1. Scaling into dimensional variables

This note aims to show how to formally display Benney-Luke equations (1.5) with dimensional variables. We start from the potential flow equations with dimensional variables (t, x, y, z):

$$\Delta\phi + \partial_{zz}\phi = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^3, \tag{1.1a}$$

$$\partial_t\eta + \nabla\phi\cdot\nabla\eta - \partial_z\phi = 0 \quad \text{at} \quad z = h, \tag{1.1b}$$

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}(\partial_z\phi)^2 + g\eta = 0 \quad \text{at} \quad z = h, \tag{1.1c}$$

$$\mathbf{n}\cdot\nabla\phi = 0 \quad \text{on} \quad \partial\Omega_b, \tag{1.1d}$$

$$\partial_z\phi = 0 \quad \text{at} \quad z = 0, \tag{1.1e}$$

where g is the gravitational constant, $9.8m/s^2$. Note that unit of t and x, y, z, η are a second and a meter, respectively.

Now, we formally derive Benney-Luke equations from (1.1). First, let us introduce non-dimensional variables

$$\widehat{x} = \frac{\sqrt{\mu}}{H_0}x, \quad \widehat{y} = \frac{\sqrt{\mu}}{H_0}y, \quad \widehat{z} = \frac{1}{H_0}z, \quad \widehat{t} = \frac{\sqrt{gH_0\mu}}{H_0}t, \quad \widehat{\eta} = \frac{1}{\epsilon H_0}\eta, \quad \widehat{\phi} = \frac{\sqrt{\mu}}{\epsilon H_0\sqrt{\epsilon H_0}}\phi. \tag{1.2}$$

where H_0 is rest depth, the amplitude parameter $\varepsilon = a/H_0 \ll 1$, and the dispersion parameter $\mu = (H_0/\lambda)^2 \ll 1$. After that, let Φ denote the seabed potential at $\widehat{z} = 0$,

$$\Phi(\widehat{x}, \widehat{y}, \widehat{t}) := \widehat{\phi}(\widehat{x}, \widehat{y}, 0, \widehat{t}). \tag{1.3}$$

And then, $\widehat{\phi}$ is assumed to have Taylor expansion in terms of z as

$$\widehat{\phi}(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{z}) = \Phi(\widehat{t}, \widehat{x}, \widehat{y}) - \frac{\mu}{2}\widehat{z}^2\widehat{\Delta}\Phi + \frac{\mu^2}{24}\widehat{\Delta}^2\widehat{z}^4\Phi + O(\mu^3\widehat{z}^3). \tag{1.4}$$

(For more detail, [1]) Substituting (1.2) and (1.4) into (1.1) and truncating terms up to order $O(\varepsilon^2\mu, \varepsilon^3)$ yields Benney-Luke equations (BLE) with non-dimensional variables $(\widehat{t}, \widehat{x}, \widehat{y}) \in \mathbb{R}^3$:

$$\partial_{\widehat{t}}\Phi - \frac{\mu}{2}\partial_{\widehat{t}}\widehat{\nabla}^2\Phi + \frac{\epsilon}{2}\left|\widehat{\nabla}\Phi\right|^2 + \widehat{\eta} = 0 \quad \text{in} \quad \Omega, \tag{1.5a}$$

$$\partial_{\widehat{t}}\widehat{\eta} - \frac{\mu}{2}\partial_{\widehat{t}}\widehat{\nabla}^2\widehat{\eta} + \widehat{\nabla}\cdot\left((1+\epsilon\widehat{\eta})\,\widehat{\nabla}\Phi\right) - \frac{2}{3}\mu\widehat{\nabla}^4\Phi = 0 \quad \text{in} \quad \Omega, \tag{1.5b}$$

$$\mathbf{n}\cdot\widehat{\nabla}\Phi = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5c}$$

$$\mathbf{n}\cdot\widehat{\nabla}(\widehat{\nabla}^2\Phi) = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5d}$$

As a result, after solving (1.5), the BLE solutions are formally displayed with dimensional variables by substituting (1.2) into the solutions.

References

[1] Bokhove, Onno and Kalogirou, Anna, *Variational water wave modelling: from continuum to experiment*, (Bridges, T., Groves, M., and Nicholls, D.), LMS Lecture Note Series,Cambridge University Press,426,226–260 2016. 1