_{air} C_{p_{air}}}{k_{air}} \; ; \quad Nu(y) = \frac{h\,y}{k_{air}}$$

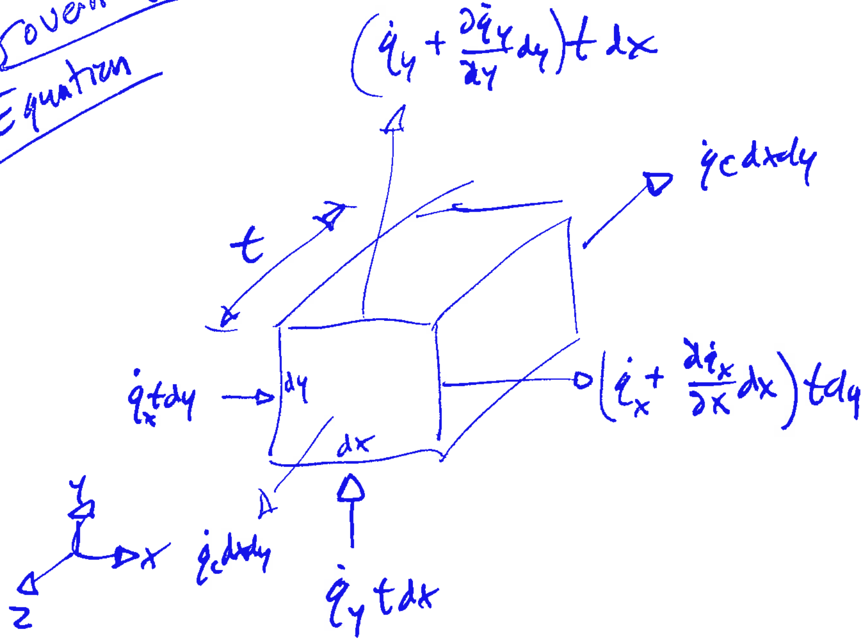
$$\dot{q}_{conv} = -k_{air}\left.\frac{\partial T}{\partial z}\right|_{wall} = h\left(T_w - T_\infty\right)$$

$$h(y) = 0.332 \cdot k_{air} \cdot Pr^{1/3}\left(\frac{\rho_{air} V_\infty}{\mu_{air}\,y}\right)^{1/2}$$

For air: $\rho_{air} = 1.05\ \frac{kg}{m^3}$ $C_{p_{air}} = 1.008\ \frac{kJ}{kg\cdot K}$ $\mu_{air} = 1.95E\text{-}5\ \frac{N\cdot s}{m^2}$

$k_{air} = 2.8E\text{-}2\ \frac{W}{m\,K}$ $Pr_{air} = 0.704$

Governing Equation



$$\left[\dot{q}_x - \left(\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx\right)\right] t\, dy + \left[\dot{q}_y - \left(\dot{q}_y + \frac{\partial \dot{q}_y}{\partial y} dy\right)\right] t\, dx - 2\dot{q}_c\, dx\, dy = 0$$

$$-\frac{\partial \dot{q}_x}{\partial x} t\, dx\, dy - \frac{\partial \dot{q}_y}{\partial y} t\, dx\, dy = 2\dot{q}_c\, dx\, dy$$

$$\dot{q}_x = -K_{Metal} \frac{\partial T}{\partial x} \qquad \dot{q}_y = -K_{metal} \frac{\partial T}{\partial y}$$

$$K_{metal} \simeq \text{const}$$

$$K_{metal}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{2\dot{q}_c}{t}$$

but $\dot{q}_c = h(y)\,(T - T_\infty)$

$$\therefore \quad K_{metal}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{2}{t} h(y)\,(T - T_\infty)$$

$\Rightarrow T(x,y)$

* Unsteady Formulation (for iterating to steady-state)

$$\frac{\partial T}{\partial t} - \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = S$$

$$\alpha = \frac{k}{\rho C_p} \quad \text{(for the metal)}$$

$$-\rho C_p \frac{\partial T}{\partial t} + k\left(\vec{\nabla}^2 T\right) = -\rho C_p S$$

from before (steady): $k\left(\vec{\nabla}^2 T\right) = \frac{2}{t} h(y)\left(T - T_\infty\right)$

$$\therefore \quad -\rho C_p S = \frac{2}{t} h(y)\left(T - T_\infty\right)$$

$$S = -\frac{2 h(y)}{\rho C_p t}\left(T - T_\infty\right)$$

Governing Eqn. Becomes:

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{time}} - \alpha_{Metal}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{-2 h(y)}{\rho_{Metal}\, C_{p_{Metal}}\, \underbrace{t}_{\text{thickness}}}\left(T - T_\infty\right)$$

Stability Criteria: $\Delta t \leq \frac{1}{4} \frac{\Delta x \Delta y}{\alpha_{Metal}}$

or $\Delta t = CFL \frac{\Delta x \Delta y}{\alpha_{Metal}} \qquad CFL \leq \frac{1}{4}$

Possible Boundary Conditions

1) Dirichlet (fixed temperature): e.g., $T_{1,j} = T_{BC}$

(applied at $x=0$ or $i=1$)

2) Neumann (fixed derivative)

A) Adiabatic: e.g., $\dot{q}_x\big|_{x=0} = 0$

$$-k_{metal}\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \quad \therefore \quad \frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

Discretization: ($i=1$)

$$\frac{-T_{i+2} + 4T_{i+1} - 3T_i}{2\Delta x} = 0$$

or

$$T_{1,j} = \frac{4T_{2,j} - T_{3,j}}{3}$$

B) Constant heat flux: e.g., $\dot{q}_x\big|_{x=0} = \dot{q}_{BC} = \text{const}$

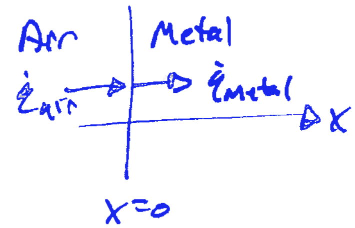
$$-k_{metal}\frac{\partial T}{\partial x}\bigg|_{x=0} = \dot{q}_{BC}$$

Discretization: ($i=1$)

$$\frac{-T_{i+2} + 4T_{i+1} - 3T_i}{2\Delta x} = -\frac{\dot{q}_{BC}}{k_{metal}}$$

or

$$T_{1,j} = \left(4T_{2,j} - T_{3,j} + \frac{\dot{q}_{BC}\,2\Delta x}{k_{metal}}\right)\frac{1}{3}$$



3) Robin (e.g., convective)

$$\dot{q}_{air}\big|_{x=0} = \dot{q}_{metal}\big|_{x=0}$$

$$\dot{q}_{air}\big|_{x=0} = -h(y)\left(T - T_\infty\right)$$

$$\dot{q}_{metal}\big|_{x=0} = -k_{metal}\,\frac{\partial T}{\partial x}\bigg|_{x=0}$$

$$\therefore \quad k_{metal}\,\frac{\partial T}{\partial x}\bigg|_{x=0} = h(y)\left(T_{x=0} - T_\infty\right)$$

Discretization: $(i=1)$

$$\frac{-T_{i+2} + 4T_{i+1} - 3T_i}{2\Delta x} = \frac{h(y)}{k_{metal}}\left(T_i - T_\infty\right)$$

$$-T_{i+2} + 4T_{i+1} - 3T_i = \underbrace{\frac{2\Delta x\, h(y)}{k_{metal}}}_{=\varphi(y)} T_i - \frac{2\Delta x\, h(y)}{k_{metal}} T_\infty$$

$$\left[\varphi(y) + 3\right] T_{1,j} = \varphi(y)\, T_\infty + 4T_{2,j} - T_{3,j}$$

$$T_{1,j} = \frac{1}{\varphi(y)+3}\left[\varphi(y)\, T_\infty + 4T_{2,j} - T_{3,j}\right]$$