

# Response to Referee

July 31, 2018

We would like to thank the reviewer for their remarks. We have addressed each of their points as follows:

## Main Points:

1. This was a good suggestion. We have added equations (3)-(7) on page 7, and equations (9)-(13) on page 8, and we refer to these equations often. In the same vein we have added equations (26)-(29) on pages 13-14, which are used frequently in Propositions 3-7.
2. Since the Hamiltonian depends on the scalars  $\bar{\mu}$  and  $\bar{\nu}$ , instead of the full measures, the derivative in the measure argument greatly simplifies. On page 8 line 6, we refer to section 4 in [5], where they write what this condition should be in the linear quadratic case. On page 8 lines 7 and 8, the derivative is a standard derivative and does not need explaining further. There was a typo in the previous version that may have caused confusion, where we mistakenly wrote  $\partial_{\mu}$  instead of  $\partial_{\bar{\mu}}$ .
3. We have made many modifications in the computation of  $\int_0^T (\bar{\eta}_t^{MFG})^2 (\bar{x}_t^{MFG})^2 dt$  on pages 10-11 to make it much more readable. Whereas we previously dropped the superscripts *MFG* and *MKV* to address both simultaneously, we now show the computation fully for the *MFG* case. Then after the computation, we state on page 11 line -5: ‘Repeating the calculation for  $\int_0^T (\bar{\eta}_t^{MKV})^2 (\bar{x}_t^{MKV})^2 dt...$  we arrive at...’ and we provide the result for the *MKV* case. We now only use the notation introduced thus far, instead of mixing in the notation from the appendix, which could cause confusion. We also take the suggestion from the reviewer of explicitly writing the end result of our computation. Also, we now give an explanation for each step of the calculation. Finally, we separated the equations in the former equation (7) and give more explanation for how they are derived. They are now equations (18) and (19) in the final version. And yes, there should not have been a  $t$  in  $h_{var}$ . We believe pages 10-11 should be much easier to follow in their present form.
4. Yes, it is correct that there is a square in the definitions of  $C^w$  and  $D^w$  that doesn’t appear in the definitions of  $C^u$  and  $C^w$ . This was overlooked by

the authors. If we were to rewrite Corollary 3 with the correct condition to have  $(A^u, B^u, C^u, D^u) = (A^w, B^w, C^w, D^w)$ , it would not be substantially different from Corollary 1 and thus, we dropped Corollary 3.

- 5.
- 6.
7. We added a sentence at the beginning of section 2.4 on page 27 stating that we use a simple rectangular integration rule to numerically compute integrals.

**Minor Points:**

1. We agree that this notation was awkward. We have changed all of the function definitions to the more standard notation.
2. We changed ‘its’ to ‘their’ to have consistent pronoun usage.
3. We added ‘game’.
4. We have used the pronoun ‘their’ throughout the text, so we will continue to use it here.
5. The missing parenthesis have been added.
6. We agree. Now page 6 line 10 reads ‘According to the Pontryagin stochastic maximum principle, a sufficient condition for optimality is...’
7. Yes, this was a typo. On page 6 line 13, we have replaced  $x$  by  $X_t$ . At this point, the fixed point is not addressed, and the control is the best response when the mean field is fixed. So we should not yet replace  $\bar{\mu}$  and  $\bar{\nu}$ .
8. On page 6 line -6, we have added: ‘Note  $c^{MFG}(t) = a^{MFG}(t) + b^{MFG}(t)$ .’ We also added ‘Note  $c^{MKV}(t) = a^{MKV}(t) + b^{MKV}(t)$ .’ on page 8 line -12.
9. The extra parenthesis has been removed.
10. We have added equation (6) and refer to it in the proof of Corollary 2, page 13 line 2. And yes, it should have been  $\bar{x}_t^{MFG}$ .
11. We changed ‘tends’ to ‘tend’.
12. In the proof of Proposition 3, page 15 line -6, we replaced the limit of  $v_t^r$  with our new notation  $v_t^{r \rightarrow \infty}$ .
13. Yes, this was a typo. These values were indeed after taking the limit. The limits in a slightly different presentation now appear in the proof of Proposition 5 Case 2, page 19 line 12.

In addition to the suggested revisions, we have also made the following minor modifications:

1.

Finally, we would like to acknowledge that the length of our paper has increased from 26 pages at submission to 33 pages for the final version. The issues in our proofs raised by the reviewer required us to write many more details, and in some cases, new approaches altogether. We feel that the increase in length was necessary to address all of the issues pointed out by the reviewer. (ToDo: Is this comment necessary?)