Math 137 Dr. Ian Payne

"solving" absolute values & intro to sequence

Triangle inequality $|a+b| \le |a| + |b|$

$$|x-y| \le |x-z| + |z-y|$$

set $x=a$, $z=0$, $y=-b$

then $|x-y| = |a-(-b)| = |a+b|$

so $|a+b| = |x-z| + |z-y|$

$$= |a-0| + |0-(-b)|$$

$$= |a| + |b|$$

Examples: Find all real numbers which satisfy each of follow:

(1) $|x-3| = 5$

(2) $0 < |x-a| < 3$

(3) $|2x+11| \geq 3$

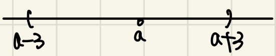
Solution:

① we have either $x-3=5$ or $x-3=-5$

$x=8$ or $x=-2$

(numbers which are 5 away from 3)

②



$(a-3, a+3) - \{a\}$

↑ interval

set difference

③ $|2x+11| \geq 3$ so

$2x+11 \geq 3$ or $2x+11 \le -3$

$2x \geq -8$ $\quad$ $2x \le -14$

$x \geq -4$ $\quad$ $x \le -7$

In interval notation

$x \in [-4, \infty)$ or $x \in (-\infty, -7]$

some notation

$x \in [-4, \infty) \cup (-\infty, -7]$

Ex.2 Find all x satisfying $|x| + |x+1| \geq 2$

case 1 $x < -1$ which means $x+1 < 0$

so $|x| + |x+1| = -x - (x+1)$

$$= -2x - 1$$

therefore, we want $-2x - 1 \geq 2$

or $2x \le -3$ which gives $x \le -\frac{3}{2}$

$(-\infty, -\frac{3}{2})$ you should verify that all such x satisfy the inequality

Case 2 $-1 \le x < 0$

which means $0 \le x+1 < 1$

$|x| + |x+1| = -x + x + 1 = 1$

there are no x in this range.

Case 3 $0 \le x$ which means $x+1 \ge 0$

$|x| + |x+1| = x + x + 1 = 2x+1$

For $2x+1 \ge 2$, we need $x \ge \frac{1}{2}$ or $x \in [\frac{1}{2}, \infty)$

The solution set is $(-\infty, -\frac{3}{2}] \cup [\frac{1}{2}, \infty)$

Section 1.2

Definition: A sequence is an infinite list of (real) numbers with a different order.

$a_1, a_2, a_3, a_4 \ldots$

Examples:

① $a_n = n$ for all $n \ge 1$ $1, 2, 3, 4, 5, 6 \ldots$ (natural #s)

② $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_n = \frac{1}{n}$ for all $n \ge 1$ harmonic sequence

③ 3.7 3.7 3.7 $3.7 \ldots$

$a_n = 3.7$ for all $n \ge 1$ constant sequence

④ $-1, 1, -1, 1, -1, 1 \ldots$

$a_n = (-1)^n$ for all $n \ge 1$

⑤ $a_n = \sqrt{n-7}$ for all $n \ge 7$

$0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4}$

sequence can be defined explicitly ($a_n = f(n)$) or recursively. For example, $a_1 = 1$ $a_2 = 1$ $a_n = a_{n-1} + a_{n-2}$ for $n \ge 3$

$1, 1, 2, 3, 5, 8, 13, 21 \ldots$

Fibonacci sequence

$a_1 = 1$ $a_n = \frac{a_{n-1} + 2}{a_{n-1} + 1}$ for $n \ge 2$

$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$

Notation for sequences.

$\{a_n\}_{n=1}^{\infty}$ $\{a_n\}_{n=38}^{\infty}$ $\{a_n\}$ induce the implicit

$(a_1, a_2, a_3, a_4 \ldots)$

$\{a_1, a_2, a_3 \ldots\}$

We'd like to plot sequence: suppose $a_n = \frac{1}{n}$ for $n \ge 1$

Sept 13^{rd}, 2019
Math 137

Arithmetic of limits

$a_n > 0$ for all n

$\lim_{n\to\infty} a_n = L$

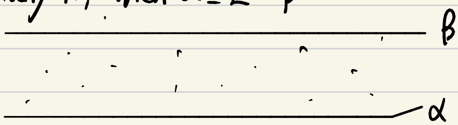
$a_n = \frac{1}{n} > 0$

Facts: Suppose $\lim_{n\to\infty} a_n = L$

(1) If $a_n > 0$, for all n, then $L \geq 0$

(2) If there are real numbers α and β such that $\alpha \leq a_n \leq \beta$ for all n, then $\alpha \leq L \leq \beta$

(3) If there are $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq a_n \leq \beta$, for all but finitely many n, then $\alpha \leq L \leq \beta$



Divergence

$\lim_{n\to\infty} (-1)^n$ does not exist

$\lim_{n\to\infty} n$ "=" ∞ (diverges)

$\lim_{n\to\infty} \sqrt{n+1} = \infty$ (diverges)

Definition:

Let $\{a_n\}$ be a sequence, we say that $\{a_n\}$ diverges to infinity and write $\lim_{n\to\infty} a_n = \infty$, if for any M, there exists N such that $a_n > M$ for any $n \geq N$.

do not say "the sequence converges to ∞"

Exercise: write down a definition for $\lim_{n\to\infty} a_n = -\infty$

Example: show $\lim_{n\to\infty} \sqrt{n+1} = \infty$

Aside: want $\sqrt{n+1} > M$

$n+1 > M^2$

$n > M^2 - 1$

take $n > M^2 > M^2 - 1$

proof: let $M > 0$ be given, take $N > M^2$ ($N = \lfloor M^2 \rfloor + 1$)

If $n \geq N$, then

$$
\begin{aligned}
a_n &= \sqrt{n+1} \\
&\geq \sqrt{N+1} \\
&> \sqrt{M^2+1} \\
&> \sqrt{M^2} = M
\end{aligned}
$$

$\therefore$ so, $\lim_{n\to\infty} \sqrt{n+1} = \infty$

Theorem: let $\{a_n\}$ and $\{b_n\}$ be sequence and suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$

(1) For any $c \in \mathbb{R}$, if $a_n = c$ for all n, then $L = c$

(2) For any $c \in \mathbb{R}$, $\lim_{n\to\infty} c\, a_n = cL$

(3) $\lim_{n\to\infty} (a_n + b_n) = L + M$

(4) $\lim_{n\to\infty} a_n b_n = LM$

(5) If $M \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$

(6) If $a_n \geq 0$ for all n and $\alpha > 0$, is a real number then $\lim_{n\to\infty} a_n^{\alpha} = L^{\alpha}$

(7) For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$

(8) If $\alpha > 0$, then $\lim_{n\to\infty} n^{\alpha} = \infty$

(9) If $\alpha < 0$, then $\lim_{n\to\infty} n^{\alpha} = 0$

Computing limits (1, 2, 6 mostly) Sept. 18, 2019

Example: show that $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

divide through by n (since $n \geq 1$)

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{3}{n}} = \lim_{n\to\infty} = \frac{1+\frac{1}{n}}{2+\frac{3}{n}}$$

$$\lim_{n\to\infty} 2+\frac{3}{n} = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{3}{n}$$

$$= 2+3\lim_{n\to\infty}\frac{1}{n}$$

$$= 2+3(0)$$

$$= 2.$$

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{\lim_{n\to\infty}(1+\frac{1}{n})}{\lim_{n\to\infty}(2+\frac{3}{n})}$$

$$= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}}{2}$$

$$= \frac{1}{2}$$

Example: $\lim_{n\to\infty} \frac{3n^3+n+10^{73}}{n^5+3n^2+1}$

$$= \lim_{n\to\infty} \frac{\frac{3n^3}{n^5}+\frac{n}{n^5}+\frac{10^{73}}{n^5}}{1+\frac{3}{n^3}+\frac{1}{n^5}}$$

$$= \lim_{n\to\infty} \frac{\frac{3}{n^2}+\frac{1}{n^4}+\frac{10^{73}}{n^5}}{1+\frac{3}{n^3}+\frac{1}{n^5}}$$

$$= \frac{0+0+0}{1+0+0} = 0$$ by limit arithmetic.

Fact: Suppose $\{a_n\}$ is a sequence with

$$a_k = \frac{b_0 + b_1 n_1 + b_2 n_2 + \dots + b_k n_k}{c_0 + c_1 n_1 + c_2 n_2 + \dots + c_k n_k}$$

where $j, k \geq 0$ and $b_k, c_j \neq 0$

then,

$$\lim_{n\to\infty} a_n = \begin{cases} \frac{b_k}{c_j} & \text{if } k = j \\ 0 & \text{if } j > k \\ \infty & \text{if } k > j \text{ and } \frac{b_k}{c_j} > 0 \\ -\infty & \text{if } k > j \text{ and } \frac{b_k}{c_j} < 0 \end{cases}$$

Part 5 of thm from last time.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \quad \text{as long as } M \neq 0$$

Suppose $b_n = \frac{1}{n}$, for $n \geq 1$, $\lim_{n\to\infty} b_n = 0$

(1) $a_n = \frac{\sqrt{13}}{n}$ ~~for~~ all $n \geq 1$

then $\lim_{n\to\infty} a_n = 0$

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{13} = \sqrt{13}$

(2) $a_n = \frac{1}{n^2}$ ($\lim_{n\to\infty} a_n = 0$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

(3) $a_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{n} = \infty$$

($a_n = \frac{-1}{\sqrt{n}}$, get $-\infty$)

Theorem: Suppose $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for some real number L. Then $\lim_{n\to\infty} a_n = 0$

Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n \cdot \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} \cdot b_n = L \cdot 0$
$= 0$

Example: $a_1 = 1$ and $a_n = \frac{2 + a_{n-1}}{1 + a_{n-1}}$ for $n \geq 2$

$1, \frac{3}{2}, \frac{7}{3}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169} \approx 1.41420\ldots$

Assuming the sequence converges, $\lim_{n\to\infty} a_n = L$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$
$$= \lim_{n\to\infty} \frac{2 + a_n}{1 + a_n}$$
$$= \frac{2 + \lim_{n\to\infty} a_n}{1 + \lim_{n\to\infty} a_n}$$
$$= \frac{2 + L}{1 + L}$$

so; $L = \frac{2+L}{1+L}$

$$\cancel{L} + L^2 = 2 + \cancel{L}$$
$$L = \pm\sqrt{2}$$

so, $L = \sqrt{2}$ because $a_n \geq 0$ for all a_n

$a_{n+1} = \frac{\alpha + a_n}{1 + a_n} \quad a_n \to \sqrt{\alpha}$

What is $\lim_{n\to\infty} \frac{\sin(n)}{n}$

$\frac{\sin(20)}{20} = 0.0456$

$\frac{\sin(100)}{100} = -0.0506$

guess $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

$\lim_{n\to\infty} \sin(n) \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 0 \cdot \lim_{n\to\infty} \sin(n)$ does not exist

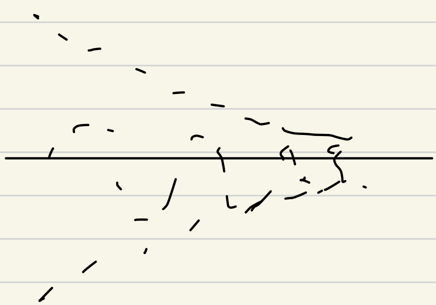
so we can not use part 4 of our thm

$-1 \le \sin(n) \le 1$

$$\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$



Theorem (Squeeze Theorem)

Suppose $\{a_n\}$ $\{b_n\}$ and $\{c_n\}$ are sequence with $a_n \le b_n \le c_n$, for all (but finitely many) n. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} m = L$

The monotone converges theorem set up (induction)

Proof of Squeeze Theorem

Suppose

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ and $a_n \le b_n \le c_n$ for all n

let $\varepsilon > 0$ be given, then since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

there exists $N_1, N_2 \in \mathbb{N}$

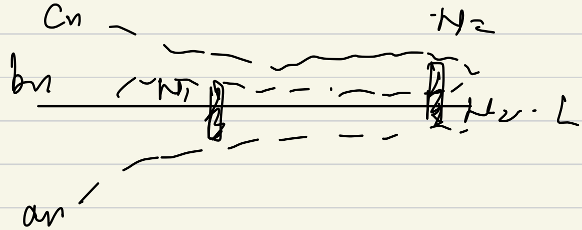
such that if $n \ge N_1$, $|a_n - L| < \varepsilon$, and if $n \ge N_2$, $|c_n - L| < \varepsilon$

set $N = \max\{N_1, N_2\}$, If $n \ge N$, then $a_n \in (L-\varepsilon, L+\varepsilon)$ and $c_n \in (L-\varepsilon, L+\varepsilon)$, so

$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$, so

$b_n \in (L-\varepsilon, L+\varepsilon)$ which means $|b_n - L| < \varepsilon$

Therefore, $\lim_{n\to\infty} b_n = L$.



Example: we know $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

and $-1 \le \sin(n) \le 1$, so $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$

therefore, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ by the squeeze theorem

Notation: for $n \in \mathbb{N}$, $n! = 1\times 2\times 3\times \cdots (n-1)\times n$

Example: prove that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

Since 2^n and $n! > 0$, $\frac{2^n}{n!} \ge 0$ for all n

n	1	2	3	4	5	6	7	8	9	10
$\frac{2^n}{n!}$	2	2	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{8}{45}$	$\frac{13}{315}$	$\frac{4}{315}$	$\frac{8}{2835}$	$\frac{8}{14175}$
$\frac{1}{2^n}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$

$\frac{2^n}{n!} < \frac{1}{2^n}$

$a_{n+1} = \frac{2^n}{n!} \cdot \frac{2}{n+1} < \frac{1}{2^n} \cdot \frac{1}{2} = \frac{1}{2^{n+1}}$

for all but finitely many n,

$\frac{2^n}{n!} < \frac{1}{2^n}$

so $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{2^n}$

by the squeeze theorem

Induction: Let a_n be the sequence a_1 and $a_{n+1} = \frac{a_n + 2}{a_n + 1}$. prove that $a_n \ge 0$ for all $n \in \mathbb{N}$

For the base case, take $n=1$, since $a_1 = 1 \ge 0$, the claims holds for $n=1$.

Assume for some $k \geq 1$ that $a_k \geq 0$, Then $a_{k+2} \geq 2 \geq 0$ and $a_{k+1} \geq 1 \geq 0$ So $a_{k+1} = \dfrac{a_{k+2}}{a_{k+1}} \geq 0$

Therefore, the claim is true for all $n \geq 1$.

Start of the MCT

Definition: let $\{a_n\}$ be a sequence, we say that $\{a_n\}$ is

(1) increasing if $a_n < a_{n+1}$ for all $n \geq 1$

(2) non-decreasing if $a_n \leq a_{n+1}$ for all $n \geq 1$

(3) decreasing: if $a_n > a_{n+1}$ for all $n \geq 1$

(4) non-increasing if $a_n \geq a_{n+1}$ for all $n \geq 1$

(5) monotonic if it satisfies any of (1) - (4)

Example:

(1) $\frac{1}{n}$ is decreasing

(2) $\frac{n-1}{n}$ is increasing

(3) $\{(-1)^n\}$ is not monotonic

(4) $\{\lfloor \sqrt{n} \rfloor\}$ is non decreasing

Definition Let $S \subseteq \mathbb{R}$ be a set of real numbers we say that S is bounded below if there exists $\alpha \in \mathbb{R}$ such that $x \geq \alpha$ for all $x \in S$

S is bounded if it is bounded below and above

The Monotone Covergence Theorem

Example:

(1) The natural numbers are bounded below and not bounded above. (-28)

(2) The natural numbers are neither bounded above nor bounded below.

(3) $\{-3, 28, 106, \pi\}$ is bounded

(4) Every finite set is bounded

(5) If $a<b$, then (a,b), $(a,b]$, $[a,b)$, $[a,b]$ are all bounded.

Definition: Let S be a set, we say that β is the least upper bound for S, denoted $\text{lub}(S)$ or $\sup(S)$ if

(1) β is an upper bound for S

(2) If γ is an upper bound for S, then $\beta \le \gamma$

Definition: The greatest lower bound for S is denoted $\text{glb}(S)$ or $\inf(S)$ and...

Axiom: If S is bounded above, then it has a least upper bound

Example:

(1) The least upper bound of $[0,1)$ is 1

(2) $\{-20\} \cup [-1,1] \cup \{38\} = S$

$\inf(S) = -20 \qquad \sup(S) = 38$

Important: $\inf(S)$ and $\sup(S)$ are not necessarily in S

Theorem: Monotone Convergence Theorem:

Let $\{a_n\}_{n=1}^{\infty}$ be sequence which is non-decreasing.

(1) If $\{a_n\}$ is bounded above, then $\lim_{n\to\infty} a_n = \sup(\{a_n\})$,

(2) If $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$

Proof: proof of part 1, set $\varepsilon > 0$, and set $L = \sup(\{a_n\})$

Suppose $a_n \le L-\varepsilon$ for all $n \ge 1$. Then $L-\varepsilon$ is an upper bound for $\{a_n\}$ $L-\varepsilon < L$, so this would imply L is not the least upper bound of $\{a_n\}$

So set N to be the smallest, such that $L-\varepsilon < a_N$. Then for any $n \ge N$, since $\{a_n\}$ is non-decreasing, $L-\varepsilon < a_N \le a_n < L < L+\varepsilon$

so $L-\varepsilon < a_n < L+\varepsilon$, which is the same as $|a_n - L| < \varepsilon$

Example: Define $\{a_n\}$ $a_1 = 1$ and $a_{n+1} = \frac{1+2a_n}{5}$, show that $\lim_{n\to\infty} a_n$ exists and find it:

$1, \frac{3}{5}, \frac{11}{25}$

Proof: Each term is positive, so the sequence is bounded below by 0, let's prove this formula.

$a_1 = 1 > 0$ Assume for some $(k \ge 1)$ that $a_k > 0$, then $1+2a_k > 0$,

So $a_{k+1} = \frac{1+2a_k}{5} > 0$

Therefore; all the terms are positive by the principle of mathematical induction, to see that the sequence is decreasing, note that $a_1 = 1$ and $a_2 = \frac{3}{5}$

So $a_1 > a_2$, Assume for some $k \geq 1$ that $a_k > a_{k+1}$. Then $2a_k > 2a_{k+1}$

so $1 + 2a_k > 1 + 2a_{k+1}$, and $\frac{1+2a_k}{5} > \frac{1+2a_{k+1}}{5}$, which means

$a_{k+1} > a_{k+2}$.

So, by the principle of mathematical induction, $a_n > a_{n+1}$ for all $n \geq 1$

So, by the MCT, $\lim_{n\to\infty} a_n = L$, for some L

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1+2a_n}{5}$$

$$= \frac{\lim_{n\to\infty} 1 + 2\lim_{n\to\infty} a_n}{\lim_{n\to\infty} 5} = \frac{1+2L}{5}$$

Since $L = \frac{1+2L}{5}$, $L = \frac{1}{3}$

$a_n > \frac{1}{3}$

$3a_n > 1$

$5a_n > 1 + 2a_n$

$a_n > \frac{1+2a_n}{5} > a_{n+1}$

Example: what is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$

$a_1 = \sqrt{2}$

$a_2 = \sqrt{2+\sqrt{2}}$

$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

$\vdots$

$a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2+a_n}$

$a_1 = \sqrt{2} < 2$

Assume $a_k < 2$ for some k

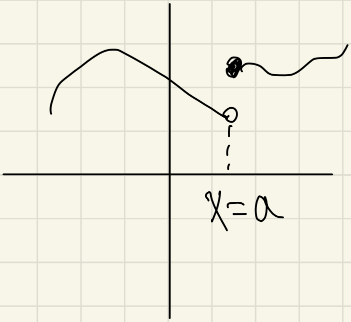
then $a_{k+1} = \sqrt{2+a_k}$

$< \sqrt{2+2} = 2$

So $a_n < 2$ for all n by induction

Limits of functions

Sept 23, 2019

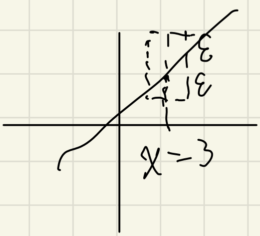


Definition: Let $f(x)$ be a function and $a \in \mathbb{R}$. We say that the limit as x approaches a of $f(x)$ is L, written $\lim_{x\to a} f(x) = L$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$

Example: $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 22 & \text{otherwise} \end{cases}$

show that $\lim_{x\to 0} f(x) = 0$

Example: let $f(x) = 3x+1$ show that $\lim_{x\to 3} f(x) = 10$



Aside: $|3x+1) - 10| < \varepsilon$

$|3x - 9| < \varepsilon$

$3|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{3}$

Proof: Let $\varepsilon > 0$ be given, choose $\delta = \frac{\varepsilon}{3}$, assume $0 < |x-3| < \delta$

Then, $3|x-3| < 3\delta$

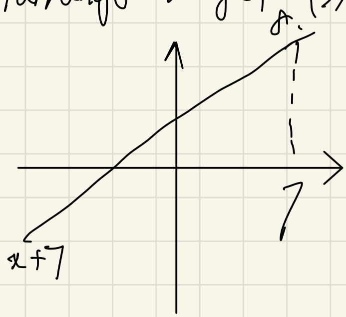
so $|3x-9| < \frac{3\delta}{3}$, which can rearrange to get $|3x+1-10| < \varepsilon$

Therefore, $\lim_{x\to 3} 3x+1 = 10$

Example: show that $\lim_{x\to 7} x^2+1 = 50$

$|x^2+1 - 50| = |x^2 - 49|$

$= |x-7||x+7|$



if $\delta < 1$, then $|x+7| = |x-7+14| \le |x-7| + |14|$

$|x+7||x-7| < \varepsilon$ $\qquad \le 1+14 = 15$

$< \delta \cdot 15 < \varepsilon$

$\delta = \frac{\varepsilon}{15}$

Proof: set $\varepsilon > 0$ arbitrarily, choose $\delta < \min\{1, \frac{15}{\varepsilon}\}$

$(= \frac{1}{2}\min\{1, \frac{15}{\varepsilon}\})$

Assume $0 < |x-7| < \delta$

$|x+7| = |x-7+14|$

$\le |x-7| + |14| < 15$

so $|x^2+1-50| = |x^2-49|$

$= |x+7||x-7|$

$< 15\delta$

$< 15 \cdot \frac{\varepsilon}{15} = \varepsilon$

Example: let $f(x) = \frac{|x|}{x}$. show that $\lim_{x \to 0} f(x)$ does not exist.

if $x < 0$, $|x| = -x$, so $\frac{|x|}{x} = \frac{-x}{x} = -1$

if $x > 0$, $f(x) = 1$

so $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

Theorem (Sequential characterization of limits)

Let $f(x)$ be a function which is defined in some open interval containing a, but not necessarily at a. The following are equivalent.

(1) $\lim_{x \to a} f(x)$ exists and it is equal to L.

(2) For all sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{x \to \infty} x_n = a$ and $x_n \neq a$ for all $n \geq 1$, we have $\lim_{n \to \infty} f(x_n) = L$

$x_n = \frac{-1}{n}$ for all n

$y_n = \frac{1}{n}$ for all n

$\lim_{x \to 0} x_n = \lim_{y \to 0} y_n = 0$

$f(x_n) = +$ $f(y_n) = 1$

So $\lim_{x \to \infty} f(x_n) = -1 \neq \lim_{n \to \infty} f(y_n)$

Theorem (Sequential Equivalence)

Let $f(x)$ be defined on an open interval around $x = a$ (possibly excluding $x = a$). then, the following are equivalent

- $\lim_{x \to a} f(x) = L$
- If for every sequence $\{x_n\}$ with $x_n \neq a$ & $\lim_{n \to \infty} x_n = a$, the sequence $\{f(x_n)\}$ satisfies $\lim_{n \to \infty} f(x_n) = L$

This is use to help determine when limits do not exist

E.x. Given $f(x) = \frac{|x-4|}{3x^2-8x-16}$, show that $\lim f(x)$ do not exist, $x \to 4$

by find 2 different sequences, $\{x_n\}$ & $\{y_n\}$ that converge to 4 as $n \to \infty$ such that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

Sol^n: let $x_n = 4 + \frac{1}{n}$ & $y_n = 4 - \frac{1}{n}$

We have $x_n \to 4$ & $y_n \to 4$ as $n \to \infty$

$$f(x_n) = \frac{|4+\frac{1}{n}-4|}{3(4+\frac{1}{n})^2-8(4+\frac{1}{n})-16} = \frac{\frac{1}{n}}{\frac{16}{n}+\frac{3}{n^2}} = \frac{1}{16+\frac{3}{n}} \to \frac{1}{16}$$

$$f(y_n) = \frac{|4-\frac{1}{n}-4|}{3(4-\frac{1}{n})^2-8(4-\frac{1}{n})-16} = \frac{\frac{1}{n}}{-\frac{16}{n}+\frac{3}{n^2}} = \frac{1}{-16+\frac{3}{n}} \to -\frac{1}{16}$$

as $n \to \infty$

$\because -\frac{1}{16} \neq \frac{1}{16}$

$\therefore$ the $\lim_{x\to 4} f(x)$ can not exist

In general, we can say, Thm Unique limits,

If $\lim_{x\to a} f(x) = L$. then L is unique

Note: to show the $\lim_{x\to a} f(x)$ does not exist

either: 1) find a sequence $\{x_n\}$. $x_n \neq a$,

such that $\{f(x_n)\}$ does not converge.

Note $\lim_{n\to\infty} x_n = a$

d) Find 2 sequences $\{x_n\}$ & $\{y_n\}$, $x_n \neq a$ & $y_n \neq a$, with $x_n \to a$, $y_n \to a$ but $f(x_n) \to L$ & $f(y_n) \to M$, with $L \neq M$

Ex. show that $\lim_{x \to 2} \frac{1}{x^2-4}$ does not exist.

Sol^n let $x_n = 2 + \frac{1}{n}$, then $x_n \to 2$ as $n \to \infty$

Also, $\frac{1}{(2+\frac{1}{n})^2 - 4} = \frac{1}{\frac{1}{n}(4+\frac{1}{n})} = \frac{n}{4+\frac{1}{n}} = \frac{n^2}{4n+1} \to \infty$ as $n \to \infty$

Thus, $\lim_{x \to 2} \frac{1}{x^2-4}$ does not exist

(Sure, it's probably going to ∞, but we haven't officially defined that yet for functions)

Limit Rules:

Given $\lim_{x \to a} f(x) = L$ + $\lim_{x \to a} g(x) = M$

i) if $f(x) = c$, for all x, then $L = c$, $c \in \mathbb{R}$

ii) $\lim_{x \to a} c f(x) = cL$, $c \in \mathbb{R}$

iii) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

iv) $\lim_{x \to a} f(x) \cdot g(x) = LM$

v) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

vi) $\lim_{x \to a} [f(x)]^p = L^p$ for $p > 0$

Thm Basic Limit

If $f(x) = x$, then $\lim_{x \to a} f(x) = a$

(can you prove this using $\epsilon - \delta$ definition?)

Thm Polynomial limit

If $p(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$, then $\lim\limits_{x\to a} p(x) = p(a)$

Limits of Rational Functions

Generally, for a function $f(x) = \frac{p(x)}{q(x)}$, p & q polynomials

$\lim\limits_{x\to\infty} f(x) = \frac{p(a)}{q(a)}$ so long as $q(a) \neq 0$

Ex. $\lim\limits_{x\to 3} \frac{x^2-2}{2x-1} = \frac{7}{5}$

- if $q(a)=0$, & $p(a)\neq 0$, then limit will not exist. It may tend to $\pm\infty$ (coming soon)
- if $q(a)=0$ & $p(a)=0$, then we should factor $x-a$ out of $p(x)$ & $q(x)$

Ex. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

$= \lim\limits_{x\to 2} \frac{(x-2)}{(x-2)} \cdot \frac{(x+3)}{(x-1)}$

$= \lim\limits_{x\to 2} \frac{x+3}{x-1}$

$= 5$

Note that $\frac{x^2+x-6}{x^2-3x+2} \neq \frac{x+3}{x-1}$

this function is not defined at $x=2$

This is a hole in the graph.

this function is defined at $x=2$

The do however have the same limiting behaviour at $x=5$

One side limits Sept 17, 2019

Def^n: The notation $\lim_{x \to a^+} f(x) = L$, means that for a $\varepsilon > 0$, there is a $\delta > 0$, such that if $a < x < a+\delta$, then $|f(x) - L| < \varepsilon$

Def^n: The notation $\lim_{x \to a^-} f(x) = L$ means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$

Ex. Given $f(x) = \frac{x - x^2}{|3x-3|}$ compute $\lim_{x \to 1^+} f(x)$ & $\lim_{x \to 1^-} f(x)$

Does $\lim_{x \to 1} f(x)$ exist?

Sol^n: For $x > 1$, we have $|3x-3| = 3x-3$, so we get $\lim_{x \to 1^+} \frac{x - x^2}{|3x-3|}$

$= \lim_{x \to 1^+} \frac{x(1-x)}{3x-3}$

$= \lim_{x \to 1^+} \frac{x(1-x)}{3(x-1)}$

$= \lim_{x \to 1^+} -\frac{x}{3}$

$= -\frac{1}{3}$

Similarly, when $x < 1$, we have

$|3x-3| = 3 - 3x$

$\lim_{x \to 1^-} \frac{x - x^2}{|3x-3|} = \lim_{x \to 1^-} \frac{x(1-x)}{-3(x-1)}$

$= \lim_{x \to 1^-} \frac{x}{3} = \frac{1}{3}$

At this point it seems unlikely that $\lim_{x \to 1} f(x)$ would exist. Since we can get arbitarily to $-\frac{1}{3}$ from one side and arbitarily close to $\frac{1}{3}$ from the other.

Indeed, $\lim_{x \to 1} \frac{x^2 - x}{|3x-3|}$ does not exist

This is generalized by the following.

Thm (Onesided / Twosided equivalence)

The following are equivalent:

- $\lim\limits_{x\to a} f(x) = L$
- $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$

This means if one of them is true/false, then the other is true/false

Thm: Squeeze theorem

Assume $f(x) \le g(x) \le h(x)$ for all x except possibly $x=a$

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$

Ex Given that $\lim\limits_{\theta\to 0} \cos(\theta) = 1$, find $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}$

Sol^n: It is tempting to say $-1 \le \sin\theta \le 1$

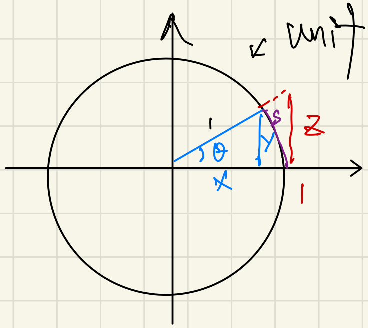
and then for $\theta > 0$, $-\frac{1}{\theta} \le \frac{\sin\theta}{\theta} \le \frac{1}{\theta}$

However, as $\theta \to 0^+$, this just says

$$-\infty \le \frac{\sin\theta}{\theta} \le \infty$$

which is not useful.

consider the diagram,



$s = r\theta$ since $r=1$ $s=\theta$.

Also $y = \sin\theta$ $x = \cos\theta$

By similar triangles,

$$\frac{1}{x} = \frac{z}{y} \Rightarrow z = \frac{y}{x}$$

$$\Rightarrow z = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Now, it turns out that $y \le s \le z$

$$\Rightarrow \sin\theta \le \theta \le \tan\theta$$

divide by $\sin\theta$

$$\Rightarrow 1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$ (for $\theta > 0$, $\sin\theta > 0$)

Since $\lim_{\theta\to 0} \cos\theta = 1$, then $\lim_{\theta\to 0} \frac{1}{\cos\theta} = 1$

$$1 \le \lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} \le 1$$

so by the squeeze theorem, $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$

A special argument works, for $\theta < 0$, we thus have,

special limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Ex. Compute $\lim_{x\to 0} \frac{1-\cos(x)}{x}$ (see pg. 57-58 for more ex.)

Sol^n: At the moment, this is undefined at $x=0$

Rewrite as $\frac{1-\cos(x)}{x}\left(\frac{1+\cos(x)}{1+\cos(x)}\right) = \frac{1-\cos^2 x}{x(1+\cos x)}$

$$= \frac{\sin^2 x}{x(1+\cos(x))}$$

$$= \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}.$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{\sin x}{1+\cos x} = \frac{0}{2} = 0$$

Limits as infinity (i.e. $x \to \pm\infty$)

When $x \to \infty$ or $x \to -\infty$ where ready identical of definition for those of sequence.

Notation	For any $\varepsilon > 0$, there is a N such that
$\lim_{x\to\infty} f(x) = L$	$x > N \Rightarrow \lvert f(x) - L\rvert < \varepsilon$
$\lim_{x\to-\infty} f(x) = L$	$x < N \Rightarrow \lvert f(x) - L\rvert < \varepsilon$

In case where a function remains finite as $x \to \pm\infty$, we further define:

Def^n: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ $(L \neq \pm\infty)$, then we call $y = L$ a horizontal asymptote of $y = f(x)$

Ex. Find the horizontal asymptote of

$$f(x) = \frac{2e^x - e^{-x}}{e^x + 2e^{-x}}$$

Sol^n: We compute $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$ separately

We have

$$f(x) = \frac{2e^x - e^{-x}}{e^x + 2e^{-x}}$$

$$= \frac{e^x(2 - e^{-2x})}{e^x(1 + 2e^{-2x})}$$

$$= \frac{2 - e^{-2x}}{1 + 2e^{-2x}} \to \frac{2 - 0}{1 + 0} = 2 \text{ as } x \to \infty$$

Similarly, $f(x) = \dfrac{2e^{x} - e^{-x}}{e^{x} + 2e^{-x}} = \dfrac{e^{-x}(2e^{2x} - 1)}{e^{-x}(e^{2x} + 2)}$

$$= \frac{2e^{2x} - 1}{e^{2x} + 2} \to \frac{0 - 1}{0 + 2} \text{ as } x \to -\infty$$

$$= -\frac{1}{2}$$

Thus, both $y = 2$ and $y = -\frac{1}{2}$ are horizontal asymptotes

Note: any rational function $f(x)$ equals $\dfrac{p(x)}{q(x)}$ will have a horizontal asymptote of:

- $y = 0$, if $\deg(q(x)) > \deg(p(x))$
- $y = k \neq 0$ if $\deg(q(x)) = \deg(p(x))$

e.g. $f(x) = \dfrac{x + x^2}{1 - x^3}$ statisfies $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$

e.g. $f(x) = \dfrac{2x^4 - x^3 + 1}{x - 3x^4}$ satisfies $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = -\frac{2}{3}$

Thm Sequeeze Theorem

Assume $f(x) \leq g(x) \leq h(x)$, If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L$

then $\lim_{x \to \infty} g(x) = L$

if $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} h(x) = L$, then $\lim_{x \to -\infty} g(x) = L$

Note the condition $f(x) \leq g(x) \leq h(x)$ only has to hold "eventually" i.e. for $x >$ some N when $x \to \infty$

for $x <$ some N when $x \to -\infty$

Ex. Given $\frac{1}{x} < \frac{\ln(x)}{x} < \frac{1}{\sqrt{x}}$

for $x > e$, compute $\lim_{x\to\infty} \frac{\ln(x)}{x}$

Sol^n: since $\lim_{x\to\infty} \frac{1}{x} = 0$, & $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

then by the squeeze theorem, $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

special limit $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

Ex. Compute $\lim_{x\to\infty} \frac{[\ln(x)]^3}{x}$

Sol^n: $\frac{[\ln(x)]^3}{x} = \left(\frac{\ln(x)}{x^{\frac{1}{3}}}\right)^3 = \left(\frac{3\ln(x^{\frac{1}{3}})}{x^{\frac{1}{3}}}\right)^3 = 27\left(\frac{\ln(x^{\frac{1}{3}})}{x^{\frac{1}{3}}}\right)^3$

Now, let $u = x^{\frac{1}{3}}$, so that $u\to\infty$ as $x\to\infty$

and we have,

$$\lim_{x\to\infty} \frac{[\ln(x)]^3}{x} = \lim_{x\to\infty} 27\left(\frac{\ln(u)}{u}\right)^3 = 27\cdot 0^3 = 0$$

More generally, for $a, b \in \mathbb{R}$, $b > 0$, $\lim_{x\to\infty} \frac{[\ln(x)]^a}{x^b} = 0$.

Exercise: Compute $\lim_{x\to\infty} \frac{x^p}{e^x}$ for $p > 0$, Hint: let $u = e^x$

Ex. Compute $\lim_{x\to 0^+} x\cdot\ln(x)$

Sol^n: let $u = \frac{1}{x}$, then $u\to\infty$ as $x\to 0^+$ and so $\lim_{x\to 0^+} x\cdot\ln =$

$$\lim_{u\to\infty} x\ln(x) = \lim_{u\to\infty} \frac{\ln(\frac{1}{u})}{u} = \lim_{x\to\infty} \frac{-\ln(u)}{u} = -0 = 0$$

Infinite limits (i.e. $f(x) \to \pm\infty$)

Similar to $\lim_{n\to\infty} a_n = \pm\infty$

we have

Notation	For any M, there is an N such that
$\lim_{x\to\infty} f(x) = \infty$	$x > N \Rightarrow f(x) > M$
$\lim_{x\to\infty} f(x) = -\infty$	$x > N \Rightarrow f(x) < M$
$\lim_{x\to-\infty} f(x) = \infty$	$x < N \Rightarrow f(x) > M$
$\lim_{x\to-\infty} f(x) -\infty$	$x < N \Rightarrow f(x) < M$

e.g. $\lim_{x\to-\infty} 1-2x^3 = \infty$

because for any M, choose $N = \sqrt[3]{\frac{1-M}{2}}$

then when $x < \sqrt[3]{\frac{1-M}{2}}$, we get $1-2x^3 > M$

A more interesting situation is when a function tends to $\pm\infty$ at a specific value $x = a$

e.g. The function $f(x) = \frac{1+x^2}{2-x}$ is undefined at 2

Since $\lim_{x\to2} 1+x^2 = 5$, and $\lim_{x\to2} 2-x = 0$

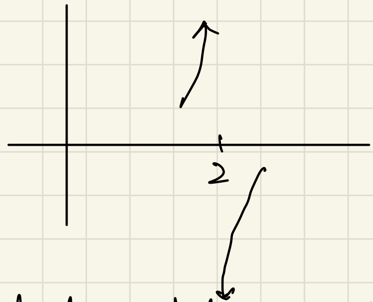
the size of $\frac{1+x^2}{2-x}$ is getting larger & larger as $x \to 2$

To determine it's positive or negative note that $1+x^2 > 0$ for all x. So we investigate the sign of $2-x$

For $x<2$, $2-x>0$, so $f(x)=\dfrac{1+x^2}{2-x}>0$

$x>2$, $2-x<0$ so $f(x)=\dfrac{1+x^2}{2-x}<0$

This suggests the graph near $x=2$ looks like



we get the feeling that

$$\lim_{x\to 2^+}\frac{1+x^2}{2-x}=-\infty$$

$$\lim_{x\to 2^-}\frac{1+x^2}{2-x}=\infty$$

Indeed, we define the following:

Notation	For any M, there is a $\delta>0$, such that
$\lim_{x\to a^+} f(x)=\infty$	$a<x<a+\delta \Rightarrow f(x)>M$
$\lim_{x\to a^+} f(x)=-\infty$	$a<x<a+\delta \Rightarrow f(x)<M$
$\lim_{x\to a^-} f(x)=\infty$	$a-\delta<x<a \Rightarrow f(x)>M$
$\lim_{x\to a^-} f(x)=-\infty$	$a-\delta<x<a \Rightarrow f(x)<M$

More generally, we say

$\lim_{x\to a} f(x)=\infty$ if both $x\to a^+$ & $x\to a^-$ imply $f(x)\to\infty$

$\lim_{x\to a} f(x)=-\infty$ if . . . . $f(x)=-\infty$

Def^n: The line $x=a$ is a vertical asymptote of $f(x)$ if $f(x)\to\infty$ or $f(x)\to-\infty$ for $x\to a^+$ or $x\to a^-$

E.x. Find the vertical asymptotes of $f(x)=\dfrac{x^2-1}{x^2+5x+4}$

Sol^n: We need to investigate where the denominator $=0$ and the numerator $\neq 0$ (see caution)

$$f(x)=\frac{x^2-1}{x^2+5x+4}=\frac{(x+1)(x-1)}{(x+1)(x+4)}=\frac{x-1}{x+4},\ x\neq -1$$

so $x=-4$ is a vertical asymptote

Caution: Be careful for "stronger" roots

e.g. $f(x)=\dfrac{x^2-9}{x+6x+9}=\dfrac{(x-3)(x+3)}{(x+3)^2}$ has both numerator and denominator $=0$ at $x=-3$

However, $x=-3$ is still a vertical asimptote since the $(x+3)$ term "sticks around" after cancelling the numerator

Continuity:

Generally speaking, a function is countious is is they are no holes or breaks in its graph. More specifically, we say,

Def^n: $\boxed{\text{Contiouns as a point (Version I)}}$

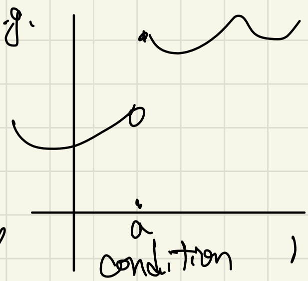
A function $f(x)$ is contious at $x=a$ if

$$\lim_{x\to a} f(x)=f(a)$$

Note that this implies,

1. $\lim_{x\to a} f(x)$ exists
2. $f(x)$ is defined at $x=a$
3. The limit actually equals $f(a)$

e.g.



condition 1 fails
2. fails
3 fails

cond'f. 1 holds
2 holds
3 fails

All 3 holds

Def^n: Continuity at a point (version 2)

A function $f(x)$ is contiouns at $x=a$ if for any $\varepsilon>0$, we can find a $\delta>0$, such that when $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$

Note that we don't have $0<|x-a|$ since we do care about what happens at $x=a$

Exercise: show that $f(x)=3x-4$ is contiouns at $x=-2$ using $\varepsilon-\delta$

Common contious functions:

Recall that a function is contiouns at $x=a$ if $\lim_{x\to a} f(x)=f(a)$

Alternatively, if we let $x=a+h$ $h\neq 0$. we can rewrite this condition

as $\lim_{h\to 0} f(a+h)=f(a)$ means f is cts at $x=a$

Ex. Given that $f(x)=2^x$ is cts at $x=0$, show that it is cts for any $x=a$

Sol^n: We want to show ~~for~~ any $x=a$, $\lim_{h\to 0} f(a+h)=f(a)$ [or $\lim_{x\to a} f(x)=f(a)$]

Since $f(x)=2^x$, we get the $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} 2^{a+h}$

$= \lim_{h\to 0} 2^a \cdot 2^h$

$= 2^a \lim_{h\to 0} 2^h$ since 2^a is constant

$= 2^a \cdot 2^0$ (since 2^x is continous at $x=0$

$= 2^a$

$= f(a)$

Thus, we have shown that $f(x) = 2^x$ is continous everywhere.
Using a similar arguement, we can show that $\cos x, \sin x + a^x$ ($a>0$) are cts for all x

Futhermore, since we already established that for a polynomial $p(x)$ we have $\lim_{x\to a} p(x) = p(a)$ then every polynomial is cts

Finally, we also claim that $\ln(x)$ is cts. on its domain. To show this we require

Thm [Contious Inverse]

Assume $y = f(x)$ is continous at $x=a$, with $f(a) = b$,
If $f(x)$ has inverse $f^{-1}(x)$, then $f^{-1}(x)$ is continous at $x=b$

Thus since e^x is cts for all x, and $e^x > 0$, then $f(x) = \ln(x)$ is cts for all $x > 0$.

In summary, $\sin x, \cos x, a^x$ $\ln x$ polynomials are all cts on their domains

We can build more interesting cts functions by using the following:

Continuity Rules:

If f & g are continous, at $x=a$, then:

1) $f \pm g$ is cts at $x=a$
2) fg is cts at $x=a$.
3) $\frac{f}{g}$ is cts at $x=a$ if $g(a) \neq 0$

These can all be proven from the limit rules (i.e. replace lim. with $f(a) \pm g(a)$)

Ex. Determine where the function $f(x)$ is cts

$$f(x) = \begin{cases} x^2+1 & x<0 \\ e^x - x & 0<x<1 \\ \frac{1}{x} & x \geq 1 \end{cases}$$

Sol^n: note that each piece x^2+1, e^x-x, $\frac{1}{x}$ is cts over their defined intervals. We thus only need to investigate $x=0$ & $x=1$

$x=0$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2+1 = 1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x - x = e^0 - 0 = 1$$

Therefore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$$\lim_{x\to 0} f(x) = 1$$

however, $f(0)$ is not defined

$$\lim_{x\to 0} f(x) \neq f(0)$$

so f is not cts at $x=0$

$x=1$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^x - x = e-1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = 1$$

Also $f(1)=1$

since $\lim_{x\to 1} f(x)$ DNE, it is not cts there.

Thus, $f(x)$ is continous for $x \neq 0, 1$

Pls read section 2.5-1 on types of discontinuities

Thm Sequential Continuity

Every sequence $\{x_n\}$ with $x_n \to a$ satisfies $\lim_{n\to\infty} f(x_n) = f(a)$ if and only if f is continuous at $x=a$

Note we do not have the restriction $x_n \neq a$ here.

This theorem is useful in proving countinuity rules as well as:

Thm continuity of composition

If $f(x)$ is cts at $x=a$, and $g(x)$ is cts at $f(a)$, then $g(f(x))$ is cts at $x=a$.

The theorem allows us to make much more interesting functions

Ex. find where $f(x) = \ln(\sin(x)+1)$ is cts, justify your answer.

Sol^n: The function $\sin(x)$ is cts for all x

The polynomial $g(x) = 1$ is cts for all x

By continuity rules, $\sin(x)+1$ is cts for all x

The function $\ln(x)$ is cts for $x>0$

By composition, we need to find all x such that

$$\sin(x)+1>0$$

$$\Rightarrow \sin(x) > -1$$

Since $-1 \leq \sin(x) \leq 1$ and $\sin(x) = -1$ only when

$$x = \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

Thus, $\sin(x)+1>0$. for all

$$x \neq \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

$\therefore$ by composition theorem $f(x) = \ln(\sin(x)+1)$ is cts for all $x \neq \frac{3}{2}\pi + 2\pi k,\ k \in \mathbb{Z}$

Continuity on an interval

We have so far only defined continuity at a point $x=a$. Base on our recent work, it makes sense to define:

Defn: Continuity on (a,b)

A function $f(x)$ is cts on (a,b) if it is cts at each point in (a,b)

Note: if $(a,b) = (-\infty,\infty)$ then we just say $f(x)$ is cts

Defn: Continuity on $[a,b]$

A function $f(x)$ is cts on $[a,b]$ if:

1) it is cts on (a,b)

2) $\lim_{x \to a^+} f(x) = f(a)$ & $\lim_{x \to b^-} f(x) = f(b)$

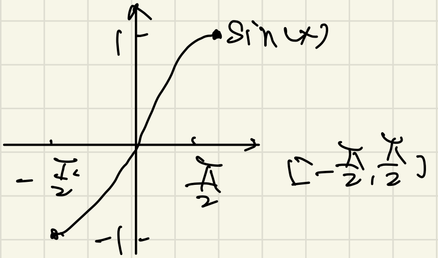
There is often used in cases where a function is not defined for all $\mathbb{R}$..

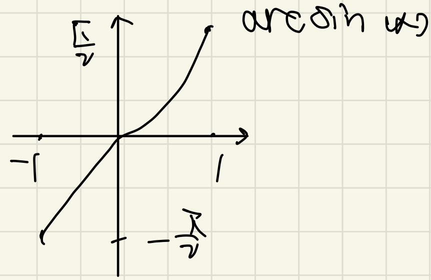
e.g. The function $\sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has inverse $\arcsin(x)$ which is cts on $x \in [-1,1]$ (by continuity inverse)

Note: the function $\sin(x)$ does not have an inverse for all x.

The same is true for $\cos(x)$ & $\tan(x)$ as they are not one to one. (i.e. don't pass the horizontal line test)

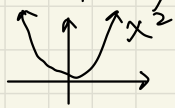
If we restrict their domain however then we can create an invertible function.





Exercise: Determine where $f(x) = \sqrt{4 - x^2}$ is cts

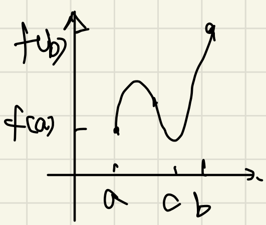
Hint: $f(x) = \sqrt{x}$ is cts by inverse theorem for x^2 restricted to $x \geq 0$



Thm: Intermediate Value Theorem IVT

If f is cts on $[a,b]$ and $f(a) < D < f(b)$ or $f(a) > D > f(b)$

Then there is $c \in (a,b)$ such that $f(c) = D$



e.g. Given $f(x) = x - \frac{2x^2}{1-x}$, since $f(2) = 10$ & $f(3) = 12$ and since f is cts on $[2,3]$, there is some $c \in (2,3)$ such that $f(c) = 11$.

Note: We could have also said $f(c) = 10.5$ or $f(c) = 11.7$ basically any number between 10 & 12.

Exercise: Can you think of a function that is defined on $[a,b]$ cts on (a,b) with $f(a) < 0$ & $f(b) > 0$, but with no point x with $f(x) = 0$

Root Finding

There are many algorithms for finding roots of a function $f(x)$
One of the simplest is the Bisection Method.

Bisection Method

Given a cts function $f(x)$ on $[a,b]$ such that $f(a)<0$ and $f(b)>0$

- calculate $m=\frac{a+b}{2}$
- if $f(m)>0$ then repeat on $[a,m]$
- if $f(m)<0$ then repeat on $[m,b]$
- if $f(m)=0$ you're done

Note: You are unlikely to ever reach $f(m)=0$. So you will need to stop at some point when you're satisfied

- the process also works if $f(a)>0$ and $f(b)<0$, just adjust the split accordingly.

Ex. show that $f(x)=x^2-\sqrt{x}-1$ has a root between $[0,4]$. Find an interval of length $\frac{1}{2}$ where the root lies.

Sol^n. Since $f(0)=-1$ & $f(4)=13$ AND since f is cts on $[0,4]$
then by IVT, there must be a $c\in(0,4)$, with $f(c)=0$

Let $m_1=\frac{0+4}{2}=2$ then $f(2)=3-\sqrt{2}>0$

Since $f(2)>0$ & $f(0)<0$, root must be in $[0,2]$

Let $m_2=\frac{0+2}{2}=1$ then $f(1)=-1$

since $f(1)<0$ & $f(2)>0$ root must be in $[1,2]$

Let $m_3=\frac{1+2}{2}=1.5$, then $f(1.5)=2.25-\sqrt{1.5}-1>0$

$(\sqrt{1.5}<1.23)$

Since $f(1.5)>0$, $f(1)<0$, root must be in $[1,1.5]$

Let $m_4=\ldots$, etc.

Actual solution to $f(x)=0$ is $x=1.490216$

Absolute max & min

Defn: Let I be an interval of real numbers. If $f(c) \geq f(x)$ for all $x \in I$ then $f(x)$ has an absolute/global max at c.

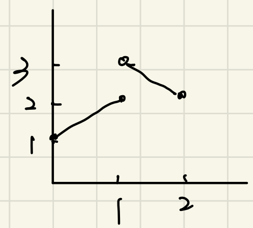
If $f(c) \leq f(x)$ for all $x \in I$, then $f(x)$ has an absolute/global min at c.

In either case we also call $(c, f(c))$ an absolute/global extremum.

Ex. Given $f(x) = \begin{cases} 1+x & 0 \leq x \leq 1 \\ 4-x & 1 < x \leq 2 \end{cases}$

Find the absolute max/min on $[0, 2]$ if possible.

Soln.



Based on our diagram it seems like f achieves an absolute min of 1 at $x=0$

It looks like f has a max of 3, however this is no value $x \in [0,2]$ such that $f(x) = 3$. It turns out there is no absolute max of the function.

Q: Can we guarantee that a function will achieve a max/min?

Thm: Extreme Value Theorem.

If $f(x)$ is cts on $[a,b]$ then there are points $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$. That is f will achieve both max & min on $[a,b]$

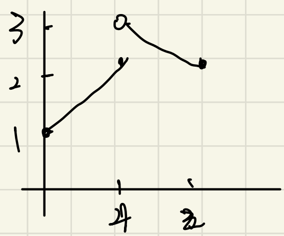
Note that continuity and a closed interval are sufficient for a function to attain a max/min. They are not necessary.

e.g. The function $f(x) = \frac{3}{x}$ on $[1,5]$ is cts. Since $[1,5]$ is closed, the f achieves a max/min by EVT.

e.g. The function

$$f(x) = \begin{cases} 1+x & 0 \leq x < 1 \\ 4-x & 1 \leq x \leq 2 \end{cases}$$

is not cts.



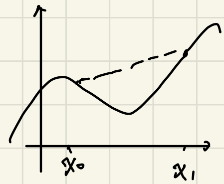
However, it does achieve a max of 3 at $x=1$ and a min of 1 at $x=0$

e.g. The function $f(x) = \sin(x)$ on $(0, 2\pi)$ is cts but $(0, 2\pi)$ is not closed
However, it still attains a max at $x = \frac{\pi}{2}$ and min at $x = \frac{3}{2}\pi$

Average Change

Given a function $f(x)$, the average change of f between 2 points x_0 & x_1 is given by

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



This "Newton Quotient" gives the slope of the line between $(x_0, f(x_0))$, $(y_0, f(y_0))$

e.g. If $s(t) = 2t + t^2 + 30$ represents the position of an object in metres at time t in seconds, then the average velocity of the object between $t=0$ & $t=10$ is

$$\frac{s(10) - s(0)}{10 - 0} = \frac{150 - 30}{10} = 12$$

That is the object travels from 30 m to 150 m in 10s thus travels an average of 12 m/s
Note: the object is not traveling at a constant at 12 m/s.
An interesting question is whether it hits 12 m/s exactly during its trip.

Instantaneous Change

We can take this idea one step further by applying a limit as $x_1 \to x_0$ to get the instantaneous change of $f(x)$ at x_0

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This gives the slope of the tangent line (coming soon) to $f(x)$ at x_0
More commonly, we find the instantaneous change of f at $x = a$ by writing.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Ex. if $s(t) = 2t + t^2 + 30$
Find the instantaneous change (i.e. velocity) at $t = 3$
Sol^n: here $a = 3$, we have $\lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$

$$= \lim_{h \to 0} \frac{\cancel{6} + 2h + \cancel{9} + 6h + h^2 + \cancel{30} - \cancel{6} - \cancel{9} - \cancel{30}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 8h}{h}$$

$$= \lim_{h \to 0} h + 8 = 8$$

An important question is whether we can always find this limit. For now, we define:

Defn: Derivative

Given a function $f(x)$ if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists then we denote it by $f'(a)$ and call it the derivative of $f(x)$ at $x=a$

When $f'(a)$ exists, we say f is differentiable

If we treat $x=a$ as arbitrary we can get a function that calculates the derivative at any point.

Ex. Find the derivative of $s(t) = 2t + t^2 + 30$ at $t=a$. Determine when the instaneous change is 12 m/s.

Soln: $s'(a) = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h}$

$= \lim_{h\to 0} \frac{2(a+h) + (a+h)^2 + 30 - 2a - a^2 + 30}{h}$

$= \lim_{h\to 0} 2 + 2a + h$

$= 2 + 2a$

Thus, $s'(a) = 2 + 2a$ for any $t=a$.

To find when the instantaneous change is 12 we set

$12 = 2 + 2a$

$a = 5$

Thus at $t=5$, the object moves at 12m/s, note this happens once on the interval $t\in[0,10]$

Will this always happen? Will it always be at the halfway point?

Defn: Derivative function.

Given $f(x)$, if f is differentiable on some interval I, i.e. that

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists for all $x\in I$

$\boxed{\text{Notation}}$

Other notations for calculating the derivative include

$\dot{f}(t)$ - Newton

$\frac{df}{dt}$ or $\frac{d[f]}{dt}$ ← Leibniz highly recommand you use this

$f'(t)$ - Lagrange

The notation $\frac{d}{dt}$ or $\frac{d}{dx}$ is called a differential operator. Think of it sort of like $\sqrt{}$, "the square root operator" We apply $\frac{d}{dt}$ or $\frac{d}{dx}$ to generate the instantaneous rate of change of the given relationship with respect to an independent variable.

e.g. Given $y = 3x + \frac{2}{x}$ we can apply $\frac{d}{dx}$ on both sides to get

$$\frac{d}{dx}[y] = \frac{d}{dx}[3x + \frac{2}{x}]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[3x + \frac{2}{x}]$$

This means using the limit definition

i.e. $$\lim_{h \to 0} \frac{3(x+h) + \frac{2}{x+h} - (3x + \frac{2}{x})}{h}$$

we will find an easier way

Tangent Line

Def^n: If a given function $f(x)$ is differentiable at $x=a$, then we call the line $y=f(a)+f'(a)(x-a)$ the tangent line to the graph of f at $x=a$

Note: a possibly simpler way to remember this is to write

$$\frac{y-f(a)}{x-a}=f'(a)$$

Ex. Find the tangent line to the function $f(x)=\frac{1}{x}$ at $x=2$

Sol^n: We need to compute $f(2)$ & $f'(2)$

$$f(2)=\frac{1}{2}$$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{\cancel{2}-\cancel{2}-h}{2(2+h)}}{h}$$

$$=-\frac{1}{4}$$

the tangent line is $y=\frac{1}{2}-\frac{1}{4}(x-2)$

Thm Differentiability $\Rightarrow$ Continuity

If $f(x)$ is differentiable at $x=a$, then $f(x)$ is cts at $x=a$.

Proof: Since f is differentiable at $x=a$, we know $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists and is equal to $f'(a)$

We need to show that $\lim_{h\to 0}f(a+h)=f(a)$

$$\lim_{h\to 0}f(a+h)=\lim_{h\to 0}f(a+h)-f(a)+f(a)$$

$$=\lim_{h\to 0}[f(a+h)-f(a)]\left[\frac{h}{h}\right]+f(a)$$

$$=\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]\cdot h+f(a)$$

$$=f'(a)\lim_{h\to 0}h+\lim_{h\to 0}f(a)$$

$$=0+f(a)=f(a)$$

Note the converse is not true in general

e.g. the function $|x|$ is cts at $x=0$, but $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ does not exist. Try it!

Ex. Consider the function

$$f(x)=\begin{cases}cx+d & x<0\\ \sqrt{x+1} & x\geq 0\end{cases}$$

What value of c & d will make the function

a) cts at $x=0$

b) differentiable at $x=0$

Sol^n a) To make it cts, we need
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$
Since $\sqrt{x+1}$ is cts at $x=0$
we only require $\lim_{x\to 0^-} cx+d = \sqrt{0+1} = 1$
$$\Rightarrow d = 1$$
Thus, $c \in \mathbb{R}$, $d=1$, will make $f(x)$ cts at $x=0$

b) If we are clever, we should use the fact that differentiability $\Rightarrow$ continuity
This means if there is any hope of making f differentiable at $x=0$, we require $d=1$. However, let's assume we're not clever.
Differentiable at $x=0$ means we need $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ to exist.
Since $f(x)$ has different pieces on either side of $x=0$, we split the limit.
$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt{h+1}-1}{h}$$
$$= \lim_{h\to 0^+} \frac{(\sqrt{h+1}-1)(\sqrt{h+1}+1)}{h(\sqrt{h+1}+1)}$$
$$= \lim_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}(\sqrt{h+1}+1)}$$
$$= \frac{1}{2}$$
$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{ch+d-1}{h}$$
$$= \lim_{h\to 0^-} \frac{ch}{h} + \lim_{h\to 0^-} \frac{d-1}{h}$$
$$= c + \lim_{h\to 0^-} \frac{d-1}{h}$$
At this point, we once again see that if this left hand limit is to exist, we require $d=1$

So if $d=1$, we get $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = c$
whereas
$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \frac{1}{2}$$
for the overall limit to exist we require $c=\frac{1}{2}$ & $d=1$

Derivative Rules

Thm If f & g are differentiable on an open interval I, then on I

A. $\frac{d}{dx}[cf] = c\frac{df}{dx}$ $\in \mathbb{R}$

B. $\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx}$ sum rule

C. $\frac{d}{dx}[f \cdot g] = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$ product rule

d. $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{1}{g^2}\left(\frac{df}{dx} \cdot g - f\frac{dg}{dx}\right)$ Quotient rule

Proof A - C exercises

Proof D by definition,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{g(x)g(x+h)}\left[g(x)\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right]$$

Since g is differentiable then it is cts so $\lim_{h\to 0} g(x+h) = g(x)$ and we get

$$= \frac{1}{g(x)^2}\left[\frac{g(x)\,df(x)}{dx} - \frac{f(x)\,d\,g(x)}{dx}\right]$$

Basic Derivatives

We have

1. $\frac{d}{dx}k = 0$ for $k \in \mathbb{R}$
2. $\frac{d}{dx}x^n = nx^{n-1}$ $n \in \mathbb{R}$ power rule
3. $\frac{d}{dx}\sin x = \cos x$
4. $\frac{d}{dx}\cos x = -\sin x$
5. $\frac{d}{dx}a^x = a^x \ln a$ for $a \in \mathbb{R}, a > 0$ (specifically, $\frac{d}{dx}e^x = e^x$)

Proof of 2, for $n \in \mathbb{Z}$, By induction:

step 1, for $n=1$, $\frac{d}{dx}x = \lim_{h\to 0}\frac{x+h-x}{h} = 1 = 1x^{1-1} = nx^{1-1}$ ✓

step 2 assume $\frac{d}{dx}x^k = kx^{k-1}$, for some fixed $k \in \mathbb{N}$

step 3

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x^k \cdot x)$$

$$= \left(\frac{d}{dx}x^k\right)\cdot x + x^k\frac{dx}{dx}$$

$$= (kx^{k-1})\cdot x + x^k \quad \text{by step 2}$$

$$= kx^k + x^k = (k+1)x^k = nx^{n-1}$$

i.e. $\frac{d}{dx} x^{k+1} = (k+1)x^k$

By induction, $\frac{d}{dx} x^n = nx^{n-1}$ for $n \in \mathbb{N}$

For $n<0$, let $m=-n$ so $m>0$

then $x^n = \frac{1}{x^{-n}} = \frac{1}{x^m}$

By Quotient rule & rule 1, we get

$$\frac{d x^n}{dx} = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{0 \cdot x^m - \frac{d}{dx}x^m}{x^{2m}}$$

$$= \frac{-mx^{m-1}}{x^{2m}}$$

$$= -m x^{-m-1}$$

$$= n x^{n-1}$$

For $n=0$, $x^n = 1$

thus $\frac{d}{dx} 1 = 0 = nx^{n-1}$

$\therefore$ For all $n \in \mathbb{Z}$, $\frac{d}{dx} x^n = nx^{n-1}$

For $n \in \mathbb{R}$, we require more tools.

Proof of 3. by definition,

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

Recall: We established

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \quad \& \quad \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Exercises Prove 1 by definition

Prove 4 by definition.

Use the fact that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ to prove that $\lim_{h\to 0}\frac{a^h - 1}{h} = \ln a$, then show $\frac{d}{dx}a^x = a^x \ln a$

Hint: let $u = a^h - 1$ so that $u \to 0$ as $h \to 0$

Also $\log_a(u+1) = h$

Finally, let $u = \frac{1}{n}$ and use $\log_b a = \frac{\log_c a}{\log_c b}$

Chain Rule

The chain rule is often used in practice according to the phrase "derivative of the outside times the derivative of the inside."

For example-

$$\frac{d}{dx}[e^{3x}] = 3e^{3x}$$
$$\frac{d}{dx}[\sin(x^2+\frac{1}{x})] = \cos(x^2+\frac{1}{x})\cdot(2x-\frac{1}{x^2})$$

However, it has a much more powerful national property.

let $f=f(x)$ & $x=x(t)$ then

$$\frac{d}{dt}[f] = \frac{d[f]}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{ds}\cdot\frac{ds}{dt}$$

That is, we can rewrite the $\frac{d}{dt}$ operator by chaining together as many derivatives as needed until we end at a derivative with respect to t.

eg. $\frac{d\square}{dt} = \frac{d\square}{dx}\cdot\frac{dx}{dt}$

Note: Proving the chain rule is ugly and not enlighting

Ex. If $f=\sqrt{1+\sin(\frac{1}{\sqrt{x}})}$, find $\frac{df}{dx}$

Sol^n. let $p=\frac{1}{\sqrt{x}}$ & $u=1+\sin(p)$

then $f=\sqrt{u}$

and $\frac{d\square}{x} = \frac{d\square}{du}\cdot\frac{du}{dp}\cdot\frac{dp}{dx}$ that is.

$$\frac{df}{x} = \frac{df}{du}\cdot\frac{du}{dp}\cdot\frac{dp}{dx}$$

Since $\frac{d\sqrt{u}}{du} = \frac{1}{2\sqrt{u}}$

$$\frac{du}{dp} = \cos(p)$$

$$\frac{dp}{dx} = -\frac{1}{2x^{3/2}}$$

we get $\frac{df}{dx} = (\frac{1}{2\sqrt{u}})\cdot\cos(p)\cdot(-\frac{1}{2x^{3/2}})$

$$= \frac{1}{2\sqrt{1+\sin(\frac{1}{\sqrt{x}})}}\cdot\cos(\frac{1}{\sqrt{x}})\cdot(-\frac{1}{2x^{3/2}})$$

The chain rule is very useful when differentiating implicit relationship

Find $\frac{dy}{dx}$ of $x^2\cdot\sin(y)+y^3=8x$

assuming $y=y(x)$

Sol^n: Since we're looking for $\frac{dy}{dx}$, apply $\frac{d}{dx}$ to both side.

$$\frac{d}{dx}[x^2\sin(y)+y^3]=\frac{d}{dx}\cdot 8x$$

$$\Rightarrow \frac{d}{dx}\cdot x^2\cdot\sin(y)+\frac{d}{dx}y^3=8$$

$$\frac{d}{dx}[x^2]\sin(y)+x^2\cdot\frac{d}{dx}\sin y+\frac{d}{dy}y^3\cdot\frac{dy}{dx}=8$$

$$2x\cdot\sin(y)+x^2\cdot\cos y\cdot\frac{dy}{dx}+3y^2\cdot\frac{dy}{dx}=8.$$

$$\frac{dy}{dx}=\frac{8-2x\cdot\sin y}{3y^2+x^2\cos y}$$

Higher derivatives.

Def^n Given $f(x)$ if $f'(x)$ exists and is also differentiable, we call $\frac{d}{dx}f'(x)=f''(x)$ the second derivative of f

Similarly, $f'''(x)$ or $f^{(3)}x$ is the third derivative

Generally, $f^{(n)}(x)$ is the n^{th} derivative.

Using Leibinz notation.

$$f^{(n)}(x)=\frac{d^nf}{dx^n}\ ,\ \text{eg } f''(x)=\frac{d^2f}{dx^2}$$

Note: when using libniz notation we write

$$\left.\frac{df}{dx}\right|_a=f'(a)\ \text{ or }\ \left.\frac{d^nf}{dx^n}\right|_a=f^{(n)}(a)$$

use when evaluating a derivative at $x=a$.

Other derivatives

Ex. Find $\frac{d}{dx}[\tan x]$

Sol^n $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

Thus, $\frac{d}{dx}\tan x = \sec^2 x$

Similarly, we can find

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\cot x = -\csc^2 x$

Ex. Find $\frac{d}{dx}\ln x$

Sol^n let $y = \ln x$ so we seek $\frac{dy}{dx}$ Note that $y = \ln x \Leftrightarrow e^y = x$, $x > 0$

Now apply $\frac{d}{dx}$ to $e^y = x$ to get

$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$

$\frac{d}{dx}[e^y]\cdot\frac{dy}{dx} = 1$ (by chain rule)

$e^y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

thus, $\frac{d}{dx}\ln x = \frac{1}{x}$

Using the change of base formula $\log_b a = \frac{\log_c a}{\log_c b}$ we get that

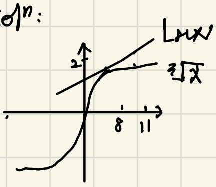
$\frac{d}{dx}\log_b x = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{x}\cdot\frac{1}{\ln b}$ $\quad x > 0$

Linear Approximation

Given a differentiable function $f(x)$ we can use the tangent line to approximate values near the point of angency.

Ex. Find the tangent line to $f(x) = \sqrt[3]{x}$ at $x = 8$ & call it $L_8(x)$. Use $L_8(x)$ to approximate $\sqrt[3]{11}$

Sol^n:



Tangent line equation at $x = a$ is $y = f(a) + f'(a)(x-a)$

That is $L_8(x) = f(8) + f'(8)(x-8)$

Since $f(8) = 2$

and $f'(x) = \frac{1}{3x^{2/3}} \Rightarrow f'(8) = \frac{1}{12}$

we get $L_8(x) = 2 + \frac{1}{12}(x-8)$

Note that $L(x) \approx f(x)$ (especially near 8)

so $L_8(11) = 2 + \frac{1}{12}(11-8) = 2.25$

$\therefore f(11) = \sqrt[3]{11} \approx 2.25$ } not bad

Note $\sqrt[3]{11} = 2.22398..$

Defn: If $f(x)$ is differentiable at $x=a$ then we call

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

the linear approximation to f at $x=a$

Note: if it is obvious what $f(x)$ is, we write $L_a(x)$

- L_a^f is also called the linearization of f

Error estimates

In the previous example, the error was

$$|\sqrt[3]{11} - 2.25| = 0.02602...$$

In general the error at $x=b$ is $|f(b) - L_a(b)|$

(note that $|f(a) - L_a(a)| = 0$ i.e. no error at $x=a$)

Usually however we don't know what $f(b)$ is, that's why we are approximating it!!!

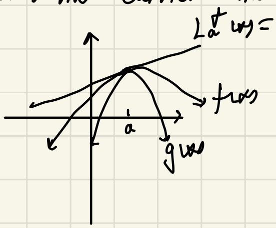
Remarkably, if f behaves "nicely" we can still find an upper bound on the error.

Thm Error in $L_a^f(x)$

Let $a \in I$, if $|f''(x)| < M$, for all $x \in I$. then $|f(x) - L_a^f(x)| \leq \frac{M}{2}(x-a)^2$

Generally, the further away from $x=a$ the bigger the error [due to $(x-a)^2$]

Also, the "curvier" the function the bigger the error [due to $|f''(x)| \leq M$]



Ex. Use $L_1(x)$ to approximate $\ln(\frac{3}{2})$ by using $f(x)=\ln x$ What is an upper bound on the error?

Sol^n: $L_1(x) = f(1) + f'(1)(x-1)$

Since $f(1) = \ln 1 = 0$

& $f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$ then

$L_1(x) = x-1$

$\therefore \ln(\frac{3}{2}) \approx L_1(\frac{3}{2}) = \frac{1}{2}$

To find an upper bound on the error, we need to find an upper bound on $|f''(x)|$ on the interval $[1, \frac{3}{2}]$.

Since $f'' = -\frac{1}{x^2}$, then on $[1, \frac{3}{2}]$, $|f''(x) \le 1|$ ← M

So $|f(x) - L_1(x)| \le \frac{1}{2}(x-1)^2$ (for $x \in [1, \frac{3}{2}]$)

If $x = \frac{3}{2}$, we get $|f(\frac{3}{2}) - L_1(\frac{3}{2})| = |\ln\frac{3}{2} - \frac{1}{2}|$

$\le \frac{1}{2}(\frac{3}{2}-1)^2$ (by Thm)

$= \frac{1}{8}$

This means

$$\frac{1}{2} - \frac{1}{8} \le \ln\frac{3}{2} \le \frac{1}{2} + \frac{1}{8}$$

i.e. $\ln\frac{3}{2} \in [0.375, 0.625]$

We can also use $L_a^f(x)$ to estimate the change of a function as x goes from a to b, i.e. $\Delta x = b-a$.

$\Delta f = f(b) - f(a) \approx L_a(b) - f(a)$

actual change

$= f(a) + f'(a)(b-a) - f(a)$

$= f'(a)(b-a)$

$= f'(a)\Delta x$

$\Delta f \approx f'(a)\Delta x$ ← linear change

Ex. Given $F(r) = \frac{GmM}{r^2}$ if r increases by 2% approximate the change in F (i.e. $\frac{\Delta F}{F}$)

Sol^n: $\Delta r = 1.02r - r = 0.02r$

$\Delta F = F(1.02r) - F(r)$

$F'(r) = -\frac{2GmM}{r^3}$ so $\Delta F \approx F'(r)\Delta r = (-\frac{2GmM}{r^3})(0.02r)$

$= -0.04\left(\frac{GmM}{r^2}\right)$

$= -0.04F$

Thus $\frac{\Delta F}{F} \approx -\frac{0.04F}{F} = -0.04$

i.e. $F(r)$ decreases by approximatly 4%

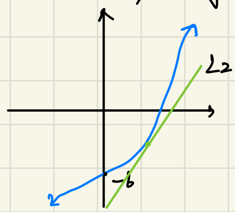
Newton's Method

We can use $L_a^f(x)$ to help us find roots of functions.

Ex. Approximate the root of $f(x) = x^3 - x^2 - 6$ by using $L_2(x)$

Sol^n $L_2(x) = f(2) + f'(2)(x-2)$

$= -2 + 8(x-2)$



The root is approximated by the x-intercept of $L_2(x)$

$$0 = -2 + 8(x-2) \Rightarrow x = \frac{9}{4}$$

Note the actual root of $f(x)$ is $x = 2.21871\ldots$ whereas $x = \frac{9}{4} = 2.25$, not bad

Idea!!! Since $x = \frac{9}{4}$ is a better guess than $x = 2$, why not compute $L_{\frac{9}{4}}(x)$ and find the x-intercept?

$$L_{\frac{9}{4}}(x) = f(\tfrac{9}{4}) + f'(\tfrac{9}{4})(x - \tfrac{9}{4})$$

$$= \frac{21}{64} + \frac{171}{16}(x - \tfrac{9}{4})$$

Find x-int

$$0 = \frac{21}{64} + \frac{171}{16}(x - \tfrac{9}{4}) \Rightarrow x = \frac{253}{114} = 2.21929$$ ← even better.

We can generalize this method as follows: **Newton's Method.**

To find a root of $f(x)$ (i.e. solve $f(x) = 0$) when f is differentiable

1. Guess a starting value x_1
2. Use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ until you're happy.

(if you want k decimal places of accuracy, stop when first k digits don't change)

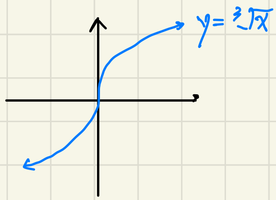
Notes: The recursive formula comes from finding the x-int. of L_{x_n}

If the slope of the function is too flat, the sequence might not converge at all.

Ex. The function $f(x) = \sqrt[3]{x}$ has a root at $x = 0$. Show that for any $x_1 \neq 0$ Newton's Method will not find the root.

Sol^n

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$ so we get

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3x_n^{2/3}}} = x_n - 3x_n = -2x_n$$

i.e. $|x_{n+1}| = 2|x_n| = 2(2|x_{n-1}|) \ldots = 2^n |x_1|$

And so far for any $x_1 \neq 0$, the sequence is unbounded. (i.e. it will keep growing and so can't converge.)

Derivatives of Inverses.

We already showed how to calculate $\frac{d}{dx}\ln x$ using $y = e^x$

A similar procedure can be used for a general invertiable differentiable function $f(x)$

Assuming f is invertible let $y = f^{-1}(x)$ we seek $\frac{dy}{dx}$

we have $f(y) = x$, apply $\frac{d}{dx}$

$$\frac{d}{dx} f(y) = \frac{dx}{dx}$$

$$\Rightarrow \quad \frac{d}{dx} f(y) \cdot \frac{dy}{dx} = 1 \quad \text{(by chain rule)}$$

$$\frac{dy}{dx} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(x))}$$

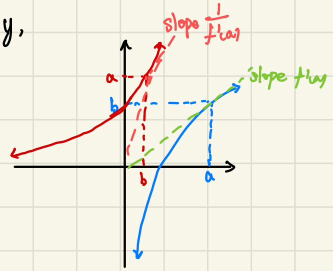
Thus, $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

Thm Inverse function theorem I.F.T.

let $y = f^{-1}(x)$ be the inverse of a differentiable function $f(x)$ such that $f(a) = b$, & $f^{-1}(b) = a$

For all points where $f'(a) \neq 0$, we have $(f^{-1})'(b) = \frac{1}{f'(b)}$

Graphically,



Ex. Give $f(x) = \sqrt{x} + \frac{x}{4}$. $f(4) = 3$, compute $L_3^{f^{-1}}(x)$

Sol^n: Recall

$$L_a^g(x) = g(a) + g'(a)(x-a)$$

Though we don't have $f^{-1}(x)$ explicitly we can still find $f^{-1}(3)$ & $(f^{-1})'(3)$ and thus $L_3^{f^{-1}}(x)$

we know $f(4) = 3$, so $f^{-1}(3) = 4$

Also, by I.F.T., $(f^{-1})'(3) = \frac{1}{f'} = \frac{1}{f'(4)}$

Since $f(x) = \sqrt{x} + \frac{x}{4}$ then $f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{4}$, so $f'(4) = \frac{2}{2}$ and thus $\frac{1}{f'(4)} = 2$

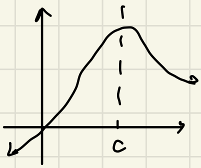
$(f^{-1})'(3) = 2$

We thus have $L_3^{f^{-1}}(x) = f^{-1}(3) + (f^{-1})'(3)(x-3)$

$$= 4 + 2(x-3)$$

Note the IFT requires $f'(x) \neq 0$ which should make sense since a slope of 0 and f would convert to an infinite slope in f^{-1}.

Also, when $f'(x) = 0$ this often concides with a local max or min through which a function is not one to one and thus not invertiable



↓ not invertible on any invertible that includes c (not at an endpoint)

Inverse Trig Derivatives

In a previous example, (See continuity on $[a, b]$) we discussed how $f(x) = \sin x$ is not invertible for all x.

If we define $f(x) = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then it does have an inverse $f^{-1}(x) = \arcsin(x)$

$x \in [-1, 1] \longrightarrow \arcsin(x) \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

input is a ratio $\qquad$ output is an angel

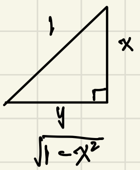
If $y = \arcsin(x)$ to find $\frac{dy}{dx}$ we write, $\sin(y) = x$, apply $\frac{d}{dx}$ to both sides $\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$

$\Rightarrow \frac{d}{dy}[\sin(y)]\frac{dy}{dx} = 1$ (by chain rule)

$\Rightarrow \cos(y)\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now if $\sin(y) = x$, we get the triangle,



so $\cos(y) = \sqrt{1-x^2}$

Thus, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ i.e. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

Note the triangle is sort of assuming $x > 0$ but even if $x < 0$, the value of $\cos(y)$ does not change since $\cos(y) > 0$, when $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Similarly, we make the following restrictions on our standard trig functions to make them invertible.

$f(x)$	Domain restriction
$\sin(x)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos(x)$	$[0, \pi]$
$\tan(x)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
* $\csc(x)$	$(0, \frac{\pi}{2}] \cup (-\pi, -\frac{\pi}{2}]$
* $\sec(x)$	$[0, \frac{\pi}{2}) \cup [-\pi, -\frac{\pi}{2})$
* $\cot(x)$	$(0, \pi)$

* not always the convention used.

Exercise. Find $\frac{d}{dx}\arccos(x)$

Ex. Find $\frac{d}{dx}\arctan(x)$

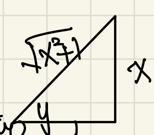
Sol^n Let $y = \arctan(x) \Rightarrow \tan(y) = x$

Note $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow \frac{d}{dx}[\tan(y)] = \frac{d}{dx}[x]$

$\sec^2(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y)$



$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

i.e. $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Implicit Differentiation

This is not a special kind of derivative. We simply apply a derivative operator to an implicit relationship (e.g. $f(x,y)=0$)

Note: we already did an example of this in the chain rule lesson.

Ex. Find the equation of the tangent line to the graph of $xy+x^2y^2=2$ at $(-1,2)$

We don't have an explicit function $y=f(x)$ however if we can find $\frac{dy}{dx}$ we can create a tangent line using $\frac{y-2}{x+1}=\frac{dy}{dx}\Big|_{(-1,2)}$

Apply $\frac{d}{dx}$ to $xy+x^2y^2=2$ to get $\frac{d}{dx}[xy+x^2y^2]=\frac{d}{dx}[2]$

$$\Rightarrow \frac{d}{dx}[xy]+\frac{d}{dx}[x^2y^2]=0$$

$$y+x\frac{dy}{dx}+2xy^2+x^2(2y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-y-2xy^2}{x+2yx^2}$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(-1,2)}=\frac{-2-(2)(-1)(4)}{-1+(2)(2)(-1)^2}=2$$

$\therefore$ The equation of the tangent line is thus, $\frac{y-2}{x+1}=2 \Rightarrow y=2+2(x+1)$

Note that every implicit relationship will be meaningful. For example, there are no points (x,y) satisfying $x^2+y^2=-1$ though we can still incorrectly calculate $\frac{dy}{dx}=-\frac{x}{y}$ Be careful !!!

Logarithmic Differentiation

This is also not a special kind of derivative but a technique that is required.

When dealing with functions of the form $y=f(x)^{g(x)}$

Ex. Find $\frac{dy}{dx}$ if $y=(\cos^2x)^x$

Note that $\frac{dy}{dx}\neq x(\cos^2x)^{x-1}[2\cos x(-\sin x)]$

To do this properly, we take log of both sides.

$$\ln y=\ln(\cos^2x)^x$$

$$\ln y=x\ln\cos^2x$$

Now apply $\frac{d}{dx}$ to both side

$\vdots$

$$\frac{dy}{dx}=(\cos^2x)^x[\ln(\cos^2x)-2x\tan x]$$

Local Extrema

Def^n Let $c \in (a,b)$. Given $f(x)$, if for all $x \in (a,b)$

- $f(x) \le f(c)$ then c is a local max
- $f(x) \ge f(c)$ then c is a local min

local max @ x_2, x_4, x_6

local min @ x_3, x_5

Global max @ x_4

global min @ x_1

End point cannot be local extrema
(not everyone agrees with this)

Thm Local Extrema theorem

If $f(x)$ has a local extrema at $x=c$ then either $f'(c)=0$, or $f'(c)$ does not exist

Proof Note that at any point, c either $f'(c)$ exists or it doesn't. So we show that $f'(c)$ exists it must be 0.

assume f has a local min at c & $f'(c)$ exists

By definition this means $f(x) \ge f(c)$ for all x in some open interval around c.

That is for small enough h we have $f(x+c) \ge f(c) \Rightarrow f(c+h) - f(c) \ge 0$

case 1 $h > 0$ then $\dfrac{f(c+h)-f(c)}{h} \ge 0$

case 2 $h < 0$ then $\dfrac{f(c+h)-f(c)}{h} \le 0$

Since $f'(c)$ exists $f'(c) = \lim_{h \to 0} \dfrac{f(c+h)-f(c)}{h}$

(i.e. both left and right side limits have to exist and be equal)

Thus, $f'(c) = \lim_{h \to 0^+} \dfrac{f(c+h)-f(c)}{h} \ge 0$ (by case 1)

and $f'(c) = \lim_{h \to 0^-} \dfrac{f(c+h)-f(c)}{h} \le 0$ (by case 2)

Therefore, $0 \le f'(c) \le 0 \Rightarrow f'(c) = 0$

A similar argument can be used for local max.

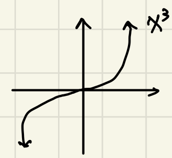
Based on the previous theorem, we defined,

Def^n. We call $x=c$ in the domain of f a critical point of f is either $f'(c)=0$ or $f'(c)$ DNE

In other words, the previous theorem states:

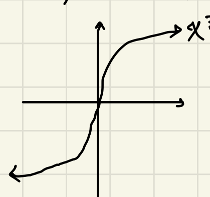
If $x=c$ is a local extremum of f then $x=c$ is a critical point

Note the converse is not true in general.



e.g. the function $f(x) = x^3$ satisfies $f'(0)=0$. However, $x=0$ is not a local max/min

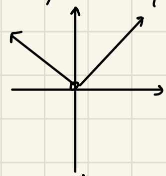
e.g. the function $f(x) = x^{\frac{1}{3}}$ do not have a derivative at $x=0$ (thus $x=0$ is a critical point). However, $x=0$ is not a local max/min.



f must be defined at $x=c$ to be a critical point e.g. $f(x)=\frac{|x^3|}{x^2}$ techically doesn't have any critical points.

Note : $f(x)=|x|$ does have a critical point at $x=0$.

$y=\frac{|x^3|}{x^2}$



Find global max & min on $[a,b]$

Combining EVT with the previous information, we get that, for a cts function f on $[a,b]$ the global max & min will occur at either:

- the endpoint $x=a$ or $x=b$
- the critical points

Ex. Find the global max & min of the function $f(x)=\frac{x}{1+x^2}$ on $[-\frac{1}{2},3]$

Sol^n: $f(-\frac{1}{2})=-\frac{2}{5}$

$f(3)=10$

$f'(x)=\frac{1-x^2}{(1+x^2)^2}$

$f'(x)$ exists everywhere, so only critical points happen when $f'(x)=0 \Rightarrow 1-x^2=0$

$x=\pm 1$

Since $x\in[-\frac{1}{2},3]$ we only look at $x=1$, $f(1)=\frac{1}{2}$

Since $\frac{1}{2}>\frac{3}{10}>-\frac{2}{5}$

global max is $\frac{1}{2}$ @ $x=1$, global is $-\frac{2}{5}$ at $x=-\frac{1}{2}$

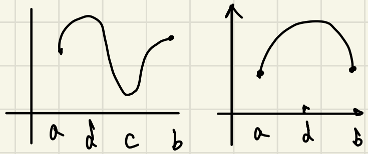
Thm $\boxed{\text{Rolle's Theorem}}$

If f is cts on $[a,b]$ and differetiable on (a,b) and $f(a)=f(b)$, then there is a $c\in(a,b)$ such that $f'(c)=0$

Proof since f is cts on $[a,b]$ the EVT tells us that there are c & d in $[a,b]$ such that

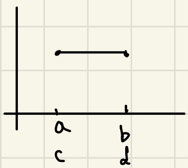
$$f(c)\le f(x)\le f(d) \text{ for all } x\in[a,b]$$

case 1 at least one of c or d is not an endpoint:



In this case, we have either $c\in(a,b)$ or $d\in(a,b)$.
Either way get a local extrema (let's just say it's at $x=c$) and by the local extrema theorem, $f'(c)=0$ or $f'(c)$ DNE. Since f is differentiable on (a,b), we must have $f'(c)=0$.

case 2 both c & d are endpoint



(recall $f(a)=f(b)$)
In this case the global max and min are at the end points. So either $f(a)\le f(x)\le f(b)$ or $f(b)\le f(x)\le f(a)$

However, since $f(a)=f(b)$ we get $f(a)\le f(x)\le f(a)$ for all $x\in[a,b]$ so $f(x)$ is constant
Thus $f'(x)=0$

Thm $\boxed{\text{Mean Value Theorem}}$

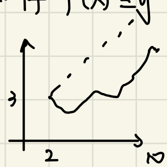
This is one of the most important theorem in calculus, it essentially says that,
smooth enough function the instantaneous rate of change will have to equal the average rate of change at least once.

Statement: If f is cts on $[a,b]$ and differentiable on $[a,b]$ then there is a $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Proof: let $g(x)=f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$
Note $g(a)=g(b)=0$
Then Rolle's theorem says there is a $c\in(a,b)$ where $g'(c)=0$ i.e. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Ex. If $f'(x)\le 4$ for all x, and $f(2)=3$, how big could $f(10)$ possibly be?



Sol^n By MVT, $\dfrac{f(10)-f(2)}{10-2}=f'(c)$ for some $c\in(2,10)$

$$\Rightarrow \frac{f(10)-3}{8}=f'(c)\le 4$$

$$f(10)\le(4)(8)+3=35$$

i.e. if you start at $(2,3)$ and move with a slope of 4 for 8 units you reach $(10,35)$

Thm $\boxed{\text{Constant Function Theorem}}$

Given an interval I, if $f'(x)=0$ for all $x \in I$ then $f(x)=k$, $k \in \mathbb{R}$ (i.e. f is constant)

Proof exercise Tip use MVT

Thm Antiderivative Theorem

If $f'(x)=g'(x)$ for all $x \in I$ then $f(x)=g(x)+c$

Proof let $h(x)=f(x)-g(x)$

then, $h'(x)=f'(x)-g'(x)=0$ (since $f'=g'$)

By the previous theorem, $h(x)$ is constant, $h(x)=c$

i.e. $f(x)=g(x)+c$

E.x. Show that if $f'(x)=f(x)$ for all $x \in \mathbb{R}$, then $f(x)=Ae^x$ for any $A \in \mathbb{R}$

Sol^n: Note: the converse is also true: If $f(x)=Ae^x$ then $f'(x)=Ae^x=f(x)$

Assume $f'(x)=f(x)$ let $g(x)=\frac{f(x)}{e^x}$ (we are "hoping" this will be constant)

Monotonicity

Def^n: Given an interval I, if for all $x \in I$ with $x_1 < x_2$

Condition	f is said to be
$f(x_1) < f(x_2)$	increasing.
$f(x_1) \le f(x_2)$	non-decreasing
$f(x_1) > f(x_2)$	decreasing
$f(x_1) \ge f(x_2)$	non-increasing

If f is any of these we say f is monotonic on I.

Thm $\boxed{\text{Increasing / Decreasing function theorem}}$

Given an interval I and differentiable function f

For $x \in I$

If	Then f is
$f'(x) > 0$	increasing
$f'(x) \ge 0$ $\Leftrightarrow$	non-decreasing
$f'(x) < 0$	decreasing
$f'(x) \le 0$ $\Leftrightarrow$	non-increasing

Proof of $f'(x) \le 0 \Rightarrow$ non-increasing (others are similar)

Assume $f'(x) \le 0$, given two points x_1 & x_2 in I with $x_1 < x_2$

By MVT, $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$. for some $c \in (x_1, x_2)$

Since $f'(x) \le 0$, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le 0 \Rightarrow f(x_2) \le f(x_1)$$

Thus f is non-increasing.

Ex. Find where $f(x) = x - 2\arctan(x)$ is increasing / decreasing.

Sol^n: Find where $f'(x) = 0$ and determine the sign of $f'(x)$ outside of these point.

$$f'(x) = 1 - \frac{2}{1+x^2}, \quad f'(x) = 0 \Rightarrow 0 = \frac{1+x^2-2}{1+x^2}$$

$$\Rightarrow x = \pm 1$$

$f'(x)$	$+$	0	$-$	0	$+$
		-1		1	

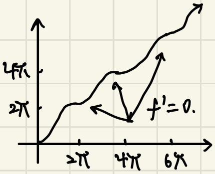
Thus, f is increasing on $(-\infty, -1] \cup [1, \infty)$ and f is decreasing on $[-1, 1]$

Note for differentiable f, f is non-decreasing / non-increasing if and only if $f'(x) \ge 0$ / $f'(x) \le 0$

However, the statement if f is differentiable & increasing then $f'(x) > 0$ is NOT TRUE.

(similar, for decreasing & $f'(x) < 0$)

e.g. the function $f(x) = x - \sin x$ is differentiable & increasing but $f'(x) = 1 - \cos x \geq 0$

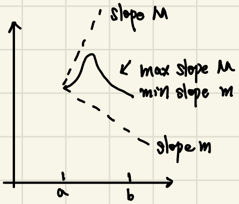


Thm Bounded Derivative Theorem

Given f is cts on $[a,b]$ & differentiable on (a,b) with $m \leq f'(x) \leq M$ for $x \in (a,b)$

then $f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a)$ for $x \in [a,b]$

(also $f(b) + M(x-b) \leq f(x) \leq f(b) + m(x-b)$)



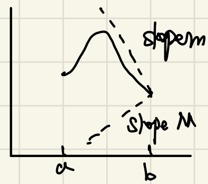
That is, starting at $x=a$, $f(x)$ lies between a line with min slope m and max slope M.

Proof: Pick an $x \in [a,b]$ then use MVT on $[a,x]$

$$\frac{f(x) - f(a)}{x-a} = f'(c) \text{ for } c \in (a,x)$$

Since $m \leq f'(x) \leq M$, we get $m \leq \frac{f(x) - f(a)}{x-a} \leq M$

Note second version starts at $x=b$ and goes backwards.



Ex. Show that $1.25 \leq \sqrt{2} \leq 1.5$

Sol'n let $f(x) = \sqrt{x}$, for $x \in [1,4]$, we have $f'(x) = \frac{1}{2\sqrt{x}}$,

so that $\frac{1}{4} \leq f'(x) \leq \frac{1}{2}$

when $x=4$ when $x=2$.

by previous theorem, $f(1) + \frac{1}{4}(x-1) \leq \sqrt{x} \leq f(1) + \frac{1}{2}(x-1)$

$$1 + \frac{1}{4}(2-1) \leq \sqrt{2} \leq 1 + \frac{1}{2}(2-1)$$

$$1.25 \leq \sqrt{2} \leq 1.5$$

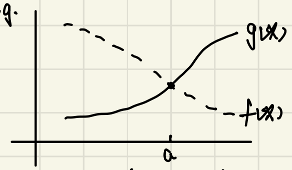
Thm Comparison Theorem

Given f & g both differetiable with $f(a) = g(a)$

if $f'(x) \le g'(x)$ for $x>a$ then $f(x) \le g(x)$ for $x \ge a$

if $f'(x) \le g'(x)$ for $x<a$ then $f(x) \ge g(x)$ for $x<a$

e.g.



Proof let $h(x) = f(x) - g(x)$

case 1 $f'(x) \le g'(x)$ for $x > a$

on the interval $[a, x]$ use MVT to get

$$\frac{h(x) - h(a)}{x-a} = h'(c) \quad \text{for } c \in (a, x)$$

Note $h(a) = 0 = f(a) - g(a)$

so we get $f(x) - g(x) = [f'(c) - g'(c)][x-a]$ (replacing $h(x)$ with $f-g$)

$\Rightarrow f(x) - g(x) \le 0$

(since $x>a$ & $f'(c) \le g'(c)$)

$\Rightarrow f(x) \le g(x)$

case 2 $f'(x) \le g'(x)$ for $x<a$

similar except $x-a<0$ which will lead to $f(x) \ge g(x)$

Note: we can replace $\le$ with $<$ & $\ge$ with $>$ without issue as the proof doesn't rely on equality anywhere.

Ex. show that $x > \ln(1+x)$ for $x>0$

Sol^n let $f(x) = x$ & $g(x) = \ln(1+x)$ at $x=0$

we have $f(0) = g(0) = 0$

Now $f'(x) = 1$ & $g'(x) = \frac{1}{1+x} < 1$ for $x>0$

$g'(x) < f'(x)$ for $x>0$

By the previous theorem, $g(x) < f(x)$ for $x>0$

i.e. $\ln(1+x) < x$

Exercise show that $x - \frac{x^2}{2} < \ln(1+x)$ for $x>0$

Hint: $1 - x^2 < 1$

{Cool Alert} Combining the last 2 examples we have that $x - \frac{x^2}{2} < \ln(1+x) < x$

$$\Rightarrow 1 - \frac{x}{2} < \frac{\ln(1+x)}{x} < 1 \quad \text{for } x>0$$

let $x = \frac{1}{n}$ so that

$$1 - \frac{1}{2n} < n \ln(1 + \tfrac{1}{n}) < 1$$

If we let $n \to \infty$, then $1 - \frac{2}{n} \to$ so by sequeeze theorem

$$\lim_{n\to\infty} n \ln(1+\tfrac{1}{n}) = 1 \Rightarrow \lim_{n\to\infty} \ln(1+\tfrac{1}{n})^n = 1$$

Recall for a cts function f if $\lim_{n\to\infty} x_n = a$

then $\lim_{n\to\infty} f(x_n) = f(a)$

let $f(x) = e^x$ & $x_n = \ln(1+\frac{1}{n})^n$

Then we get

$\lim_{n\to\infty} e^{\ln(1+\frac{1}{n})^n} = e^1$

$\Rightarrow \lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Generally for any $x \in \mathbb{R}$

$e^x = \lim_{n\to\infty} (1+\frac{x}{n})^n = \lim_{n\to-\infty} (1+\frac{x}{n})^n$

Inderterminate forms

Often, when working with limits we will encounter expressions that don't have a single meaning. We call these inderterminate forms.

Specifically, the expressions $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, \infty\cdot 0, 0^0, \infty^0, 1^\infty$ don't have a set value

(i.e. they could be any real number or tend to $\pm\infty$ depending on how they're formed)

e.g. $\lim_{x\to 0} \frac{\sin(x)}{x}$ has the form $\frac{0}{0}$ but we know $\frac{\sin x}{x} = 1$ as $x \to 0$

$\lim_{x\to 1} \frac{x^2-1}{x-1}$ has the form $\frac{0}{0}$ but we know $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

Note that $\frac{0}{\infty}, \frac{\infty}{0}, \infty+\infty, \infty\cdot\infty, 0^\infty, \infty^\infty$ are not indeterminate forms as they will always approach the same value. (either 0 or $\pm\infty$

L'Hopital's Rule

If: $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$f'(x)$ & $g'(x)$ exists around $x=a$

$g'(x) \neq 0$ around $x=a$, except at $x=a$

$\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists / tends to $\pm\infty$

then, $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

if $x\to\pm\infty$, it still works as long as we adjust = around $x=a$ $\Rightarrow$ as $x\to\pm\infty$ in hypo

e.g. $\lim\limits_{x\to 0} \frac{\sin x}{x} \to \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{\sin x}{x} \overset{LHR}{=\!=} \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$

e.g. $\lim\limits_{x\to\infty} \frac{e^x}{x^3} \to \frac{\infty}{\infty}$

$\lim\limits_{x\to\infty} \frac{e^x}{x^3} \overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{e^x}{3x^2} \to \frac{\infty}{\infty}$

$\overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{e^x}{6x} \to \frac{\infty}{\infty}$ (exist / ∞)

$\overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{e^x}{6} = \infty$

form: $0\cdot\infty$ / $\infty-\infty$ rearrange

e.g. $\lim\limits_{x\to\infty} x\cdot\sin(\frac{1}{x}) \to \infty\cdot 0$

rewrite: $\lim\limits_{x\to\infty} \frac{\sin(\frac{1}{x})}{(\frac{1}{x})} \to \frac{0}{0}$

then $\lim\limits_{x\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{\cos(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{1}{n^2}} = 1$

exercise: compute $\lim\limits_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$ $(\infty-\infty)$

Mind the hypothesis:

$\boxed{\text{Caution 1}}$ Don't over do it!

use LHR $(\frac{0}{0} / \frac{\infty}{\infty})$

e.g. $\lim\limits_{x\to 1} \frac{x\ln x - x + 1}{e^x - e}$

wrong: $\overset{LHR}{=\!=} \lim\limits_{x\to 1} \frac{\frac{1}{x}\ln x + \ln x - 1}{e^x}$

$\frac{0}{e} = 0$ ✓

not $\frac{0}{0}$ / $\pm\frac{\infty}{\infty}$

$\overset{LHR}{=\!=} \lim\limits_{x\to 1} \frac{\frac{1}{x}}{e^x}$

$= \frac{1}{e}$

$\boxed{\text{Caution 2}}$ $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ must exist / $\pm\infty$

eg. $\lim\limits_{x\to\infty} \frac{\cos(x) + 2x^3}{x^3+1} \to \frac{\infty}{\infty}$

$\overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{6x^2 - \sin(x)}{3x^2} \to \frac{\infty}{\infty}$

$\overset{LHR}{=\!=} \lim\limits_{x\to\infty} \frac{12x - \cos(x)}{6x} \to \frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{12 + \sin(x)}{6} \to$ DNE

✗ conclude $\lim\limits_{x\to\infty} \frac{\cos(x) + 2x^3}{x^3+1}$ DNE

✓ $\frac{\cos(x) + 2x^3}{x^3+1} = \frac{x^3(\frac{\cos x}{x^3} + 2)}{x^3(1 + \frac{1}{x^3})}$

$\lim\limits_{x\to\infty} \to 2$

Forms 0^0 1^∞ ∞^0

Expressions $f(x)^{g(x)}$ that yelt these form can be handle by

① $e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$

② computing limits of $g(x)\ln f(x)$

e.g. $\lim_{x\to\infty} x^{\frac{1}{2\ln x+1}} \to \infty^0$

$x^{\frac{1}{2\ln x+1}} = e^{\frac{1}{2\ln x+1}\cdot \ln x}$

then $\lim_{x\to\infty} \frac{\ln(x)}{2\ln(x)+1} \to \frac{\infty}{\infty}$

$\overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{x}} = \frac{1}{2}$

$\Rightarrow x^{\frac{1}{2\ln x+1}} = e^{\frac{\ln x}{2\ln x+1}} \to e^{\frac{1}{2}}$ as $x\to\infty$

exer. Find $\lim_{x\to 0}(1-x)^{\frac{1}{x}} \to 1^\infty$

Ex. compute $\lim_{x\to 0^+} x^{\frac{\ln(a)}{1+\ln x}}$ $(\frac{0}{0})$

$f(x) = x$ & $g(x) = \frac{\ln a}{1+\ln x}$

$\lim_{x\to 0^+} g(x)\ln f(x) = \lim_{x\to 0^+} (\ln x)\left(\frac{\ln(a)}{1+\ln x}\right) \to \frac{\infty}{\infty}$

⋮

$\overset{LHR}{=} \ln(a)$

$e^{g(x)\cdot\ln f(x)} = e^{\ln(a)} = a$ as $x\to 0^+$

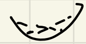
Concavity

Generally speaking, a function is concave up on an interval when it looks something like ↺

and, concave down when it looks something like ↷

Mathematically, there are many equivalent definitions:

Concave up

- if the secant line joining any 2 points on the graph lies above the graph 
- for a differentiable function, if every tangent line lies below the graph 
- for all $a, b \in I$,

  $\underbrace{f(ta+(1-t)b)}_{\text{value of } f} < tf(a) + (1-t)f(b)$ for $t \in (0,1)$

Concave down

Similar to concave up but swap above and below, and switch $<$ to $>$

For differentiable function, concave up basically means the derivative is increasing

i.e. $f'(x)$ is an increasing function

(similar for concave down by using decreasing)

Thm The concavity test

For a twice differentiable function f on an interval I

- if $f''(x) > 0$, then f is concave up
- if $f''(x) < 0$ then f is concave down

i.e. $f''(x) > 0 \iff \frac{d}{dx} f'(x) > 0$

$\Rightarrow f'$ is increasing

by the increasing function function.

Similar, for $f''(x) < 0$

If we relax the "strictness" of the inequality in the definitions

(i.e. allow the second line to lie on or above; allow the tangent to lie on or below; change $<$ to $\le$)

then we get

$f''(x) \ge 0$ if and only if f is concave up

$f''(x) \le 0$ if and only if f is concave down

This however does have the side effect of letting straight line $y = mx + b$ be both concave up & concave down

Unless told otherwise, stick to the original definition

Ex. Find where $f(x) = (x^4 - 16)^3$ is concave up/down

Sol^n We find where $f''(x) > 0$ or $f''(x) < 0$

$f''(x) = 12x^2(x^4-16)(11x^4-48)$

set $f''(x) = 0$ to get

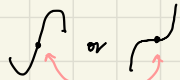
$x = 0, \pm 2, \pm\left(\frac{48}{11}\right)^{\frac{1}{4}}$ ← call this $p \simeq 1.445$

x		-2		$-p$		0		p		2	
$f''(x)$	$+$	0	$-$	0	$+$	0	$+$	0	$-$	0	$+$

Thus, it is concave up on $(-\infty,-2] \cup [-p,p] \cup [p,\infty)$

and concave down on $[-2,-p] \cup [p,2]$

Graphically, when a function changes concavity, it usually looks something like



concavity changes here.

We give these a special name

Defn. if

$f(x)$ is cts at $x=c$

the concavity changes at $x=c$

then we call $(c, f(c))$ an inflection point on f.

In the previous example,

$(-2,0)$ $(-p, f(-p))$ $(p, f(p))$, $(2,0)$

and inflection points but $(0, (-16)^3)$ is not !!!

i.e. $f''(x)=0$ MIGHT be an inflection point

In fact:

Thm $\boxed{\text{Inflection point requirement.}}$

If f'' is cts at $x=c$ and $x=c$ is an inflection point of f, then $f''(c)=0$

Note $f''(c)=0$ is a necessary condition but it is not sufficient (as seen in last examples

Classifying critical points

We previously learned that for a function f, if $x=c$ is a local extremum $\Rightarrow x=c$ is a critical point.
However, this only works in one-direction as the function $f(x)=x^3$ has a critical point at $x=0$ but it is not a local extremum.

We will discuss 2 methods for classifying critical points.

Method 1 First derivative test

Given a function f, cts at c with $c \in (a,b)$

if $f'(x)<0$, for $x \in (a,c)$

& $f'(x)>0$ for $x \in (c,b)$

then $x=c$ is a local minimum of f

(and thus a critical point by the local extrema theorem)

Similarly, if $f'(x)>0$ for $x \in (a,c)$

& $f'(x)<0$ for $x \in (c,b)$

then $x=c$ is a local maximum of f.

Ex. Classify the critical points of $f(x) = x - x^{\frac{2}{3}}$

Sol^n Find critical points by identifying where $f'(x)=0$ or $f'(x)$ DNE

$f'(x) = 1 - \frac{2}{3x^{\frac{1}{3}}}$

$f'(x)=0 \quad 1 = \frac{2}{3x^{\frac{1}{3}}} \Rightarrow x^{\frac{1}{3}} = \frac{2}{3} \Rightarrow x = \frac{8}{27}$

$f'(x)$ DNE $\Rightarrow x=0$

Note f is cts at both $x=0$ & $x=\frac{8}{27}$

$f'(x)$ +   dne   −   0   +

0   $8/27$

Thus, $(0,0)$ is a local max

$(\frac{8}{27}, -\frac{4}{27})$ is a local max.

Method 2 Second derivative test

This method is a bit more limited than method 1, however, it can be easier to use.

<u>Test</u> Given a function with $f'(c)=0$

- if $f''(c)>0$, then $x=c$ is a local min
- if $f''(c)<0$, then $x=c$ is a local max.

Ex. Classify the critical points of $f(x) = x(x-2)^2-1$

Sol^n $f'(x) = (x-2)^2 + 2x(x-2)$

$= (3x-2)(x-2)$

$f'(x)=0$ when $x=\frac{2}{3}$, $x=2$

$f''(x) = 6x-8$

Since $f''(\frac{2}{3}) = -4 < 0$

$f''(2) = 4 > 0$

Thus, $(2, f(2))$ is a local min

$(2/3, f(\frac{2}{3}))$ is a local max.

Proof of first derivative test

Local min

Assume $f'(x) < 0$ on (a, c) & $f'(x) > 0$ on (c, b) & f is cts at $x = c$

Goal: get to $f(x) \geq f(c)$ for $x \in (a, b)$ (i.e. definition of local min)

On (a, c) $f'(x) < 0 \Rightarrow f$ is decreasing by decr. fcn. thm For any $x_1 < x_2$, both in (a, c) $f(x_1) > f(x_2) \Rightarrow f(x_1) > \lim_{x_2 \to c} f(x_2)$

$\Rightarrow f(x_1) > f(c)$ since f is cts

Similarly for $x \in (c, b)$ we have $f(x_1) < f(x_2)$ (for $x_1, x_2 \in (c, b)$)

as $x_1 \to c$ $f(c) < f(x_2)$ (since f is cts)

Since x_1 & x_2 are arbitrary $f(c) \leq f(x)$ for all $x \in (a, b)$

$\boxed{\text{Local max}}$ Similar argument

Proof of Second derivative test

Local min Assume $f'(c) = 0$ & $f''(c) > 0$

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h} > 0 \quad (\text{since } f''(c) > 0)$$

$\Rightarrow f'(c+h) > 0$ when $h > 0$

$f'(c+h) < 0$ when $h < 0$

i.e. to the right of c, $f' > 0$

to the left of c, $f' < 0$ $\Rightarrow x = c$ is a local min by the first derivative test.

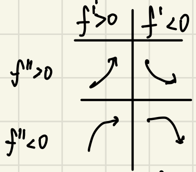
Local max similar

Curve Sketching

Putting together all the various tests and theorems from the past several lectures we create the following procedure to sketch the graph of a function $y = f(x)$

1. Find the domain of $f(x)$
2. Find x & y intercepts if any
3. Find vertical & horizontal asymptotes (if any) or end behaviour (i.e. $f \to \pm\infty$)
4. Find $f'(x)$, critical points and where $f(x)$ is discontinuous.
5. Find $f''(x)$ and potential inflection points
6. Classify points from steps 4 & 5 to identify local extrema & actual inflection points with these and any discontinuities determine intervals of increase/decrease and concavity.
7. sketch and label

Tip: Shape Grid



Ex. Sketch $f(x) = \dfrac{x}{\sqrt[3]{x^2-1}}$

Sol^n: 1. f is defined for all $x \neq \pm 1$

2. When $x = 0$ we get $y = 0$ (and vice versa)

3. Vertical asymptotes at $x = \pm 1$ since the numerator $\neq 0$

Horizontal asymptotes

$$\lim_{x\to\infty} \frac{x}{(x^2-1)^{\frac{1}{3}}} = \lim_{x\to\infty} \frac{x}{x^{\frac{2}{3}}(1-\frac{1}{x^2})^{1/3}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}}{(1-\frac{1}{x^2})^{\frac{1}{3}}} = \infty$$

Similarly, $\displaystyle\lim_{x\to-\infty} \frac{x}{(x^2-1)^{\frac{1}{3}}} = \lim_{x\to-\infty} \frac{x^{\frac{1}{3}}}{(1-\frac{1}{x^2})^{\frac{1}{3}}} = -\infty$

4. $f'(x) = \dfrac{1}{3}\dfrac{x^2-3}{(x^2-1)^{4/3}}$

Critical points at $x = \pm\sqrt{3}$

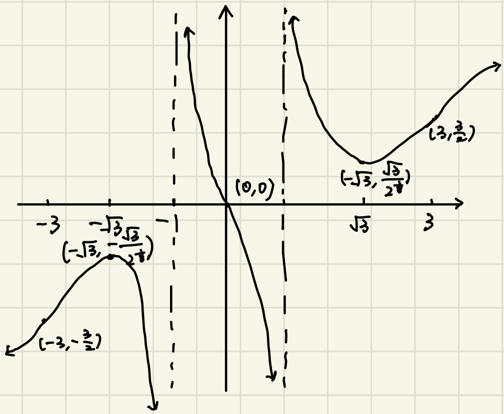
Note $x = \pm 1$ are not in the domain of f so technically aren't critical points

5. $f''(x) = -\dfrac{2}{9}\cdot\dfrac{x(x^2-9)}{(x^2-1)^{7/3}}$

possible inflection points at $x = 0$ & $x = \pm 3$

Once again, $x = \pm 1$ are not in the domain but are none-the less "point of interest."

		-3		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		3	
f''	$+$		$-$				$+$		$-$		$+$				$-$
f'	$+$				$-$		$-$				$-$		$+$		
f															
Key points		P.O.I $(-3, -\frac{3}{2})$		local max $(-\sqrt{3}, -\frac{\sqrt{3}}{2^{1/3}})$		VA		P.O.I $(0,0)$		VA		local min $(\sqrt{3}, \frac{\sqrt{3}}{2^{1/3}})$		P.O.I $(3, \frac{3}{2})$	

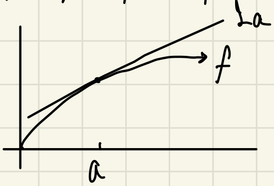


(0,0)
(−√3, √3/2)
(3, 3/2)
(−√3, −√3/2)
(−3, −3/2)
−3
−√3
−1
√3
3

Higher order approximation

Recall that for a differentiable function f, we can approximate the value of $f(x)$ by using the line.

$L_a(x) = f(a) + f'(a)(x-a)$

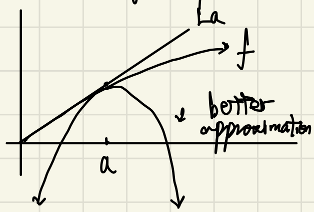


We had also established an upper bound on the error

$$|f(x) - L_a(x)| \le \frac{M}{2}(x-a)^2$$

as long as $|f''| < M$, between a & x.

Q: Can we go further? That is can we build a "tangent parabola"



For $L_a(x)$ we had that $L_a(a) = f(a)$ AND $L_a'(a) = f'(a)$

A general parabola "centered at $x=a$" is given by

$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2$

We want $p(a) = f(a)$ starts the same

$p'(a) = f'(a)$ grows the same.

$p''(a) = f''(a)$ curves the same

We use these to find c_0, c_1 and c_2

Since $p(a) = c_0$

$p'(a) = c_1$

$p''(a) = 2c_2$

Combining with what "we want" gives

$c_0 = f(a)$, $c_1 = f'(a)$ $c_2 = \frac{f''(a)}{2}$

and we have $p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

We call this the second degree Taylor polynomial of f centered at $x=a$ & write $T_{2,a}(x)$

Ex. Find $T_{2,1}(x)$ of $f(x) = \sqrt{x}$

Sol^n. $f(x) = \sqrt{x}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x^{3/2}}$ $\quad f''(1) = -\frac{1}{4}$

Thus, $T_{2,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$

$= 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8}$

Try it on Desmos!!!

More generally we get the following pattern:

Def^n: Given an n-times differentiable function f, we call

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

the n^{th} degree Taylor polynomial of f centered at $x=a$

(Note: $n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1$)

Ex. Find $T_{5,0}(x)$ of $\sin x$

Sol^n:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x \quad \Rightarrow & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

In general for $f(x) = \sin x$

$$f^{(2k+1)}(0) = (-1)^k \quad k \in \mathbb{Z}^+$$
$$f^{(2k)}(0) = 0$$

We have,

$$T_{5,0}(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f^{(3)}(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Note that, for $\sin(x)$, $T_{2,0}(x) = T_{1,0}(x)$ and $T_{4,0}(x) = T_{3,0}(x) \dots\dots T_{2n,0}(x) = T_{2n-1,0}(x)$ due to the even derivatives being zero

Exercise Show that for $f(x) = \cos(x)$ $T_{5,0}(x) = T_{4,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$

Ex. Find $T_{n,0}(x)$ for $f(x) = e^x$

Sol^n: Since $f(x) = f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$ then $f(0) = f'(0) = \dots f^{(n)}(0) = 1$

and we get

$$T_{n,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

High Order error

Recall for $L_a(x)$, we defined the error to be $|f(x) - L_a(x)|$

Similarly, for the Taylor polynomial, $T_{n,a}(x)$ we define the error as $|f(x) - T_{n,a}(x)|$.

The expression $|f(x) - T_{n,a}(x)|$ is often called the n^{th} degree Taylor Remainder and denoted by $R_{n,a}(x)$

That is, $R_{n,a}(x) = f(x) - T_{n,a}(x)$

so that error $= |R_{n,a}(x)|$

Again for the linear approximation $L_a(x)$ we had the result

$$|f(x) - L_a(x)| \le \frac{M}{2}(x-a)^2$$

when $|f''| \le M$

For the Taylor polynomial approximation we have Thm

$\boxed{\text{Taylor's Theorem}}$ For an $n+1$ times differentiable function f on an interval I

if $x \in I$, and $a \in I$, then there is a c between x & a such that $R_{n,a}(x) = f(x) - T_{n,a}(x) = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$

i.e. when $n=0$, we get $f(x) - T_{0,a}(x) = f'(x)(x-a)$

Recall: $T_{n,a}(x) = f(a) + f'(a)(x-a) + \ldots + \dfrac{f^{(n)}(a)(x-a)^n}{n!}$

so that $T_{0,a}(x) = f(a)$ and we get $f(x) - f(a) = f'(c)(x-a)$ i.e. the MVT

i.e. when $n=1$ we get $f(x) - T_{1,a}(x) = \dfrac{f''(c)(x-a)^2}{2}$

Since $T_{1,a}(x) = f(a) + f'(a)(x-a) = L_a(x)$ we have $f(x) - L_a(x) = \dfrac{f''(c)(x-a)^2}{2}$

which is our previous result (when you take absolute values and assume $|f''| \le M$)

This leads to:

Thm Taylor inequality

Assuming the conditions of Taylor's Theorem, if $|f^{(n+1)}(c)| \le M$ then

error $= |R_{n,a}(x)| \le \dfrac{M|x-a|^{n+1}}{(n+1)!}$

That is, for a given x, if we can find an upper bound on the $(n+1)$st derivative between x & a then we will have an upper bound on the error

Ex. Approximate $\sqrt{2}$ using $T_{2,1}(x)$ with $f(x) = \sqrt{x}$. Find an upper bound on the error

Sol^n. We previously calculated $T_{2,1}(x) = f(1) + f'(1)(x-1) + \dfrac{f''(1)(x-1)^2}{2}$

$$= 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{8}$$

So, $\sqrt{x} \approx T_{2,1}(x)$

$$\Rightarrow \sqrt{2} \approx T_{2,1}(2) = 1 + \frac{1}{2} - \frac{1}{8} = 1.375$$

Taylor's Theorem says, error $= |f(x) - T_{2,1}(x)| = \left|\dfrac{f^{(3)}(c)(x-1)^3}{3!}\right|$

for some c between x & 1

Now, $f(x) = \sqrt{x}$ $\quad f^{(3)}(x) = \dfrac{3}{8x^{5/2}}$

Since we will let $x=2$ then $c \in (1,2)$ and $f^{(3)}(c) = \dfrac{3}{8c^{5/2}} \le \dfrac{3}{8}$ for $c \in (1,2)$

Therefore we have, error $= |f(2) - T_{2,1}(2)|$

$$= |\sqrt{2} - T_{2,1}(2)| \le \left| \frac{\frac{3}{8}(2-1)^3}{3!} \right|$$

$$= \frac{1}{16}$$

Note that since $c \in (1, x)$, $\frac{f^{(3)}(c)}{3!}(x-1)^3 > 0$

so Taylor's Theorem says, $f(x) - T_{2,1}(x) > 0$

$$\Rightarrow f(x) > T_{2,1}(x)$$

i.e. $T_{2,1}$ will under estimate $f(x)$ for many $x > 1$

Thus, a better final interval for $\sqrt{2}$ would be $\sqrt{2} \in [1.375, 1.4375]$

E.X. Calculate $T_{3,0}(x)$ of $\arctan(x)$. Approximate π and determine an upper bound on the error using Taylor's Theorem.

Sol^n

$$f(x) = \arctan(x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \quad \Rightarrow \quad f''(0) = 0$$

$$f^{(3)}(x) = \frac{6x^2-2}{(1+x^2)^3} \qquad f^{(3)}(0) = -2$$

so $T_{3,0}(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f^{(3)}(0)x^3}{3!}$

$= x - \frac{x^3}{3}$

Recall that $\arctan(1) = \frac{\pi}{4}$ so $\arctan(1) \approx T_{3,0}(1)$

$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3}$

$\Rightarrow \pi \approx 4 - \frac{4}{3} = 2.6\overline{6}$

Taylor's Theorem says, $|f(x) - T_{3,0}(x)| = \left|\frac{f^{(4)}(c)x^4}{4!}\right|$ for c between 0 and x

we chose $x=1$ so $c \in (0,1)$

Here $f^{(4)}(c) = -24\left[\frac{c-c^3}{(1+c^2)^4}\right]$

For $c \in (0,1)$

$1+c^2 > 1$

and $c - c^3 < \frac{2}{3\sqrt{3}} < \frac{1}{2}$ ← this is just to make it nice

↑ local max at $c = \frac{1}{\sqrt{3}}$

$\therefore |f^{(4)}(c)| \leq 24\left[\frac{\frac{1}{2}}{1}\right] = 12$

so that,

$|f(1) - T_{3,0}(1)| \leq \frac{12(1)^4}{4!}$

$\Rightarrow \left|\frac{\pi}{4} - \frac{2}{3}\right| \leq \frac{1}{2}$

$\Rightarrow |\pi - \frac{8}{3}| \leq 2$

i.e. $\pi \in (2.6\overline{6}-2,\ 2.6\overline{6}+2)$ not the best approximation

Note: if we found the actual max of $f^{(4)}(c)$ we would have obtained $f^{(4)}(c) \leq 5$ which would have ultimately led to

$|\pi - \frac{8}{3}| \leq \frac{5}{6}$

Alternatively, for $f(x) = \arctan(x)$, $f^{(4)}(0) = 0$.

so $T_{4,0}(x) = T_{3,0}(x) = x - \frac{x^3}{3}$

and we could get a smaller error bound by using.

$|f(x) - T_{4,0}(x)| = \left|\frac{f^{(5)}(c)x^5}{5!}\right|$

It turns out $f^{(5)}(c) \leq 24$ on $(0,1)$

so $\left|\frac{f^{(5)}(c)}{5!}\right| \leq \frac{1}{5}$ & $|\pi - \frac{8}{3}| \leq \frac{4}{5}$

the n^{th} degree Taylor Polynomial for $\arctan(x)$ centered at $x=0$ is $T_{n,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \frac{(-1)^k x^{2k+1}}{2k+1}$

and the series $T_{n,0}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$ does in fact converge to $\frac{\pi}{4}$ (albeit very slowly)

Big O

Essentially, big O picks out the dominant behaviour of an expression (ignoring scaling constants)

e.g. $x^2+3x+5x^3 = O(x^3)$ as $x\to\infty$

e.g. $x^2+3x+5x^3 = O(x)$ as $x\to\infty$

In calculus we are often more concerned with $x\to 0$ whereas in computer science you worry about $x\to\infty$

Def^n: Given f and g, we write $f(x)=O(g(x))$ as $x\to a$ (and say f is of order g as x approaches a) if there exists a constant k such that

$|f(x)| \le k|g(x)|$ as $x\to a$

i.e. for some open interval around $x=a$ except possibly $x=a$.

e.g. $x^3+3x+5x^3 = O(x)$ as $x\to 0$

Since $|x^2| \le |x|$ as $x\to 0$

$|x^3| \le |x|$ as $x\to 0$

so $|x^2+3x+5x^3| \le |x^2| + |3x| + |5x^3|$ $\triangle$ inequality

$\le |x| + |3x| + 5|x|$

$\le 9|x|$