Combinatorics

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Contents

1 Combinatorial Methods **3**

1.1 Counting Principles 3

1.2 Permutations 5

1.3 Combinations 7

1.4 Inclusion and Exclusion Principle 10

2 Binomial Coefficients **13**

2.1 The Binomial Theorem 13

2.2 title 13

3 Fibonacci Numbers and Recursion **15**

4 Graph Theory **17**

4.1 Basic Definitions and Properties 17

4.2 Properties of Degree 19

4.3 Graph Structure and Equality 20

4.4 Paths and Connectivity 21

4.5 Eulerian Graphs 23

Chapter 1

Combinatorial Methods

1.1 Counting Principles

Theorem 1 (Addition Principle). Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets, so that

$$A_i \cap A_j = \varnothing \quad \text{for } i \neq j.$$

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|. \tag{1.1}$$

Proof. Since the sets $A_1, \ldots, A_n$ are pairwise disjoint, each element of $\bigcup_{i=1}^{n} A_i$ belongs to exactly one of the sets A_i. Thus no element is counted more than once when summing their cardinalities. Therefore,

$$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1| + |A_2| + \cdots + |A_n|.$$

■

Theorem 2 (Multiplication Principle). Let an experiment consist of k stages, where stage i admits n_i possible outcomes for each outcome of the previous stages. Then the total number of possible outcomes of the experiment is

$$n_1 n_2 \cdots n_k.$$

Proof. Let S_i denote the finite set of possible outcomes at stage i. An outcome of the experiment is determined by choosing one element from each S_i, so the sample space is the Cartesian product

$$S = S_1 \times S_2 \times \cdots \times S_k.$$

By the definition of Cartesian product,

$$|S| = |S_1|\,|S_2| \cdots |S_k| = n_1 n_2 \cdots n_k.$$

■

In general we may depict the multiplication principle through a choice tree that grows exponentially.

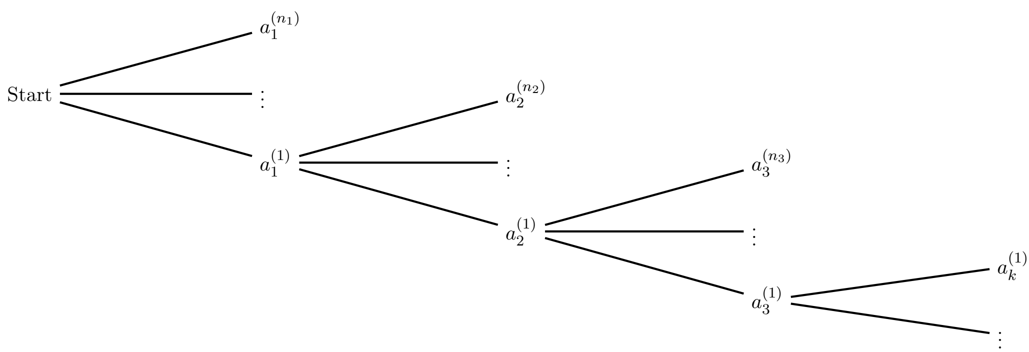


Figure 1.1: The Choice Tree

1.2 Permutations

Definition 1. The **symmetric group** S_n is a set containing all such permutations of a set $\{1, 2, ..., n\}$.

Remark 1. In cycle notation we commonly see permutations of S_n as (12),(23), etc. For our purposes, each element will appear as an ordered list. That is,

$$f(1)f(2)f(3)f(4) = (12)$$

where $f(1) = 2$, $f(2) = 1$, and $f(3) = 3, f(4) = 4$.

Proposition 1. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. Then

$$|S_n| = n!$$

Proof. By definition, a permutation is a bijection

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

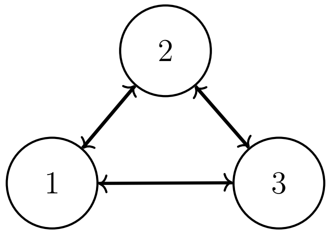
We construct a bijection by specifying the values of $f(1), f(2), ..., f(n)$ sequentially. There are n possible choices for $f(1)$. Once $f(1)$ is chosen, by way of injectivity, any subsequent selection must have $n - k$ possible choices for $f(k + 1)$, and for the final selection of $f(n)$ there exists exactly one element to choose from. Then, by the multiplication rule, the total number of such sequences of choices is

$$n \cdot (n - 1) \cdot (n - 2) \cdots = n!$$

Thus, $|S_n| = n!$. ■

Remark 2. The number of permutations of n objects contains permutations that don't necessarily preserve symmetry. The *Dihedral Group* D_n does.

Example 1. Take a triangle D_n does though, with 3 distinct vertices, each labeled 1,2,3 respectively. The symmetric group S_3 of all permutations of the triangle will have 3! total permutations.



Example 2. Take a 52 card deck with no jokers. Shuffle the deck randomly. The total number of possible permutations of the deck is exactly 52!.

Example 3. Consider the word MISSISSIPPI. If we want to find out how many distinct arrangements of the word there are, then we have to adjust for any over-counting that occurred. For this particular problem, the over-counting stems from the fact that we 3 letters with repeats, meaning in the permutation their internal arrangements will be treated as unique even though they are *not*. So, we have 4 letter S, 3 letter I, and 2 letter P. So the total distinct arrangements is $\frac{11!}{4!3!2!}$.

Definition 2 (Rotational Permutation)**.** For n distinct objects arranged in a circle, there exists

$$(n-1)!$$

total arrangements.

Example 4. Consider 5 couples seated around a round table. There exists $(5-1)! = 4!$ total permutations. Visualize as fixing one individual and permuting the rest.

Definition 3 (Partial Permutation). The number of ways to choose and order k elements from n is

$$P(n,k) = \frac{n!}{(n-k)!} \tag{1.2}$$

Example 5. Consider a race of 10 total individuals. Determine the possible outcomes for the top 3 finishers. By the multiplication principle, we have $10 \cdot 9 \cdot 8$ selections for 3 stages, so we can write it as

$$\frac{10!}{(10-3)!} = \frac{(10)(9)(8)(7)\cdots(1)}{(7)\cdots(1)} = (10)(9)(8)$$

There are also scenarios when we are concerned with arranging elements that come as a pairs. In other words, we arrange blocks and elements together.

Example 6 (Block Counting). Roll a six sided die 8 times. How many total outcomes where two 4s occur consecutively? First, we adjust our scope to reflect the amount of objects we're actually arranging. This generally can be solved as

$$\text{Total units} - \text{Units in the block} + \text{Number of blocks} = \text{New amount of units}$$

For this problem we end up with $8-2+1=7$. So the total selections for where our block can go is 7. For every object that is not the block, we also have 6 possible numbers. So in total, the number of outcomes is

$$7 \cdot 5^6$$

1.3 Combinations

We derive the formula for $\binom{n}{k}$ by separating the notions of <u>order</u> and <u>choice</u>. Return to the original partial permutation formulation

$$P(n,k) = \frac{n!}{(n-k)!}$$

Each unordered k-element subset is counted once for every permutation of its elements. Since a set of k elements has $k!$ orderings, we divide by $k!$ to correct for overcounting/collapse all orderings to one count.

Definition 4. For integers n, k with $0 \le k \le n$, the **binomial coefficient** is defined by the collection of all k-element subsets of a given set.

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}. \tag{1.3}$$

Example 7. Consider the set $\{A, B, C\}$ and suppose we want to choose 2 letters, where order does not matter. The ordered selections of 2 distinct letters will look like:

$$\begin{gathered} (A, B), (B, A) \\ (A, C), (C, A) \\ (B, C), (C, B) \end{gathered}$$

There are exactly 6 pairs. Notice that for each selection of 2 letters we have 2! permutations. Now using 2! to collapse ordered pairs to a subset

$$\begin{gathered} (A, B), (B, A) \to \{A, B\} \\ (A, C), (C, A) \to \{A, C\} \\ (B, C), (C, B) \to \{B, C\} \end{gathered}$$

so only 3 distinct subsets remain.

Example 8. Roll a six sided dice 10 total times. The total outcomes where exactly two 1s are present is $\binom{10}{2} \cdot 5^8$. We came to this conclusion by treating each roll (1-10) as a slot for which the 1s can go. But now consider if we require that two 1s are present and consecutive in the order of the rolls. We treat the two 1s as a block. The total unique block slots can be calculated as $n - k + 1$ which yields 7. Then the outcomes with this condition is now $\binom{7}{2} \cdot 5^8$.

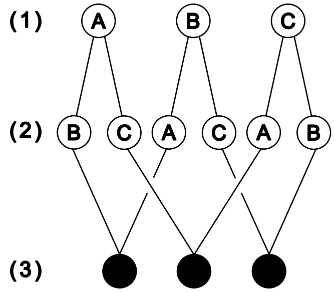


Figure 1.2: (1) Initial selection (2) Results after permutations decided (3) After collapsing redundancies

Example 9. Consider a bus with 30 seats. Suppose 10 passengers enter the bus and we are only interested in which seats are occupied, not which passenger sits in which seat. Treating the passengers as indistinguishable, each seating configuration corresponds to a choice of 10 occupied seats from the 30 available. Hence, the total number of configurations is $\binom{30}{10}$.

Theorem 3 (Identities). Binomial coefficients satisfy the following identities,

i) Symmetry - $\binom{n}{k} = \binom{n}{n-k}$

ii) Pascals Identity - $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

iii) Power Set - $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Proof.

i) Let S be a set with $|S| = n$. Choosing a subset of S with k elements is equivalent to choosing which $n - k$ elements are excluded. Hence there is a bijection between k-element subsets and

$(n-k)$-element subsets of S, so

$$\binom{n}{k} = \binom{n}{n-k}.$$

ii) Let S be a set with $|S| = n$ and fix an element $x \in S$. Every k-element subset of S either contains x or does not contain x. If it does not contain x, the k elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k}$ possibilities. If it does contain x, the remaining $k-1$ elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k-1}$ possibilities. Since these two cases are disjoint and exhaustive, we obtain

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

iii) Let S be a set with $|S| = n$. The number of subsets of S with exactly k elements is $\binom{n}{k}$. Summing over all possible values of k counts all subsets of S:

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Alternatively, each element of S may be either included or excluded from a subset, giving 2 choices per element and hence 2^n subsets in total. Therefore,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

■

1.4 Inclusion and Exclusion Principle

Earlier, in section 1 we discussed counting principles under the assumption that sets were explicitly disjoint. Such conditions created scenarios for which *over-counting* wasn't possible.

Theorem 4 (Inclusion-Exclusion Principle). Let $A_1, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i\le j\le n} |A_i \cap A_j| + \cdots + (-1)^{n-1}|A_1 \cap \cdots \cap A_n| \tag{1.4}$$

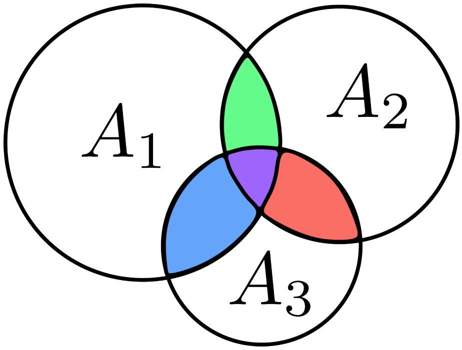


Figure 1.3: The Inclusion-Exclusion Principle

Remark 3. The Inclusion Exclusion Principle is applicable to both permutations and combinations.

The proof will be left up to exercise. Visually we may represent the principle with a small family first.

Example 10. Consider the number of integers from 1 to 100 are divisible by 2 or 3. Let

$$A = \{n : 2|n\}, \quad B = \{n : 3|n\}$$

Then we have that $|A| = 50$, $|B| = 33$ and $|A \cap B| = 16$. The final count is 67.

Example 11. How many functions

$$f : \{1, 2, 3, 4\} \to \{a, b, c\}$$

are surjective? There are 3^4 total functions. For each $x \in \{a, b, c\}$, let

$$A_x = \{\text{functions that omit } x\}.$$

Then

$$|A_x| = 2^4 \quad \text{and} \quad |A_x \cap A_y| = 1^4 \quad (x \neq y).$$

By the Inclusion–Exclusion Principle,

$$\begin{aligned} \text{Number of surjections} &= 3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4 \\ &= 81 - 48 + 3 \\ &= 36. \end{aligned}$$

Chapter 2

Binomial Coefficients

2.1 The Binomial Theorem

2.2 title

Chapter 3

Fibonacci Numbers and Recursion

Chapter 4

Graph Theory

4.1 Basic Definitions and Properties

Definition 5. A ***graph*** $G = (V, E)$ consists of a set of ***vertices*** V and a set of ***edges*** E. If two vertices are connected by an edge we call them ***adjacent***. The number of vertices adjacent to a given vertex is called the ***degree*** of a vertex denoted $\deg(v)$.

Remark 4. Adjacency is not a transitive property i.e $e_1 \circ e_2$ does not induce adjacency.

Definition 6. A graph $H = (V_H, E_H)$ is a ***subgraph*** of $G = (V, E)$ if

$$V_H \subseteq V \text{ and } E_H \subseteq E$$

and every edge in E_H has both vertice endpoints in V_H.

Example 12. Consider the following graph G consisting of vertice set $S = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_4, v_2\}$, $e_4 = \{v_1, v_4\}$, and a subgraph H with vertice set $S = \{v_1, v_2, v_4\}$ and edge set $E = \{e_1, e_4\}$.

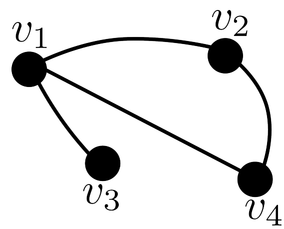


Figure 4.1: Graph G

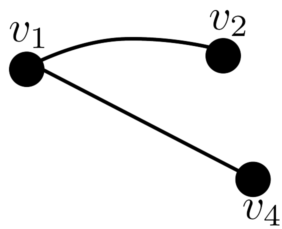


Figure 4.2: Subgraph H

Example 13 (Graph Types). In this next example, we provide a set of diagrams alongside informal definitions of particular graph types.

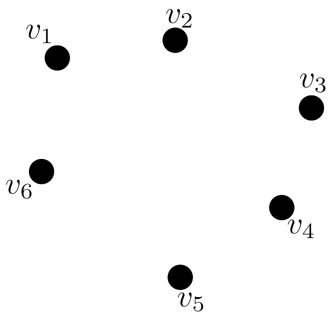


Figure 4.3: Empty Graph

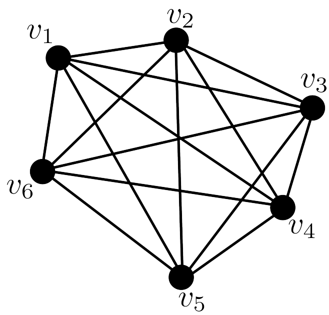


Figure 4.4: Complete Graph

i.e $|E| = \varnothing$ for empty, and $|E| = \binom{n}{2}$ for complete.

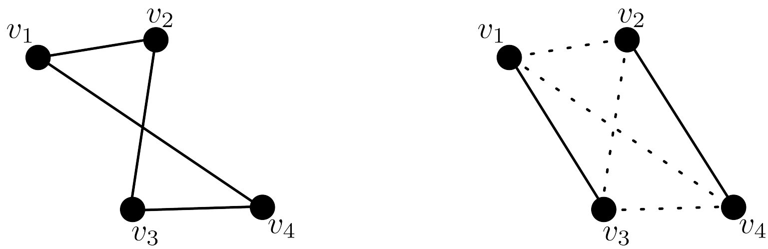


Figure 4.5: Complement Graphs

i.e G^c implies $|E^c|$.

Definition 7. A ***simple graph*** is an unweighted undirected graph containing no loops or multiple edges between two vertices.

Remark 5. All graphs we've seen so far have been simple. More complex graphs do not obey these rules.

4.2 Properties of Degree

Theorem 5. For any graph G, the number of vertices with odd degree is even.

Proof. Let G be a graph. The proof follows by finite induction, modeling a graph being constructed from scratch. The outline is:

1. Begin with graph of vertices with no edges i.e number of even vertices is $|V|$ (all 0 deg). Begin connecting vertices.

2. If both vertices are even degree, we increase the number of odd degree nodes by 2

3. If both vertices are odd degree, we decrease the number of odd degree vertices by 2

4. If one is even and one is odd degree, nothing changes.

■

Remark 6. Think about combinatorically what we're doing. The idea of choosing which becomes even, directly chooses which must be odd!

Theorem 6. The sum of degrees of all vertices in a graph is twice the number of edges

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof. Each edge of an undirected graph is incident to exactly two vertices, so counting vertex–edge incidences by summing degrees counts every edge twice, yielding ∎

Theorem 7 (Havel–Hakimi Algorithm). Let

$$d_1 \geq d_2 \geq \cdots \geq d_n$$

be a nonincreasing sequence of nonnegative integers. Then this sequence is the degree sequence of a finite simple graph if and only if $d_1 \leq n-1$ and the sequence

$$d_2 - 1,\ d_3 - 1,\ \ldots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ \ldots,\ d_n$$

obtained by deleting d_1 and subtracting 1 from the next d_1 terms is itself graphical.

Remark 7. *Graphical* just refers to the property of being able to construct a degree sequence

Proof. c ∎

Example 14. content...

4.3 Graph Structure and Equality

Definition 8. A ***labeled graph*** is a graph G with unique identifiers assigned to each vertice.

Definition 9 (Graph Isomorphism). Two graphs G_1 and G_2 are ***isomorphic*** if there exists a bijection between vertex sets

$$\varphi : V(G) \to V(H)$$

such that

$$\{u, v\} \in E(G) \iff \{\varphi(u), \varphi(v)\} \in E(H)$$

Remark 8 (Relabelings). A relabeling of a graph is a bijective renaming of its vertices that transports each edge to the corresponding edge with renamed endpoints. Hence every relabeling defines a graph isomorphism, and relabeled graphs are structurally identical (i.e., equal up to isomorphism).

Remark 9. Unlabeled graphs correspond to equivalence classes of labeled graphs under relabeling i.e they represent absolute structure in the absence of labelings. Also, any relabeling of a graph is always isomorphic to the original graph.

Example 15. Consider two graphs G and H
We can construct a graph isomorphism one by anchoring one point and determining the edges that must satisfy:

$$\varphi(v_1) = v'_1, \quad \varphi_(v_2) = v'_6, \quad \varphi(v_6) = v'_4, \quad \varphi(v_4) = v'_5, \quad \cdots$$

4.4 Paths and Connectivity

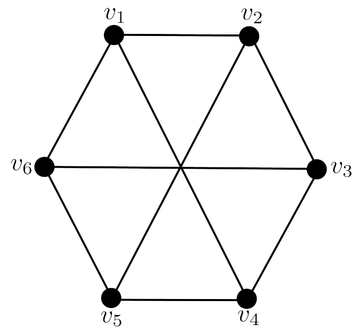


Figure 4.6: Graph G

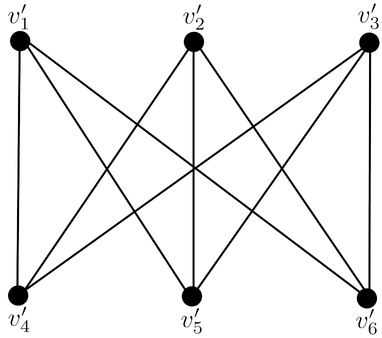


Figure 4.7: Graph H

Definition 10. A ***path*** in G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that for each $i = 0, \ldots, k-1$, we have that $\{v_i, v_{i+1}\} \in E$.

Remark 10. In directed graphs, paths are a symmetric property.

Definition 11. A graph is ***connected*** if for any two vertices v_i and v_j, there exists some path with endpoints v_i and v_j. A ***connected component*** of a graph G is a maximal subgraph that is connected.

Remark 11. Connectivity is an equivalence relation on the vertex set V seen through the relation $v_1 \sim v_2$ if and only if there exists a path from v_1 to v_2.

Remark 12. Vertices with degree 0 are trivial examples of connected components.

Example 16. Consider the following unlabeled graph G
While the graph itself as a whole is not connected because of the singular vertex with degree 0, the three particular subgraphs we've highlighted are connected and therefore are connected componenents of G. In particular, the blue subgraph admits 8 total paths, the red graph

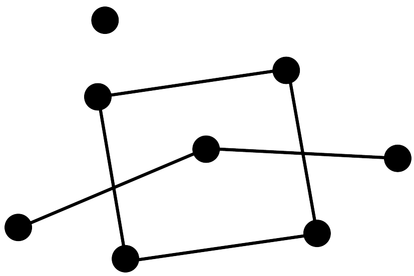


Figure 4.8: Graph G

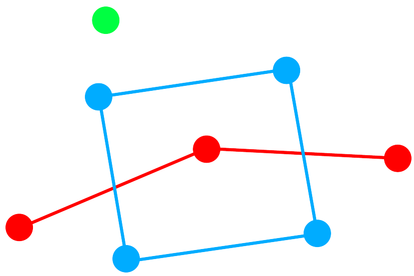


Figure 4.9: Components of G

admits 3 paths, and the green one. Note that these are not the *only* connected components, as there exists *finer* subgraphs of G.

4.5 Eulerian Graphs