Classical Probability

Let Ω be a finite sample space with equally likely outcomes.

1. **Classical probability**
   $P(A) = \dfrac{|A|}{|\Omega|}$
2. **Complement**
   $P(A) = 1 - P(A^c)$
3. **Union of two events**
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. **Union of three events**
   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
5. **Mutually exclusive events**
   $A \cap B = \varnothing \ \Rightarrow \ P(A \cup B) = P(A) + P(B)$
6. **At least one**
   $P(\text{at least one}) = 1 - P(\text{none})$

Axioms of Probability

Let $P : \mathcal{F} \to [0, 1]$.

1. **Nonnegativity**
   $P(A) \geq 0$
2. **Normalization**
   $P(\Omega) = 1$
3. **Countable additivity**
   If $\{A_i\}$ are pairwise disjoint,
   $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
4. **Empty set**
   $P(\varnothing) = 0$
5. **Monotonicity**
   $A \subseteq B \ \Rightarrow \ P(A) \leq P(B)$
6. **Difference of events**
   $P(A \setminus B) = P(A) - P(A \cap B)$
7. **Union bound**
   $P(A \cup B) \leq P(A) + P(B)$

Conditional Probability

Assume $P(B) > 0$.

1. **Definition**
   $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
2. **Multiplication rule**
   $P(A \cap B) = P(A \mid B)\, P(B)$
3. **Chain rule**
   $P(A_1 \cap \cdots \cap A_n) = P(A_1) \prod_{k=2}^{n} P(A_k \mid A_1 \cap \cdots \cap A_{k-1})$
4. **Independence**
   $A \perp B \iff P(A \cap B) = P(A)P(B)$
5. **Independence via conditional**
   $A \perp B \iff P(A \mid B) = P(A)$
6. **Law of Total Probability**
   If $\{B_i\}$ partitions Ω,
   $P(A) = \sum_i P(A \mid B_i) P(B_i)$
7. **Bayes' Theorem**
   $P(B_j \mid A) = \dfrac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}$

Things to Remember / Common Phrasings

- **Complement shortcut:** “At least one” problems $\Rightarrow P(\text{at least one}) = 1 - P(\text{none})$.
- **Mutually exclusive vs independent:**
  - Mutually exclusive $\implies P(A \cap B) = 0$.
  - Independent $\implies P(A \cap B) = P(A)P(B)$.
  - *Important:* mutually exclusive events cannot be independent (except trivial cases).
- **Counting tips:**
  - Unordered selection: $\binom{n}{k}$.
  - Ordered selection: $n!/(n-k)!$.
  - Permutations with repeated items: $n!/(n_1!n_2!\dots)$.
  - Round table arrangements: $(n-1)!$ total permutations (fix one position to remove rotational symmetry).
- **Conditional probability phrasing:**
  - “Given that ...” $\Rightarrow$ use $P(A \mid B)$.
  - “Probability of ... if ... happens” $\Rightarrow$ multiplication / chain rule: $P(A \cap B) = P(A \mid B)P(B)$.
  - “Partition of cases” $\Rightarrow$ Law of Total Probability.
  - “Update your belief / posterior” $\Rightarrow$ Bayes’ Theorem.
- **Quick checks:**
  - Probabilities always between 0 and 1.
  - Check whether order matters and whether repeats are allowed.
  - For unions: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  - Union bound / Boole’s inequality: $P(A_1 \cup \dots \cup A_n) \leq \sum_i P(A_i)$.
- **Mnemonic phrasings:**
  - “None” $\Rightarrow$ complement.
  - “Exactly one” $\Rightarrow$ sum over mutually exclusive cases.
  - “At least / at most” $\Rightarrow$ usually easier via complement.
  - “Independent” $\Rightarrow$ treat events separately in multiplication.
- **General Reminders**
  - For problems involving inclusion exclusion principle, you can use Binomial Coefficient for determining multiplicity if possible.
  - ”At most one separated = none separated + exactly one separated”
  - For round tables, compute UNITS for outcomes: UNITS = Total people - (number of blocks)(people in blocks) + (number of blocks treated as objects)
  - Consider cases where you may have to compute outcomes - bad outcomes to find good outcomes. i.e Probability group sits together but one group doesn’t.
  - At least k implies use complement : 1- P(¡k) example P(at least 3 sixes) = 1 - P(0 sixes) - P(1 sixes) - P(2 sixes)
  - At most k implies summing the probabilities for all outcomes 0 up to k. ex Flip coin 5 times. P of at most 2 heads: P(at most 2 heads) = P(0 heads)+P(1 heads)+ P(2 heads)