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Data Structures and Algorithms
CS F211

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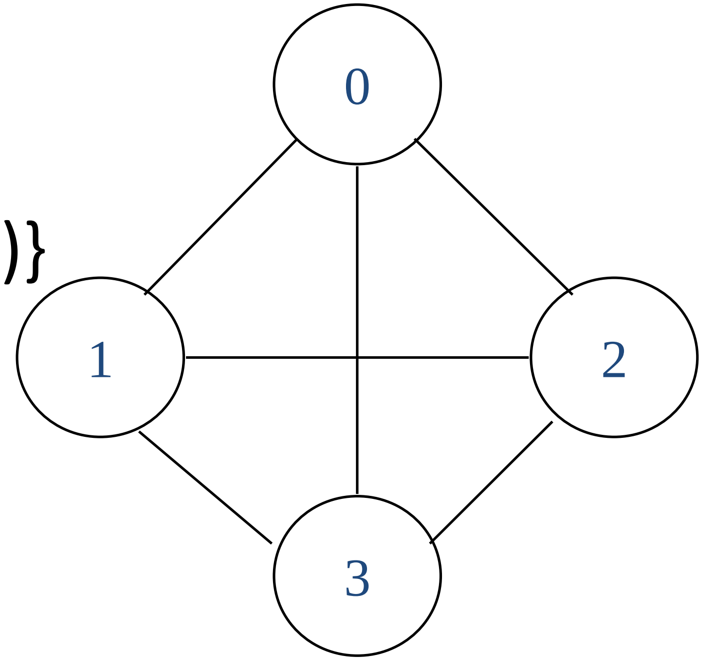
Agenda: Elementary Graph Algorithms

Definitions

- A graph G=(V,E), V and E are two sets
  - V: finite non-empty set of vertices
  - E: set of pairs of vertices, edges
- Undirected graph
  - The pair of vertices representing any edge is unordered. Thus, the pairs (u,v) and (v,u) represent the same edge
- Directed graph
  - each edge is represented by a ordered pairs <u,v>

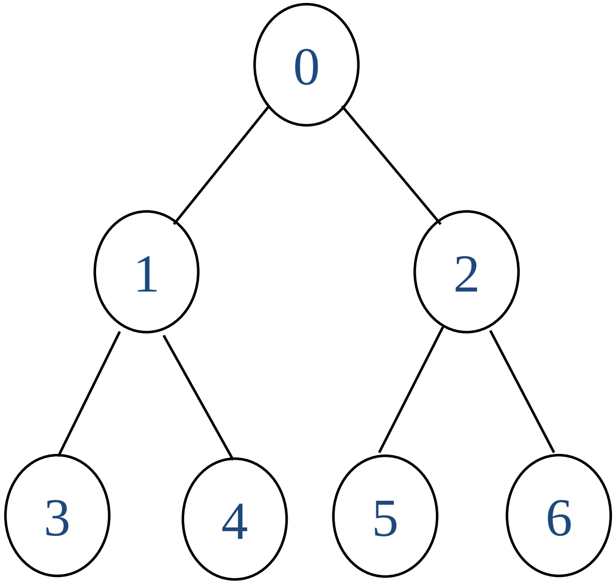
Examples of Graph G_1

- G_1
  - $V(G1)=\{0,1,2,3\}$
  - $E(G1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$



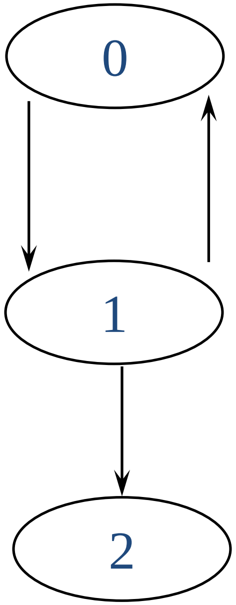
Examples of Graph G_2

- G_2
  - V(G2)={0,1,2,3,4,5,6}
  - E(G2)={(0,1),(0,2), (1,3),(1,4),(2,5),(2,6)}
- G_2 is also a tree
  - Tree is a special case of graph



Examples of Graph G_3

- G_3
  - V(G3)={0,1,2}
  - E(G3)={<0,1>,<1,0>,<1,2>}
- Directed graph (digraph)

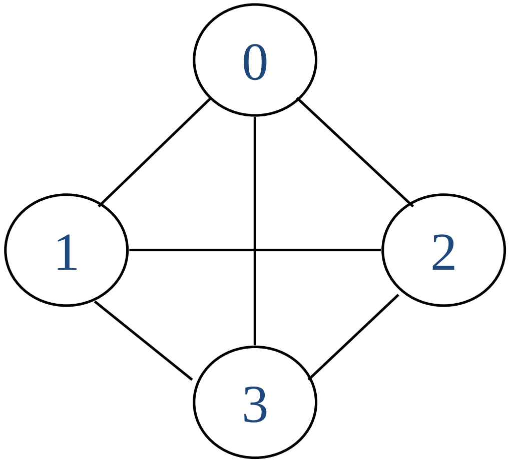


Adjacent and Incident

- If (u,v) is an edge in an undirected graph,
  - Adjacent: u and v are adjacent
  - Incident: The edge (u,v) is incident on vertices u and v
- If <u,v> is an edge in a directed graph
  - Adjacent: u is adjacent to v, and vu is adjacent from v
  - Incident: The edge <u,v> is incident on u and v

Cycle

- Cycle
  - a simple path, first and last vertices are same.
- 0,1,2,0 is a cycle
- Acyclic graph
  - No cycle is in graph



Degree

- Degree of Vertex
  - is the number of edges incident to that vertex
- Degree in directed graph
  - Indegree
  - Outdegree

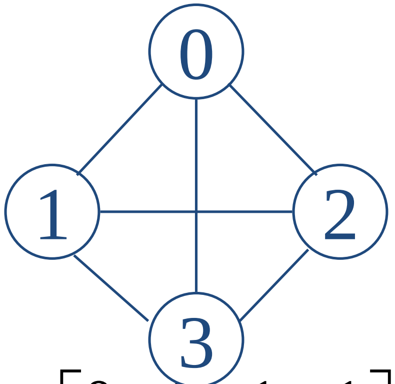
Graph Representations

- Adjacency Matrix
- Adjacency Lists

Adjacency Matrix

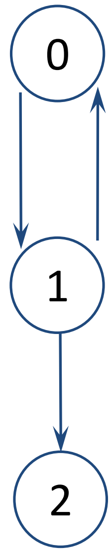
- Adjacency Matrix : let G = (V, E) with n vertices, $n \geq 1$. The adjacency matrix of G is a 2-dimensional $n \times n$ matrix, A
  - A(i, j) = 1 iff $(v_i, v_j) \in E(G)$ ($\langle v_i, v_j \rangle$ for a diagraph)
  - A(i, j) = 0 otherwise.
- The adjacency matrix for an undirected graph is symmetric; the adjacency matrix for a digraph need not be symmetric

Example



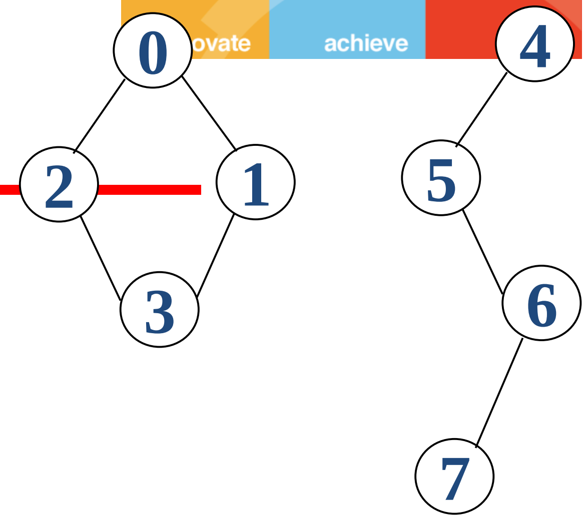
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

G2



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

G_4

symmetric

undirected: $n^2/2$
directed: n^2

Merits of Adjacency Matrix

- From the adjacency matrix, to determine the connection of vertices is easy
- The degree of a vertex is $\sum_{j=0}^{n-1} adj\_mat[i][j]$
- For a digraph, the row sum is the out_degree, while the column sum is the in_degree

$$ind(vi) = \sum_{j=0}^{n-1} A[j,i] \quad outd(vi) = \sum_{j=0}^{n-1} A[i,j]$$

Adjacency Lists

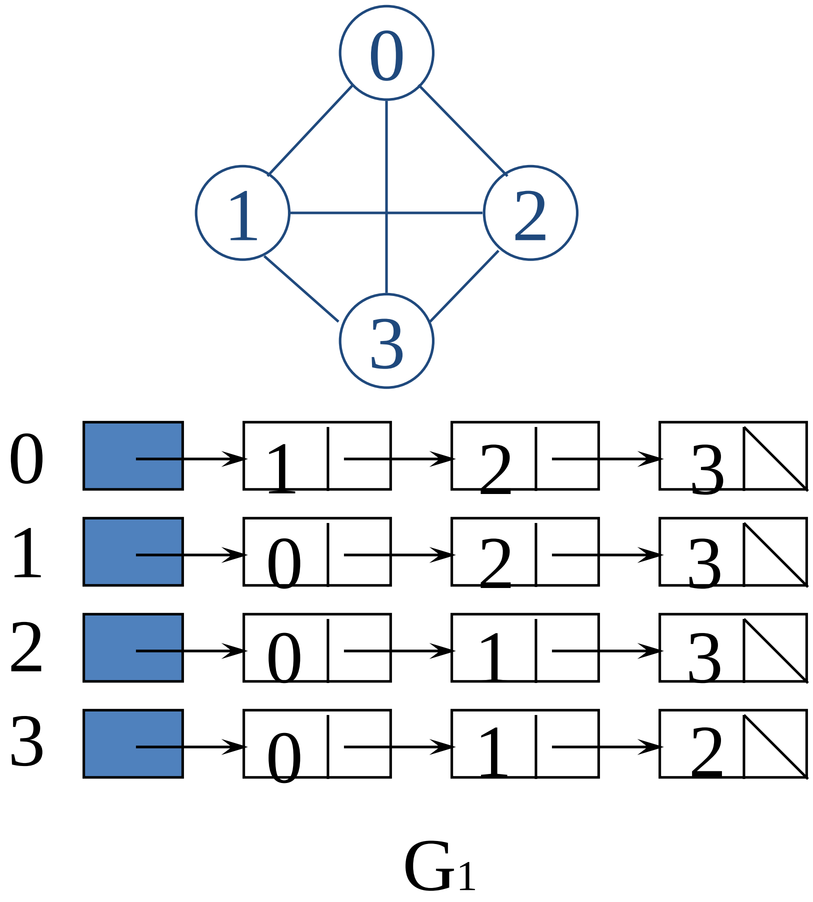
- Replace n rows of the adjacency matrix with n linked list

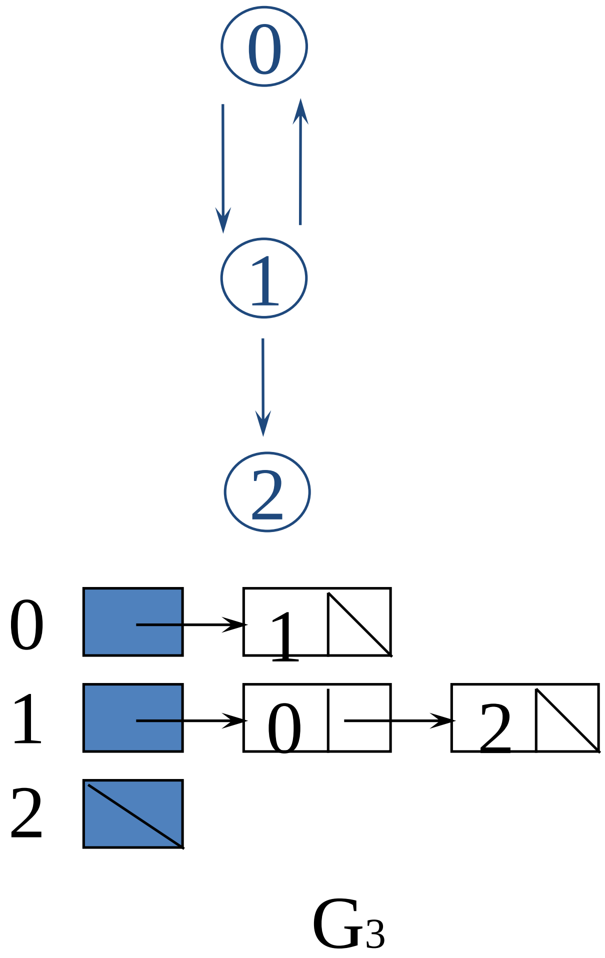
Data Structures for Adjacency Lists

Each row in adjacency matrix is represented as an adjacency list.

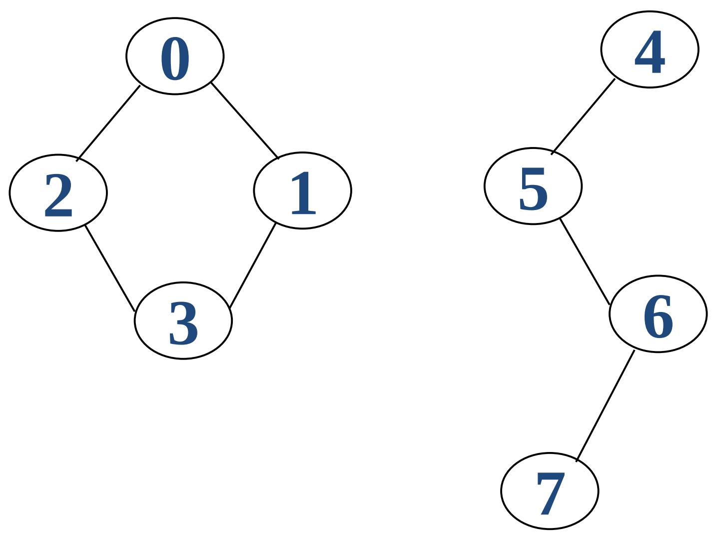
```
#define MAX_VERTICES 50
typedef struct node *node_pointer;
typedef struct node {
    int vertex;
    struct node *link;
};
node_pointer graph[MAX_VERTICES];
int n=0; /* vertices currently in use */
```

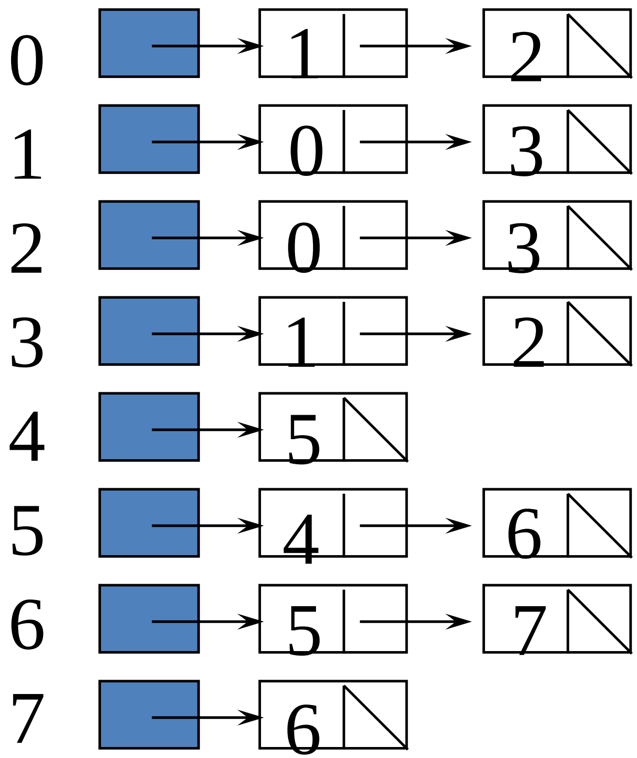
Example(1)





Example(2)

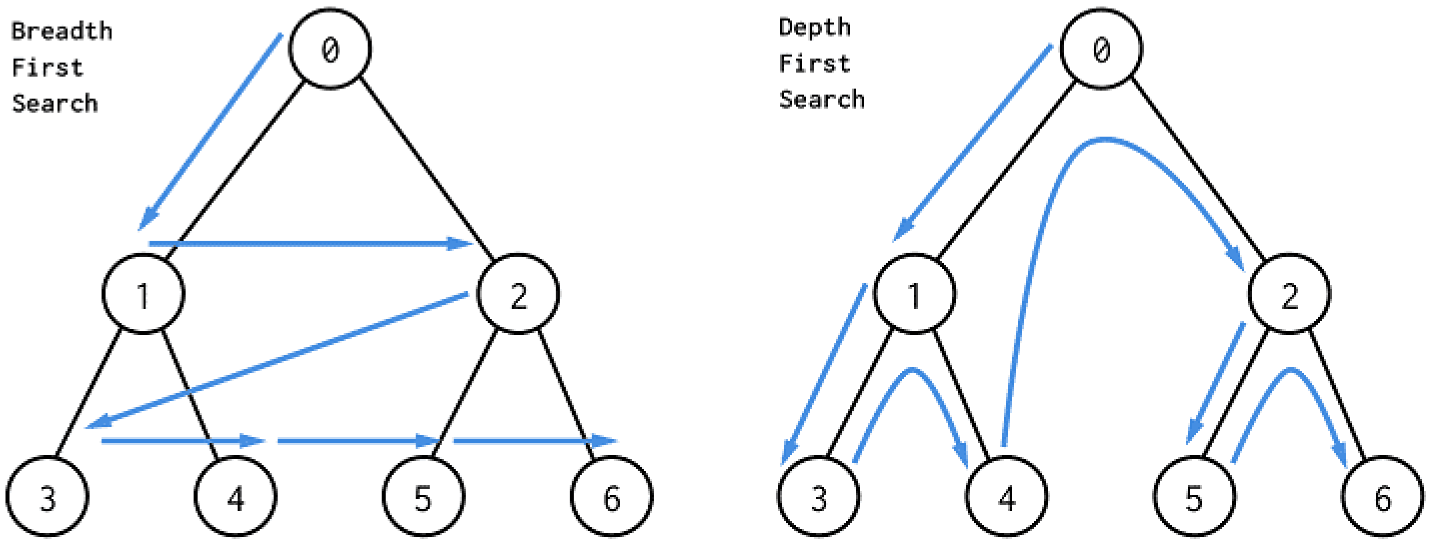




G_4

Interesting Operations

- degree of a vertex in an undirected graph
  - # of nodes in adjacency list
- # of edges in a graph
  - determined in O(n+e)
- out-degree of a vertex in a directed graph
  - # of nodes in its adjacency list
- in-degree of a vertex in a directed graph
  - traverse the whole data structure


Breadth
First
Search
0
1
2
3
4
5
6
Depth
First
Search
0
1
2
3
4
5
6

Breadth-first search

Breadth First Search

- ***Breadth-first search*** is one of the simplest algorithms for searching a graph.
- Dijkstra's single-source shortest-paths algorithm Prim's minimum-spanning-tree algorithm use ideas similar to those in BFS.

Breadth First Search

- Given a graph $G = (V, E)$ and a distinguished ***source*** vertex s, BFS systematically explores the edges of G to "discover" every vertex that is reachable from s.
- It computes the distance (fewest number of edges) from s to all such reachable vertices.
- It also produces a "breadth-first tree" with root s that contains all such reachable vertices.
- BFS discovers all vertices at distance k from s before discovering any vertices at distance $k + 1$.

Breadth First Search

```
Breadth_First_Search(G, s)
    for each vertex u ∈ G.V – {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
    s.color = GRAY
    s.d = 0
    s.π = NIL
    Q = ϕ
    ENQUEUE (Q, s)
```

```
While (Q ≠ ϕ)
{
    u = DEQUEUE (Q)
    for each v ∈ G.Adj[u]
    {
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE (Q, v)
    }
    u.color = BLACK
}
```

Depth first search

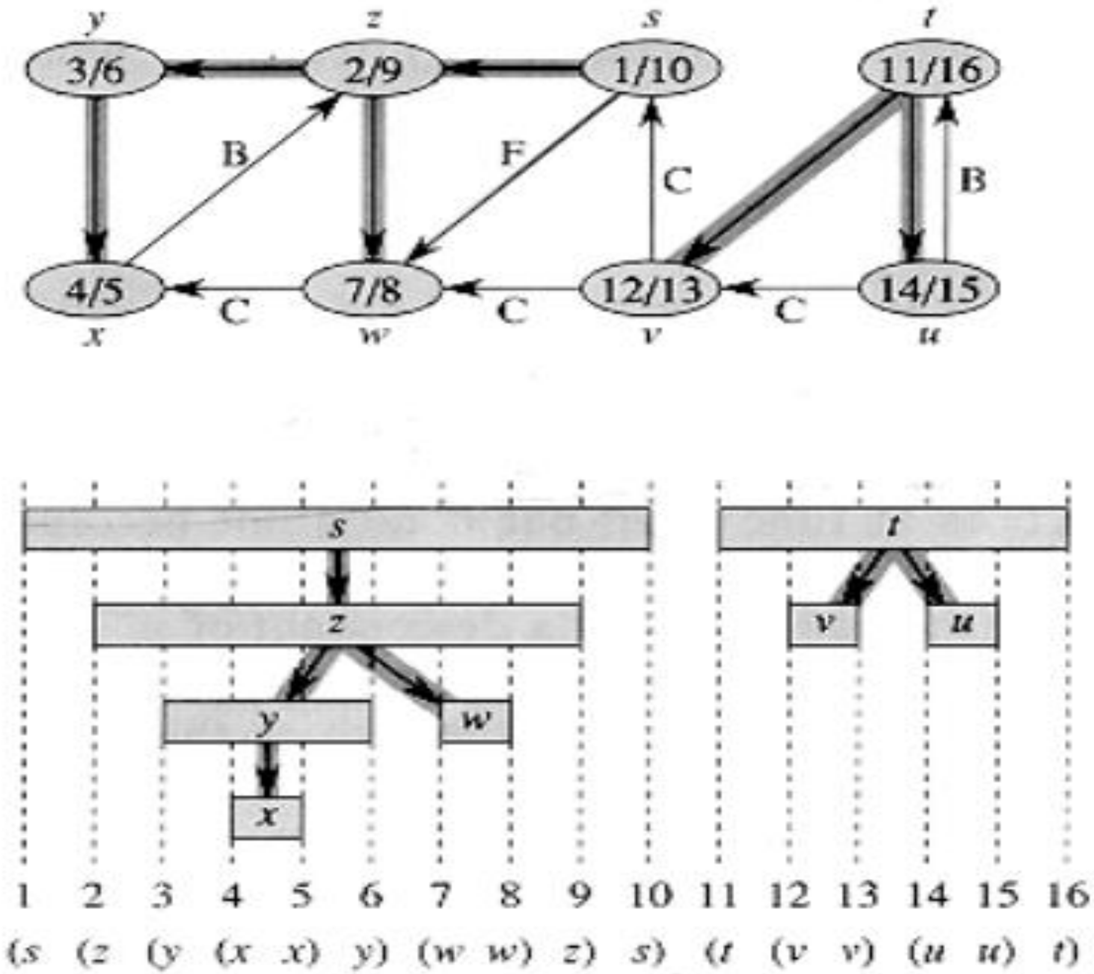
Global Variable: time

```
DEPTH_FIRST_SEARCH
   for each vertex u ∈ G.V
   {
       u.color = WHITE
       u.π =NIL
   }
   time = 0
   for each vertex u ∈ G.V
   {
        if u.color == WHITE
        DFS-VISIT (G, u)
   }
```

```
DFS-VISIT (G, u)
    time = time+1
    u.d = time
    u.color = GRAY
    for each v ∈ G.Adj[u]
    {
        if v.color == WHITE
            v.π = u
            DFS-VISIT (G, v)
    }
    u.color = BLACK
    time = time+1
    u.f = time
```

DFS: Properties

- discovery and finishing times have ***parenthesis structure***.



Parenthesis Theorem

In any depth first search of G = (V, E), for any two vertices u and v, exactly one of the following three conditions holds:

a) The intervals [u.d, u.f] and [v.d, v.f] are entirely disjoint, and neither u nor v is a descendant of the other in the depth first forest.
b) The interval [u.d, u.f] is contained entirely within the interval [v.d, v.f], and u is a descendent of v in a depth first tree, or
c) The interval [v.d, v.f] is contained entirely within the interval [u.d, u.f], and v is a descendent of u in a depth first tree

Corollary: Vertex v is a proper descendant of vertex u in the depth first forest for a (directed or undirected) graph G if and only if

$$u.d < v.d < v.f < u.f$$

Classification of Edges

- **DFS can be used to classify the edges of G into:**

a) **Tree Edge**: Edge (u, v) is a tree edge if v was first discovered by exploring edge (u, v).

b) **Back edges:** Edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree. Self-loops, which may occur in directed graphs, are considered to be back edges.

c) **Forward edges:** Those non-tree edges (u, v) connecting a vertex u to a descendant v in a depth-first tree.

d) **Cross edges** are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

- **Edge (u, v) can be classified by the color of the vertex v that is reached when the edge is first explored**

1. **WHITE** indicates a tree edge,
2. **GRAY** indicates a back edge, and
3. **BLACK** indicates a forward or cross edge.

Properties

- In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.
- A directed graph is acyclic if and only if a depth-first search yields no “back” edges.
- For a weighted graph, DFS traversal of the graph produces the minimum spanning tree and all pair shortest path tree.
- We can specialize the DFS algorithm to find a path between two given vertices u and z.
    i. Call DFS(G, u) with u as the start vertex.
    ii. Use a stack S to keep track of the path between the start vertex and the current vertex.
    iii. As soon as destination vertex z is encountered, return the path as the contents of the stack

Properties

- In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.
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Topological sort

- A topological sort of a DAG G = (V, E) is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.
- A topological sort of a graph can be viewed as an ordering of its vertices along a horizontal line so that all directed edges go from left to right.
- If the graph is not acyclic, then no linear ordering is possible.

TOPOLOGICAL-SORT (G)

1. **call DFS(G) to compute finishing times u.f for each vertex u**
2. **As each vertex is finished, insert it onto the front of a linked list**
3. **Return the linked list of vertices**

Topological sort