Combinatorics

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1 Combinatorial Methods

1.1 Counting Principles

Theorem 1.1 (Addition Principle). Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets, so that

$$A_i \cap A_j = \varnothing \quad \text{for } i \neq j.$$

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|. \tag{1.1.1}$$

Proof. Since the sets $A_1, \ldots, A_n$ are pairwise disjoint, each element of $\bigcup_{i=1}^{n} A_i$ belongs to exactly one of the sets A_i. Thus no element is counted more than once when summing their cardinalities. Therefore,

$$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1| + |A_2| + \cdots + |A_n|.$$

■

If the sets $A_1, \ldots, A_n$ are not pairwise disjoint, the formula $|\bigcup_i A_i| = \sum_i |A_i|$ can fail because overlapping elements would be counted twice. However, modification to the principle via compensation of the over-count corrects such issues:

$$|A| + |B| - |A \cap B|$$

Theorem 1.2 (Multiplication Principle). Let an experiment consist of k stages, where stage i admits n_i possible outcomes for each outcome of the previous stages. Then the total number of possible outcomes of the experiment is

$$n_1 n_2 \cdots n_k.$$

Proof. Let S_i denote the finite set of possible outcomes at stage i. An outcome of the experiment is determined by choosing one element from each S_i, so the sample space is the Cartesian product

$$S = S_1 \times S_2 \times \cdots \times S_k.$$

By the definition of Cartesian product,

$$|S| = |S_1|\,|S_2| \cdots |S_k| = n_1 n_2 \cdots n_k.$$

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In general we may depict the multiplication principle through a choice tree that grows exponentially.

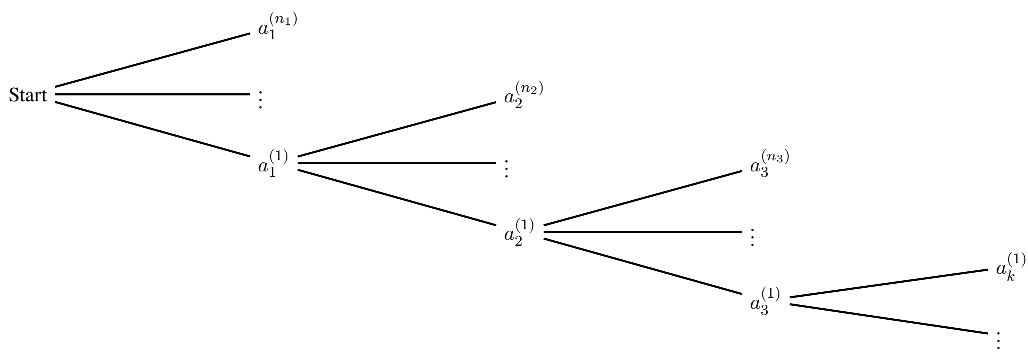


Figure 1: The Choice Tree

1.2 Permutations

Consider a set $\{1, 2, ..., n\}$ of n distinct elements. Then we define a *permutation* of $\{1, 2, ..., n\}$ as a bijection

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

Definition 1.1. The **symmetric group** S_n is a set containing all such permutations of a set $\{1, 2, ..., n\}$.

In cycle notation we commonly see permutations of S_n as (12),(23), etc. For our purposes, we'll each element will appear as an ordered list. That is,

$$f(1)f(2)f(3)f(4) = (12)$$

where $f(1) = 2$, $f(2) = 1$, and $f(3) = 3$, $f(4) = 4$.

Proposition 1.1. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. Then

$$|S_n| = n!$$

Proof. By definition, a permutation is a bijection

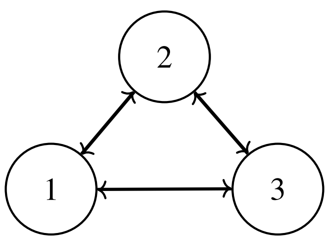
$$f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

We construct a bijection by specifying the values of $f(1), f(2), ..., f(n)$ sequentially. There are n possible choices for $f(1)$. Once $f(1)$ is chosen, by way of injectivity, any subsequent selection must have $n - k$ possible choices for $f(k + 1)$, and for the final selection of $f(n)$ there exists exactly one element to choose from. Then, by the multiplication rule, the total number of such sequences of choices is

$$n \cdot (n - 1) \cdot (n - 2) \cdots = n!$$

Thus, $|S_n| = n!$. ■

Example 1.1. Take a triangle D_n does though, with 3 distinct vertices, each labeled 1,2,3 respectively. The symmetric group S_3 of all permutations of the triangle will have $3!$ total permutations.



Example 1.2. Take a 52 card deck with no jokers. Shuffle the deck randomly. The total number of possible permutations of the deck is exactly 52!.

Up to this point, we have considered permutations of all n elements. We now ask how many permutations can be formed using only k of the n distinct elements. Rather than permuting all n distinct elements, we may instead consider permutations that involve only k elements of the set.

Definition 1.2 (Partial Permutation). The number of ways to choose and order k elements from n is

$$P(n,k) = \frac{n!}{(n-k)!} \tag{1.2.1}$$

Example 1.3. Consider a race of 10 total individuals. Determine the possible outcomes for the top 3 finishers. By the multiplication principle, we have $10 \cdot 9 \cdot 8$ selections for 3 stages, so we can write it as

$$\frac{10!}{(10-3)!} = \frac{(10)(9)(8)(7)\cdots(1)}{(7)\cdots(1)} = (10)(9)(8)$$

1.3 Combinations

Previously, for permutations we specifically cared about the ordering of the selections and distinguished them that way. Now, we look at the case for when we *do not* consider the order. We derive the formula for $\binom{n}{k}$ by separating the notions of *order* and *choice*. Return to the original partial permutation formulation

$$P(n,k) = \frac{n!}{(n-k)!}$$

Each unordered k-element subset is counted once for every permutation of its elements. Since a set of k elements has $k!$ orderings, we divide by $k!$ to correct for overcounting/collapse all orderings to one count.

Definition 1.3. For integers n, k with $0 \leq k \leq n$, the **binomial coefficient** is defined by the collection of all k-element subsets of a given set.

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}. \tag{1.3.1}$$

Example 1.4. Consider the set $\{A, B, C\}$ and suppose we want to choose 2 letters, where order does not matter. The ordered selections of 2 distinct letters will look like:

$$(A, B), (B, A)$$
$$(A, C), (C, A)$$
$$(B, C), (C, B)$$

There are exactly 6 pairs. Notice that for each selection of 2 letters we have 2! permutations. Now using 2! to collapse ordered pairs to a subset

$$(A, B), (B, A) \to \{A, B\}$$
$$(A, C), (C, A) \to \{A, C\}$$
$$(B, C), (C, B) \to \{B, C\}$$

so only 3 distinct subsets remain.

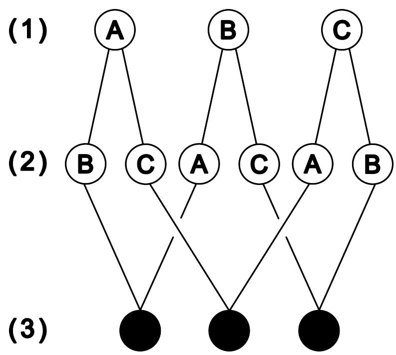


Figure 2: (1) Initial selection (2) Results after permutations decided (3) After collapsing redundancies

Example 1.5. Roll a six sided dice 10 total times. The total outcomes where exactly two 1s are present is $\binom{10}{2} \cdot 5^8$. We came to this conclusion by treating each roll (1-10) as a slot for which the 1s can go. But now consider if we require that two 1s are present and consecutive in the order of the rolls. We treat the two 1s as a block. The total unique block slots can be calculated as $n - k + 1$ which yields 7. Then the outcomes with this condition is now $\binom{7}{2} \cdot 5^8$.

Theorem 1.3 (Identities). Binomial coefficients satisfy the following identities,

i) $\binom{n}{k} = \binom{n}{n-k}$

ii) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

iii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

Proof.

i) Let S be a set with $|S| = n$. Choosing a subset of S with k elements is equivalent to choosing which $n-k$ elements are excluded. Hence there is a bijection between k-element subsets and $(n-k)$-element subsets of S, so

$$\binom{n}{k} = \binom{n}{n-k}.$$

ii) Let S be a set with $|S| = n$ and fix an element $x \in S$. Every k-element subset of S either contains x or does not contain x. If it does not contain x, the k elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k}$ possibilities. If it does contain x, the remaining $k-1$ elements are chosen from the remaining $n-1$ elements, giving $\binom{n-1}{k-1}$ possibilities. Since these two cases are disjoint and exhaustive, we obtain

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

iii) Let S be a set with $|S| = n$. The number of subsets of S with exactly k elements is $\binom{n}{k}$. Summing over all possible values of k counts all subsets of S:

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Alternatively, each element of S may be either included or excluded from a subset, giving 2 choices per element and hence 2^n subsets in total. Therefore,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

■

1.4 Inclusion and Exclusion Principles

Earlier, in section 1 we discussed counting principles under the assumption that sets were explicitly disjoint. Such conditions created scenarios for which *over-counting* wasn't possible.

Theorem 1.4 (Inclusion-Exclusion Principle). Let $A_1, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| \tag{1.4.1}$$

Remark 1.1. The Inclusion Exclusion Principle is applicable to both permutations and combinations.

The proof will be left up to exercise. Visually we may represent the principle with a small family first.

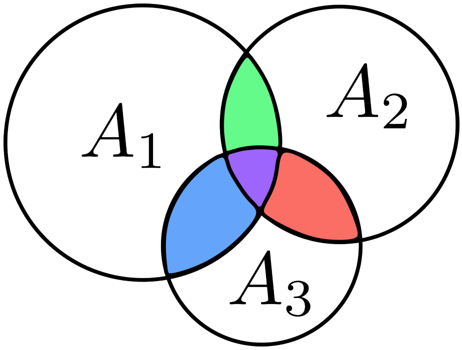


Figure 3: The Inclusion-Exclusion Principle

In Figure 3, all points in each A_i are first counted once. So we add all to the count individually as $|A_i|$. Now consider the double intersections. The points in each intersection (RGB) are counted again. So we have to remove them through double intersections $|A_i \cap A_j|$. However, during that process we also removed the points in the triple intersection (purple) completely ("uncounted three times"). Thus, we compensate by adding $|A_1 \cap A_2 \cap A_3|$. In the end, we have an accurate count.

Example 1.6. Consider the number of integers from 1 to 100 are divisible by 2 or 3. Let

$$A = \{n : 2|n\}\,, \quad B = \{n : 3|n\}$$

Then we have that $|A| = 50$, $|B| = 33$ and $|A \cap B| = 16$. The final count is 67.

Example 1.7. How many functions

$$f : \{1, 2, 3, 4\} \to \{a, b, c\}$$

are surjective? There are 3^4 total functions. For each $x \in \{a, b, c\}$, let

$$A_x = \{\text{functions that omit } x\}.$$

Then

$$|A_x| = 2^4 \quad \text{and} \quad |A_x \cap A_y| = 1^4 \quad (x \neq y).$$

By the Inclusion–Exclusion Principle,

$$\begin{aligned}\text{Number of surjections} &= 3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4 \\ &= 81 - 48 + 3 \\ &= 36.\end{aligned}$$

2 Binomial Coefficients and Pascals Triangle

2.1 Binomial Theorem