1. L_{SSM}

We have $L_{SSM}(\theta) = \mathbb{E}_{x\sim p(x)}\|s(x;\theta)\|^2 + \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}[2v^T \nabla_x (v^T s(x;\theta))]$

We want to show $L_{SSM}(\theta) = \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}[\|v^T s(x;\theta)\|^2 + 2v^T \nabla_x (v^T s(x;\theta))]$

For any fixed x, $s(x;\theta) \in \mathbb{R}^d$ is a constant, $v \in \mathbb{R}^d$ is a random vector satisfies $\mathbb{E}_{v\sim p(v)}(vv^T) = I$

$\mathbb{E}_{v\sim p(v)}\|v^T s(x;\theta)\|^2 = \mathbb{E}_{v\sim p(v)}(v^T s(x;\theta))^T(v^T s(x;\theta)) = \mathbb{E}_{v\sim p(v)}[\,s(x;\theta)^T v v^T s(x;\theta)]$

$= s(x;\theta)^T \mathbb{E}_{v\sim p(v)}(vv^T)\, s(x;\theta) = s(x;\theta)^T s(x;\theta) = \|s(x;\theta)\|^2$

Thus, $\mathbb{E}_{x\sim p(x)}\|s(x;\theta)\|^2 = \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}\|v^T s(x;\theta)\|^2$

$L_{SSM}(\theta) = \mathbb{E}_{x\sim p(x)}\|s(x;\theta)\|^2 + \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}[2v^T \nabla_x (v^T s(x;\theta))]$

$= \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}[\|v^T s(x;\theta)\|^2 + 2v^T \nabla_x (v^T s(x;\theta))]$ #

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2. SDE

$dx_t = \underbrace{f(x_t,t)\,dt}_{\text{drift}} + \underbrace{G(x_t,t)\,dW_t}_{\text{diffusion}}$, $x(0)=x_0$, $x_t \in \mathbb{R}^d$, $f \in \mathbb{R}^d$, $G \in \mathbb{R}^{d\times d}$

Drift term: 沒有干擾時的趨勢

Diffusion term: 隨機性的影響 (噪音、不確定性)

W_t: Standard Brownian motion, $W_0 = 0$

特性: 1. Independent Increments.

For any $0 \le t_1 < t_2 < t_3 < t_4$, $W_{t_2} - W_{t_1}$ 和 $W_{t_4} - W_{t_3}$ 是相互獨立的隨機變量

也就是說過去、現在、未來的運動變化是不相關的

2. Gaussian Increments.

For any $0 \le s \le t$. $W_t - W_s \sim \mathcal{N}(0, t-s)$

$E[W_t - W_s] = 0$ 表示它沒有特定的漂移方向

3. Nowhere Differentiable

W_t 的軌跡連續但 $\frac{dW_t}{dt}$ 不存在

3. 布朗運動在任何時間間隔內的軌跡長度是無限的嗎?

這在物理上如何解釋?