Practice Midterm Questions - Solutions

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1 The Foundations - Logic and Proofs

Sec - 1.1 - Ex. 24

a) If I am to remember to send you the address, then you will have to send me an email message.
b) If you were born in United States, then you are a citizen of this country.
c) If you keep your textbook, then it will be useful reference in your future.
d) If their goaltender plays well, then the Red Wings will win the Stanley cup.
e) If you get he job, then you had the best credentials.
f) If there is a storm, then you had a valid password.
g) If you log on to the server, then you have a valid password.
h) If you do not begin your climb too late, then you will reach the summit.

Sec - 1.1- Ex - 28

a) Converse: If I stay home, then it will snow tonight. Contrapositive: If I do not stay at home, then it will not snow tonight. Inverse: If it does not snow tonight, then I will not stay home.
b) Converse: Whenever I go to the beach, it is a sunny summer day. Contrapositive: Whenever I do not goto the beach, it is not a sunny summer day. Inverse: Whenever it is not a sunny day, I do not go to the beach.
c) Converse: If I sleep until noon, then I stayed up late. Contrapositive: If I do not sleep until noon, then I did not stay up late. Inverse: If I dont stay up late, then I don't sleep until noon.

Sec 1.3- Ex - 30

The conclusion $q \vee r$ will be true in every case except when q and r are both false. But if q and r are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of p or $\neg p$ is false. Thus, in this case the hypothesis $(p \vee q) \wedge (p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false is true, and that completes the argument. (You could also use truth tables here.)

Sec 1.3- Ex - 32

We just need to find an assignment of truth values that makes one of these propositions true and the other false. We can let p be true and the other two variables be false. Then the first statement will be $F \to F$,which is true, but the second will be $F \wedge T$, which is false.

Sec 1.4- Ex - 42

There are many ways to write these, depending on what we use for predicates.
a) Let $A(x)$ be 'User x has access to an electronic mailbox.' Then we have $\forall x A(x)$.
b) Let $A(x, y)$ be 'Group member x can access resource y ,' and let $S(x, y)$ be 'System x is in state y.' Then we have $S(file\ system,\ locked) \to \forall x A(x, system\ mailbox)$.
c) Let $S(x, y)$ be "System x is in state y." Recalling that "only if" indicates a necessary condition, we have $S(firewall, diagnostic) \to S(proxy\ server, diagnostic)$.
d) Let $T(x)$ be "Throughput is atleast x kbps," where the domain of discourse is positive numbers, let $M(x, y)$ be "Resource x is in mode y," and let $S(x, y)$ be "The router x is in state y." Then we have $(T(100) \wedge \neg T(500) \wedge \neg M(proxy\ server, diagnostic)) \to \exists x S(x, normal)$.

Sec 1.5- Ex - 36

In each case we need to specify some predicates and identify the domain of discourse.
a) Let $L(x, y)$ mean that person x has lost y dollars playing the lottery. The original statement is then $\neg\exists x\exists y(y > 1000 \wedge L(x, y))$. Its negation of course is $\exists x\exists y(y > 1000 \wedge L(x, y))$; someone has lost more than 1000 playing the lottery.
b) Let $C(x, y)$ mean that person x has chatted with person y. The given statement is $\exists x\exists y(y \neq x \wedge \forall z(z \neq x \to (z = y \leftrightarrow C(x, z))))$. The negation is therefore $\forall x\forall y(y \neq x \to \exists z(z \neq x \wedge \neg(z = y \leftrightarrow C(x, z))))$. In English, everybody in this class has either chatted with no one else or has chatted with two or more others.
c) Let $E(x, y)$ mean that person x has sent e-mail to person y . The given statement is $\neg\exists x\exists y\exists z(y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w(w \neq x \to (E(x, w) \leftrightarrow (w = y \vee w = z))))$. The negation is obviously $\exists x\exists y\exists z(y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w(w \neq x \to (E(x, w) \leftrightarrow (w = y \vee w = z))))$. In English, some student in this class has sent e-mail to exactly two other students in this class.
d) Let $S(x, y)$ mean that student x has solved exercise y . The statement is $\exists x\forall y S(x, y)$. The negation is $\forall x\exists y\neg S(x, y)$. In English, for every student in this class, there is some exercise that he or she has not solved.
(One could also interpret the given statement as asserting that for every exercise, there exists a student perhaps a different one for each exercise who has solved it. In that case the order of the quantifiers would be reversed. Word order in English sometimes makes for a little ambiguity.)
e) Let $S(x, y)$ mean that student x has solved exercise y , and let $B(y, z)$ mean that exercise y is in section z of the book. The statement is $\neg\exists x\forall z\exists y(B(y, z) \wedge$

$S(x, y))$. The negation is of course $\exists x \forall z \exists y(B(y, z) \wedge S(x, y))$. In English, some student has solved at least one exercise in every section of this book.

Sec 1.5- Ex - 48

We need to show that each of these propositions implies the other. Suppose that $\forall x P(x) \vee \forall x Q(x)$ is true. We want to show that $\forall x \forall y(P(x) \vee Q(y))$ is true. By our hypothesis, one of two things must be true. Either P is universally true, or Q is universally true. In the first case, $\forall x \forall y(P(x) \vee Q(y))$ is true, since the first expression in the disjunction is true, no matter what x and y are; and in the second case, $\forall x \forall y(P(x) \vee Q(y))$ is also true, since now the second expression in the disjunction is true, no matter what x and y are. Next we need to prove the converse. So suppose that $\forall x \forall y(P(x) \vee Q(y))$ is true. We want to show that $\forall x P(x) \vee \forall x Q(x)$ is true. If $\forall x P(x)$ is true, then we are done. Otherwise, $P(x_0)$ must be false for some x_0 in the domain of discourse. For this x_0 , then, the hypothesis tells us that $P(x_0) \vee Q(y)$ is true, no matter what y is. Since $P(x_0)$ is false, it must be the case that $Q(y)$ is true for each y . In other words, $\forall y Q(y)$ is true, or, to change the name of the meaningless quantified variable, $\forall x Q(x)$ is true. This certainly implies that $\forall x P(x) \vee \forall x Q(x)$ is true, as desired.

Sec 1.6 - Ex 24

Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

Problems 9.6, 9.8

- Solutions are in the tutorial itself.

Section 1.7 Ex - 38.

We must find a number that cannot be written as the sum of the squares of three integers. We claim that 7 is such a number (in fact, it is the smallest such number). The only squares that can be used to contribute to the sum are 0, 1, and 4. We cannot use two 4s, because their sum exceeds 7. Therefore we can use at most one 4, which means that we must get 3 using just 0s and 1s. Clearly three 1s are required for this, bringing the total number of squares used to four. Thus 7 cannot be written as the sum of three squares.

Section 1.7 - Ex - 42

We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k + 1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \rightarrow (ii)$ we see that $1 - n = 1 - (2k + 1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1 - n = 1 - 2m = 2(-m) + 1$, so $1 - n$ would be odd, and this completes the proof by contraposition that $(ii) \rightarrow (v)$. For $(v) \rightarrow (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \rightarrow (v)$. Finally, for $(v) \rightarrow (iv)$, we see that $n^2 + 1 = (2k+1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2+1 = 2(2m^2)+1$, so $n^2 + 1$ would be odd, and this completes the proof by contraposition that (iv) $\rightarrow$ (v).

Section 1.8 - Ex - 6

Because x and y are of opposite parities, we can assume, without loss of generality, that x is even and y is odd. This tells us that $x = 2m$ for some integer m and $y = 2n + 1$ for some integer n. Then $5x + 5y = 5(2m) + 5(2n + 1) = 10m + 10n + 1 = 10(m + n) + 1 = 2 \cdot 5(m + n) + 1$, which satisfies the definition of being an odd number.

Section 1.8 - Ex - 18

Given r , let a be the closest integer to r less than r , and let b be the closest integer to r greater than r . In the notation to be introduced in Section 2.3, $a = \lfloor r \rfloor$ and $b = \lceil r \rceil$. In fact, $b = a + 1$. Clearly, the distance between r and any integer other than a or b is greater than 1 so cannot be less than 1/2. Furthermore, since r is irrational, it cannot be exactly half-way between a and b, so exactly one of $r - a < 1/2$ and $b - r < 1/2$ holds.

Section 2.1 - Ex - 10

1. T
2. T
3. F
4. T
5. T
6. T
7. F

2 Basic Structures - Sets, Functions, Sequences, Sums and Matrices.

Section 2.1 - Ex - 26

We need to show that every element of $A \times B$ is also an element of $C \times D$. By definition, a typical element of $A \times B$ is pair (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$, we know that $a \in C$; similarly, $b \in D$. Therefore, $(a, b) \in C \times D$.

Section 2.2 - Ex - 4

Note that $A \subseteq B$.
a) $\{a, b, c, d, e, f, g, h\} = B$
b) $\{a, b, c, d, e\} = A$
c) $\emptyset$
d) $\{f, g, h\}$

Section 2.2 - Ex - 36

There are precisely two ways that an item can be in either A or B but not both. it can be either in A but not in B (which is equivalent to saying that is in $A - B$), or it can be in B not in A (which is equivalent to saying that it is in $B - A$). Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.

Section 2.3 - Ex - 10

a) This is one-to-one.
b) This is not one-to-one, since b is the image of both a and b.
c) This not one-to-one, since d is the image of both a and d.

Section 2.3 - Ex - 14

a) This is clearly onto, since $f(0 - n) = f(n)$ for every n.
b) This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m + n)(m - n) = 2$, them m and n must have the same parity (both even or odd). In either case, both m+n and m-n should be divisible 2, so the expression is divisible by 4 and hence cannot be equal to 2.
c) This is clearly onto, since $f(0, n - 1) = n$ for every integer n.
d) This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.
e) This is not onto, for the same reason as (b). In fact, the range here is clearly a subset of the range in that part.

Section 2.3 - Ex - 26

a) Let $f : R \to R$ be the given function. We are told that $f(x_1) < f(x_2)$. We need to show that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. This follows immediately from the given conditions, because without loss of generality, we may assume that $x_1 < x_2$.
b) We need to make the function increasing, but not strictly increasing, so, for example we could take the trivial function $f(x) = 12$. If we want the range to be all R, we could define f in parts this way: $f(x) = x$ for $x < 0$; $f(x) = 0$ for $0 \leq x \leq 1$; and $f(x) = x - 1$ for $x > 1$.

Section 2.4 - Ex - 10

(For verification purposes, I have only written down the final term.)
a) $a_5 = 32$
b) $a_5 = 3$
c) $a_5 = 3^{31}$
d) $a_5 = 74$
e) $a_5 = 1$

Section 2.4 - Ex - 26

a) $T(n) = n^2 + 2$, 123, 146, 171,
b) $T(n) = 4n + 3$, 47, 51, 55.
c) $T(n) =$ binary expansion of n. 1100, 1101, 1110.
d) 8,8,8.
e) $T(n) = 3^n - 1$, 59048, 177146, 531440.
f) $T(n) = (2n - 1)!$, 654729075, ...
g) 0,0,0.
h) $T(n) = T(n-1)^2$

Section 2.4 - Ex - 34

a) 3, b) 78, c) 9, d) 180. (Explicitly evaluate these sums.)

3 Induction and Recursion

Section 5.1 - Ex - 5

Solution should be pretty straight forward and should be available in the student manual. If you still need help with this send me an email.

Section 5.1 - Ex - 34

The statement is true for the base case n =0, since $6|0$. Suppose that $6|(k^3+2k)$. We must show that $6|((k+1)^3 + 2(k+1))$. If we expand the expression in the question, we obtain $(k^3+2k)+3(k^2+k+1)$. By inductive hypothesis, 3 divides k^3+2k, and certainly 3 divides $3(k^2+k+1)$, so 3 divides their sum, and we are done.

Section 5.2 - Ex - 4

This is a graded homework problem. Solution to this problem will be posted in the homework solutions after the submission due date.

-Good Luck -