CHAPTER ONE

Counting Methods

1.1 Counting Principles

Theorem 1 (Addition Principle)**.** Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets, so that

$$A_i \cap A_j = \varnothing \quad \text{for } i \neq j.$$

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|.$$

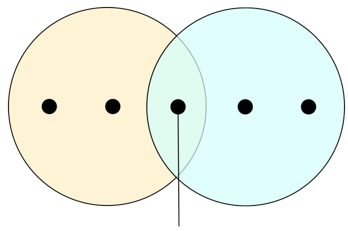


Figure 1.1: Two sets A and B with overlap. In the specific figure, the principle would produce a 6 count when there are only really 5

.

Remark 1. If the sets $A_1, \ldots, A_n$ are not pairwise disjoint, the formula $\left|\bigcup_i A_i\right| = \sum_i |A_i|$ can fail because overlapping elements would be counted twice. Observe figure 1.2 for specific example. However, modification to the principle via compensation of the over-count corrects such issues:

$$|A| + |B| - |A \cap B|$$

Proof. Since the sets $A_1, \ldots, A_n$ are pairwise disjoint, each element of $\bigcup_{i=1}^{n} A_i$ belongs to exactly one of the sets A_i. Thus no element is counted more than once when summing their cardinalities. Therefore,

$$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1| + |A_2| + \cdots + |A_n|.$$

□

Theorem 2 (Multiplication Principle)**.** Let an experiment consist of k stages, where stage i admits n_i possible outcomes for each outcome of the previous stages. Then the total number of possible outcomes of the experiment is

$$n_1 n_2 \cdots n_k.$$

Proof. Let S_i denote the finite set of possible outcomes at stage i. An outcome of the experiment is determined by choosing one element from each S_i, so the sample space is the Cartesian product

$$S = S_1 \times S_2 \times \cdots \times S_k.$$

By the definition of Cartesian product,

$$|S| = |S_1|\,|S_2| \cdots |S_k| = n_1 n_2 \cdots n_k.$$

□

In general, we can depict the multiplication principle by a tree representing selection at each stage.

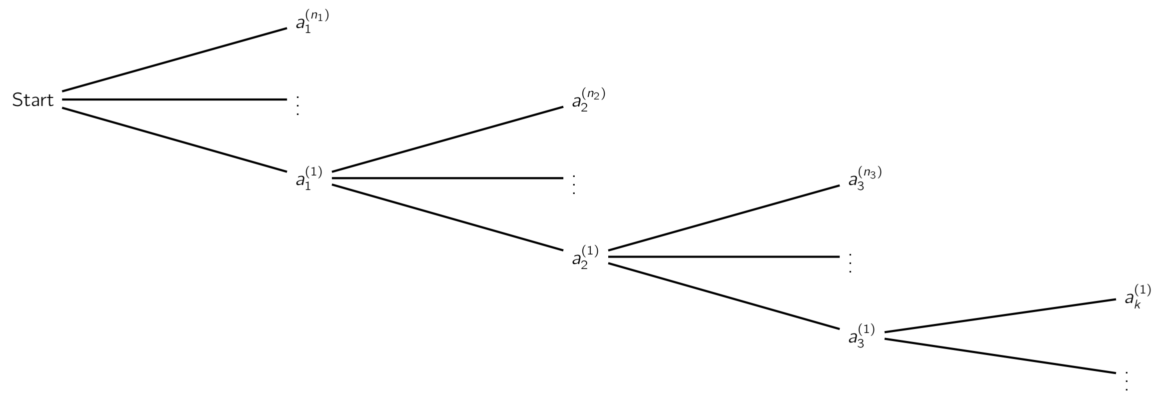


Figure 1.2: A horizontal choice tree illustrating the Multiplication Principle. Each level represents a stage of choice, and each root-to-leaf path corresponds to a unique outcome.

1.2 Permutations

Consider a set $\{1, 2, ..., n\}$ of n distinct elements. Then a permutation of $\{1, 2, ..., n\}$ is a bijection

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

Definition 1. The **symmetric group** S_n is a set containing all such permutations of a set $\{1, 2, ..., n\}$.

Remark 2. In cycle notation we commonly see permutations of S_n as (12),(23), etc. For our purposes, we'll each element will appear as an ordered list. That is,

$$f(1)f(2)f(3)f(4) = (12)$$

where $f(1) = 2$, $f(2) = 1$, and $f(3), f(4) = I_X$

Proposition 1. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. Then

$$|S_n| = n!$$

Note that each permutation is treated as unique

Proof. By definition, a permutation is a bijection

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

We construct a bijection[1] by specifying the values of $f(1), f(2), ..., f(n)$ sequentially. There are n possible choices for $f(1)$. Once $f(1)$ is chosen, by way of injectivity, any subsequent selection must have $n - k$ possible choices for $f(k + 1)$, and for the final selection of $f(n)$ there exists exactly one element to choose from. Then, by the multiplication rule, the total number of such sequences of choices is

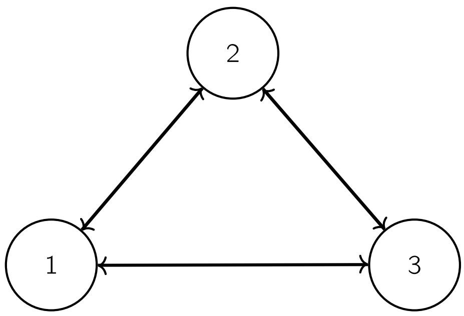
$$n \cdot (n - 1) \cdot (n - 2) \cdots = n!$$

Thus, $|S_n| = n!$. □

[1] Note that, in this proof, we're focusing on the method in which a bijection is constructed because it allows us to directly compute the number of possible permutations i.e the number of $f \in S_n$.

Example 1. Take a triangle [2], with 3 distinct vertices, each labeled 1,2,3 respectively. The symmetric group S_3 of all permutations of the triangle will have 3! total permutations.

[2] Permutations of geometric configurations do not guarantee symmetry after transformation. The *dihedral group* D_n does though.



Example 2. Take a 52 card deck with no jokers. Shuffle the deck randomly. The total number of possible permutations of the deck is exactly 52!.

Up to this point, we have considered permutations of all n elements. We now ask how many permutations can be formed using only k of the n distinct elements. Rather than permuting all n distinct elements, we may instead consider permutations that involve only k elements of the set.

Definition 2 (Partial Permutation)**.** The number of ways to choose and order k elements from n is

$$P(n, k) = \frac{n!}{(n - k)!}$$

Example 3. Consider a race of 10 total individuals. Determine the possible outcomes for the top 3 finishers. By the multiplication principle, we have $10 \cdot 9 \cdot 8$ selections for 3 stages, so we can write it as

$$\frac{10!}{(10 - 3)!} = \frac{(10)(9)(8)(7) \cdots (1)}{(7) \cdots (1)} = (10)(9)(8)$$

1.3 Combinations

Previously, for permutations we specifically cared about the ordering of the selections and distinguished them that way. Now, we look at the case for when we *do not* consider the order. We derive the formula for $\binom{n}{k}$ by separating the notions of *order* and *choice*. Return to the original partial permutation formulation

$$P(n,k) = \frac{n!}{(n-k)!}$$

Key idea: count something *with* order, then divide the order out.

Each unordered k-element subset is counted once for every permutation of its elements. Since a set of k elements has $k!$ orderings, we divide by $k!$ to correct for overcounting/collapse all orderings to one count.

Definition 3. For integers n, k with $0 \leq k \leq n$, the **binomial coefficient**

$$\binom{n}{k}$$

is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

Another useful way to interpret combinations is as the *cardinality* of the collection of all k-element subsets of a given set.

Example 4. Consider the set $\{A, B, C\}$ and suppose we want to choose 2 letters, where order does not matter. The ordered selections of 2 distinct letters will look like:

$$(A,B),(B,A)$$
$$(A,C),(C,A)$$
$$(B,C),(C,B)$$

There are exactly 6 pairs. Notice that for each selection of 2 letters we have 2! permutations. Now using 2! to collapse ordered pairs to a subset

$$(A,B),(B,A) \to \{A,B\}$$
$$(A,C),(C,A) \to \{A,C\}$$
$$(B,C),(C,B) \to \{B,C\}$$

so only 3 distinct subsets remain.

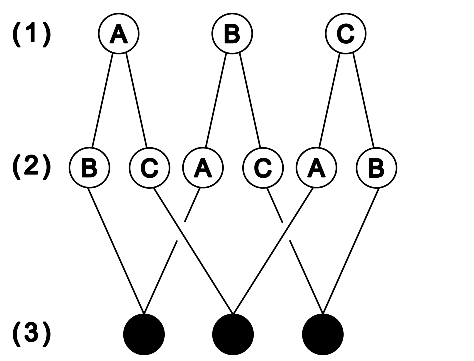


Figure 1.3: Illustration of Example 4. (1) being initial selection. (2) being results after permutations decided. (3) After collapsing redundancies.

Theorem 3 (Identities)**.** Binomial coefficients satisfy the following identities,

i) $\binom{n}{k} = \binom{n}{n-k}$

ii) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

iii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

Note that part (iii) can be viewed as a count of all subsets of all sizes. Take $X = \{1,2\}$ for example. Each binomial coefficient corresponds as the following

$$\binom{2}{0} \to \varnothing$$
$$\binom{2}{1} \to \{1\}, \{2\}$$
$$\binom{2}{2} \to \{1,2\}$$

Proof.

i) Let S be a set with $|S| = n$. Choosing a subset of S with k elements is equivalent to choosing which $n-k$ elements are excluded. Hence

there is a bijection between k-element subsets and $(n-k)$-element subsets of S, so
$$\binom{n}{k} = \binom{n}{n-k}.$$

ii) Let S be a set with $|S| = n$ and fix an element $x \in S$. Every k-element subset of S either contains x or does not contain x. If it does not contain x, the k elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k}$ possibilities. If it does contain x, the remaining $k-1$ elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k-1}$ possibilities. Since these two cases are disjoint and exhaustive, we obtain
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

iii) Let S be a set with $|S| = n$. The number of subsets of S with exactly k elements is $\binom{n}{k}$. Summing over all possible values of k counts all subsets of S:
$$\sum_{k=0}^{n} \binom{n}{k}.$$
Alternatively, each element of S may be either included or excluded from a subset, giving 2 choices per element and hence 2^n subsets in total. Therefore,
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

□

1.4 Inclusion and Exclusion Principles

In the first section, we specifically examined pairwise disjoint sets to ensure proper behavior of counting. Now we expand on the addition principle to examine less ideal families of sets.

Theorem 4 (Inclusion-Exclusion Principle)**.** Let $A_1, \ldots, A_n$ be finite sets. Then
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq \mathcal{I} \subseteq \{1,2,\ldots,n\}} (-1)^{|\mathcal{I}|+1} \left|\bigcap_{i \in \mathcal{I}} A_i\right|.$$

Remark 3. The expanded version of the formula above;
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof. We'll use mathematical induction. Assume that A_i are non-pairwise disjoint sets. First, considering the trivial/base case for $n = 2$, we know from the addition principle [3]
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A \cap B|$$

□

[3] Also verify that this is true through directly plugging $n = 2$ into equation.

CHAPTER TWO

Binomial Coefficients and Identities

2.1 Pascal's Triangle

2.2 Binomial Theorem

2.3 Combinatorial Identities