different sampling variances

Building on Mate's last document, we consider a case where

$$\tilde{x}_1 \sim \mathcal{N}(x_1, \tilde{\sigma}_1)$$
$$\tilde{x}_2 \sim \mathcal{N}(x_2, \tilde{\sigma}_2)$$

with different $\tilde{\sigma}$ for the two observations. This changes a few things, but primarily the covariance matrices for the two causes:

$$\Sigma_0 = \begin{pmatrix} \sigma^2 + \tilde{\sigma}_1^2 & 0 \\ 0 & \sigma^2 + \tilde{\sigma}_2^2 \end{pmatrix}$$
$$\Sigma_1 = \begin{pmatrix} \sigma^2 + \tilde{\sigma}_1^2 & \sigma^2 \\ \sigma^2 & \sigma^2 + \tilde{\sigma}_2^2 \end{pmatrix}$$

this changes some of the quantities used by Eq. 20 in the original (although sometimes this has no effect in the limit $\sigma \to \infty$).

determinants and ratio of determinants does not change in the limit:

$$det(\Sigma_0) = (\sigma^2 + \tilde{\sigma}_1^2)(\sigma^2 + \tilde{\sigma}_2^2)$$
$$det(\Sigma_1) = (\sigma^2 + \tilde{\sigma}_1^2)(\sigma^2 + \tilde{\sigma}_2^2) - \sigma^4$$
$$\frac{det(\Sigma_1)}{det(\Sigma_0)} = \frac{\tilde{\sigma}_1^2\tilde{\sigma}_2^2 + \sigma^2\tilde{\sigma}_1^2 + \sigma^2\tilde{\sigma}_2^2}{\sigma^4 + \tilde{\sigma}_1^2\tilde{\sigma}_2^2 + \sigma^2\tilde{\sigma}_1^2 + \sigma^2\tilde{\sigma}_2^2} \to \frac{1}{\sigma^2}$$

nor does Σ_0^{-1}:

$$\Sigma_0^{-1} = \begin{pmatrix} \frac{1}{\sigma^2+\tilde{\sigma}_1^2} & 0 \\ 0 & \frac{1}{\sigma^2+\tilde{\sigma}_2^2} \end{pmatrix} \to \frac{1}{\sigma^2}\mathbf{I} \to 0$$

but there is a slight change in Σ_1^{-1}:

$$\begin{aligned}
\Sigma_1^{-1} &= \frac{1}{(\sigma^2+\tilde{\sigma}_1^2)(\sigma^2+\tilde{\sigma}_2^2)-\sigma^4}\begin{pmatrix}\sigma^2+\tilde{\sigma}_1^2 & -\sigma^2\\ -\sigma^2 & \sigma^2+\tilde{\sigma}_2^2\end{pmatrix}\\
&= \frac{1}{(\sigma^2+\tilde{\sigma}_1^2)(\sigma^2+\tilde{\sigma}_2^2)-\sigma^4}(\begin{pmatrix}2\sigma^2+\tilde{\sigma}_1^2 & 0\\ 0 & 2\sigma^2+\tilde{\sigma}_2^2\end{pmatrix}-\sigma^2\mathbf{1}\mathbf{1}^\top)\\
&= \begin{pmatrix}\frac{2\sigma^2+\tilde{\sigma}_1^2}{\tilde{\sigma}_1^2\tilde{\sigma}_2^2+\sigma^2\tilde{\sigma}_1^2+\sigma^2\tilde{\sigma}_2^2} & 0\\ 0 & \frac{2\sigma^2+\tilde{\sigma}_2^2}{\tilde{\sigma}_1^2\tilde{\sigma}_2^2+\sigma^2\tilde{\sigma}_1^2+\sigma^2\tilde{\sigma}_2^2}\end{pmatrix}-\frac{\sigma^2}{\tilde{\sigma}_1^2\tilde{\sigma}_2^2+\sigma^2\tilde{\sigma}_1^2+\sigma^2\tilde{\sigma}_2^2}\mathbf{1}\mathbf{1}^\top\\
&\rightarrow \frac{2}{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}\mathbf{I}-\frac{1}{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}\mathbf{1}\mathbf{1}^\top=\frac{1}{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}
\end{aligned}$$

Which makes sense: we replace a $2\tilde{\sigma}^2$ term with $\tilde{\sigma}_1^2+\tilde{\sigma}_2^2$

Skipping some intermediate steps, plugging this into Eq. 32:

$$\ell(\tilde{x}) = -\frac{1}{2(\tilde{\sigma}_1^2+\tilde{\sigma}_2^2)}\left[(\delta\tilde{x})^2-2(\tilde{\sigma}_1^2+\tilde{\sigma}_2^2)ln\frac{\rho}{1-\rho}\right]$$

and inflection point

$$\delta(\tilde{x}) = \sqrt{2(\tilde{\sigma}_1^2+\tilde{\sigma}_2^2)}\sqrt{ln\frac{\rho}{1-\rho}}$$