# multivariable calculus

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## 1 Dot Product

$x + 2y + 3z = 0$ equation of a plane
Let $\overrightarrow{OP} = (x, y, z)$,$\overrightarrow{A} = (1, 2, 3)$,then $\overrightarrow{OP} \cdot \overrightarrow{A} = 0 \Rightarrow P$ on a plane going through the origin point($O$). **Remember to use vectors in multivariable calculus!**

## 2 Deteminants

### 2.1 In a plane

Area of a triangle:

$$\pm Area = \frac{1}{2}\left|\overrightarrow{A}\right| \cdot \left|\overrightarrow{B}\right| \sin\theta = \frac{1}{2}\left|\overrightarrow{A'}\right| \cdot \left|\overrightarrow{B}\right| \cos\theta' = \frac{1}{2}\overrightarrow{A'} \cdot \overrightarrow{B} = \frac{1}{2}(-a_2, a_1) \cdot (b_1, b_2)$$

$$= \frac{1}{2}(a_1b_2 - a_2b_1) = det(A, B) = \frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

($\overrightarrow{A'}$:$\overrightarrow{A}$ rotated by 90° anticlockwise)

### 2.2 In a space

#### 2.2.1 volume of solids

$\boxed{Theorem}$Geomatrically, $det(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}) = \pm Volume\,of\,parallelepiped$
**cross product** of 2 vectors in 3d space:
$\boxed{Def}$

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(3×3 only for remembering)
In particular,$\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$, and$\overrightarrow{A} \times \overrightarrow{A} = 0$
$\boxed{Theorem}$ $\left|\overrightarrow{A} \times \overrightarrow{B}\right|$ is the area of the parallelogram spanned by $\overrightarrow{A}$ and $\overrightarrow{B}$.
$dir(\overrightarrow{A} \times \overrightarrow{B})$ is perpendicular to the plane of the parallelogram.

use **right-hand rule** to determine the direction of the cross product.
For example, $\hat{i} \times \hat{j} = \hat{k}$

Another loook at volumn:

$$Volumn = area(base) \cdot height = \left|\overrightarrow{B} \times \overrightarrow{C}\right| \cdot (\overrightarrow{A} \cdot \vec{n}) = \left|\overrightarrow{B} \times \overrightarrow{C}\right| \cdot (\overrightarrow{A} \cdot \frac{\overrightarrow{B} \times \overrightarrow{C}}{\left|\overrightarrow{B} \times \overrightarrow{C}\right|})$$

$$= \overrightarrow{A} \cdot (\overrightarrow{B} \times \overrightarrow{C}) = det(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C})$$

which makes sense with formulas.

To decide that $P$ is on the plane$P_1P_2P_3$:

$$Volumn = det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$$

Other solution($\overrightarrow{N}$is a normal vector of the plane.):

$$\overrightarrow{P_1P} \perp \overrightarrow{N}$$

How to find a normal vector? Use the cross product of 2 vectors on the plane!

$$\overrightarrow{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$$

So the two solutions are equivalent.(triple product=det.)

$$\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$$

# 3 Matrices

Ex.exchange of coordinate systems.
multiplying matrices
$(AB)X$
$AB$represents: do transformation $B$ first, then do transformation $A$.
Identity matrix $I$

Example:in the plane,rotation by 90° counterclockwise:

$$use \quad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Inverse matrix:for a square matrix $A$,$AA^{-1} = A^{-1}A = I$
So a linear system $AX = b$can be solved.
To find the invert matrix: adjoint matrix is introduced.

# 4 Square Systems