Combinatorics

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1 Counting Methods

1.1 Counting Principles

Theorem 1.1 (Addition Principle). Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets, so that

$$A_i \cap A_j = \varnothing \quad \text{for } i \neq j.$$

Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|. \tag{1.1.1}$$

Proof. Since the sets $A_1, \ldots, A_n$ are pairwise disjoint, each element of $\bigcup_{i=1}^{n} A_i$ belongs to exactly one of the sets A_i. Thus no element is counted more than once when summing their cardinalities. Therefore,

$$\left|\bigcup_{i=1}^{n} A_i\right| = |A_1| + |A_2| + \cdots + |A_n|.$$

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If the sets $A_1, \ldots, A_n$ are not pairwise disjoint, the formula $|\bigcup_i A_i| = \sum_i |A_i|$ can fail because overlapping elements would be counted twice. However, modification to the principle via compensation of the over-count corrects such issues:

$$|A| + |B| - |A \cap B|$$

Theorem 1.2 (Multiplication Principle). Let an experiment consist of k stages, where stage i admits n_i possible outcomes for each outcome of the previous stages. Then the total number of possible outcomes of the experiment is

$$n_1 n_2 \cdots n_k.$$

Proof. Let S_i denote the finite set of possible outcomes at stage i. An outcome of the experiment is determined by choosing one element from each S_i, so the sample space is the Cartesian product

$$S = S_1 \times S_2 \times \cdots \times S_k.$$

By the definition of Cartesian product,

$$|S| = |S_1|\,|S_2| \cdots |S_k| = n_1 n_2 \cdots n_k.$$

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In general we may depict the multiplication principle through a choice tree that grows exponentially.

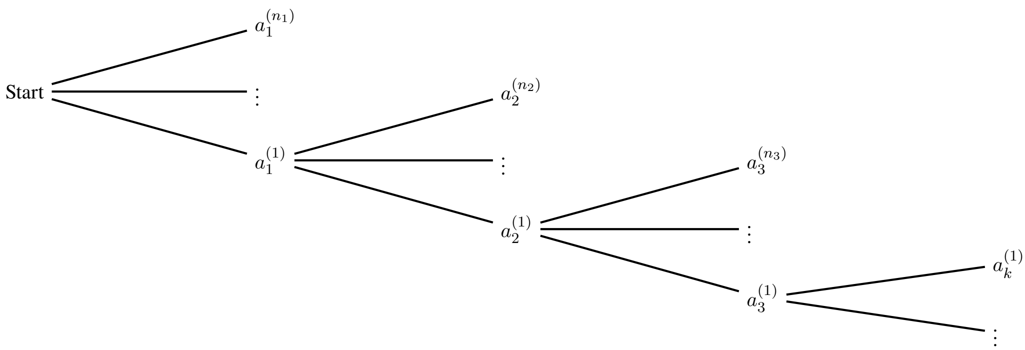


Figure 1: The Choice Tree

1.2 Permutations

Consider a set $\{1, 2, ..., n\}$ of n distinct elements. Then a permutation of $\{1, 2, ..., n\}$ is a bijection

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

Definition 1.1. The **symmetric group** S_n is a set containing all such permutations of a set $\{1, 2, ..., n\}$.

In cycle notation we commonly see permutations of S_n as (12),(23), etc. For our purposes, we'll each element will appear as an ordered list. That is,

$$f(1)f(2)f(3)f(4) = (12)$$

where $f(1) = 2$, $f(2) = 1$, and $f(3) = 3, f(4) = 4$.

Proposition 1.1. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. Then

$$|S_n| = n!$$

Proof. By definition, a permutation is a bijection

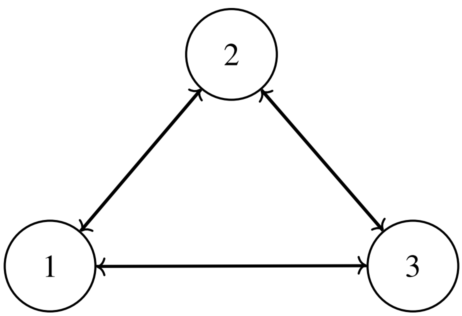
$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

We construct a bijection by specifying the values of $f(1), f(2), ..., f(n)$ sequentially. There are n possible choices for $f(1)$. Once $f(1)$ is chosen, by way of injectivity, any subsequent selection must have $n - k$ possible choices for $f(k + 1)$, and for the final selection of $f(n)$ there exists exactly one element to choose from. Then, by the multiplication rule, the total number of such sequences of choices is

$$n \cdot (n - 1) \cdot (n - 2) \cdots = n!$$

Thus, $|S_n| = n!$. ■

Example 1.1. Take a triangle D_n does though, with 3 distinct vertices, each labeled 1,2,3 respectively. The symmetric group S_3 of all permutations of the triangle will have 3! total permutations.



Example 1.2. Take a 52 card deck with no jokers. Shuffle the deck randomly. The total number of possible permutations of the deck is exactly $52!$.

Up to this point, we have considered permutations of all n elements. We now ask how many permutations can be formed using only k of the n distinct elements. Rather than permuting all n distinct elements, we may instead consider permutations that involve only k elements of the set.

Definition 1.2 (Partial Permutation). The number of ways to choose and order k elements from n is

$$P(n,k) = \frac{n!}{(n-k)!} \tag{1.2.1}$$

Example 1.3. Consider a race of 10 total individuals. Determine the possible outcomes for the top 3 finishers. By the multiplication principle, we have $10 \cdot 9 \cdot 8$ selections for 3 stages, so we can write it as

$$\frac{10!}{(10-3)!} = \frac{(10)(9)(8)(7)\cdots(1)}{(7)\cdots(1)} = (10)(9)(8)$$

1.3 Combinations

Previously, for permutations we specifically cared about the ordering of the selections and distinguished them that way. Now, we look at the case for when we *do not* consider the order. We derive the formula for $\binom{n}{k}$ by separating the notions of *order* and *choice*. Return to the original partial permutation formulation

$$P(n,k) = \frac{n!}{(n-k)!}$$

Each unordered k-element subset is counted once for every permutation of its elements. Since a set of k elements has $k!$ orderings, we divide by $k!$ to correct for overcounting/collapse all orderings to one count.

Definition 1.3. For integers n, k with $0 \leq k \leq n$, the **binomial coefficient** is defined by the collection of all k-element subsets of a given set.

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}. \tag{1.3.1}$$

Example 1.4. Consider the set $\{A, B, C\}$ and suppose we want to choose 2 letters, where order does not matter. The ordered selections of 2 distinct letters will look like:

$$\begin{gathered}(A, B), (B, A)\\(A, C), (C, A)\\(B, C), (C, B)\end{gathered}$$

There are exactly 6 pairs. Notice that for each selection of 2 letters we have 2! permutations. Now using 2! to collapse ordered pairs to a subset

$$\begin{gathered}(A, B), (B, A) \to \{A, B\}\\(A, C), (C, A) \to \{A, C\}\\(B, C), (C, B) \to \{B, C\}\end{gathered}$$

so only 3 distinct subsets remain.

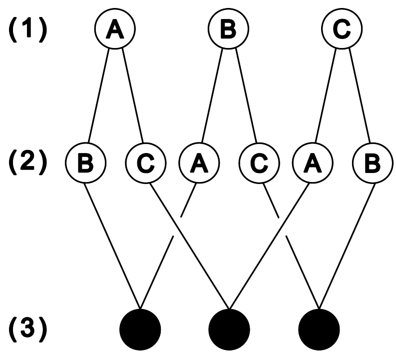


Figure 2: (1) Initial selection (2) Results after permutations decided (3) After collapsing redundancies

Example 1.5. Roll a six sided dice 10 total times. The total outcomes where two 1s are present is $\binom{10}{2}$. But now consider if we require that two 1s are present and consecutive in the order of the rolls. We treat the two 1s as a block. The total unique block slots out of 10 slots is 7. Then the outcomes with this condition is now $\binom{7}{2}$.

Theorem 1.3 (Identities). Binomial coefficients satisfy the following identities,

i) $\binom{n}{k} = \binom{n}{n-k}$

ii) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

iii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

Proof.

i) Let S be a set with $|S| = n$. Choosing a subset of S with k elements is equivalent to choosing which $n-k$ elements are excluded. Hence there is a bijection between k-element subsets and $(n-k)$-element subsets of S, so

$$\binom{n}{k} = \binom{n}{n-k}.$$

ii) Let S be a set with $|S| = n$ and fix an element $x \in S$. Every k-element subset of S either contains x or does not contain x. If it does not contain x, the k elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k}$ possibilities. If it does contain x, the remaining $k-1$ elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k-1}$ possibilities. Since these two cases are disjoint and exhaustive, we obtain

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

iii) Let S be a set with $|S| = n$. The number of subsets of S with exactly k elements is $\binom{n}{k}$. Summing over all possible values of k counts all subsets of S:

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Alternatively, each element of S may be either included or excluded from a subset, giving 2 choices per element and hence 2^n subsets in total. Therefore,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

■

1.4 Inclusion and Exclusion Principles

Earlier, in section 1 we discussed counting principles under the assumption that sets were explicitly disjoint. Such conditions created scenarios for which *over-counting* wasn't possible.

Theorem 1.4 (Inclusion-Exclusion Principle). Let $A_1, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\leq i\leq j\leq n} |A_i \cap A_j| + \cdots + (-1)^{n-1}|A_1 \cap \cdots \cap A_n| \tag{1.4.1}$$

The proof will be left up to exercise. Visually we may represent the principle with a small family first.

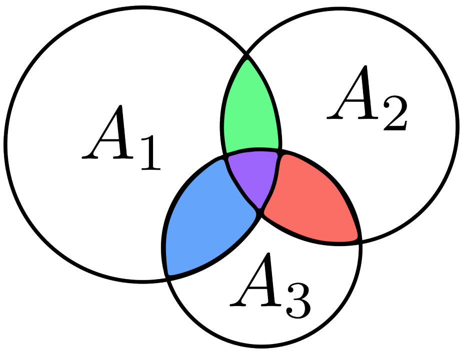


Figure 3: The Inclusion-Exclusion Principle

In Figure 3, all points in each A_i are first counted once. So we add all to the count individually as $|A_i|$. Now consider the individual intersections. The points in each intersection (RGB) are counted again. So we have to remove them through double intersections $|A_i \cap A_j|$. However, during that process we also removed the points in the triple intersection (purple) completely ("uncounted three times"). Thus, we compensate by adding $|A_1 \cap A_2 \cap A_3|$. In the end, we have an accurate count.

Remark 1.1. The Inclusion Exclusion Principle is applicable to both permutations and combinations.

Example 1.6. Consider the number of integers from 1 to 100 are divisible by 2 or 3. Let

$$A = \{n : 2|n\}, \quad B = \{n : 3|n\}$$

Then we have that $|A| = 50$, $|B| = 33$ and $|A \cap B| = 16$. The final count is 67.