Price of Anarchy for Mean Field Games

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Abstract

The price of anarchy, originally introduced to quantify the inefficiency of selfish behavior in routing games, is extended to mean field games. The price of anarchy is defined as the ratio of a worst case social cost computed for a mean field game equilibrium to the optimal social cost as computed by a central planner. We illustrate properties of such a price of anarchy on linear quadratic extended mean field games, for which explicit computations are possible. Various asymptotic behaviors of the price of anarchy are proved for limiting behaviors of the coefficients in the model and numerics are presented.

1 Introduction

The concept of the 'price of anarchy' was introduced to quantify the inefficiency of selfish behavior in finite player games [8][9][12][17][18][19]. In this report, we extend the notion of price of anarchy to mean field games (MFG). Mean field games were introduced by Lasry and Lions [13] and Caines and his collaborators [11] to describe the limiting regime of large symmetric games when the number of players, N, tends to infinity. A mean field game equilibrium characterizes the analogue of a Nash equilibrium in the $N = \infty$ regime. Thus, as in the finite player case, it is possible that the mean field game equilibrium is inefficient. In fact, in the paper of Balandat and Tomlin [2], they present a numerical example that shows that mean field game equilibria are not efficient, in general. The suboptimality of a mean field game equilibrium is also illustrated numerically for a congestion model in a paper of Achdou and Laurière [1]. More recently Cardaliaguet and Rainer gave in [4] a partial differential equation based thorough analysis of the *(in)efficiency* of the mean field game equilibria.

In this report, the goal is to define the price of anarchy in the context of mean field games, and to compute it for a class of linear quadratic mean field game models, which can be solved explicitly. In fact, we consider an even more general class of games by allowing for interaction between the players through their controls, in addition to interaction through their states. This is often referred in the literature as extended mean field game, or mean field game of control. We

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compare the social cost of a mean field game equilibrium to the cost incurred when the players execute a strategy computed centrally.

We consider a system of N players whose private states are denoted at time t by X_t^1, X_t^2, $\cdots$, X_t^N. To keep the presentations simple, we assume the state space is $\mathbb{R}$. We denote by μ_t^N the empirical distribution of the states, namely

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}.$$

We assume that these states evolve in continuous time under the influences of controls α_t^1, α_t^2, $\cdots$, $\alpha_t^N \in \mathbb{A}$, where the set of admissible controls, $\mathbb{A}$, will be defined later. Let ν_t^N denote the empirical measure of the controls:

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_t^i}.$$

We also assume that if and when interactions between these states and controls are present, they are of a mean field type, i.e. through μ_t^N and ν_t^N. The time evolution of the state for player i is given by the Itô dynamics:

$$dX_t^i = b(t, X_t^i, \mu_t^N, \alpha_t^i, \nu_t^N)dt + \sigma dW_t.$$

We work over the interval $[0, T]$ limited by a finite time horizon $T \in \mathbb{R}^+$. We assume the drift function $b : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{A} \times \mathcal{P}(\mathbb{A}) \ni (t, x, \mu, \alpha, \nu) \to \mathbb{R}$ is Lipschitz in each of it's inputs. For the sake of simplicity, we assume that the volatility, σ, is a positive constant.

Cost Functionals

We assume that we are given two functions $f : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{A} \times \mathcal{P}(\mathbb{A}) \ni (t, x, \mu, \alpha, \nu) \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \ni (x, \mu) \to \mathbb{R}$ which we call running and terminal cost functions, respectively. We assume f and g are Lipschitz in each of their arguments. The goal of player i is to minimize its expected cost as given by:

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \mu_t^N, \alpha_t^i, \nu_t^N)\, dt + g(X_T^i, \mu_T^N)\right].$$

Social Cost

We restrict ourselves to Markovian control strategies $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ given by feedback functions in the form $\alpha_t = \phi(t, X_t)$ and we let $\mathbb{A}$ denote the set of such controls. If the N players use distributed Markovian control strategies of the form $\alpha_t^i = \phi(t, X_t^i)$, we define the cost (per player) to the system as the quantity $J_\phi^{(N)}$:

$$J_\phi^{(N)} = \frac{1}{N}\sum_{i=1}^{N} J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N).$$

We shall compute this social cost in the limit $N \to \infty$ when all the players use the distributed control strategies given by the same feedback function ϕ identified by solving an optimization problem in the limit $N \to \infty$. We take the social cost to be the limit as $N \to \infty$ of $J_\phi^{(N)}$, namely

$$\begin{aligned}
\lim_{N\to\infty} J_\phi^{(N)} &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} J^i(\boldsymbol{\alpha}^1,\cdots,\boldsymbol{\alpha}^N) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\bigg[\int_0^T f(t,X_t^i,\mu_t^N,\phi(t,X_t^i),\nu_t^N)\,dt + g(X_T^i,\mu_T^N)\bigg], \\
&= \lim_{N\to\infty} \mathbb{E}\bigg[\int_0^T < f(t,\,\cdot\,,\mu_{X_t}^N,\phi(t,\,\cdot\,),\nu_t^N),\mu_t^N > \,dt + < g(\,\cdot\,,\mu_{X_T}^N),\mu_T^N > \bigg],
\end{aligned}$$

if we use the notation $< \varphi,\rho >$ for the integral $\int \varphi(z)\rho(dz)$ of the function φ with respect to the measure ρ. Now if we assume that in the limit $N \to \infty$ the empirical distributions μ_t^N converge toward a measure μ_t, and thus $\nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\phi(t,X_t^i)}$ also converges toward a measure ν_t, then the social cost of the feedback function ϕ becomes:

$$SC(\phi) = \int_0^T < f(t,\,\cdot\,,\mu_t,\phi(t,\,\cdot\,),\nu_t),\mu_t > \,dt + \,< g(\,\cdot\,,\mu_T),\mu_T >$$

with the expectation, $\mathbb{E}$, disappearing when the limiting flows $\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}$ and $\boldsymbol{\nu} = (\nu_t)_{0\le t\le T}$ are deterministic.

We would like to evaluate $SC(\phi)$ in the $N = \infty$ regime directly, without having to construct the deterministic measure flows $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as limits of the finite player empirical measures. To do this, we assume that propagation of chaos holds and that the states of the N players become asymptotically independent in the limit as $N \to \infty$. We consider a representative agent whose state is given by $\boldsymbol{X}^{\boldsymbol{\phi}} = (X_t^\phi)_{0\le t\le T}$, the continuous time solution of the stochastic differential equation of McKean-Vlasov type:

$$dX_t^\phi = b(t,X_t^\phi,\mathcal{L}(X_t^\phi),\phi(t,X_t^\phi),\mathcal{L}(\phi(t,X_t^\phi))dt + \sigma dW_t \tag{1}$$

controlled by ϕ. Then we can identify $\boldsymbol{\mu}$ as the law of a representative agent using the feedback function ϕ, i.e. $\mu_t = \mathcal{L}(X_t^\phi)$, and similarly, we can identify $\boldsymbol{\nu}$ as the law of the control, such that $\nu_t = \mathcal{L}(\phi(t,X_t^\phi))$. Thus, in the $N = \infty$ regime, we rewrite the social cost as

$$SC(\phi) = \int_0^T < f(t,\,\cdot\,,\mathcal{L}(X_t^\phi),\phi(t,\,\cdot\,),\mathcal{L}(\phi(t,X_t^\phi))),\mathcal{L}(X_t^\phi) > \,dt + < g(\,\cdot\,,\mathcal{L}(X_T^\phi)),\mathcal{L}(X_T^\phi) >$$

where $\boldsymbol{X}^{\boldsymbol{\phi}}$ satisfies equation (1). For the remainder of the paper, we work in the $N = \infty$ regime. As mentioned earlier, ϕ should be identified by solving an optimal control problem. We consider two distinct problems:

- ϕ is a feedback function providing a mean field game equilibrium. We detail more precisely what is meant by ϕ providing a mean field game equilibrium in Section 1.1.
- ϕ is the feedback function minimizing the social cost $SC(\phi)$, without having to be a mean field game equilibrium, in which case we use the notation SC^{MKV} for $SC(\phi)$. This is a control problem of McKean-Vlasov type, which is detailed more precisely in Section 1.2.

The two problems are detailed more precisely in Sections 1.1 and 1.2. In Section 1.3, we define the price of anarchy based on these two problem formulations. The class of linear quadratic models is explored in Section 2, where we provide some theoretical results on the price of anarchy for this class of games, show numerical results, and detail a particular example of flocking. We conclude in Section 3.

1.1 Nash Equilibrium: Mean Field Game Formulation

The goal of this subsection is to articulate what is meant by a feedback function providing a mean field game equilibrium. To begin, we define what we call the *mean field environment*. By symmetry of the players, we suppose all of the players in the mean field game use the same feedback function, ϕ. Then the *mean field environment* specified by ϕ is characterized by $\mathcal{L}(X_t^\phi)_{0\le t\le T}$ and $\mathcal{L}(\phi(t, X_t^\phi))_{0\le t\le T}$ where the dynamics of $(X_t^\phi)_{0\le t\le T}$ are given by equation (1). Since we search for a Nash equilibrium, we consider a representative agent who wishes to find their best response, ϕ', to the mean field environment specified by ϕ, in which case their state is given by $\boldsymbol{X}^{\boldsymbol{\phi}',\boldsymbol{\phi}} = (X_t^{\phi',\phi})_{0\le t\le T}$ solving the standard stochastic differential equation:

$$dX_t^{\phi',\phi} = b(t, X_t^{\phi',\phi}, \mathcal{L}(X_t^\phi), \phi'(t, X_t^{\phi',\phi}), \mathcal{L}(\phi(t, X_t^\phi)))dt + \sigma dW_t.$$

Consider the function:

$$\mathcal{S}(\phi', \phi) = \left[\int_0^T < f(t, \cdot, \mathcal{L}(X_t^\phi), \phi'(t, \cdot), \mathcal{L}(\phi(t, X_t^\phi)), \mathcal{L}(X_t^{\phi',\phi})) > dt + < g(\cdot, \mathcal{L}(X_T^\phi)), \mathcal{L}(X_t^{\phi',\phi}) >\right].$$

The best response for the representative agent in the mean field environment specified by ϕ is the feedback function minimizing this cost, namely $\phi^* = \arg\inf_{\phi'} \mathcal{S}(\phi', \phi)$. Assuming the minimizer is unique (which will be the case for the models we consider), this defines a mapping $\Phi : \phi \to \phi^*$. If there is a $\hat{\phi}$ such that $\Phi(\hat{\phi}) = \hat{\phi}$, then the players are in a mean field game equilibrium.

Thus, the search for a feedback function providing a mean field game equilibrium can be summarized as the following set of two successive steps:

1. For each feedback function $\phi : [0, T] \times \mathbb{R} \ni (t, x) \to \mathbb{R}$, solve the optimal control problem

   $$\phi^* = \arg\inf_{\phi'} \mathcal{S}(\phi', \phi).$$

   Define the mapping $\Phi(\phi) := \phi^*$.

2. Find a fixed point $\hat{\phi}$ of Φ such that $\Phi(\hat{\phi}) = \hat{\phi}$.

When these two steps can be taken successfully, we say that $\hat{\phi}$ provides a mean field game equilibrium. Note that $\boldsymbol{X}^{\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\phi}}} = \boldsymbol{X}^{\hat{\boldsymbol{\phi}}}$ and therefore $\mathcal{S}(\hat{\phi}, \hat{\phi}) = SC(\hat{\phi})$ gives the social cost for the mean field game equilibrium provided by $\hat{\phi}$. Notice that there could possibly be many feedback functions providing a mean field game equilibrium. Let $\mathcal{N}$ denote the set of all such feedback functions providing mean field game equilibria, as detailed above, i.e.

$$\mathcal{N} = \{\phi : [0, T] \times \mathbb{R} \ni (t, x) \to \mathbb{R} \mid \Phi(\phi) = \phi\}.$$

1.2 Centralized Control: Optimal Control of McKean-Vlasov Type

The goal of this subsection is to articulate how to compute the cost associated with the control problem of McKean-Vlasov type, SC^{MKV}. The central planner considers the following control problem:

$$\begin{aligned}\hat{\phi} &= \arg\inf_{\phi} SC(\phi) \\ &= \arg\inf_{\phi} \left[\int_0^T < f(t, \cdot, \mathcal{L}(X_t^\phi), \phi(t, \cdot), \mathcal{L}(\phi(t, X_t^\phi))), \mathcal{L}(X_t^\phi) > dt + < g(\cdot, \mathcal{L}(X_T^\phi)), \mathcal{L}(X_T^\phi) >\right].\end{aligned}$$

Thus, the cost of the solution to the optimal control problem of McKean-Vlasov is given by

$$SC^{MKV} = SC(\hat{\phi}).$$

Remark 1. *We are not concerned with uniqueness for the control of McKean-Vlasov type problem, because $SC^{MKV} = SC(\phi_1) = SC(\phi_2)$ is still well defined even if there are two different optimal feedback functions ϕ_1 and ϕ_2 minimizing $SC(\phi)$.*

1.3 Price of Anarchy

We have described two approaches to compute the optimal feedback function ϕ. In the mean field game formulation, we require $\phi \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of feedback functions providing mean field game equilibria. In the optimal control of McKean-Vlasov type formulation, the optimal control to be adopted by all players is computed by a central planner, who optimizes the social cost function $SC(\phi)$ directly. Thus, we necessarily have:

$$SC^{MKV} \leq SC(\phi), \ \forall \phi \in \mathcal{N}.$$

In other words, there is a 'price of anarchy' associated with allowing players to choose their controls selfishly. We thus define the price of anarchy (denoted PoA) as the ratio between the worst case cost for a mean field game equilibrium and the optimal cost computed by a central planner:

$$PoA = \frac{\sup_{\phi \in \mathcal{N}} SC(\phi)}{SC^{MKV}}.$$

2 Price of Anarchy for Linear Quadratic Extended Mean Field Games

The class of linear quadratic extended mean field games is a class of problems for which explicit solutions can be computed analytically, and thus, we can compute the price of anarchy explicitly. To the best of our knowledge, the case of linear quadratic extended mean field games has not been explored in the literature, as well as computing the price of anarchy for this class of games.

To begin, we need to describe in more detail the two problems that will be used to compute the price of anarchy: the linear quadratic extended mean field game, and the linear quadratic control problem of McKean-Vlasov type with dependence on the law of the control. To specify the problems, we only need to specify the drift and cost functions, b, f, and g introduced in Section 1. For the linear quadratic models, we take the drift to be linear:

$$b(t, x, \mu, \alpha, \nu) = b_1(t)x + \bar{b}_1(t)\bar{\mu} + b_2(t)\alpha + \bar{b}_2(t)\bar{\nu},$$

where $\bar{\mu}$ denotes the mean of the measure μ, namely, $\bar{\mu} = \int_{\mathbb{R}} x d\mu(x)$, and similarly for $\bar{\nu}$. We take the running and terminal costs to be quadratic:

$$\begin{aligned} f(t, x, \mu, \alpha, \nu) &= \frac{1}{2}\left(q(t)x^2 + \bar{q}(t)(x - s(t)\bar{\mu})^2 + r(t)\alpha^2 + \bar{r}(t)(\alpha - \bar{s}(t)\bar{\nu})^2\right), \\ g(x, \mu) &= \frac{1}{2}\left(q_T x^2 + \bar{q}_T(x - s_T\bar{\mu})^2\right). \end{aligned}$$

Remark 2. *If $\bar{b}_2(t) \equiv 0$ and $\bar{r}(t) \equiv 0$, then we have the standard mean field game or control problem of McKean-Vlasov type. (See Theorem 1 for assumptions that provide existence and uniqueness.)*

2.1 Linear Quadratic Extended Mean Field Games

To solve the linear quadratic extended mean field game (LQEMFG), we begin by considering the reduced Hamiltonian for this problem:

$$\begin{aligned}H(t,x,\bar{\mu},\alpha,\bar{\nu},y) =&(b_1(t)x+\bar{b}_1(t)\bar{\mu}+b_2(t)\alpha+\bar{b}_2(t)\bar{\nu})y\\ &+\frac{1}{2}\left(q(t)x^2+\bar{q}(t)(x-s(t)\bar{\mu})^2+r(t)\alpha^2+\bar{r}(t)(\alpha-\bar{s}(t)\bar{\nu})^2\right),\end{aligned}$$

and whenever the flows $\bar{\boldsymbol{\mu}}=(\bar{\mu}_t)_{0\le t\le T}$ and $\bar{\boldsymbol{\nu}}=(\bar{\nu}_t)_{0\le t\le T}$ are fixed, we consider for each control process $\boldsymbol{\alpha}=(\alpha_t)_{0\le t\le T}$ the adjoint equation:

$$\begin{aligned}dY_t &= -\partial_x H(t,X_t,\bar{\mu}_t,\alpha_t,\bar{\nu}_t,Y_t)dt+Z_t dW_t\\ Y_T &= \partial_x g(X_T,\mathcal{L}(X_T)).\end{aligned}$$

According to the Pontryagin stochastic maximum principle, a sufficient condition for optimality is $\partial_\alpha H(t,X_t,\bar{\mu}_t,\hat{\alpha}_t,\bar{\nu}_t,y)=0$. We introduce the function:

$$\hat{\alpha}(t,x,\bar{\mu},\bar{\nu},y)=\frac{\bar{r}(t)\bar{s}(t)\bar{\nu}-b_2(t)y}{r(t)+\bar{r}(t)},$$

and use the control $\hat{\alpha}_t=\hat{\alpha}(t,X_t,\bar{\mu},\bar{\nu},Y_t)$. When solving the fixed point step, we identify $\bar{\nu}_t=\mathbb{E}(\hat{\alpha}_t)$. By taking the expectation, we find:

$$\mathbb{E}(\hat{\alpha}_t)=c^{MFG}(t)\mathbb{E}(Y_t)$$

with:

$$c^{MFG}(t)=-\frac{b_2(t)}{r(t)+\bar{r}(t)(1-\bar{s}(t))}.$$

Thus, necessarily we must have:

$$\hat{\alpha}_t=a^{MFG}(t)Y_t+b^{MFG}(t)\mathbb{E}(Y_t)$$

with:

$$a^{MFG}(t)=-\frac{b_2(t)}{r(t)+\bar{r}(t)},$$

and:

$$b^{MFG}(t)=-\frac{\bar{r}(t)\bar{s}(t)b_2(t)}{(r(t)+\bar{r}(t))(r(t)+\bar{r}(t)(1-\bar{s}(t)))}.$$

Note that $c^{MFG}(t)=a^{MFG}(t)+b^{MFG}(t)$. The solution of the mean field game equilibrium problem is given by the solution to the FBSDE system:

$$\begin{aligned}dX_t &= \left(b_1(t)X_t+\bar{b}_1(t)\mathbb{E}X_t+a^{MFG}(t)b_2(t)Y_t+(b^{MFG}(t)b_2(t)+c^{MFG}(t)\bar{b}_2(t))\mathbb{E}Y_t\right)dt+\sigma dW_t\\ dY_t &= -\left((q(t)+\bar{q}(t))X_t-\bar{q}(t)s(t)\mathbb{E}X_t+b_1(t)Y_t\right)dt+Z_t dW_t\end{aligned} \tag{2}$$

with initial condition $X_0=\xi$, a random variable with finite mean and variance, and terminal condition $Y_T=(q_T+\bar{q}_T)X_T-\bar{q}_T s_T\mathbb{E}X_T$.

This is a linear FBSDE of McKean-Vlasov type, which can be solved explicitly under mild assumptions (or at least in the case of time-independent coefficients which we will consider later. See Appendix A). Let $\bar{\eta}_t^{MFG}$, η_t^{MFG}, $\bar{x}_t^{MFG}$, and v_t^{MFG} denote the solutions for this problem as described in the appendix so that $Y_t = \eta_t^{MFG} X_t + (\bar{\eta}_t^{MFG} - \eta_t^{MFG})\bar{x}_t^{MFG}$, $\mathbb{E}(Y_t) = \bar{\eta}_t^{MFG}\bar{x}_t^{MFG}$, $\mathbb{E}(X_t) = \bar{x}_t^{MFG}$, and $Var(X_t) = v_t^{MFG}$ provide a solution to the LQEMFG problem. Then from the appendix, we have:

$$\begin{aligned} \dot{\bar{\eta}}_t^{MFG} + c^{MFG}(t)(b_2(t) + \bar{b}_2(t))(\bar{\eta}_t^{MFG})^2 + (2b_1(t) + \bar{b}_1(t))\bar{\eta}_t^{MFG} + q(t) + \bar{q}(t)(1 - s(t)) = 0, \\ \bar{\eta}_T^{MFG} - (q_T + \bar{q}_T(1 - s_T)) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{\eta}_t^{MFG} + a^{MFG}(t)b_2(t)(\eta_t^{MFG})^2 + 2b_1(t)\eta_t^{MFG} + q(t) + \bar{q}(t) = 0, \\ \eta_T^{MFG} - (q_T + \bar{q}_T) = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{\bar{x}}_t^{MFG} &= (b_1(t) + \bar{b}_1(t) + c^{MFG}(t)(b_2(t) + \bar{b}_2(t))\bar{\eta}_t^{MFG})\bar{x}_t^{MFG}, \\ \bar{x}_0^{MFG} &= \mathbb{E}(\xi), \end{aligned} \tag{5}$$

where the dot is the standard ODE notation for a derivative. And thus,

$$\bar{x}_t^{MFG} = \mathbb{E}(\xi) e^{\int_0^t (b_1(u) + \bar{b}_1(u) + c^{MFG}(u)(b_2(u) + \bar{b}_2(u))\bar{\eta}_u^{MFG})du}, \tag{6}$$

$$v_t^{MFG} = Var(\xi) e^{\int_0^t 2(b_1(s) + a^{MFG}(s)b_2(s)\eta_s^{MFG})ds} + \sigma^2 \int_0^t e^{2\int_s^t (b_1(u) + a^{MFG}(u)b_2(u)\eta_u^{MFG})du} ds. \tag{7}$$

Let $SC^{MFG} := SC(\phi)$ for the feedback function specified by this solution, i.e.

$$\phi(t,x) = a^{MFG}(t)\eta_t^{MFG} x + \left(a^{MFG}(t)(\bar{\eta}_t^{MFG} - \eta_t^{MFG}) + b^{MFG}(t)\bar{\eta}_t^{MFG}\right)\bar{x}_t^{MFG}.$$

Then we can compute the social cost as described in Section 1.1:

$$\begin{aligned} SC^{MFG} = &\frac{1}{2}[(q_T + \bar{q}_T)v_T^{MFG} + (q_T + \bar{q}_T(1 - s_T)^2)(\bar{x}_T^{MFG})^2 \\ &+ \int_0^T (q(t) + \bar{q}(t) + (r(t) + \bar{r}(t))(a^{MFG}(t)\eta_t^{MFG})^2)v_t^{MFG} \\ &+ (q(t) + \bar{q}(t)(1 - s(t))^2 + (r(t) + \bar{r}(t)(1 - \bar{s}(t))^2)(c^{MFG}(t)\bar{\eta}_t^{MFG})^2)(\bar{x}_t^{MFG})^2 dt], \end{aligned}$$

where we have used the fact that:

$$\mathbb{E}(\phi(t, X_t)) = c^{MFG}(t)\bar{\eta}_t^{MFG}\bar{x}_t^{MFG},$$

and:

$$Var(\phi(t, X_t)) = (a^{MFG}(t)\eta_t^{MFG})^2 v_t^{MFG}.$$

2.2 Linear Quadratic Control of McKean-Vlasov Type Involving the Law of the Control

To solve the linear quadratic optimal control problem of McKean-Vlasov type involving the law of the control (LQEMKV), we begin with the reduced Hamiltonian, which is the same as in the LQEMFG problem:

$$\begin{aligned} H(t, x, \bar{\mu}, \alpha, \bar{\nu}, y) = &(b_1(t)x + \bar{b}_1(t)\bar{\mu} + b_2(t)\alpha + \bar{b}_2(t)\bar{\nu})y \\ &+ \frac{1}{2}\left(q(t)x^2 + \bar{q}(t)(x - s(t)\bar{\mu})^2 + r(t)\alpha^2 + \bar{r}(t)(\alpha - \bar{s}(t)\bar{\nu})^2\right). \end{aligned}$$

Since we require $\bar{\nu}_t$ to be equal to $\mathbb{E}(\alpha_t)$ throughout the optimization, it is not sufficient to minimize the Hamiltonian with respect to the α input alone in order to guarantee optimality. A sufficient condition for control problems of McKean-Vlasov type involving the law of the control is derived in [5]. Since we consider a Hamiltonian that depends on the means of $\bar{\mu}$ and $\bar{\nu}$ instead of the full distributions, the sufficient condition reduces to the following (see section 4 in [5]): $\hat{\alpha}(t, X_t, \bar{\mu}, \bar{\nu}, Y_t)$ should satisfy:

$$\partial_\alpha H(t, X_t, \mathbb{E}(X_t), \hat{\alpha}_t, \mathbb{E}(\hat{\alpha}_t), Y_t) + \tilde{\mathbb{E}}\left[\partial_{\bar{\nu}} H(t, \tilde{X}_t, \mathbb{E}(X_t), \hat{\alpha}_t, \mathbb{E}(\hat{\alpha}_t), \tilde{Y}_t)\right] = 0,$$

where the adjoint equation is given by:

$$\begin{aligned} dY_t &= -\left[\partial_x H(t, X_t, \bar{\mu}_t, \alpha_t, \bar{\nu}_t, Y_t) + \tilde{\mathbb{E}}\left[\partial_{\bar{\mu}} H(t, \tilde{X}_t, \bar{\mu}_t, \tilde{\alpha}_t, \bar{\nu}_t, \tilde{Y}_t)\right]\right] dt + Z_t dW_t \\ Y_T &= \partial_x g(X_T, \mathcal{L}(X_T)) + \tilde{\mathbb{E}}\left[\partial_{\bar{\mu}} g(\tilde{X}_T, \mathcal{L}(X_T))(X_T)\right], \end{aligned}$$

and $(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}, \tilde{\boldsymbol{\alpha}})$ denotes an independent copy of $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\alpha})$. In the present LQ case, the sufficient condition can be used to solve for:

$$\hat{\alpha}_t = a^{MKV}(t) Y_t + b^{MKV}(t)\mathbb{E}(Y_t),$$

and:

$$\mathbb{E}(\hat{\alpha}_t) = c^{MKV}(t)\mathbb{E}(Y_t),$$

with:

$$\begin{aligned} a^{MKV}(t) &= -\frac{b_2(t)}{r(t) + \bar{r}(t)} \\ b^{MKV}(t) &= -\frac{1}{r(t) + \bar{r}(t)}\left(\bar{b}_2(t) - \frac{\bar{r}(t)\bar{s}(t)(\bar{s}(t) - 2)(b_2(t) + \bar{b}_2(t))}{r(t) + \bar{r}(t)(1 - \bar{s}(t))^2}\right) \\ c^{MKV}(t) &= -\frac{b_2(t) + \bar{b}_2(t)}{r(t) + \bar{r}(t)(1 - \bar{s}(t))^2}. \end{aligned}$$

So the solution of the optimal control problem of McKean-Vlasov type is given by the solution to the FBSDE system:

$$\begin{aligned} dX_t &= \big(b_1(t) X_t + \bar{b}_1(t)\mathbb{E}X_t + a^{MKV}(t) b_2(t) Y_t + (b^{MKV}(t) b_2(t) + c^{MKV}(t)\bar{b}_2(t))\mathbb{E}Y_t\big)\, dt + \sigma dW_t \\ dY_t &= -\big((q(t) + \bar{q}(t)) X_t + s(t)\bar{q}(t)(s(t) - 2)\mathbb{E}X_t + b_1(t) Y_t + \bar{b}_1(t)\mathbb{E}Y_t\big)\, dt + Z_t dW_t \end{aligned} \tag{8}$$

with initial condition $X_0 = \xi$, and terminal condition $Y_T = (q_T + \bar{q}_T) X_T + s_T \bar{q}_T (s_T - 2)\mathbb{E}X_T$.

As in the previous section, this is a linear FBSDE of McKean-Vlasov type, which can be solved explicitly under mild assumptions (or at least in the case of time-independent coefficients which we will consider later. See Appendix A). Let $\bar{\eta}_t^{MKV}$, η_t^{MKV}, $\bar{x}_t^{MKV}$, and v_t^{MKV} so that $Y_t = \eta_t^{MKV} X_t + (\bar{\eta}_t^{MKV} - \eta_t^{MKV})\bar{x}_t^{MKV}$, $\mathbb{E}(Y_t) = \bar{\eta}_t^{MKV}\bar{x}_t^{MKV}$, $\mathbb{E}(X_t) = \bar{x}_t^{MKV}$, and $Var(X_t) = v_t^{MKV}$ provide a solution to the LQEMKV problem. Then from the appendix, we have:

$$\begin{aligned} \dot{\bar{\eta}}_t^{MKV} + c^{MKV}(t)(b_2(t) + \bar{b}_2(t))(\bar{\eta}_t^{MKV})^2 + 2(b_1(t) + \bar{b}_1(t))\bar{\eta}_t^{MKV} + q(t) + \bar{q}(t)(1 - s(t))^2 &= 0, \\ \bar{\eta}_T^{MKV} - (q_T + \bar{q}_T(1 - s_T)^2) &= 0, \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{\eta}_t^{MKV} + a^{MKV}(t)b_2(t)(\eta_t^{MKV})^2 + 2b_1(t)\eta_t^{MKV} + q(t) + \bar{q}(t) = 0, \\ \eta_T^{MKV} - (q_T + \bar{q}_T) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{\bar{x}}_t^{MKV} &= (b_1(t) + \bar{b}_1(t) + c^{MKV}(t)(b_2(t) + \bar{b}_2(t))\bar{\eta}_t^{MKV})\bar{x}_t^{MKV}, \\ \bar{x}_0^{MKV} &= \mathbb{E}(\xi). \end{aligned} \tag{11}$$

And thus,

$$\bar{x}_t^{MKV} = \mathbb{E}(\xi)e^{\int_0^t (b_1(u)+\bar{b}_1(u)+c^{MKV}(u)(b_2(u)+\bar{b}_2(u)\bar{\eta}_u^{MKV})du}, \tag{12}$$

$$v_t^{MKV} = Var(\xi)e^{\int_0^t 2(b_1(s)+a^{MKV}(s)b_2(s)\eta_s^{MKV})ds} + \sigma^2 \int_0^t e^{2\int_s^t (b_1(u)+a^{MKV}(u)b_2(u)\eta_u^{MKV})du} ds. \tag{13}$$

Then $SC^{MKV} = SC(\phi)$ where ϕ is the feedback function specified by this solution, i.e.

$$\phi(t,x) = a^{MKV}(t)\eta_t^{MKV} x + \left(a^{MKV}(t)(\bar{\eta}_t^{MKV} - \eta_t^{MKV}) + b^{MKV}(t)\bar{\eta}_t^{MKV}\right)\bar{x}_t^{MKV}.$$

Then we can compute the social cost, denoted SC^{MKV}, as described in Section 1.2:

$$\begin{aligned} SC^{MKV} =& \frac{1}{2}[(q_T + \bar{q}_T)v_T^{MKV} + (q_T + \bar{q}_T(1-s_T)^2)(\bar{x}_T^{MKV})^2 \\ &+ \int_0^T (q(t) + \bar{q}(t) + (r(t) + \bar{r}(t))(a^{MKV}(t)\eta_t^{MKV})^2)v_t^{MKV} dt \\ &+ \int_0^T (q(t) + \bar{q}(t)(1-s(t))^2 + (r(t) + \bar{r}(t)(1-\bar{s}(t))^2)(c^{MKV}(t)\bar{\eta}_t^{MKV})^2)(\bar{x}_t^{MKV})^2 dt], \end{aligned}$$

where we have used the fact that:

$$\mathbb{E}(\phi(t, X_t)) = c^{MKV}(t)\bar{\eta}_t^{MKV}\bar{x}_t^{MKV},$$

and:

$$Var(\phi(t, X_t)) = (a^{MKV}(t)\eta_t^{MKV})^2 v_t^{MKV}.$$

2.3 Theoretical Results

For the remainder of the paper, we assume the coefficients are independent of time and nonnegative:

$$(b_1(t), \bar{b}_1(t), b_2(t), \bar{b}_2(t), q(t), \bar{q}(t), r(t), \bar{r}(t), s(t), \bar{s}(t)) = (b_1, \bar{b}_1, b_2, \bar{b}_2, q, \bar{q}, r, \bar{r}, s, \bar{s}) \in (\mathbb{R}^+)^{10}$$

and therefore,

$$\begin{aligned} (a^{MFG}(t), b^{MFG}(t), c^{MFG}(t)) &= (a^{MFG}, b^{MFG}, c^{MFG}) \\ (a^{MKV}(t)b^{MKV}(t), c^{MKV}(t)) &= (a^{MKV}, b^{MKV}, c^{MKV}). \end{aligned}$$

Theorem 1. *Assume the following:*

$$\begin{array}{rclrclrcl} b_2 & > & 0 & b_2 + \bar{b}_2 & > & 0 & & & \\ r + \bar{r} & > & 0 & r + \bar{r}(1-\bar{s}) & > & 0 & r + \bar{r}(1-\bar{s})^2 & > & 0 \\ q + \bar{q} & > & 0 & q + \bar{q}(1-s) & > & 0 & q + \bar{q}(1-s)^2 & > & 0 \\ q_T + \bar{q}_T & \geq & 0 & q_T + \bar{q}_T(1-s_T) & \geq & 0 & q_T + \bar{q}_T(1-s_T)^2 & \geq & 0 \end{array} \tag{14}$$

then there exists a unique solution to the LQEMFG problem, and there exists a unique solution to the LQEMKV problem. And therefore, $PoA = \frac{SC^{MFG}}{SC^{MKV}}$ where $SC^{MFG} := SC(\phi)$ for ϕ given by the explicit solution constructed in Appendix A.

Remark 3. *Note that existence in Theorem 1 follows from the explicit construction in Appendix A, because the above conditions provide existence to the solutions of the Riccati equations. Uniqueness comes from the connection between LQEMFG or LQEMKV and deterministic LQ optimal control. (See Section 3.5.1 in [7]).*

To compute the price of anarchy, it is useful to make the following observations:

$$\begin{aligned} a^{MFG} &= a^{MKV} =: a \\ \eta_t^{MFG} &= \eta_t^{MKV} =: \eta_t \\ v_t^{MFG} &= v_t^{MKV} =: v_t \end{aligned}$$

Proposition 1. *Assuming (14), if furthermore,*

$$\begin{aligned} \left(s\bar{q}(s-1) + \bar{b}_1\bar{\eta}_t^{MKV}\right)\bar{x}_t^{MKV} &= 0, \ \forall t \in [0,T] \\ \bar{\eta}_t^{MKV}\bar{x}_t^{MKV}\left[(b^{MFG} - b^{MKV})b_2 + (c^{MFG} - c^{MKV})\bar{b}_2\right] &= 0, \ \forall t \in [0,T] \\ s_T\bar{q}_T(s_T-1)\bar{x}_T^{MKV} &= 0, \end{aligned}$$

then $PoA = 1$.

Proof. Comparing the FBSDE systems (2) and (8), the result is clear. □

Corollary 1. *Assuming (14), if furthermore, $\bar{b}_1 = 0$, $s\bar{q}(s-1) = 0$, and $s_T\bar{q}_T(s_T-1) = 0$ and at least one of the following holds: $b_2 = \bar{b}_2 = 0$ or $\frac{b_2(r+\bar{r}(1-\bar{s})^2)}{(b_2+\bar{b}_2)(r+\bar{r}(1-\bar{s}))} = 1$ then $PoA = 1$.*

Remark 4. *The only result similar to Proposition 1 that we are aware of is Remark 6.1 in [16].*

Using the above observations, we can rewrite:

$$\begin{aligned} SC^{MFG} =& \frac{1}{2}(q_T + \bar{q}_T)v_T + \frac{1}{2}\left(q_T + \bar{q}_T(1-s_T)^2\right)(\bar{x}_T^{MFG})^2 + \frac{1}{2}\int_0^T (q + \bar{q} + (r+\bar{r})(a\eta_t)^2)v_t dt \\ &+ \frac{1}{2}\int_0^T \left[q + \bar{q}(1-s)^2 + (r + \bar{r}(1-\bar{s})^2)(c^{MFG}\bar{\eta}_t^{MFG})^2\right](\bar{x}_t^{MFG})^2 dt, \end{aligned} \tag{15}$$

and:

$$\begin{aligned} SC^{MKV} =& \frac{1}{2}(q_T + \bar{q}_T)v_T + \frac{1}{2}\left(q_T + \bar{q}_T(1-s_T)^2\right)(\bar{x}_T^{MKV})^2 + \frac{1}{2}\int_0^T (q + \bar{q} + (r+\bar{r})(a\eta_t)^2)v_t dt \\ &+ \frac{1}{2}\int_0^T [q + \bar{q}(1-s)^2 + (r + \bar{r}(1-\bar{s})^2)(c^{MKV}\bar{\eta}_t^{MKV})^2)](\bar{x}_t^{MKV})^2 dt. \end{aligned} \tag{16}$$

In the following, we intend to simplify the explicit solutions (15) and (16) for the social costs in the LQEMFG and LQEMKV problems. First, consider the quantity $\int_0^T (\bar{\eta}_t^{MFG})^2(\bar{x}_t^{MFG})^2 dt$. Using equation (3), we have:

$$\begin{aligned} &\int_0^T (\bar{\eta}_t^{MFG})^2(\bar{x}_t^{MFG})^2 dt \\ &= -\frac{1}{c^{MFG}(b_2+\bar{b}_2)}\left[\int_0^T \dot{\bar{\eta}}_t^{MFG}(\bar{x}_t^{MFG})^2 dt + \int_0^T \left[(2b_1+\bar{b}_1)\bar{\eta}_t^{MFG} + (q + \bar{q}(1-s))\right](\bar{x}_t^{MFG})^2 dt\right]. \end{aligned}$$

Then we use integration by parts:

$$
= -\frac{1}{c^{MFG}(b_2+\bar{b}_2)}\Bigg[\bar{\eta}_T^{MFG}(\bar{x}_T^{MFG})^2 - \bar{\eta}_0^{MFG}(\bar{x}_0^{MFG})^2 - 2\int_0^T \bar{\eta}_t^{MFG}\bar{x}_t^{MFG}\dot{\bar{x}}_t^{MFG}dt
+ \int_0^T \left[(2b_1+\bar{b}_1)\bar{\eta}_t^{MFG} + (q+\bar{q}(1-s))\right](\bar{x}_t^{MFG})^2 dt\Bigg].
$$

Then by using equation (5), we have:

$$
=2\int_0^T (\bar{\eta}_t^{MFG})^2(\bar{x}_t^{MFG})^2 dt - \frac{1}{c^{MFG}(b_2+\bar{b}_2)}\Bigg[\bar{\eta}_T^{MFG}(\bar{x}_T^{MFG})^2 - \bar{\eta}_0^{MFG}(\mathbb{E}(\xi))^2
+ \int_0^T \left[-\bar{b}_1\bar{\eta}_t^{MFG} + (q+\bar{q}(1-s))\right](\bar{x}_t^{MFG})^2 dt\Bigg].
$$

Finally, we solve:

$$
\int_0^T (\bar{\eta}_t^{MFG})^2(\bar{x}_t^{MFG})^2 dt = \frac{1}{c^{MFG}(b_2+\bar{b}_2)}\Bigg[\bar{\eta}_T^{MFG}(\bar{x}_T^{MFG})^2 - \bar{\eta}_0^{MFG}(\mathbb{E}(\xi))^2
+ \int_0^T \left[-\bar{b}_1\bar{\eta}_t^{MFG} + (q+\bar{q}(1-s))\right](\bar{x}_t^{MFG})^2 dt\Bigg].
$$

If we denote:

$$
\lambda := \frac{c^{MFG}}{c^{MKV}} = \frac{b_2}{b_2+\bar{b}_2}\cdot\frac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})},
$$

$$
h_{var} := \frac{1}{2}\int_0^T (q+\bar{q}+(r+\bar{r})(a\eta_t)^2)v_t dt + \frac{1}{2}(q_T+\bar{q}_T)v_T,
$$

and use the terminal condition for $\bar{\eta}_T^{MFG}$, then equation (15) can be rewritten as:

$$
\begin{aligned}
SC^{MFG} =& h_{var} + \frac{1}{2}\int_0^T \left[\bar{b}_1\lambda\bar{\eta}_t^{MFG} + (q+\bar{q}(1-s)^2) - \lambda(q+\bar{q}(1-s))\right](\bar{x}_t^{MFG})^2 dt, \\
&+ \frac{1}{2}\lambda\left(\bar{\eta}_0^{MFG}(\mathbb{E}(\xi))^2 - (q_T+\bar{q}_T(1-s_T))(\bar{x}_T^{MFG})^2\right) + \frac{1}{2}(q_T+\bar{q}_T(1-s_T)^2)(\bar{x}_T^{MFG})^2.
\end{aligned}
\tag{17}
$$

Repeating the calculation for $\int_0^T (\bar{\eta}_t^{MFG})^2(\bar{x}_t^{MFG})^2 dt$ (using equations (9) and (11)), we arrive at:

$$
\int_0^T (\bar{\eta}_t^{MKV})^2(\bar{x}_t^{MKV})^2 dt = \frac{1}{c^{MKV}(b_2+\bar{b}_2)}\Bigg[\bar{\eta}_T^{MKV}(\bar{x}_T^{MKV})^2 - \bar{\eta}_0^{MKV}(\mathbb{E}(\xi))^2
+ \int_0^T (q+\bar{q}(1-s)^2)(\bar{x}_t^{MKV})^2 dt\Bigg].
$$

Using the terminal condition for $\bar{\eta}_T^{MKV}$, equation (16) can be rewritten as:

$$
SC^{MKV} = h_{var} + \frac{1}{2}\bar{\eta}_0^{MKV}(\mathbb{E}(\xi))^2. \tag{18}
$$

Let's denote the (weighted) difference between the solutions of the Riccati equations associated with $\bar{\eta}_t^{MFG}$ and $\bar{\eta}_t^{MKV}$ by:

$$\Delta\bar{\eta}_t = \lambda\bar{\eta}_t^{MFG} - \bar{\eta}_t^{MKV}. \tag{19}$$

Proposition 2. *Under assumption (14), the difference in the social costs in the LQEMFG and LQEMKV problems can be represented by:*

$$\Delta SC := SC^{MFG} - SC^{MKV} = \frac{1}{2} \cdot \frac{(b_2 + \bar{b}_2)^2}{r + \bar{r}(1-\bar{s})^2} \int_0^T (\Delta\bar{\eta}_t \cdot \bar{x}_t^{MFG})^2 dt.$$

Proof. The solutions $\bar{\eta}_t^{MFG}$ and $\bar{\eta}_t^{MKV}$ for the Riccati equations (3) and (9), respectively, are well defined under assumption (14) (see Appendix A). We notice that $\Delta\bar{\eta}_t$ defined in (19) satisfies the following linear first-order differential equation:

$$\frac{d(\Delta\bar{\eta}_t)}{dt} = \gamma_t \Delta\bar{\eta}_t + \beta_t, \qquad \Delta\bar{\eta}_T = \lambda\bar{\eta}_T^{MFG} - \bar{\eta}_T^{MKV}$$

with coefficients:

$$\begin{cases} \gamma_t &= -2b_1 - 2\bar{b}_1 + \dfrac{(b_2 + \bar{b}_2)^2}{r + \bar{r}(1-\bar{s})^2} \left(\lambda\bar{\eta}_t^{MFG} + \bar{\eta}_t^{MKV}\right), \\ \beta_t &= \bar{b}_1 \lambda \bar{\eta}_t^{MFG} + (q + \bar{q}(1-s)^2) - \lambda(q + \bar{q}(1-s)). \end{cases}$$

Since $q_T + \bar{q}_T(1-s_T) = \bar{\eta}_T^{MFG}$, $q_T + \bar{q}_T(1-s_T)^2 = \bar{\eta}_T^{MKV}$ and $\lambda\bar{\eta}_0^{MFG} - \bar{\eta}_0^{MKV} = \Delta\bar{\eta}_0$, we deduce from equations (17) and (18) that:

$$\begin{aligned} &SC^{MFG} - SC^{MKV} \\ &= \frac{1}{2}\left[\Delta\bar{\eta}_0(\mathbb{E}(\xi))^2 - \Delta\bar{\eta}_T(\bar{x}_T^{MFG})^2 + \int_0^T \beta_t (\bar{x}_t^{MFG})^2 dt\right] \\ &= \frac{1}{2}\int_0^T \left[-\frac{d(\Delta\bar{\eta}_t(\bar{x}_t^{MFG})^2)}{dt} + \left(\frac{d(\Delta\bar{\eta}_t)}{dt} - \gamma_t\Delta\bar{\eta}_t\right)(\bar{x}_t^{MFG})^2\right] dt \\ &= \frac{1}{2}\int_0^T \left[-2\Delta\bar{\eta}_t \bar{x}_t^{MFG}\dot{\bar{x}}_t^{MFG} - \gamma_t\Delta\bar{\eta}_t(\bar{x}_t^{MFG})^2\right] dt \\ &= \frac{1}{2}\int_0^T \Delta\bar{\eta}_t(\bar{x}_t^{MFG})^2 \left[-2\left(b_1 + \bar{b}_1 - \frac{(b_2 + \bar{b}_2)^2}{r + \bar{r}(1-\bar{s})^2}\lambda\bar{\eta}_t^{MFG}\right) - \gamma_t\right] dt \\ &= \frac{1}{2} \cdot \frac{(b_2 + \bar{b}_2)^2}{r + \bar{r}(1-\bar{s})^2} \int_0^T (\Delta\bar{\eta}_t \cdot \bar{x}_t^{MFG})^2 dt, \end{aligned}$$

where we use equation (5) for the fourth equality.

□

Remark 5. *We can see directly from Proposition 2 that the social cost in the LQEMFG problem is larger than (or possibly equal to) the social cost in the LQEMKV problem. This result is consistent with the definition of the price of anarchy in Section 1.3.*

Therefore, under assumption (14), the price of anarchy for the LQ model is given by:

$$PoA = 1 + \frac{\Delta SC}{SC^{MKV}} = 1 + \frac{\frac{(b_2+\bar{b}_2)^2}{r+\bar{r}(1-\bar{s})^2}\displaystyle\int_0^T (\Delta\bar{\eta}_t \cdot \bar{x}_t^{MFG})^2 dt}{\displaystyle\int_0^T \left[q+\bar{q}+\frac{b_2^2}{r+\bar{r}}\eta_t^2\right] v_t dt + (q_T+\bar{q}_T)v_T + \bar{\eta}_0^{MKV}(\mathbb{E}(\xi))^2}. \tag{20}$$

Corollary 2. *Assuming (14), if the initial condition ξ is such that $\mathbb{E}(\xi)=0$, then $PoA=1$.*

Proof. By equation (6), $\mathbb{E}(\xi)=0$ implies $\bar{x}_t^{MFG}=0$, $\forall t\in[0,T]$. Therefore by Proposition 2, $\Delta SC=0$. From equation (7), which we recall is equivalent to equation (13), $v_t>0$ which implies $h_{var}>0$. Therefore by equation (18), $SC^{MKV}>0$. We conclude that $PoA=1$. □

We study in the following the variation of PoA by letting one of the coefficients tend to zero or to infinity. It will be useful for us to note here the scalar Riccati equations associated with $u_t := \lambda\bar{\eta}_t^{MFG}$, $w_t := \bar{\eta}_t^{MKV}$ and η_t:

$$\dot{u}_t - 2A^u u_t - Bu_t^2 + C^u = 0 \qquad u_T = D^u \tag{21}$$

$$\dot{w}_t - 2A^w w_t - Bw_t^2 + C^w = 0 \qquad w_T = D^w \tag{22}$$

$$\dot{\eta}_t - 2A^\eta \eta_t - B^\eta\eta_t^2 + C^\eta = 0 \qquad \eta_T = D^\eta \tag{23}$$

where:

$$\begin{aligned}
&A^u = -\left(b_1+\frac{\bar{b}_1}{2}\right), && A^w = -(b_1+\bar{b}_1), && A^\eta = -b_1,\\
&B^u = \frac{(b_2+\bar{b}_2)^2}{r+\bar{r}(1-\bar{s})^2}, && B^w = \frac{(b_2+\bar{b}_2)^2}{r+\bar{r}(1-\bar{s})^2}, && B^\eta = \frac{b_2^2}{r+\bar{r}},\\
&C^u = \lambda(q+\bar{q}(1-s)), && C^w = q+\bar{q}(1-s)^2, && C^\eta = q+\bar{q},\\
&D^u = \lambda(q_T+\bar{q}_T(1-s_T)), && D^w = q_T+\bar{q}_T(1-s_T)^2, && D^\eta = q_T+\bar{q}_T.
\end{aligned}$$

If $B^u\neq 0$, $B^uD^u\geq 0$ and $B^uC^u>0$, we have (see equation (45) in Appendix A) the existence and uniqueness for u_t which can be expressed by:

$$u_t = \frac{C^u(1-e^{-(\delta_u^+-\delta_u^-)(T-t)}) + D^u(\delta_u^+ - \delta_u^- e^{-(\delta_u^+-\delta_u^-)(T-t)})}{B^uD^u(1-e^{-(\delta_u^+-\delta_u^-)(T-t)}) + \delta_u^+ e^{-(\delta_u^+-\delta_u^-)(T-t)} - \delta_u^-}, \tag{24}$$

where $\delta_u^\pm = -A^u \pm \sqrt{(A^u)^2+B^uC^u}$. Under assumption (14), the above conditions on B^u, C^u, and D^u are satisfied, and we have $\delta_u^- < 0 < \delta_u^+$, $u_t>0$ for all $t\in[0,T)$, and $u_T\geq 0$. We have analogous expressions for w_t and η_t, in terms of $\delta_w^\pm$ and $\delta_\eta^\pm$, respectively. Note that $B^u = B^w =: B$.

To make the following computations easier to follow, we repeat equations (20), (6), and (7), which we recall is equivalent to equation (13), using the above notations:

$$PoA = 1+\frac{\Delta SC}{SC^{MKV}} = 1 + \frac{B\displaystyle\int_0^T (u_t-w_t)^2\cdot(\bar{x}_t^{MFG})^2 dt}{\displaystyle\int_0^T \left[q+\bar{q}+B^\eta\eta_t^2\right]v_t dt + (q_T+\bar{q}_T)v_T + w_0(\mathbb{E}(\xi))^2}, \tag{25}$$

$$\bar{x}_t^{MFG} = \mathbb{E}(\xi) e^{\int_0^t (b_1 + \bar{b}_1 - Bu_s) ds}, \tag{26}$$

$$v_t = Var(\xi) e^{\int_0^t 2(b_1 - B^\eta \eta_s) ds} + \sigma^2 \int_0^t e^{2\int_s^t (b_1 - B^\eta \eta_u) du} ds. \tag{27}$$

Also for convenience, recall the definition:

$$\lambda = \frac{b_2}{b_2 + \bar{b}_2} \cdot \frac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})}.$$

It will also be useful to compute the derivative of u_t with respect to time t from the explicit form in equation (24):

$$\frac{du_t}{dt} = \frac{\left(B(D^u)^2 + 2A^u D^u - C^u\right) \cdot (\delta_u^+ - \delta_u^-)^2 \, e^{-(\delta_u^+ - \delta_u^-)(T-t)}}{\left[BD^u(1 - e^{-(\delta_u^+ - \delta_u^-)(T-t)}) + \delta_u^+ e^{-(\delta_u^+ - \delta_u^-)(T-t)} - \delta_u^-\right]^2}. \tag{28}$$

Note that u_t is increasing if $B(D^u)^2 + 2A^u D^u - C^u > 0$, and likewise, decreasing if $B(D^u)^2 + 2A^u D^u - C^u < 0$.

In the following propositions, we utilize the following assumption.

Assumption 1. *Assume (14). In addition, assume:* $b_1 > 0$, $D^u > 0$, $D^w > 0$, $D^\eta > 0$ *and the initial condition satisfies* $\mathbb{E}(\xi) \neq 0$.

Proposition 3. *Assuming assumption 1, then:*

$$\lim_{r\to\infty} PoA = 1 \qquad \text{and} \qquad \lim_{\bar{r}\to\infty} PoA = 1.$$

Proof. First, we consider $r \to \infty$. For every given $r > 0$, we have existence and uniqueness of the solutions u_t^r, w_t^r and η_t^r to the scalar Riccati equations (21)-(23). Note that we have added the superscript r to emphasize the dependence on this parameter.

When $r \to \infty$, we have:

$$\lambda^r \longrightarrow \lambda^{r\to\infty} := \frac{b_2}{b_2 + \bar{b}_2},$$

and

$$B^r \longrightarrow 0, \quad B^{\eta,r} \longrightarrow 0,$$
$$C^{u,r} \longrightarrow C_u^{r\to\infty} := \lambda^{r\to\infty}(q + \bar{q}(1-s)), \quad D^{u,r} \to D_u^{r\to\infty} := \lambda^{r\to\infty}(q_T + \bar{q}_T(1-s_T)).$$

Let $u^{r\to\infty} : [0,T] \to \mathbb{R}$ be the solution to the linear first-order differential equation:

$$(u_t^{r\to\infty})' - 2A^u u_t^{r\to\infty} + C_u^{r\to\infty} = 0, \qquad u_T^{r\to\infty} = D_u^{r\to\infty}.$$

Then we have:

$$u_t^{r\to\infty} = \left(D_u^{r\to\infty} - \frac{C_u^{r\to\infty}}{2A^u}\right) e^{-2A^u(T-t)} + \frac{C_u^{r\to\infty}}{2A^u}.$$

It is easy to show directly from their explicit solutions that for every time $t \in [0,T]$,

$$\lim_{r\to\infty} u_t^r = u_t^{r\to\infty} \quad \text{and thus,} \quad \lim_{r\to\infty} B^r u_t^r = 0.$$

Next, our goal is to bound the u_t^r uniformly over $t \in [0,T]$ for large r. Note that $A^u < 0$, B^r, $C^{u,r}$, $\lambda^{r\to\infty}$, $C_u^{r\to\infty}$, $D^{u,r}$, $D_u^{r\to\infty} > 0$, and $\delta_u^{-,r} < 0 < \delta_u^{+,r}$. Let $\epsilon > 0$. Then there exists $r^* > 0$ such that $\max\{B^r, C^{u,r}, D^{u,r}\} \le \max\{C_u^{r\to\infty}, D_u^{r\to\infty}\} + \epsilon =: c_1$ for $r \ge r^*$. Thus, we can deduce that for $r \ge r^*$, and for every $t \in [0,T]$:

$$|u_t^r| \le \frac{C^{u,r} + D^{u,r}(\delta_u^{+,r} - \delta_u^{-,r})}{\delta_u^{+,r} e^{-(\delta_u^{+,r} - \delta_u^{-,r})(T-t)}} \le \frac{c_1 + 2c_1\sqrt{(A^u)^2 + c_1^2}}{-2A^u e^{-2T\sqrt{(A^u)^2+c_1^2}}}.$$

From equation (26) and by the bounded convergence theorem, we have for every $t \in [0,T]$:

$$\lim_{r\to\infty} \bar{x}_t^{MFG,r} = \mathbb{E}(\xi) e^{(b_1+\bar{b}_1)t} =: \bar{x}_t^{MFG,r\to\infty}.$$

Moreover, $\bar{x}_t^{MFG,r}$ is uniformly bounded for $t \in [0,T]$. From the non-negativity of u_t^r, we have:

$$\left|\bar{x}_t^{MFG,r}\right| \le |\mathbb{E}(\xi)| e^{(b_1+\bar{b}_1)T}, \quad \forall t \in [0,T].$$

Similarly, for every $t \in [0,T]$,

$$\lim_{r\to\infty} w_t^r =: w_t^{r\to\infty}, \qquad \lim_{r\to\infty} \eta_t^r =: \eta_t^{r\to\infty},$$

and the functions w_t^r and η_t^r are uniformly bounded over $t \in [0,T]$ and large r. By the bounded convergence theorem we have for every $t \in [0,T]$:

$$\lim_{r\to\infty} \int_0^T (u_t^r - w_t^r)^2 (\bar{x}_t^{MFG,r})^2 dt = \int_0^T (u_t^{r\to\infty} - w_t^{r\to\infty})^2 (\bar{x}_t^{MFG,r\to\infty})^2 dt \ < \infty,$$

and thus,

$$\lim_{r\to\infty} \Delta SC^r = \lim_{r\to\infty} \frac{1}{2} \cdot B^r \int_0^T (u_t^r - w_t^r)^2 (\bar{x}_t^{MFG,r})^2 dt = 0.$$

From equation (27) and by the bounded convergence theorem, we have for every $t \in [0,T]$:

$$\lim_{r\to\infty} v_t^r = Var(\xi) e^{2b_1 t} + \sigma^2 \int_0^t e^{2b_1(t-s)} ds =: v_t^{r\to\infty}.$$

The variance function v_t^r is also uniformly bounded over $t \in [0,T]$. We also have $w_0^{r\to\infty} \ge 0$ and $v_t^{r\to\infty} > 0$ for $t > 0$. Hence,

$$\lim_{r\to\infty} SC^{MKV,r} = \frac{1}{2}\left(\int_0^T (q+\bar{q}) v_t^{r\to\infty} dt + (q_T + \bar{q}_T) v_T^{r\to\infty} + w_0^{r\to\infty} (\mathbb{E}(\xi))^2\right) > 0.$$

Therefore, from equation (25), we have

$$\lim_{r\to\infty} PoA^r = 1.$$

By replacing $\lambda^{r\to\infty}$ with $\lambda^{\bar{r}\to\infty} := \frac{b_2}{b_2+\bar{b}_2}(1-\bar{s})$, the proof can be repeated, and we obtain $\lim\limits_{\bar{r}\to\infty} PoA^{\bar{r}} = 1$. □

Proposition 4. *Assuming assumption 1, and if*

$$\frac{q+\bar{q}(1-s)}{r+\bar{r}(1-\bar{s})}=\frac{q+\bar{q}(1-s)^2}{r+\bar{r}(1-\bar{s})^2},$$

then

$$\lim_{b_2\to\infty} PoA = 1.$$

Proof. When $b_2\to\infty$, we have:

$$\begin{aligned}&\lambda^{b_2}\to \tfrac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})} =: \lambda^{b_2\to\infty}, \quad B^{b_2}\to\infty, \qquad B^{\eta,b_2}\to\infty\\ &C^{u,b_2}\to\lambda^{b_2\to\infty}(q+\bar{q}(1-s)) =: C^{u,b_2\to\infty}, \qquad D^{u,b_2}\to\lambda^{b_2\to\infty}(q_T+\bar{q}_T(1-s_T)) =: D^{u,b_2\to\infty}>0,\end{aligned}$$

and $A^u,(A^w,C^w,D^w),(A^\eta,C^\eta,D^\eta)$ are independent of b_2. Moreover, we notice that:

$$\frac{\delta_u^{\pm,b_2}}{b_2+\bar{b}_2} = -\frac{A^u}{b_2+\bar{b}_2}\pm\sqrt{\frac{(A^u)^2}{(b_2+\bar{b}_2)^2}+\frac{C^{u,b_2}}{r+\bar{r}(1-\bar{s})^2}} \xrightarrow[b_2\to\infty]{} \pm\sqrt{\frac{q+\bar{q}(1-s)}{r+\bar{r}(1-\bar{s})}} =: \pm c_{\delta_u^{b_2\to\infty}},$$

and thus, $\lim_{b_2\to\infty}\delta_u^{+,b_2}-\delta_u^{-,b_2}=+\infty$. From equation (24), for each $t\in[0,T)$, we deduce:

$$\begin{aligned}(b_2+\bar{b}_2)u_t^{b_2} &= \frac{\left(\dfrac{C^{u,b_2}}{b_2+\bar{b}_2}+D^{u,b_2}\cdot\dfrac{\delta_u^{+,b_2}}{b_2+\bar{b}_2}\right)-\left(\dfrac{C^{u,b_2}}{b_2+\bar{b}_2}+D^{u,b_2}\cdot\dfrac{\delta_u^{-,b_2}}{b_2+\bar{b}_2}\right)e^{-(\delta_u^{+,b_2}-\delta_u^{-,b_2})(T-t)}}{\left(\dfrac{-\delta_u^{-,b_2}}{(b_2+\bar{b}_2)^2}+\dfrac{D^{u,b_2}}{r+\bar{r}(1-\bar{s})^2}\right)+\left(\dfrac{\delta_u^{+,b_2}}{(b_2+\bar{b}_2)^2}-\dfrac{D^{u,b_2}}{r+\bar{r}(1-\bar{s})^2}\right)e^{-(\delta_u^{+,b_2}-\delta_u^{-,b_2})(T-t)}}\\[2ex] &\xrightarrow[b_2\to\infty]{} \quad (r+\bar{r}(1-\bar{s})^2)\cdot c_{\delta_u^{b_2\to\infty}} =: c_u.\end{aligned}$$

Similarly, for all $t\in[0,T)$,

$$\lim_{b_2\to\infty}(b_2+\bar{b}_2)w_t^{b_2} = (r+\bar{r}(1-\bar{s})^2)\cdot c_{\delta_w^{b_2\to\infty}} =: c_w, \qquad \text{with} \qquad c_{\delta_w^{b_2\to\infty}} := \sqrt{\frac{q+\bar{q}(1-s)^2}{r+\bar{r}(1-\bar{s})^2}},$$

and

$$\lim_{b_2\to\infty} b_2\eta_t^{b_2} = (r+\bar{r})\cdot c_{\delta_\eta^{b_2\to\infty}} =: c_\eta, \qquad \text{with} \qquad c_{\delta_\eta^{b_2\to\infty}} := \sqrt{\frac{q+\bar{q}}{r+\bar{r}}}.$$

Next, we derive a strictly positive uniform lower bound for $(b_2+\bar{b}_2)u_t^{b_2}$ over $[0,T]$ and large b_2. Let $\zeta_1 := \frac{1}{2}\min\left\{c_{\delta_u^{b_2\to\infty}}, D^{u,b_2\to\infty}\right\}$. Then there exists $b_2^{*,u,lower}>0$ such that for all $b_2\geq b_2^{*,u,lower}$,

$$\max\left\{\left|\frac{\delta_u^{+,b_2}}{b_2+\bar{b}_2}-c_{\delta_u^{b_2\to\infty}}\right|,\ \left|\frac{\delta_u^{-,b_2}}{b_2+\bar{b}_2}-\left(-c_{\delta_u^{b_2\to\infty}}\right)\right|, |D^{u,b_2}-D^{u,b_2\to\infty}|, \left|\frac{1}{b_2+\bar{b}_2}\right|\right\}\leq\zeta_1,$$

and thus for all $t \in [0,T]$,

$$
\begin{aligned}
(b_2+\bar{b}_2)u_t^{b_2} &\geq (b_2+\bar{b}_2)\frac{D^{u,b_2}\delta_u^{+,b_2}}{(\delta_u^{+,b_2}-\delta_u^{-,b_2})+B^{b_2}D^{u,b_2}} \\
&\geq \frac{(D^{u,b_2\to\infty}-\zeta_1)\cdot(c_{\delta_u^{b_2\to\infty}}-\zeta_1)}{\zeta_1\cdot(2c_{\delta_u^{b_2\to\infty}}+2\zeta_1)+(D^{u,b_2\to\infty}+\zeta_1)/(r+\bar{r}(1-\bar{s})^2)} =: m_u > 0. \qquad (29)
\end{aligned}
$$

Then, by the same technique in inequality (29), there exists $b_2^{*,\eta,lower} > 0$ and $m_\eta > 0$ such that for all $b_2 \geq b_2^{*,\eta,lower}$,

$$
b_2\eta_t^{b_2} \geq m_\eta.
$$

From equation (28), we see that $t \mapsto u_t^{b_2}$ is increasing if $B^{b_2}(D^{u,b_2})^2 + 2A^uD^{u,b_2} - C^{u,b_2} > 0$. Since $\lim_{b_2\to\infty} B^{b_2}(D^{u,b_2})^2 + 2A^uD^{u,b_2} - C^{u,b_2} = \infty$, there exists $b_2^{*,u,upper} > 0$ such that for all $b_2 \geq b_2^{*,u,upper}$, we have $|D^{u,b_2} - D^{u,b_2\to\infty}| \leq 1$ and $t \mapsto u_t^{b_2}$ is increasing. Therefore,

$$
u_t^{b_2} \leq u_T^{b_2} = D^{u,b_2} \leq D^{u,b_2\to\infty} + 1, \quad \forall\, t \in [0,T],\ b_2 \geq b_2^{*,u,upper}.
$$

By the same argument for $w_t^{b_2}$ and $\eta_t^{b_2}$, there exists $b_2^{*,upper} \geq b_2^{*,u,upper}$ such that:

$$
\max\left\{|u_t^{b_2}|, |w_t^{b_2}|, |\eta_t^{b_2}|\right\} \leq M, \quad \forall\, t \in [0,T],\ b_2 \geq b_2^{*,upper}, \qquad (30)
$$

and such that the functions $t \mapsto u_t^{b_2}$, $t \mapsto w_t^{b_2}$ and $t \mapsto \eta_t^{b_2}$ are increasing on $[0,T]$.

From the assumption:

$$
\frac{q+\bar{q}(1-s)}{r+\bar{r}(1-\bar{s})} = \frac{q+\bar{q}(1-s)^2}{r+\bar{r}(1-\bar{s})^2},
$$

we have $c_{\delta_u^{b_2\to\infty}} = c_{\delta_w^{b_2\to\infty}}$ and therefore, $c_u = c_w =: c$. We would like to show in this case that $\lim_{b_2\to\infty} \frac{\Delta SC^{b_2}}{SC^{MKV,b_2}} = 0$. Our approach is to split the interval $[0,T]$ into two parts: $[0,T/2]$ and $[T/2,T]$. Since $v_t^{b_2} \geq 0$ for all $t \in [0,T]$, we have $SC^{MKV,b_2} \geq w_0^{b_2}(\mathbb{E}(\xi))^2$. For the sake of simplicity, we denote $h_u(b_2,t) := (b_2+\bar{b}_2)u_t^{b_2}$ and $h_w(b_2,t) := (b_2+\bar{b}_2)w_t^{b_2}$. We have:

$$
\begin{aligned}
\frac{\Delta SC^{b_2}}{SC^{MKV,b_2}} &\leq \frac{1}{w_0^{b_2}\mathbb{E}(\xi)^2}\left(B^{b_2}\int_0^{\frac{T}{2}}(u_t^{b_2}-w_t^{b_2})^2(\bar{x}_t^{MFG,b_2})^2dt + B^{b_2}\int_{\frac{T}{2}}^T(u_t^{b_2}-w_t^{b_2})^2(\bar{x}_t^{MFG,b_2})^2dt\right) \\
&= \frac{1}{(b_2+\bar{b}_2)w_0^{b_2}}\left(I_1^{b_2}+I_2^{b_2}\right),
\end{aligned} \qquad (31)
$$

where:

$$
\begin{aligned}
I_1^{b_2} &= \frac{(b_2+\bar{b}_2)^3}{r+\bar{r}(1-\bar{s})^2}\int_0^{\frac{T}{2}}(u_t^{b_2}-w_t^{b_2})^2e^{2(b_1+\bar{b}_1)t}\exp\left(-2B^{b_2}\int_0^t u_s^{b_2}ds\right)dt \\
&= \frac{b_2+\bar{b}_2}{r+\bar{r}(1-\bar{s})^2}\int_0^{\frac{T}{2}}[h_u(b_2,t)-h_w(b_2,t)]^2\cdot e^{2(b_1+\bar{b}_1)t}\exp\left(-\frac{2(b_2+\bar{b}_2)}{r+\bar{r}(1-\bar{s})^2}\int_0^t h_u(b_2,s)ds\right)dt,
\end{aligned}
$$

and:

$$
I_2^{b_2} = \frac{(b_2+\bar{b}_2)^3}{r+\bar{r}(1-\bar{s})^2}\int_{\frac{T}{2}}^T(u_t^{b_2}-w_t^{b_2})^2e^{2(b_1+\bar{b}_1)t}\exp\left(-\frac{2(b_2+\bar{b}_2)}{r+\bar{r}(1-\bar{s})^2}\int_0^t h_u(b_2,s)ds\right)dt.
$$

Fix $\epsilon > 0$. In the following, we show that $I_1^{b_2} \leq \epsilon$ and $I_2^{b_2} \leq \epsilon$ for large b_2. First, consider $I_1^{b_2}$. Recall that for $t \in [0, T/2]$, we have $\lim_{b_2\to\infty} h_u(b_2, t) = \lim_{b_2\to\infty} h_w(b_2, t) = c$, and for all $b_2 \geq b_2^{*,upper}$, the functions $[0, T/2] \ni s \mapsto u_s^{b_2}$ and $[0, T/2] \ni s \mapsto w_s^{b_2}$ are increasing, and thus, $[0, T/2] \ni s \mapsto h_u(b_2, t)$ and $[0, T/2] \ni s \mapsto h_w(b_2, t)$ are increasing. (Note that $T/2 < T$ is chosen arbitrarily, since the above limits do not hold at T.) Let

$$\zeta_2 := \min\left\{\frac{c}{2}, \frac{1}{2}e^{-T(b_1+\bar{b}_2)}\sqrt{\epsilon c}\right\}.$$

Then there exists $b_2^{*,I_1} \geq b_2^{*,upper}$ such that for all $b_2 \geq b_2^{*,I_1}$ and all $s \in [0, T/2]$ we have:

$$\begin{array}{ccccccccc} c - \zeta_2 & \leq & h_u(b_2, 0) & \leq & h_u(b_2, s) & \leq & h_u(b_2, T/2) & \leq & c + \zeta_2, \\ c - \zeta_2 & \leq & h_w(b_2, 0) & \leq & h_w(b_2, s) & \leq & h_w(b_2, T/2) & \leq & c + \zeta_2. \end{array}$$

Thus, for any $t \in [0, \frac{T}{2}]$ and $b_2 \geq b_2^{*,I_1}$, we have:

$$|h_u(b_2, t) - h_w(b_2, t)|^2 \leq 4\zeta_2^2 \quad \text{and} \quad \int_0^t h_u(b_2, s)ds \geq (c - \zeta_2)t \geq \frac{c}{2} \cdot t.$$

Therefore,

$$\begin{aligned} I_1^{b_2} &\leq 4\zeta_2^2 e^{2T(b_1+\bar{b}_1)} \cdot \frac{(b_2 + \bar{b}_2)}{r + \bar{r}(1 - \bar{s})^2} \int_0^{\frac{T}{2}} \exp\left(-\frac{2(b_2 + \bar{b}_2)}{r + \bar{r}(1 - \bar{s})^2} \cdot \frac{c}{2} \cdot t\right) dt \\ &= \frac{4e^{2T(b_1+\bar{b}_1)}}{c} \cdot \left(1 - e^{-\frac{(b_2+\bar{b}_2)c}{r+\bar{r}(1-\bar{s})^2}\frac{T}{2}}\right) \cdot \zeta_2^2 \leq \epsilon, \end{aligned} \tag{32}$$

where the last inequality comes from the definition of ζ_2.

Next, consider $I_2^{b_2}$. Since $u_t^{b_2}$ is positive over $[0, T]$, we know from the inequalities (29) and (30) that for all $b_2 \geq \max\{b_2^{*,upper}, b_2^{*,u,lower}\}$ and all $t \in [T/2, T]$:

$$|u_t^{b_2} - w_t^{b_2}| \leq \sup_{0\leq s\leq T} |u_s^{b_2}| + |w_s^{b_2}| \leq 2M, \qquad \text{and} \qquad \int_0^t h_u(b_2, s)ds \geq \int_0^{\frac{T}{2}} h_u(b_2, s)ds \geq \frac{T}{2}m_u > 0.$$

Hence, there exists $b_2^{*,I_2} \geq \max\{b_2^{*,upper}, b_2^{*,u,lower}\}$ such that for all $b_2 \geq b_2^{*,I_2}$:

$$\begin{aligned} I_2^{b_2} &\leq \frac{(b_2 + \bar{b}_2)^3}{r + \bar{r}(1 - \bar{s})^2} \cdot 4M^2 \cdot e^{2(b_1+\bar{b}_1)T} \int_{\frac{T}{2}}^{T} \exp\left(-\frac{T(b_2 + \bar{b}_2)}{r + \bar{r}(1 - \bar{s})^2} \cdot m_u\right) dt \\ &= \kappa_3(b_2 + \bar{b}_2)^3 e^{-\kappa_4(b_2+\bar{b}_2)} \leq \epsilon, \end{aligned} \tag{33}$$

where $\kappa_3 := \frac{2TM^2 e^{2(b_1+\bar{b}_1)T}}{r+\bar{r}(1-\bar{s})^2} > 0$ and $\kappa_4 := \frac{Tm_u}{r+\bar{r}(1-\bar{s})^2} > 0$ are constants independent of b_2.

Let $b_2^* := \max\{b_2^{*,I_1}, b_2^{*,I_2}\}$. Then inequalities (31), (32) and (33) give for $b_2 \geq b_2^*$:

$$\frac{\Delta SC^{b_2}}{SC^{MKV,b_2}} \leq \frac{I_1^{b_2} + I_2^{b_2}}{(b_2 + \bar{b}_2)w_0^{b_2}} \leq \frac{\epsilon + \epsilon}{h_w(b_2, 0)} \leq \frac{2\epsilon}{c/2} = \frac{4}{c}\epsilon.$$

Since the proof holds for arbitrary $\epsilon > 0$, and $c = \sqrt{(q + \bar{q}(1 - s)^2)(r + \bar{r}(1 - \bar{s})^2)} > 0$ is independent of b_2 and ϵ, we conclude:

$$\lim_{b_2\to\infty} \frac{\Delta SC^{b_2}}{SC^{MKV,b_2}} = 0,$$

and thus, $\lim_{b_2\to\infty} PoA^{b_2} = 1$. □

Proposition 5. *Assume assumption 1. If $\bar{b}_2 = 0$, then:*

$$\lim_{b_2 \to 0} PoA = 1,$$

whereas if $\bar{b}_2 > 0$, then:

$$\lim_{b_2 \to 0} PoA > 1.$$

Proof. **Case 1:** First, consider the case $\bar{b}_2 = 0$. As $b_2 \to 0$, we have:

$$B^{b_2} \to 0, \quad B^{\eta, b_2} \to 0,$$

and $\lambda = \frac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})}$, $(A^u, C^u, D^u), (A^w, C^w, D^w), (A^\eta, C^\eta, D^\eta)$ are all independent of b_2. We can then use the same technique shown in Proposition 3 to conclude that $\lim_{b_2 \to 0} PoA = 1$.

Case 2: Now, let's assume $\bar{b}_2 > 0$. As $b_2 \to 0$, we have:

$$\lambda \longrightarrow 0, \quad B^{b_2} \longrightarrow \frac{\bar{b}_2^2}{r + \bar{r}(1-\bar{s})^2} =: B^{b_2 \to 0} > 0, \qquad B^{\eta, b_2} \longrightarrow 0, \quad C^{u, b_2} \longrightarrow 0, \; D^{u, b_2} \longrightarrow 0,$$

and $A^u, (A^w, C^w, D^w), (A^\eta, C^\eta, D^\eta)$ are independent of b_2. Moreover, we have:

$$\lim_{b_2 \to 0} \delta_u^{+, b_2} = -2A^u > 0, \quad \lim_{b_2 \to 0} \delta_u^{-, b_2} = 0, \quad \text{and} \quad \lim_{b_2 \to 0} (\delta_u^+ - \delta_u^-) = -2A^u > 0.$$

Thus, from equation (24) we deduce that for every fixed time $t \in [0, T]$, $\lim_{b_2 \to 0} u_t^{b_2} = 0$.

Similar to Proposition 3, we can derive a uniform bound for $u_t^{b_2}$ over $[0, T]$ for small b_2. Indeed, for any fixed $\epsilon > 0$ there exists $b_2^* > 0$ such that for any $b_2 \leq b_2^*$, we have:

$$\max\{B^{b_2}, C^{u, b_2}, D^{u, b_2}\} \leq B^{b_2 \to \infty} + \epsilon =: c_1, \quad \text{and thus,} \quad |u_t^{b_2}| \leq \frac{c_1 + 2\epsilon\sqrt{(A^u)^2 + c_1^2}}{-2A^u e^{-2T\sqrt{(A^u)^2 + c_1^2}}}, \; \forall\, t \in [0, T].$$

From equation (26), the assumption $\mathbb{E}(\xi) \neq 0$, and by the bounded convergence theorem, we derive that for any fixed $t \in [0, T]$:

$$\lim_{b_2 \to 0} \bar{x}_t^{MFG, b_2} = \mathbb{E}(\xi) e^{(b_1 + \bar{b}_1)t} =: \bar{x}_t^{MFG, b_2 \to 0} \neq 0.$$

It can also be shown that $\left|\bar{x}_t^{MFG, b_2}\right| \leq |\mathbb{E}(\xi)| e^{(b_1 + \bar{b}_1)T}$ for any $t \in [0, T]$ and $b_2 > 0$.

Moreover, since $B^{b_2 \to 0} > 0$, $B^{b_2 \to 0} C^w > 0$, $B^{b_2 \to 0} D^w > 0$, we have $\lim_{b_2 \to 0} w_t^{b_2} =: w_t^{b_2 \to 0}$, and $w_t^{b_2 \to 0}$ is strictly positive over $[0, T)$. It is easy to check that $w_t^{b_2}$ is also uniformly bounded over $[0, T]$ for small b_2. Hence, from Proposition 2 and the bounded convergence theorem, we deduce:

$$\lim_{b_2 \to 0} \Delta SC^{b_2} = \frac{1}{2} B^{b_2 \to 0} \int_0^T (w_t^{b_2 \to 0} \cdot \bar{x}_t^{MFG, b_2 \to 0})^2 dt > 0.$$

Since $B^{\eta, b_2} \to 0, A^\eta < 0, C^\eta > 0$, and $D^\eta > 0$, using the same argument shown in Proposition 3, we deduce that $\eta_t^{b_2}$ is uniformly bounded over $[0, T]$ for small b_2 and for all $t \in [0, T]$:

$$\lim_{b_2 \to 0} \eta_t^{b_2} = \left(D^\eta - \frac{C^\eta}{2A^\eta}\right) e^{-2A^\eta (T-t)} + \frac{C^\eta}{2A^\eta} =: \eta_t^{b_2 \to 0}.$$

From equation (27) and the bounded convergence theorem, for each $t \in [0,T]$:

$$\lim_{b_2 \to 0} v_t^{b_2} = Var(\xi)e^{2b_1 t} + \sigma^2 \int_0^t e^{2b_1(t-s)} ds =: v_t^{b_2 \to 0} > 0,$$

and thus, $0 < \lim_{b_2 \to 0} SC^{MKV, b_2} < \infty$. We conclude $\lim_{b_2 \to 0} PoA^{b_2} > 1$. □

Proposition 6. *Assuming assumption 1, then:*

$$\lim_{\bar{b}_2 \to \infty} PoA = 1.$$

Furthermore, if $\frac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})} \neq \frac{q_T+\bar{q}_T(1-s_T)^2}{q_T+\bar{q}_T(1-s_T)}$ *then:*

$$\lim_{\bar{b}_2 \to 0} PoA > 1.$$

Proof. **Case 1:** When $\bar{b}_2 \to \infty$, we have:

$$\lambda^{\bar{b}_2} \to 0, \quad B^{\bar{b}_2} \to \infty, \; C^{u,\bar{b}_2} \to 0, \; D^{u,\bar{b}_2} \to 0,$$

and $A^u, (A^w, C^w, D^w), (A^\eta, B^\eta, C^\eta, D^\eta)$ are independent of $\bar{b}_2$. Following the same technique used in Proposition 4, we can show that:

$$\lim_{\bar{b}_2 \to \infty} \frac{\delta_u^{\pm,\bar{b}_2}}{\sqrt{b_2+\bar{b}_2}} = \pm\sqrt{\frac{b_2(q+\bar{q}(1-s))}{r+\bar{r}(1-\bar{s})}} =: \pm c_{\delta_u^{\bar{b}_2 \to \infty}}, \; \lim_{\bar{b}_2 \to \infty} \frac{\delta_w^{\pm,\bar{b}_2}}{b_2+\bar{b}_2} = \pm\sqrt{\frac{q+\bar{q}(1-s)^2}{r+\bar{r}(1-\bar{s})^2}} =: \pm c_{\delta_w^{\bar{b}_2 \to \infty}},$$

and, for all $t \in [0,T)$:

$$\begin{aligned}
&\lim_{\bar{b}_2 \to \infty} (b_2+\bar{b}_2)^{\frac{3}{2}} \cdot u_t^{\bar{b}_2} = (r+\bar{r}(1-\bar{s})^2) \cdot c_{\delta_u^{\bar{b}_2 \to \infty}} =: c_u, \\
&\lim_{\bar{b}_2 \to \infty} (b_2+\bar{b}_2) w_t^{\bar{b}_2} = (r+\bar{r}(1-\bar{s}^2)) \cdot c_{\delta_w^{\bar{b}_2 \to \infty}} =: c_w.
\end{aligned}$$

Next, we provide a uniform upper bound for $u_t^{\bar{b}_2}$ over $[0,T]$ and large $\bar{b}_2$.
Let $\zeta_1 := \frac{1}{2}\min\left\{c_{\delta_u^{\bar{b}_2 \to \infty}}, \; c_{\delta_w^{\bar{b}_2 \to \infty}}\right\}$. Then there exists $\bar{b}_2^{*,u} > 0$ such that for all $\bar{b}_2 \geq \bar{b}_2^{*,u}$,

$$\max\left\{\left|\frac{\delta_u^{+,\bar{b}_2}}{\sqrt{b_2+\bar{b}_2}} - c_{\delta_u^{\bar{b}_2 \to \infty}}\right|, \; \left|\frac{\delta_u^{-,\bar{b}_2}}{\sqrt{b_2+\bar{b}_2}} - (-c_{\delta_u^{\bar{b}_2 \to \infty}})\right|, \; \left|\frac{C^{u,\bar{b}_2}}{\sqrt{b_2+\bar{b}_2}}\right|, \; \left|D^{u,\bar{b}_2}\right|, \; \frac{1}{\sqrt{b_2+\bar{b}_2}}\right\} \leq \zeta_1.$$

Then with equation (24), for any $t \in [0,T]$ and $\bar{b}_2 \geq \bar{b}_2^{*,u}$,

$$\left|u_t^{\bar{b}_2}\right| \leq \frac{C^{u,\bar{b}_2} + D^{u,\bar{b}_2}\left(\delta_u^{+,\bar{b}_2} - \delta_u^{-,\bar{b}_2}\right)}{-\delta_u^{-,\bar{b}_2}} \leq \frac{\zeta_1 + \zeta_1(2c_{\delta_u^{\bar{b}_2 \to \infty}} + 2\zeta_1)}{c_{\delta_u^{\bar{b}_2 \to \infty}} - \zeta_1}. \tag{34}$$

By the same argument for $w_t^{b_2}$ and together with inequality (34), there exists $\bar{b}_2^{*,upper} \geq \bar{b}_2^{*,u}$ and $M > 0$ such that:

$$\max\left\{|u_t^{\bar{b}_2}|, |w_t^{\bar{b}_2}|\right\} \leq M, \quad \forall t \in [0,T], \; \bar{b}_2 \geq \bar{b}_2^{*,upper}. \tag{35}$$

Furthermore, we can get a uniform lower bound for $(b_2+\bar{b}_2)^{\frac{3}{2}} u_t^{\bar{b}_2}$.
Denote $\zeta_2 := \frac{b_2(r+\bar{r}(1-\bar{s})^2)(q_T+\bar{q}_T(1-s_T))}{r+\bar{r}(1-s)}$. Then for all $t\in[0,T]$ and $\bar{b}_2 \geq \bar{b}_2^{*,u}$ we have:

$$\left|(b_2+\bar{b}_2)^{\frac{3}{2}} u_t^{\bar{b}_2}\right| \geq \frac{(b_2+\bar{b}_2)^{\frac{3}{2}} \cdot D^{u,\bar{b}_2}\delta_u^{+,\bar{b}_2}}{(\delta_u^{+,\bar{b}_2}-\delta_u^{-,\bar{b}_2}) + B^{\bar{b}_2}D^{u,\bar{b}_2}} \geq \frac{\zeta_2(c_{\delta_u^{\bar{b}_2\to\infty}}-\zeta_1)}{\zeta_1(2c_{\delta_u^{\bar{b}_2\to\infty}}+2\zeta_1)+\zeta_2/(r+\bar{r}(1-\bar{s})^2)} =: m_u. \tag{36}$$

Now, we adapt the method used in Proposition 4 to prove $\lim_{\bar{b}_2\to\infty}\Delta SC^{\bar{b}_2}=0$. Consider the two quantities:

$$I_1^{\bar{b}_2} := \frac{1}{2}B^{\bar{b}_2}\int_0^{\frac{T}{2}}(u_s^{\bar{b}_2}-w_s^{\bar{b}_2})^2(\bar{x}_s^{MFG,\bar{b}_2})^2 ds \quad \text{and} \quad I_2^{\bar{b}_2} := \frac{1}{2}B^{\bar{b}_2}\int_{\frac{T}{2}}^T(u_s^{\bar{b}_2}-w_s^{\bar{b}_2})^2(\bar{x}_s^{MFG,\bar{b}_2})^2 ds.$$

Fix $\epsilon>0$. In the following, we will show that $\Delta SC^{\bar{b}_2} = I_1^{\bar{b}_2}+I_2^{\bar{b}_2} \leq 2\epsilon$ for large $\bar{b}_2$. First, consider $I_1^{\bar{b}_2}$. Let $\zeta_3 := c_u/2$. Because $\lim_{\bar{b}_2\to\infty} B^{\bar{b}_2}(D^{u,\bar{b}_2})^2+2A^uD^{u,\bar{b}_2}-C^{u,\bar{b}_2}>0$, there exists $\bar{b}_2^{*,0}\geq \bar{b}_2^{*,upper}$ so that for all $\bar{b}_2\geq\bar{b}_2^{*,0}$, the functions $s\mapsto u_s^{\bar{b}_2}$ and $s\mapsto w_s^{\bar{b}_2}$ are increasing, and so that for all $s\in[0,T/2]$:

$$c_u-\zeta_3 \leq (b_2+\bar{b}_2)^{\frac{3}{2}}u_0^{\bar{b}_2} \leq (b_2+\bar{b}_2)^{\frac{3}{2}}u_s^{\bar{b}_2} \leq (b_2+\bar{b}_2)^{\frac{3}{2}}u_{\frac{T}{2}}^{\bar{b}_2} \leq c_u+\zeta_3,$$
$$\left|(b_2+\bar{b}_2)u_s^{\bar{b}_2}\right| \leq \left|(b_2+\bar{b}_2)u_{\frac{T}{2}}^{\bar{b}_2}\right| \leq \zeta_3, \quad \text{and} \quad \left|(b_2+\bar{b}_2)w_s^{\bar{b}_2}\right| \leq \left|(b_2+\bar{b}_2)w_{\frac{T}{2}}^{\bar{b}_2}\right| \leq c_w+\zeta_3.$$

Thus, for any $\bar{b}_2\geq\bar{b}_2^{*,0}$ we have:

$$\begin{aligned} I_1^{\bar{b}_2} &= \frac{\mathbb{E}(\xi)^2}{2(r+\bar{r}(1-\bar{s})^2)}\int_0^{\frac{T}{2}}\left((b_2+\bar{b}_2)u_t^{\bar{b}_2}-(b_2+\bar{b}_2)w_t^{\bar{b}_2}\right)^2 e^{2(b_1+\bar{b}_1)t}\cdot e^{-\frac{2(b_2+\bar{b}_2)^{1/2}}{r+\bar{r}(1-\bar{s})^2}\int_0^t(b_2+\bar{b}_2)^{3/2}u_s^{\bar{b}_2}ds}dt \\ &\leq \kappa_1\frac{1}{\sqrt{b_2+\bar{b}_2}}\left(1-e^{-\kappa_2(b_2+\bar{b}_2)^{\frac{1}{2}}}\right) \xrightarrow[\bar{b}_2\to\infty]{} 0, \end{aligned}$$

where $\kappa_1 := \frac{\mathbb{E}(\xi)^2[\zeta_3^2+(c_w+\zeta_3)^2]e^{2(b_1+\bar{b}_1)\frac{T}{2}}}{2(c_u-\zeta_3)}$ and $\kappa_2 := \frac{(c_u-\zeta_3)T}{r+\bar{r}(1-\bar{s})^2}$ are independent of $\bar{b}_2$. Therefore, there exists $\bar{b}_2^{*,I_1}\geq\bar{b}_2^{*,0}$ such that for $\bar{b}_2\geq\bar{b}_2^{*,I_1}$, we have $I_1^{\bar{b}_1}\leq\epsilon$.

Now, we consider the quantity $I_2^{\bar{b}_2}$. Since $u_t^{\bar{b}_2}$ is positive over $[0,T]$ and from inequalities (35) and (36), we know that for all $\bar{b}_2\geq\bar{b}_2^{*,upper}\geq\bar{b}_2^{*,u}$ and $t\in[T/2,T]$,

$$|u_t^{\bar{b}_2}-w_t^{\bar{b}_2}|\leq 2M, \qquad \text{and} \qquad \int_0^t(b_2+\bar{b}_2)^{\frac{3}{2}}u_s^{\bar{b}_2}ds \geq m_u\cdot\frac{T}{2}.$$

Thus, similar to inequality (33), there exists $\bar{b}_2^{*,I_2}\geq\bar{b}_2^{*,upper}$ such that for all $\bar{b}_2\geq\bar{b}_2^{*,I_2}$:

$$I_2^{\bar{b}_2}\leq\kappa_3(b_2+\bar{b}_2)^2e^{-\kappa_4\sqrt{b_2+\bar{b}_2}}\leq\epsilon,$$

where $\kappa_3 := \frac{T\mathbb{E}(\xi)^2e^{2(b_1+\bar{b}_1)T}M^2}{r+\bar{r}(1-\bar{s})^2}$ and $\kappa_4 := \frac{Tm_u}{r+\bar{r}(1-\bar{s})^2}$ are independent of $\bar{b}_2$.

Hence, for all $\bar{b}_2\geq\bar{b}_2^* := \max\{\bar{b}_2^{*,I_1},\bar{b}_2^{*,I_2}\}$ we have:

$$\Delta SC^{\bar{b}_2} = I_1^{\bar{b}_2}+I_2^{\bar{b}_2}\leq 2\epsilon.$$

Since the proof holds for arbitrary $\epsilon > 0$, we obtain:

$$\lim_{\bar{b}_2 \to \infty} \Delta SC^{\bar{b}_2} = 0.$$

Moreover, recall that η_t and v_t are invariant with respect to $\bar{b}_2$ and $0 < v_t < \infty$ for $t > 0$. Clearly we also have $w_0^{\bar{b}_2} \geq 0$ and $\lim_{\bar{b}_2 \to \infty} w_0^{\bar{b}_2} = 0$. Thus, we obtain: $0 < \lim_{\bar{b}_2 \to \infty} SC^{MKV, \bar{b}_2} < \infty$, and conclude that:

$$\lim_{\bar{b}_2 \to \infty} PoA^{\bar{b}_2} = 1.$$

Case 2: When $\bar{b}_2 \to 0$, we have:

$$\begin{aligned}
&\lambda^{\bar{b}_2} \to \lambda^{\bar{b}_2 \to 0} := \tfrac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})}, \qquad B^{\bar{b}_2} \to \tfrac{b_2^2}{r + \bar{r}(1-\bar{s})^2} =: B^{\bar{b}_2 \to 0} > 0, \\
&C^{u, \bar{b}_2} \to \lambda^{\bar{b}_2 \to 0}(q + \bar{q}(1-s)) =: C_u^{\bar{b}_2 \to 0} > 0, \qquad D^{u, \bar{b}_2} \to \lambda^{\bar{b}_2 \to 0}(q_T + \bar{q}_T(1-s_T)) =: D_u^{\bar{b}_2 \to 0} > 0,
\end{aligned}$$

and $A^u, (A^w, C^w, D^w), (A^\eta, B^\eta, C^\eta, D^\eta)$ are independent of $\bar{b}_2$. Let $u^{\bar{b}_2 \to 0} : [0,T] \to \mathbb{R}$ be the solution to the limiting Riccati equation:

$$\left(u_t^{\bar{b}_2 \to 0}\right)' - 2A^u u_t^{\bar{b}_2 \to 0} - B^{\bar{b}_2 \to 0}(u_t^{\bar{b}_2 \to 0})^2 + C_u^{\bar{b}_2 \to 0} = 0, \qquad u_T^{\bar{b}_2 \to 0} = D_u^{\bar{b}_2 \to 0}, \tag{37}$$

which we recall has an explicit solution. It is easy to show directly from the explicit solutions that for every time $t \in [0,T]$, $\lim_{\bar{b}_2 \to 0} u_t^{\bar{b}_2} = u_t^{\bar{b}_2 \to 0}$. Next, our goal is to bound $u_t^{\bar{b}_2}$ uniformly over $t \in [0,T]$ for small $\bar{b}_2$, following the methodology of the proof of Proposition 3. For any $\epsilon > 0$, there exists a $\bar{b}_2^* > 0$ such that $\max\{B^{\bar{b}_2}, C^{u,\bar{b}_2}, D^{u,\bar{b}_2}\} < \max\{B^{\bar{b}_2 \to 0}, C_u^{\bar{b}_2 \to 0}, D_u^{\bar{b}_2 \to 0}\} + \epsilon =: c_1$ for all $\bar{b}_2 \leq \bar{b}_2^*$. Thus, for all $\bar{b}_2 \leq \bar{b}_2^*$ and for every $t \in [0,T]$:

$$\left| u_t^{\bar{b}_2} \right| \leq \frac{c_1 + 2c_1\sqrt{(A^u)^2 + c_1^2}}{-2A^u e^{-2T\sqrt{(A^u)^2 + c_1^2}}}.$$

Similarly, for every time $t \in [0,T]$, $\lim_{\bar{b}_2 \to 0} w_t^{\bar{b}_2} = w_t^{\bar{b}_2 \to 0}$, and $w^{\bar{b}_2}$ is uniformly bounded over $[0,T]$ and small $\bar{b}_2$. From equation (26), the assumption $\mathbb{E}(\xi) \neq 0$, and by the bounded convergence theorem, we have for every $t \in [0,T]$:

$$\lim_{\bar{b}_2 \to 0} \bar{x}_t^{MFG, \bar{b}_2} = \mathbb{E}(\xi) e^{\int_0^t (b_1 + \bar{b}_1 - B^{\bar{b}_2 \to 0} u_s^{\bar{b}_2 \to 0}) ds} =: \bar{x}_t^{MFG, \bar{b}_2 \to 0} \neq 0. \tag{38}$$

Moreover, $\bar{x}^{MFG, \bar{b}_2}$ is uniformly bounded for all $\bar{b}_2 \leq \bar{b}_2^*$ and for all $t \in [0,T]$. From the non-negativity of u_t, we have:

$$\left| \bar{x}_t^{MFG, \bar{b}_2} \right| \leq |\mathbb{E}(\xi)| \, e^{(b_1 + \bar{b}_1)T}, \quad \forall t \in [0,T], \ \forall \bar{b}_2 \leq \bar{b}_2^*.$$

By the assumption $\frac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})} \neq \frac{q_T + \bar{q}_T(1-s_T)^2}{q_T + \bar{q}_T(1-s_T)}$, we have $D^{u, \bar{b}_2 \to 0} \neq D^w$, and thus by continuity, $u_t^{\bar{b}_2 \to 0} \neq w_t^{\bar{b}_2 \to 0}$ on a set of positive Lebesgue measure. Thus, from Proposition 2 and by the bounded convergence theorem, we deduce:

$$\lim_{\bar{b}_2 \to 0} \Delta SC^{\bar{b}_2} = \frac{1}{2} B^{\bar{b}_2 \to 0} \int_0^T \left((u_t^{\bar{b}_2 \to 0} - w_t^{\bar{b}_2 \to 0} \right)^2 \left(\bar{x}_t^{MFG, \bar{b}_2 \to 0} \right)^2 dt \ > 0.$$

Meanwhile, η_t does not depend on $\bar{b}_2$, and therefore the variance v_t also does not depend on $\bar{b}_2$. Clearly $0 < v_t < \infty$ for $t > 0$ and $0 \leq w_0^{\bar{b}_2 \to 0} < \infty$, and thus,

$$0 < \lim_{\bar{b}_2 \to 0} SC^{MKV, \bar{b}_2} < \infty.$$

Hence, we deduce:

$$\lim_{\bar{b}_2 \to 0} PoA^{\bar{b}_2} > 1.$$

□

Remark 6. *Consider assumption 1 and the case when $\bar{b}_2$ tends to zero. We have $0 < SC^{MKV, \bar{b}_2 \to 0} < \infty$, and therefore, $\lim_{\bar{b}_2 \to 0} PoA^{\bar{b}_2} = 1$ implies $\lim_{\bar{b}_2 \to 0} \Delta SC^{\bar{b}_2} = 0$. Hence, from equation (38) and Proposition 2, if $\lim_{\bar{b}_2 \to 0} PoA^{\bar{b}_2} = 1$, then for all $t \in [0, T]$:*

$$u_t^{\bar{b}_2 \to 0} = w_t^{\bar{b}_2 \to 0}.$$

From equation (37) and an analogous Riccati equation for $w_t^{\bar{b}_2 \to 0}$, the above equality is equivalent to:

$$D^{u, \bar{b}_2 \to \infty} = D^w =: D \quad \textit{and} \quad B^{\bar{b}_2 \to 0} D^2 + 2A^u D^{u, \bar{b}_2 \to 0} - C^{u, \bar{b}_2 \to 0} = B^{\bar{b}_2 \to 0} D^2 + 2A^w D^w - C^w = 0.$$

In other words, the following three equations imply $\lim_{\bar{b}_2 \to 0} PoA^{\bar{b}_2} = 1$:

$$\frac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})} = \frac{q_T + \bar{q}_T(1-s_T)^2}{q_T + \bar{q}_T(1-s_T)}, \qquad \bar{b}_1 \frac{q_T + \bar{q}_T(1-s_T)^2}{q + \bar{q}(1-s)^2} = \frac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})} \cdot \frac{q + \bar{q}(1-s)}{q + \bar{q}(1-s)^2} - 1,$$

and

$$\frac{b_2^2}{r + \bar{r}(1-\bar{s})^2} \left(q_T + \bar{q}_T(1-s_T)^2\right)^2 - 2(b_1 + \bar{b}_1)\left(q_T + \bar{q}_T(1-s_T)^2\right) - \left(q + \bar{q}(1-s)^2\right) = 0.$$

Proposition 7. *Assuming assumption 1, the initial condition satisfies $Var(\xi) > 0$, and:*

$$\frac{b_2}{b_2 + \bar{b}_2} \cdot \frac{r + \bar{r}(1-\bar{s})^2}{r + \bar{r}(1-\bar{s})} \cdot (q_T + \bar{q}_T(1-s_T)) \neq q_T + \bar{q}_T(1-s_T)^2,$$

then:

$$\lim_{b_1 \to 0} PoA > 1 \qquad \textit{and} \qquad \lim_{\bar{b}_1 \to 0} PoA > 1,$$
$$\lim_{b_1 \to \infty} PoA = 1 \qquad \textit{and} \qquad \lim_{\bar{b}_1 \to \infty} PoA = \infty.$$

Proof. **Cases 1 and 2:** First, we consider $b_1 \to 0$. We have $A^{u, b_1} \to A_u^{b_1 \to 0}$, $\delta_u^{+, b_1} \to \delta_u^{+, b_1 \to 0} > 0$, and $\delta_u^{-, b_1} \to \delta_u^{-, b_1 \to 0} < 0$, and similarly for A^{w, b_1}, A^{η, b_1}, $\delta_w^{\pm, b_1}$, and $\delta_\eta^{\pm, b_1}$. Clearly we have for each $t \in [0, T]$, $\lim_{b_1 \to 0} u_t^{b_1} =: u_t^{b_1 \to 0}$, $\lim_{b_1 \to 0} w_t^{b_1} =: w_t^{b_1 \to 0}$, and $\lim_{b_1 \to 0} \eta_t^{b_1} =: \eta_t^{b_1 \to 0}$. Next, we show that the three sequences are uniformly bounded. Let $0 < \epsilon < -\delta_u^{-, b_1 \to 0}$. There exists a b_1^* such that $\max\left\{\left|\delta_u^{+,b_1}-\delta_u^{+,b_1\to 0}\right|,\left|\delta_u^{-,b_1}-\delta_u^{-,b_1\to 0}\right|\right\}<\epsilon$ for each $b_1\leq b_1^*$. Then for each $b_1\leq b_1^*$ and $t\in[0,T]$, we have:

$$\left|u_t^{b_1}\right|\leq\frac{C^u+D^u\left(\delta_u^{+,b_1}-\delta_u^{-,b_1}\right)}{-\delta_u^{-,b_1}}\leq\frac{C^u+D^u\left(\delta_u^{+,b_1\to 0}-\delta_u^{-,b_1\to 0}+2\epsilon\right)}{-\delta_u^{-,b_1\to 0}-\epsilon},$$

and similarly for $\left|w_t^{b_1}\right|$ and $\left|\eta_t^{b_1}\right|$. From the assumption $\frac{b_2}{b_2+\bar{b}_2}\cdot\frac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})}\cdot q_T+\bar{q}_T(1-s_T)\neq q_T+\bar{q}_T(1-s_T)^2$, we have $D^u\neq D^w$ and thus by continuity, $u_t^{b_1\to 0}\neq w_t^{b_1\to 0}$ on a set of positive Lebesgue measure.

From equation (26), the assumption $\mathbb{E}(\xi)\neq 0$, and by the bounded convergence theorem, we have for every $t\in[0,T]$:

$$\lim_{b_1\to 0}\bar{x}_t^{MFG,b_1}=\mathbb{E}(\xi)e^{\int_0^t(\bar{b}_1-Bu_s^{b_1\to 0})ds}=:\bar{x}_t^{MFG,b_1\to 0}\neq 0.$$

Moreover, $\bar{x}_t^{MFG,b_1}$ is uniformly bounded over $b_1\leq b_1^*$ and $t\in[0,T]$, i.e.

$$\left|\bar{x}_t^{MFG,b_1}\right|\leq|\mathbb{E}(\xi)|\,e^{\bar{b}_1T},\quad\forall t\in[0,T],\ b_1\leq b_1^*.$$

Therefore, by Proposition 2, and the bounded convergence theorem, $0<\lim_{b_1\to 0}\Delta SC^{b_1}<\infty$. By the bounded convergence theorem, $\lim_{b_1\to 0}v_t^{b_1}=:v_t^{b_1\to 0}$, which is bounded over $t\in[0,T]$, and thus $0<\lim_{b_1\to 0}SC^{MKV,b_1}<\infty$. Therefore, $\lim_{b_1\to 0}PoA^{b_1}>1$. The proof can be repeated to show $\lim_{\bar{b}_1\to 0}PoA^{\bar{b}_1}>1$.

Case 3: Consider $b_1\to\infty$. Since $\lim_{b_1\to\infty}A^{u,b_1}=\lim_{b_1\to\infty}A^{w,b_1}=-\infty$ we have:

$$\lim_{b_1\to\infty}B(D^u)^2+2A^{u,b_1}D^u-C^u=\lim_{b_1\to\infty}B(D^w)^2+2A^{w,b_1}D^w-C^w=-\infty.$$

Denote $h_u(b_1,t):=B\frac{u_t^{b_1}}{b_1}$ and $h_w(b_1,t):=B\frac{w_t^{b_1}}{b_1}$. For all $t\in[0,T)$, we have the limits:

$$\lim_{b_1\to\infty}h_u(b_1,t)=\lim_{b_1\to\infty}h_w(b_1,t)=2. \tag{39}$$

From equation (28), and together with equation (39), there exists $b_1^{*,upper}>0$ such that for all $b_1\geq b_1^{*,upper}$, the functions $t\mapsto u_t^{b_1}$ and $t\mapsto w_t^{b_1}$ are decreasing and such that:

$$\sup_{0\leq t\leq T}\{h_u(b_1,t),h_w(b_1,t)\}\leq\max\{h_u(b_1,0),h_w(b_1,0)\}\leq 3.$$

Fix $\epsilon>0$. There exists $b_1^{*,I_1}\geq b_1^{*,upper}$ such that for all $b_1\geq b_1^{*,I_1}$, and for all $t\in[0,3T/4]$,

$$\begin{aligned}|h_u(b_1,t)-h_w(b_1,t)|&=\max\{h_u(b_1,t)-h_w(b_1,t),h_w(b_1,t)-h_u(b_1,t)\}\\&\leq\max\{h_u(b_1,0)-h_w(b_1,3T/4),h_w(b_1,0)-h_u(b_1,3T/4)\}\\&\leq\epsilon,\end{aligned}$$

and

$$h_w(b_1,0)\geq 2-\frac{1}{3}=\frac{5}{3},\quad h_u(b_1,t)\geq h_u(b_1,3T/4)\geq\frac{5}{3}.$$

We adapt the methodology in Proposition 4 and split the interval $[0, T]$ into two parts: $[0, 3T/4]$ and $[3T/4, T]$. For all $b_1 \geq b_1^{*,I_1}$, similar to inequality (31), we have:

$$\frac{\Delta SC^{b_1}}{SC^{MKV,b_1}} \leq \frac{I_1^{b_1} + I_2^{b_1}}{h_w(b_1, 0)},$$

where

$$\begin{aligned}
I_1^{b_1} &= b_1 \int_0^{\frac{3}{4}T} (h_u(b_1,t) - h_w(b_1,t))^2 e^{2\bar{b}_1 t} \cdot \exp\left(2b_1 t - 2b_1 \int_0^t h_u(b_1,s)ds\right) dt \\
&\leq \epsilon^2 e^{2\bar{b}_1 \cdot \frac{3}{4}T} \cdot b_1 \int_0^{\frac{3}{4}T} \exp\left(2b_1 t - 2b_1 \cdot \frac{5}{3}t\right) dt \\
&= \epsilon^2 e^{\frac{3\bar{b}_1 T}{2}} \cdot \frac{3}{4}(1 - e^{-b_1 T}) \\
&\leq \kappa_1 \epsilon^2,
\end{aligned}$$

in which $\kappa_1 := \frac{3}{4} e^{\frac{3\bar{b}_1 T}{2}}$, and

$$\begin{aligned}
I_2^{b_1} &= b_1 \int_{\frac{3}{4}T}^{T} (h_u(b_1,t) - h_w(b_1,t))^2 e^{2\bar{b}_1 t} \cdot \exp\left(2b_1 t - 2b_1 \int_0^t h_u(b_1,s)ds\right) dt \\
&\leq b_1 (3+3)^2 e^{2\bar{b}_1 T} \cdot \int_{\frac{3}{4}T}^{T} \exp\left(2b_1 t - 2b_1 \int_0^{\frac{3}{4}T} h_u(b_1,s)ds\right) dt \\
&\leq 36 e^{2\bar{b}_1 T} b_1 \cdot \frac{T}{4} \exp\left(2b_1 \cdot T - 2b_1 \cdot \frac{3}{4}T \cdot \frac{5}{3}\right) \\
&= 9T e^{2\bar{b}_1 T} b_1 e^{-\frac{1}{2}b_1 T} \xrightarrow[b_1 \to \infty]{} 0.
\end{aligned}$$

Therefore, there exists $b_1^* \geq b_1^{*,I_1}$, so that for $b_1 \geq b_1^*$, we have $I_2^{b_1} \leq \epsilon$. Hence, for all $b_1 \geq b_1^*$,

$$\frac{\Delta SC^{b_1}}{SC^{MKV,b_1}} \leq \frac{I_1^{b_1} + I_2^{b_1}}{h_w(b_1,0)} \leq \frac{3}{5}(\kappa_1 \epsilon^2 + \epsilon).$$

Since the proof holds for arbitrary $\epsilon > 0$, and $\kappa_1 = \frac{3}{4} e^{\frac{3\bar{b}_1 T}{2}}$ is independent of b_1 and ϵ, we conclude that:

$$\lim_{b_1 \to \infty} PoA^{b_1} = 1.$$

Case 4: Now, consider $\bar{b}_1 \to \infty$. Since $A^{u,\bar{b}_1} < 0$ and $\lim_{\bar{b}_1 \to \infty} |A^{u,\bar{b}_1}| = \infty$, we have the following limits:

$$\delta_u^{+,\bar{b}_1} - \delta_u^{-,\bar{b}_1} = 2\sqrt{(A^{u,\bar{b}_1})^2 + BC^u} \xrightarrow[\bar{b}_1 \to \infty]{} \infty, \quad -\delta_u^{-,\bar{b}_1} = A^{u,\bar{b}_1} + \sqrt{(A^{u,\bar{b}_1})^2 + BC^u} \xrightarrow[\bar{b}_1 \to \infty]{} 0,$$

$$\frac{\delta_u^{+,\bar{b}_1}}{\bar{b}_1} = \frac{b_1 + \frac{\bar{b}_1}{2}}{\bar{b}_1} \frac{\sqrt{\left(\frac{\bar{b}_1}{2}\right)^2 + BC^u}}{\bar{b}_1} \xrightarrow[\bar{b}_1 \to \infty]{} \frac{1}{2} + \frac{1}{2} = 1.$$

We also have for $t \in [0,T)$:

$$\delta_u^{+,\bar{b}_1} e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})(T-t)} \leq (\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1}) e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})(T-t)} \xrightarrow[\bar{b}_1\to\infty]{} 0,$$

which implies: $\lim_{\bar{b}_1\to\infty} \delta_u^{+,\bar{b}_1} e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})(T-t)} = 0$. Therefore, for $t \in [0,T)$: $\lim_{\bar{b}_1\to\infty} \frac{u_t^{\bar{b}_1}}{\bar{b}_1} = \frac{1}{B}$. By the same argument, we have for $t \in [0,T)$: $\lim_{\bar{b}_1\to\infty} \frac{w_t^{\bar{b}_1}}{\bar{b}_1} = \frac{2}{B}$.

Since $\lim_{\bar{b}_1\to\infty} B(D^u)^2 + 2A^{u,\bar{b}_1}D^u - C^u = -\infty$ and $\lim_{\bar{b}_1\to\infty} B(D^w)^2 + 2A^{w,\bar{b}_1}D^w - C^w = -\infty$, from equation (28) there exists a $\bar{b}_1^{*,lower}$ such that for $\bar{b}_1 \geq \bar{b}_1^{*,lower}$, we have:

$$\max\left\{\left|\frac{u_0^{\bar{b}_1}}{\bar{b}_1}-\frac{1}{B}\right|, \left|\frac{w_{T/2}^{\bar{b}_1}}{\bar{b}_1}-\frac{2}{B}\right|\right\} < \frac{1}{4B},$$

and such that the functions $t \mapsto u_t^{\bar{b}_1}$ and $t \mapsto w_t^{\bar{b}_1}$ are decreasing. Thus, for all $t \in [0, T/2]$ we have:

$$0 < \frac{u_t^{\bar{b}_1}}{\bar{b}_1} \leq \frac{u_0^{\bar{b}_1}}{\bar{b}_1} \leq \frac{1}{B}+\frac{1}{4B} \leq \frac{2}{B}-\frac{1}{4B} \leq \frac{w_{T/2}^{\bar{b}_1}}{\bar{b}_1} \leq \frac{w_t^{\bar{b}_1}}{\bar{b}_1}.$$

Thus, for all $\bar{b}_1 \geq \bar{b}_1^{*,lower}$ and all $t \in [0, T/2]$,

$$\left|\frac{w_t^{\bar{b}_1}}{\bar{b}_1}-\frac{u_t^{\bar{b}_1}}{\bar{b}_1}\right| \geq \frac{1}{2B}. \tag{40}$$

Note that η_t, and therefore v_t, are independent of $\bar{b}_1$. Thus,

$$\frac{1}{\bar{b}_1^2} SC^{MKV,\bar{b}_1} = \frac{1}{2\bar{b}_1^2}\left[\int_0^T \left[q+\bar{q}+B^\eta \eta_t^2\right] v_t dt + (q_T+\bar{q}_T)v_T\right] + \frac{w_0^{\bar{b}_1}}{2\bar{b}_1^2}(\mathbb{E}(\xi))^2 \xrightarrow[\bar{b}_1\to\infty]{} 0.$$

Now, consider:

$$\frac{1}{\bar{b}_1^2}\Delta SC^{\bar{b}_1} \geq \frac{B}{2}(\mathbb{E}(\xi))^2 \int_0^{\frac{T}{2}} \left(\frac{u_t^{\bar{b}_1}}{\bar{b}_1}-\frac{w_t^{\bar{b}_1}}{\bar{b}_1}\right)^2 e^{2\int_0^t (b_1+\bar{b}_1-Bu_s^{\bar{b}_1})ds} dt. \tag{41}$$

Since $u_t^{\bar{b}_1}$ is decreasing for $\bar{b}_1 \geq \bar{b}_1^{*,lower}$, we have $\bar{b}_1 - Bu_t^{\bar{b}_1} \geq \bar{b}_1 - Bu_0^{\bar{b}_1}$. We have the following limits for $t \in [0,T)$:

$$\lim_{\bar{b}_1\to\infty} \bar{b}_1 - \delta_u^{+,\bar{b}_1} = -2b_1, \quad \lim_{\bar{b}_1\to\infty} -\delta_u^{-,\bar{b}_1}\bar{b}_1 = BC^u,$$

$$\lim_{\bar{b}_1\to\infty} \delta_u^{+,\bar{b}_1}\bar{b}_1 e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})(T-t)} = 0, \quad \lim_{\bar{b}_1\to\infty} (\delta_u^{-,\bar{b}_1}-\bar{b}_1) e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})(T-t)} = 0,$$

and thus,

$$\begin{aligned}\bar{b}_1 - Bu_0^{\bar{b}_1} &= \frac{-BC^u + BD^u(\bar{b}_1-\delta_u^{+,\bar{b}_1}) - \delta_u^{-,\bar{b}_1}\bar{b}_1 + (\delta_u^{+,\bar{b}_1}\bar{b}_1 + BC^u + BD^u(\delta_u^{-,\bar{b}_1}-\bar{b}_1))e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})T}}{BD^u - \delta_u^{-,\bar{b}_1} + (\delta_u^{+,\bar{b}_1}-BD^u)e^{-(\delta_u^{+,\bar{b}_1}-\delta_u^{-,\bar{b}_1})T}} \\ &\xrightarrow[\bar{b}_1\to\infty]{} -2b_1.\end{aligned}$$

Since $\lim_{\bar{b}_1\to\infty}(\bar{b}_1 - Bu_0^{\bar{b}_1}) = -2b_1 < 0$, there exists $\bar{b}_1^* \geq \bar{b}_1^{*,lower}$, such that for $\bar{b}_1 \geq \bar{b}_1^*$, $(\bar{b}_1 - Bu_0^{\bar{b}_1}) \geq -3b_1$. Returning to inequality (41), and using inequality (40) we have for $\bar{b}_1 \geq \bar{b}_1^*$:

$$\frac{1}{\bar{b}_1^2}\Delta SC^{\bar{b}_1} \geq \frac{B}{2}\mathbb{E}(\xi)^2 \cdot \frac{1}{4B^2} \cdot \int_0^{\frac{T}{2}} e^{2b_1 t + 2(\bar{b}_1 - Bu_0^{\bar{b}_1})t} dt \geq \frac{\mathbb{E}(\xi)^2}{8B} \int_0^{\frac{T}{2}} e^{-4b_1 t} dt > 0.$$

Therefore, $\lim_{\bar{b}_1\to\infty} \frac{1}{\bar{b}_1^2}\Delta SC^{\bar{b}_1} > 0$, and thus,

$$\lim_{\bar{b}_1\to\infty} \frac{\Delta SC^{\bar{b}_1}}{SC^{MKV,\bar{b}_1}} = \lim_{\bar{b}_1\to\infty} \frac{\frac{1}{\bar{b}_1^2}\Delta SC^{\bar{b}_1}}{\frac{1}{\bar{b}_1^2} SC^{MKV,\bar{b}_1}} = \infty.$$

We conclude: $\lim_{\bar{b}_1\to\infty} PoA^{\bar{b}_1} = \infty$.

□

2.4 Numerical Results

The solutions to the problems we have considered are given by the formulas derived in Appendix A, which are explicit up to evaluating integrals. Using the simple rectangle rule to evaluate integrals, we numerically compute the price of anarchy when the coefficients are time-independent, nonnegative, and satisfy assumption 1. In particular, when we allow for full interaction (i.e. through the states and the controls), we choose the following default values:

$$\begin{aligned}
&\xi \equiv 1,\ T = 1\\
&b_1 = 1,\ \bar{b}_1 = 1,\ b_2 = 1,\ \bar{b}_2 = 1,\ \sigma = 1\\
&q = 1,\ \bar{q} = 1,\ s = 0.5,\ r = 1,\ \bar{r} = 1,\ \bar{s} = 0.5\\
&q_T = 1,\ \bar{q}_T = 1,\ s_T = 0.5.
\end{aligned}$$

Unless otherwise stated, the parameters stay at these default values. For results involving only interaction through the states, we set $\bar{b}_2 = 0$ and $\bar{r} = 0$. For results involving only interaction through the controls, we set $\bar{b}_1 = 0$, $\bar{q} = 0$, and $\bar{q}_T = 0$. Figures 1-5 show the price of anarchy as we vary one parameter at a time for each of three cases: full interaction (i.e. through the states and the controls), interaction only through the states, and interaction only through the controls.

The results show various limiting behaviors, such as some of the cases proved in the previous section. In Figure 1, we note that Proposition 7 is confirmed. For all three cases, we see that $\lim_{b_1\to 0} PoA > 1$ and $\lim_{\bar{b}_1\to 0} PoA => 1$. We also see that $\lim_{b_1\to\infty} PoA = 1$ and $\lim_{\bar{b}_1\to\infty} PoA = \infty$. Propositions 4 and 5 are confirmed in Figure 2. When there is only interaction through the states, then $\bar{b}_2 = 0$ and we see that $\lim_{b_2\to 0} PoA = 1$. When there is full interaction or only interaction through the controls, then $\bar{b}_2 \neq 0$ and we see that $\lim_{b_2\to 0} PoA > 1$. For all three cases, we note that the condition $\frac{r+\bar{r}(1-\bar{s})^2}{r+\bar{r}(1-\bar{s})} = \frac{q+\bar{q}(1-s)^2}{q+\bar{q}(1-s)}$ is satisfied, and thus, $\lim_{b_2\to\infty} PoA = 1$. For Proposition 6, Figure 2 confirms that $\lim_{\bar{b}_2\to 0} PoA > 1$ and $\lim_{\bar{b}_2\to\infty} PoA = 1$. Figure 5 confirms as in Proposition 3 that $\lim_{r\to\infty} PoA = 1$ and $\lim_{\bar{r}\to\infty} PoA = 1$.

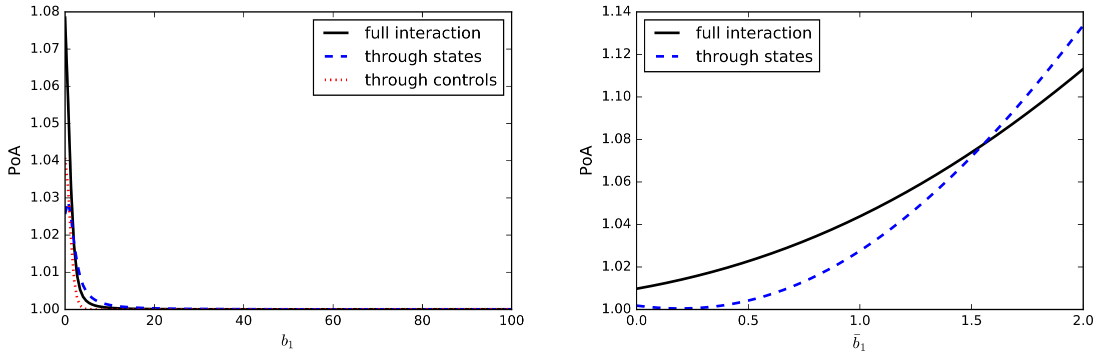

Figure 1: PoA as we vary b_1 (left) and $\bar{b}_1$ (right).

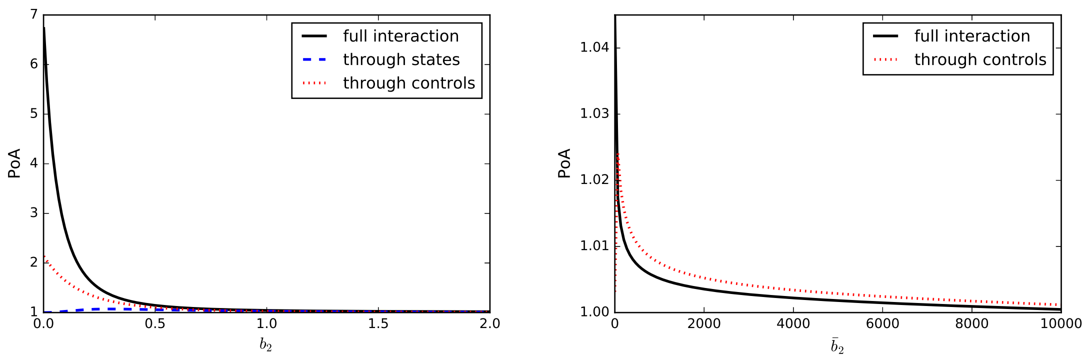

Figure 2: PoA as we vary b_2 (left) and $\bar{b}_2$ (right).

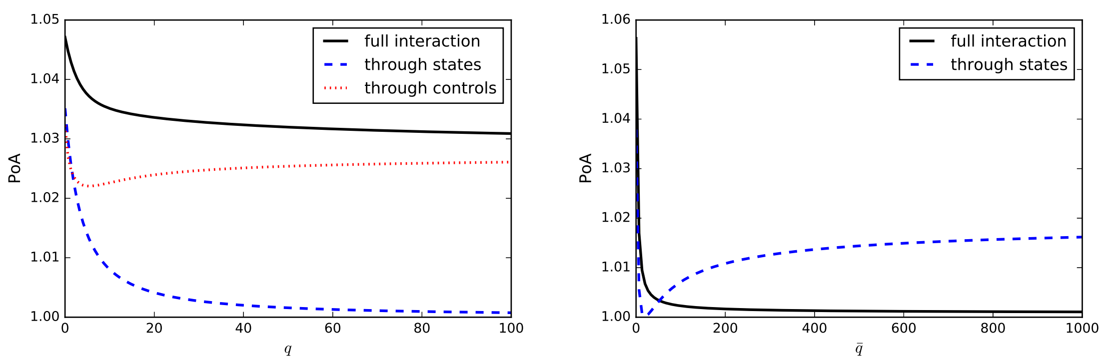

Figure 3: PoA as we vary q (left) and q_T (right).

2.5 A Particular Example: Flocking

Mean field game models of flocking have been proposed in the literature by Nourian et al [15][14]. Here we consider a slightly different formulation as described in Section 3.6.1 of the book [7]. A

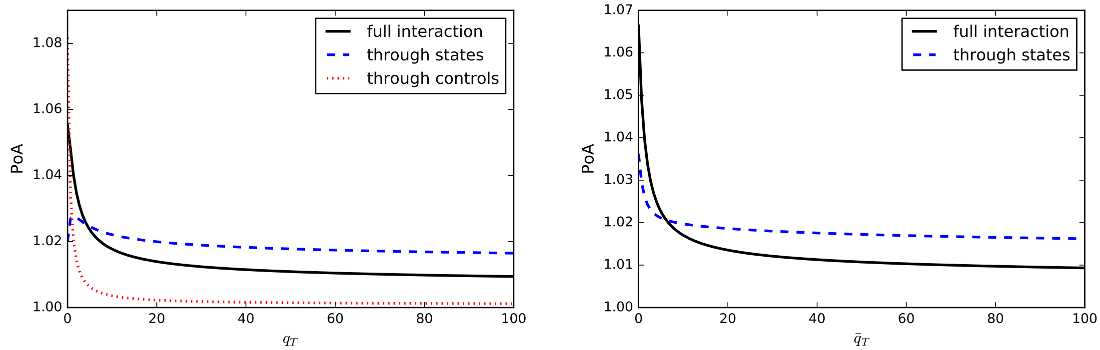

Figure 4: PoA as we vary $\bar{q}$ (left) and $\bar{q}_T$ (right).

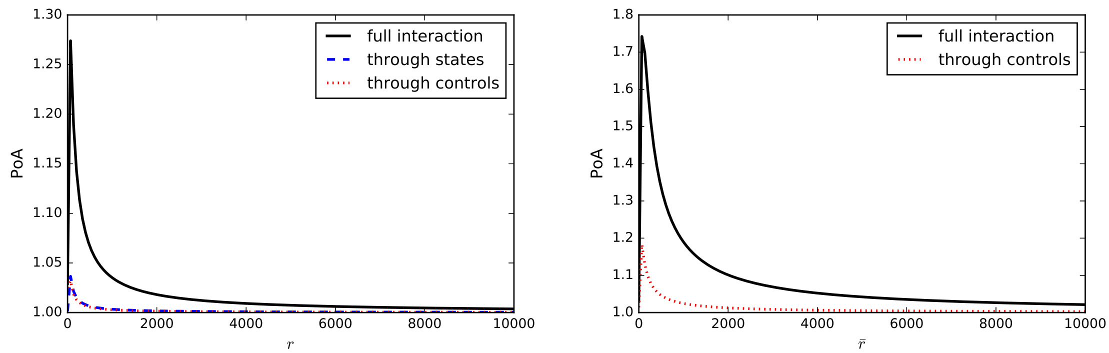

Figure 5: PoA as we vary r (left) and $\bar{r}$ (right).

representative bird in the flock controls their velocity, X_t, through the drift:

$$b(t, x, \mu, \alpha, \nu) = \alpha.$$

They choose the control with two goals in mind: to minimize their kinetic energy put into their control, and to align their velocity with the average velocity of the group. Thus, they consider the cost functions given by:

$$f(t, x, \mu, \alpha, \nu) = \frac{1}{2}\left(\bar{q}|x - s\bar{\mu}|^2 + \alpha^2\right),$$
$$g(x, \mu) = 0.$$

In our general linear quadratic formulation, this is equivalent to taking $b_1 = 0$, $\bar{b}_1 = 0$, $b_2 = 1$, $\bar{b}_2 = 0$, $q = 0$, $s = 1$, $r = 1$, $\bar{r} = 0$, and $\bar{s} = 0$. Note that for these values of the parameters, the assumptions of Theorem 1 and Corollary 1 are satisfied. Therefore, $PoA = 1$. In fact, if we took the state space to be $\mathbb{R}^d$ instead of $\mathbb{R}$, the result would still hold.

3 Conclusion

We defined the price of anarchy (PoA) in the context of extended mean field games as the ratio of the worst case social cost when the players are in a mean field game equilibrium to the social cost as computed by a central planner. Since the central planner does not require that the players be in a mean field game equilibrium, the central planner will realize a social cost that is no worse than that of a mean field game equilibrium. Thus, $PoA \geq 1$.

We computed the price of anarchy for linear quadratic extended mean field games, for which explicit computations are possible. We identify a large class of models for which $PoA = 1$ (see Proposition 1 and Corollary 1), as well as some limiting cases where $PoA \to 1$ as certain parameters tend to zero or to infinity. The numerics support our theoretical results.

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Appendices

A Solving Linear FBSDEs of McKean-Vlasov Type

Consider a linear FBSDE system of McKean-Vlasov type:

$$
\begin{aligned}
dX_t &= \left(a_t^x X_t + a_t^{\bar{x}} \mathbb{E}X_t + a_t^y Y_t + a_t^{\bar{y}} \mathbb{E}Y_t\right) dt + \sigma dW_t \\
X_0 &= \xi \\
dY_t &= \left(b_t^x X_t + b_t^{\bar{x}} \mathbb{E}X_t + b_t^y Y_t + b_t^{\bar{y}} \mathbb{E}Y_t\right) dt + Z_t dW_t \\
Y_T &= c^x X_T + c^{\bar{x}} \mathbb{E}X_T.
\end{aligned}
\tag{42}
$$

For the LQEMFG model considered in Section 2.1, the FBSDE system in equation (2) is of the form of equation (42) if we set:

$$
\begin{aligned}
&a_t^x = b_1(t),\ a_t^{\bar{x}} = \bar{b}_1(t),\ a_t^y = a^{MFG}(t) b_2(t),\ a_t^{\bar{y}} = b^{MFG}(t) b_2(t) + c^{MFG}(t) \bar{b}_2(t) \\
&b_t^x = -(q(t) + \bar{q}(t)),\ b_t^{\bar{x}} = \bar{q}(t) s(t),\ b_t^y = -b_1(t),\ b_t^{\bar{y}} = 0 \\
&c_t^x = q_T + \bar{q}_T,\ c_t^{\bar{x}} = -\bar{q}_T s_T.
\end{aligned}
$$

For the LQEMKV model considered in Section 2.2, the FBSDE system in equation (8) is of the form of equation (42) if we set:

$$
\begin{aligned}
& a_t^x = b_1(t),\ a_t^{\bar{x}} = \bar{b}_1(t),\ a_t^y = a^{MKV}(t)b_2(t),\ a_t^{\bar{y}} = b^{MKV}(t)b_2(t) + c^{MKV}(t)\bar{b}_2(t) \\
& b_t^x = -(q(t)+\bar{q}(t)),\ b_t^{\bar{x}} = -s(t)\bar{q}(t)(s(t)-2),\ b_t^y = -b_1(t),\ b_t^{\bar{y}} = -\bar{b}_1 \\
& c_t^x = q_T + \bar{q}_T,\ c_t^{\bar{x}} = s_T\bar{q}_T(s_T-2).
\end{aligned}
$$

Now we return to the general FBSDE system (42). By taking expectations in equation (42), and letting $\bar{x}_t$ and $\bar{y}_t$ denote $\mathbb{E}X_t$ and $\mathbb{E}Y_t$, respectively, we get:

$$
\begin{aligned}
\dot{\bar{x}}_t &= (a_t^x + a_t^{\bar{x}})\bar{x}_t + (a_t^y + a_t^{\bar{y}})\bar{y}_t \\
\bar{x}_0 &= \mathbb{E}(\xi) \\
\dot{\bar{y}}_t &= (b_t^x + b_t^{\bar{x}})\bar{x}_t + (b_t^y + b_t^{\bar{y}})\bar{y}_t \\
\bar{y}_T &= (c^x + c^{\bar{x}})\bar{x}_T,
\end{aligned}
\tag{43}
$$

where the dot is the standard ODE notation for a derivative. We then make the ansatz $\bar{y}_t = \bar{\eta}_t\bar{x}_t + \bar{\chi}_t$ for deterministic functions $[0,T] \ni t \mapsto \bar{\eta}_t \in \mathbb{R}$ and $[0,T] \ni t \mapsto \bar{\chi}_t \in \mathbb{R}$. By plugging in the ansatz, the system in equation (43) is equivalent to the ODE system:

$$
\begin{aligned}
\dot{\bar{\eta}}_t + (a_t^y + a_t^{\bar{y}})\bar{\eta}_t^2 + (a_t^x + a_t^{\bar{x}} - b_t^y - b_t^{\bar{y}})\bar{\eta}_t - b_t^x - b_t^{\bar{x}} &= 0 \\
\bar{\eta}_T - c^x - c^{\bar{x}} &= 0 \\
\dot{\bar{\chi}}_t + (\bar{\eta}_t(a_t^y + a_t^{\bar{y}}) - b_t^y - b_t^{\bar{y}})\bar{\chi}_t &= 0 \\
\bar{\chi}_T &= 0.
\end{aligned}
$$

The first equation is a Riccati equation. Note that $\bar{\chi}_t$ solves a first order homogeneous linear equation. Thus $\bar{\chi}_t = 0$, $\forall t \in [0,T]$. Once the equation for $\bar{\eta}_t$ is solved, we can compute $\bar{x}_t$ by solving the linear ODE:

$$
\begin{aligned}
\dot{\bar{x}}_t &= (a_t^x + a_t^{\bar{x}} + (a_t^y + a_t^{\bar{y}})\bar{\eta}_t)\bar{x}_t \\
\bar{x}_0 &= \mathbb{E}(\xi),
\end{aligned}
$$

and thus,

$$
\bar{x}_t = \mathbb{E}(\xi)e^{\int_0^t (a_u^x + a_u^{\bar{x}} + (a_u^y + a_u^{\bar{y}})\bar{\eta}_u)du}.
$$

Once we have computed $(\bar{x}_t)_{0\le t\le T}$, we can rewrite the original FBSDE system:

$$
\begin{aligned}
dX_t &= \left(a_t^x X_t + a_t^y Y_t + a_t^0\right)dt + \sigma dW_t \\
X_0 &= \xi \\
dY_t &= \left(b_t^x X_t + b_t^y Y_t + b_t^0\right)dt + Z_t dW_t \\
Y_T &= c^x X_T + c^0,
\end{aligned}
$$

with:

$$
\begin{aligned}
a_t^0 &= (a_t^{\bar{x}} + a_t^{\bar{y}}\bar{\eta}_t)\bar{x}_t \\
b_t^0 &= (b_t^{\bar{x}} + b_t^{\bar{y}}\bar{\eta}_t)\bar{x}_t \\
c^0 &= c^{\bar{x}}\bar{x}_T.
\end{aligned}
$$

Now we make the ansatz: $Y_t = \eta_t X_t + \chi_t$, which reduces the problem to the ODE system:

$$\begin{aligned}
\dot{\eta}_t + a_t^y \eta_t^2 + (a_t^x - b_t^y)\eta_t - b_t^x &= 0\\
\eta_T &= c^x,\\
\dot{\chi}_t + (-b_t^y + a_t^y \eta_t)\chi_t + a_t^0 \eta_t - b_t^0 &= 0,\\
\chi_T &= c^0,\\
Z_t &= \sigma \eta_t.
\end{aligned}$$

Again, the first equation is a Riccati equation. Note that it is not necessary to solve for χ_t because of the relationship:

$$\bar{\eta}_t \bar{x}_t = \bar{y}_t = \mathbb{E}(Y_t) = \mathbb{E}(\eta_t X_t + \chi_t) = \eta_t \bar{x}_t + \chi_t.$$

Thus,

$$\chi_t = (\bar{\eta}_t - \eta_t)\bar{x}_t.$$

In summary, the solution to the linear FBSDE of McKean-Vlasov type is reduced to solving linear ODEs and Riccati equations. It will also be useful to compute $Var(X_t)$, which we denote by v_t. After we have solved the above equations, we have:

$$\begin{aligned}
dX_t &= \left((a_t^x + a_t^y \eta_t)X_t + a_t^y \chi_t + a_t^0\right) dt + \sigma dW_t\\
X_0 &= \xi.
\end{aligned}$$

Thus,

$$v_t = Var(X_t) = Var(\xi)e^{\int_0^t 2(a_s^x + a_s^y \eta_s)ds} + \sigma^2 \int_0^t e^{2\int_s^t (a_u^x + a_u^y \eta_u)du} ds.$$

In the case where the coefficients are time-independent, the Riccati equations for $\bar{\eta}_t$ and η_t can be solved explicitly.

Scalar Riccati Equation

If the scalar Riccati equation

$$\dot{\rho}_t - B\rho_t^2 - 2A\rho_t + C = 0$$

with terminal condition $\rho_T = D$ satisfies:

$$B \neq 0, BD \geq 0, BC > 0, \tag{44}$$

then it has a unique solution:

$$\rho_t = \frac{C(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + D(\delta^+ - \delta^- e^{-(\delta^+ - \delta^-)(T-t)})}{BD(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + \delta^+ e^{-(\delta^+ - \delta^-)(T-t)} - \delta^-} \tag{45}$$

where $\delta^{\pm} = -A \pm \sqrt{(A)^2 + BC}$.

Furthermore, if $B \to 0$ and $A \neq 0$, we can deduce that the limiting solution of the scalar Riccati equation coincides with the linear first-order differential equation:

$$\dot{\rho}_t - 2A\rho_t + C = 0$$

with terminal condition $\rho_T = D$, namely:

$$\rho_t = \left(D - \frac{C}{2A}\right) e^{-2A(T-t)} + \frac{C}{2A}.$$

If $B \to 0$ and $A = 0$, the limiting solution of the scalar Riccati equation coincides with the linear first-order differential equation:

$$\dot{\rho}_t + C = 0$$

with terminal condition $\rho_T = D$, namely:

$$\rho_t = D + C(T - t).$$

Hence, returning to the linear FBSDE (42), for $\bar{\eta}_t$, we use:

$$\begin{cases} A = & -\frac{1}{2}(a^x + a^{\bar{x}} - b^y - b^{\bar{y}}) \\ B = & -(a^y + a^{\bar{y}}) \\ C = & -(b^x + b^{\bar{x}}) \\ D = & \ c^x + c^{\bar{x}}. \end{cases}$$

The conditions (44) are satisfied if:

$$\begin{cases} -(a^y + a^{\bar{y}}) & > 0 \\ -(b^x + b^{\bar{x}}) & > 0 \\ c^x + c^{\bar{x}} & \geq 0. \end{cases}$$

For η_t, we use:

$$\begin{cases} A = & -\frac{1}{2}(a^x - b^y) \\ B = & -a^y \\ C = & -b^x \\ D = & \ c^x. \end{cases}$$

The conditions (44) are satisfied if:

$$\begin{cases} -a^y & > 0 \\ -b^x & > 0 \\ c^x & \geq 0. \end{cases}$$

Returning to the LGEMFG and LGEMKV problems, if we assume the coefficients are nonnegative, we see that these conditions are exactly assumption (14).