HW7 - S16

Handed out 5-17 | Chris Troiano | 5-24, beg. of class

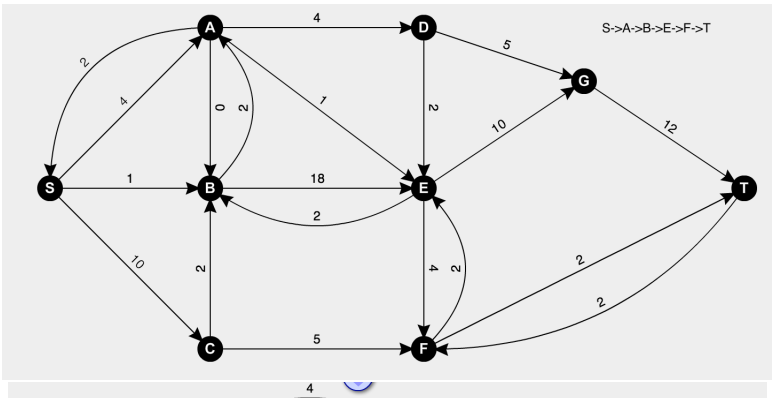
1. Consider the coin changing problem: Given an unlimited supply of coins of denominations $x_1, x_2, \ldots, x_n$, we wish to make change for a value v using minimum number of coins; that is, we wish to find a smallest set of coins whose total value is v. Define a graph over the $n \times v$ grid (plus possibly some vertices around the edges) s.t. the correct coin set corresponds to the shortest path in this graph.

   - Recall the meaning of the table entries and the recurrence.
   - Clearly describe the condition for the presence of an edge between two vertices of the grid.
   - How should the edges be labeled?
   - How do you find the shortest path?
   - Is this algorithm more efficient than the dynamic programming algorithm?

   Solution: Every node besides the source and sink is an entry from the coin change table. An edge leaves a vertex in two possible directions:

   - if we take the coin that represents the vertex we are currently observing, then an edge goes back towards the second coin we pick. The value of this edge is $v - x_i$ where x_i is the current coin we are observing.
   - if we can't take x_i, then there is an edge leaving x_i to the x_{i-1}node directly above it, the weight of this edge is v because we didn't take any coins yet. Thus using the solution of the smaller coin.

2. For the network given below, find the maximum flow from S to T using the Ford-Fulkerson algorithm. In each iteration give the residual graph, the augmenting path and the updated flow (as done in the slides). Also, show the corresponding minimum cut associated with the maximum flow.
   Solution: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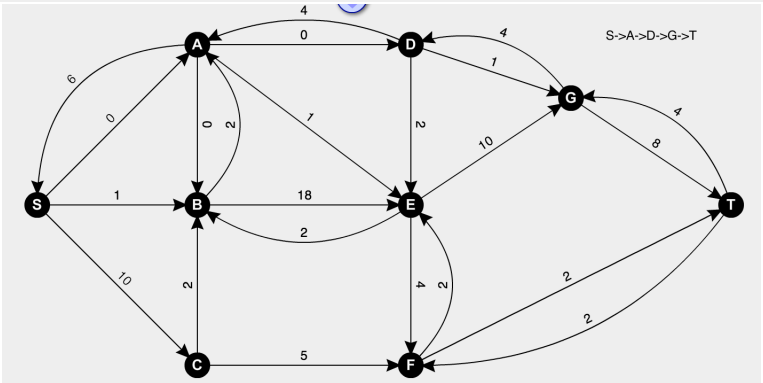
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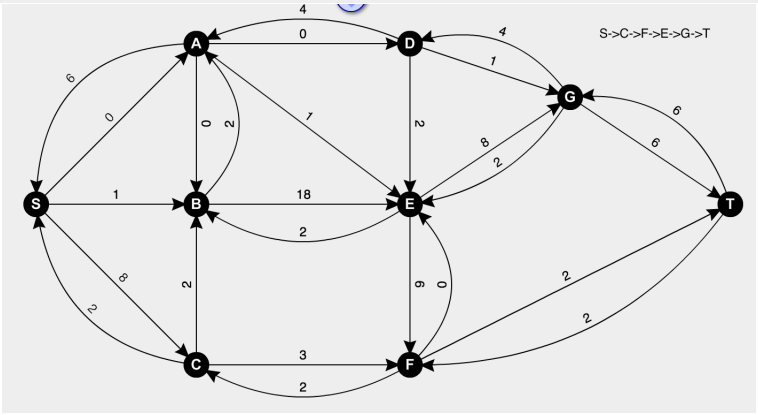
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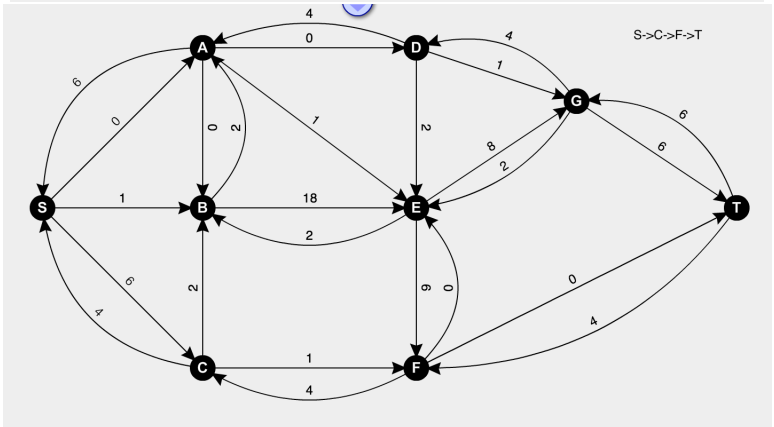
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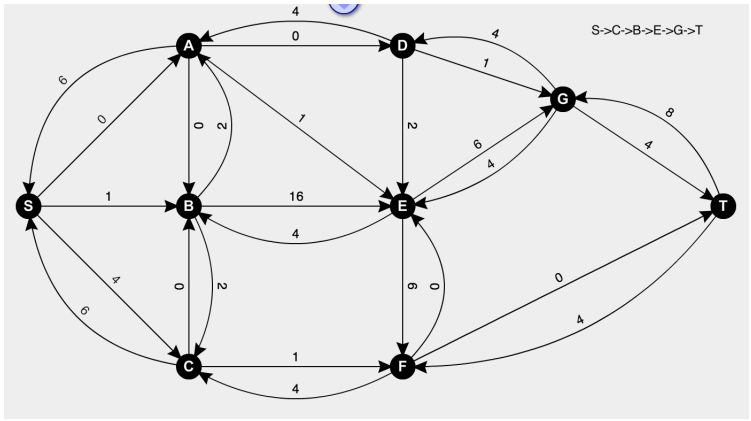
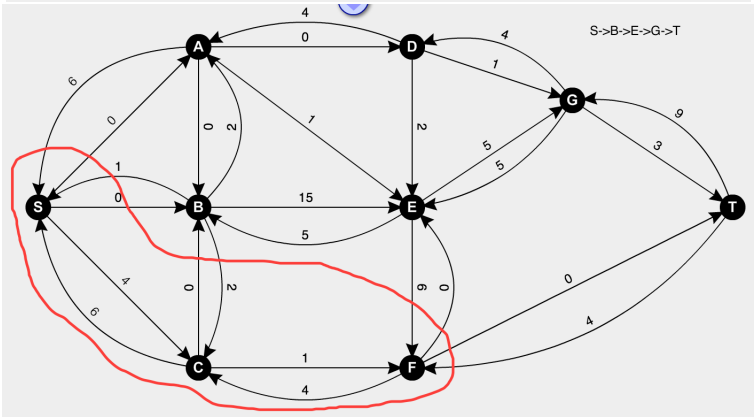




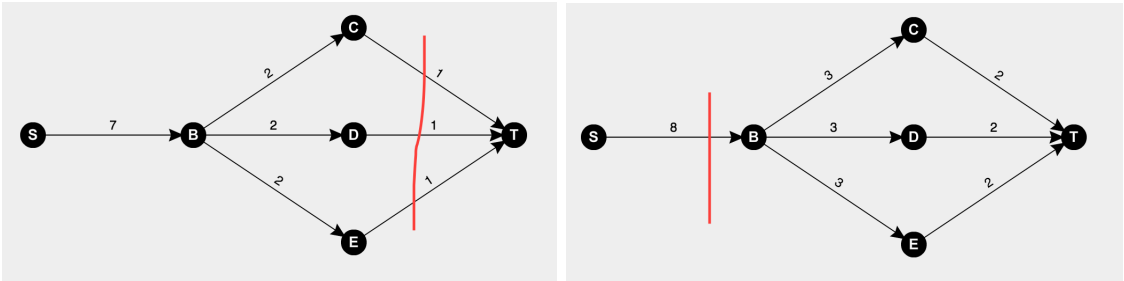
$min - cut : 6 + 1 + 2 + 4 = 13$ $max - flow : 9 + 4 = 13$

3. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e ; and let (A, B) be a mimimum $s - t$ cut with respect to these capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum $s - t$ cut with respect to these new capacities $\{1 + c_e : e \in E\}$.

   Solution:

   False, observe the two digraphs below:



4. There are many common variations of the maximum flow problem. Here are two of them:

(a) There are many sources and many sinks, and we wish to maximize the total flow from all sources to all sinks.

(b) Each vertex also has a capacity on the maximum flow that can enter it.

Both of these can be solved efficiently. Show this by reducing them to the original max-flow problem.
Solution:

(a) To reduce this to the original max-flow problem, simply add another source and connect all of the other sources to it with infinite edges going from the new source, to the other sources. We use the same fix for the sinks. Create another sink, and connect all of the other sinks to it with infinite edges going from the other sinks to the new sink. The graph is now equivalent to an original max-flow problem.

(b) To reduce this to the original max-flow problem, we need to take each vertex and split it into two; v_{in} and v_{out} and then connect them by an edge that has the weight of the vertex's capacity. Then for every edge (x, v) we change to (x, v_{in}) and (v, y) we change to (v_{out}, y). The graph is now equivalent to an original max-flow problem.

5. Suppose someone presents you with a solution to a max-flow problem on some network. Give a linear time algorithm to determine whether the solution does indeed give a maximum flow. Explain the correctness of your algorithm. By linear time we mean $O(n + m)$, where n is the number of vertices and m the number of edges of the graph.
Solution:
G' = run Ford-Fulkerson on G
run Dijkestra's algorithm on G' from S to T
if a result comes back, then the solution is incorrect
else, the solution is correct.

Correctness: After running Ford-Fulkerson, the residual graph should not allow a path from S to T. Thus, when we run Dijkestra's and get a solution, we know that the solution we have been presented with is incorrect.
Time: Ford-Fulkerson's run time is $O(mf)$ where f is the weight of the heaviest edge in the graph. Dijkestra's run time is $O(m+n \log n)$. This gives us $O(nlogn+2m) = O(n+m)$ time.