Background

Power spectrum of the Cosmic Microwave Background (CMB) was used to constrain the basic cosmological parameters of the universe. Measured parameters were the Hubble constant, density of the regular baryonic matter, density of dark matter, amplitude, and tilt of the initial power spectrum of fluctuations set in the very early universe, and the Thomson scattering optical depth between us and the CMB. This exercise used only intensity data, which would do a poor job constraining the optical depth.

Data acquired from the Planck satellite found in COM_PowerSpect_CMB-TT-full_R3.01.txt provided the variance of the sky as a function of angular scale and the uncertainty of the variance. Columns 1-4 were multipole, variance of the sky at that multipole, 1σ lower uncertainty, and 1σ upper uncertainty, respectively. Simplification purposes dictated that errors were assumed to be Gaussian and uncorrelated and that the error on each point was the average of the upper and lower errors. Additional simplifications were implemented, where applicable. Consequently, the answers would not be exactly correct, but be very close. Model power spectra as a function of input parameters were calculated.

The ordering of the parameters, where $h = H_0/100$, was as follows:

Hubble constant	H_0
Baryon density	$\Omega_b h^2$
Dark matter density	$\Omega_c h^2$
Optical depth	τ
Primordial amplitude of spectrum	A_s
Primordial tilt of spectrum	n_s

Answers

1)

The test script based on the initial parameters [60, 0.02, 0.1, 0.05, 2.00e-9, 1.0]) generated a spectrum profile shown in Figure 1. For reference, mean and variance of χ^2 were n and 2n, respectively, where n, the number of degrees of freedom, was 2501.

After the parameters were dialed into the test script, the resultant χ^2 value was determined to be **15267.9371**. In comparison with the degrees of freedom of 2501, this χ^2 value was much greater than the expected mean of 2501 by several σ; thus, the fit would be classified as **not acceptable**.

Next, χ^2 calculated based on the initial parameters [69, 0.022, 0.12, 0.06, 2.1e-9, 0.95], which are closer to the currently accepted values, equaled **3272.20536**. The computed χ^2 was determined to evidence a better fit (Figure 2) than the previous one. However, the fit would be still **not acceptable** as it was approximately 30% greater than the mean of 2501. Since the standard deviation of χ^2 should be 70.7248188, calculated χ^2=3272.20536 was far greater than several σ above the mean.

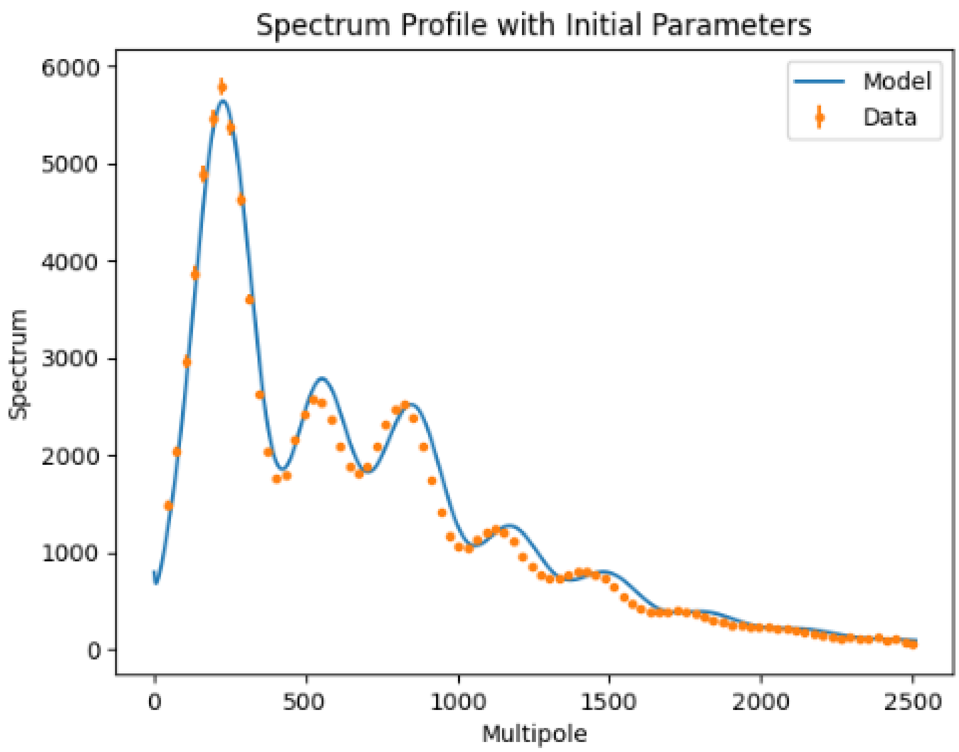

Figure 1. Spectrum profile based on test script parameters

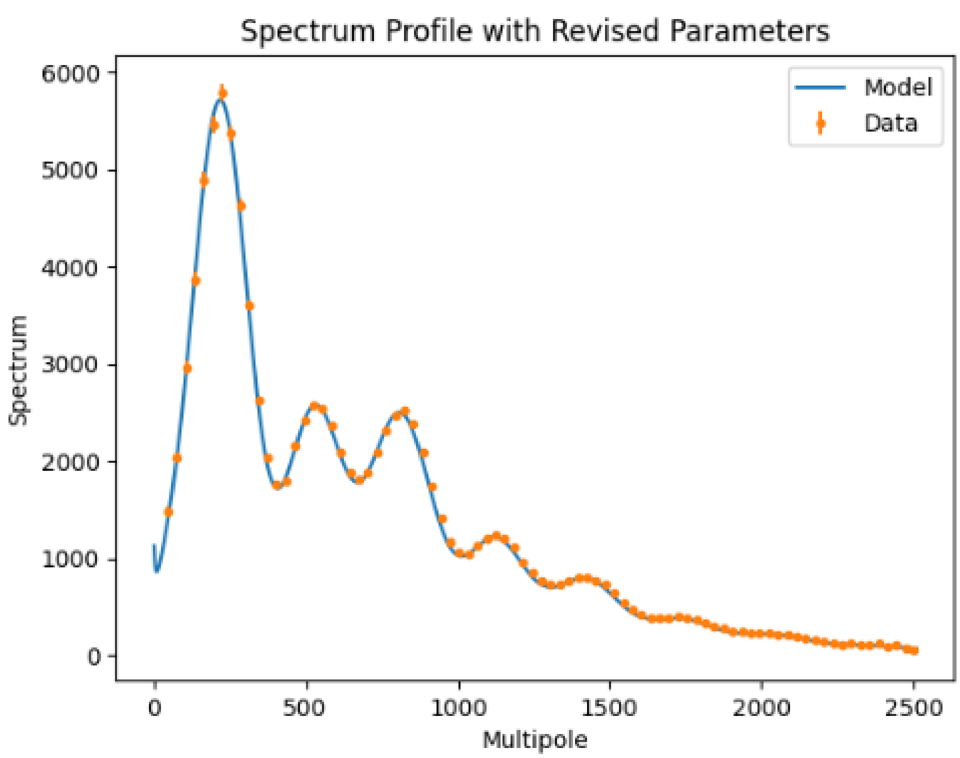

Figure 2. Spectrum profile based on revised parameter

2)

Newton's method was utilized to find the best-fit parameters, using numerical derivatives for a number of iterations equal to 20. The best-fit parameters and their errors were reported in planck_fit_params.txt. The Newton's fit generated a χ^2 of **2576.15226,** closer to the degrees of freedom, evidencing an improvement in fit (Figure 3). The resultant best-fit parameters and associated errors calculated were:

H_0	=	$68.2392347 \pm 1.18796022$
$\Omega_b h^2$	=	$0.0223632661 \pm 0.000229807587$
$\Omega_c h^2$	=	$0.117674096 \pm 0.00265533038$
τ	=	$0.0851526851 \pm 0.0341300247$
A_s	=	$2.21812693 \times 10^{-9} \pm 1.43377363 \times 10^{-11}$
n_s	=	$0.973032427 \pm 0.00658074598$

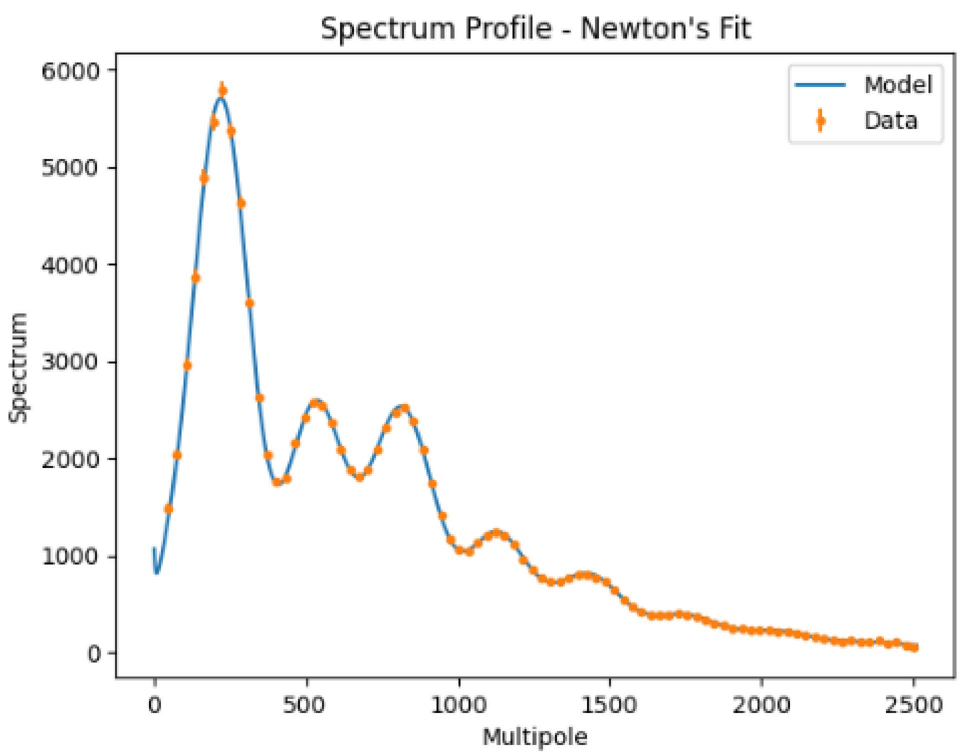

Figure 3. Spectrum profile based on Newton's fit

3)

Parameter values and uncertainties were estimated using a Markov Chain Monte Carlo (MCMC) sampler. Trial steps from the curvature matrix generated in Problem 2 proved useful for the MCMC analysis. Convergence of chains was evident as the chain progressed for each parameter with the datapoints exhibiting a near Gaussian distribution about the mean (Figure 4). The Fourier transform of each parameter within the frequency domain (Figure 5) revealed a relatively stable lead zone, further evidencing convergence. The resultant chain data was saved in planck_chain.txt with the corresponding χ^2 value for each sample in the first column.

A chain length of 5000 was determined to be adequate with the initial 1000 points discarded as the burn-in set at the beginning of the MCMC run, providing the basis of calculations for the mean parameters and the dark energy. A spectrum fit based on the mean parameters resulted in a χ^2 of **2577.83629**, whose progression along the MCMC chain is shown in Figure 6, whereas the model fit based on the mean parameters is given in Figure 7. Moreover, corner plots aided in assessment of how the parameters converged. Figure 12 demonstrates the inter-parametric agreements, providing additional support for convergence. After removal of the burn-in, the resultant mean parameters and associated errors calculated were:

H_0	=	$68.3140423 \pm 1.09932918$
$\Omega_b h^2$	=	$0.0223670858 \pm 0.000208843117$
$\Omega_c h^2$	=	$0.117551944 \pm 0.00252111013$
τ	=	$0.0893651868 \pm 0.0304036247$
A_s	=	$2.23959754\text{x}10^{-9} \pm 1.28586982\text{x}10^{-11}$
n_s	=	$0.973923827 \pm 0.00641016310$

Dark Energy:

Based on the assumption that the universe is spatially flat and $\Omega_\Lambda + \Omega_B + \Omega_C = 1$:

$\Omega_\Lambda = 1 - (\Omega_B + \Omega_C)$, where the chain-reported Baryon density $\Omega_b h^2$ and dark matter density $\Omega_c h^2$ could be substituted to solve for the dark energy Ω_Λ.

$\Omega_\Lambda = 1 - (\Omega_b h^2 + \Omega_c h^2)h^{-2}$ where $h=H_0/100$.

$\Omega_\Lambda = 1 - (\Omega_b h^2 + \Omega_c h^2)(100/H_0)^2$

Let $x= \Omega_\Lambda$, $\Omega_b h^2=b$, and $\Omega_c h^2=c$.

The uncertainty could be computed as follows:

$[\sigma_x]^2= (10^4 b/H_0^2)^2 [(\sigma_b/b)^2+(2\sigma_{H0}/H_0)^2]+ (10^4 c/H_0^2)^2 [(\sigma_c/c)^2+(2\sigma_{H0}/H_0)^2]$

The estimate on the mean value of the dark energy Ω_Λ was **0.700182611**, while the uncertainty was **0.00987351510**.

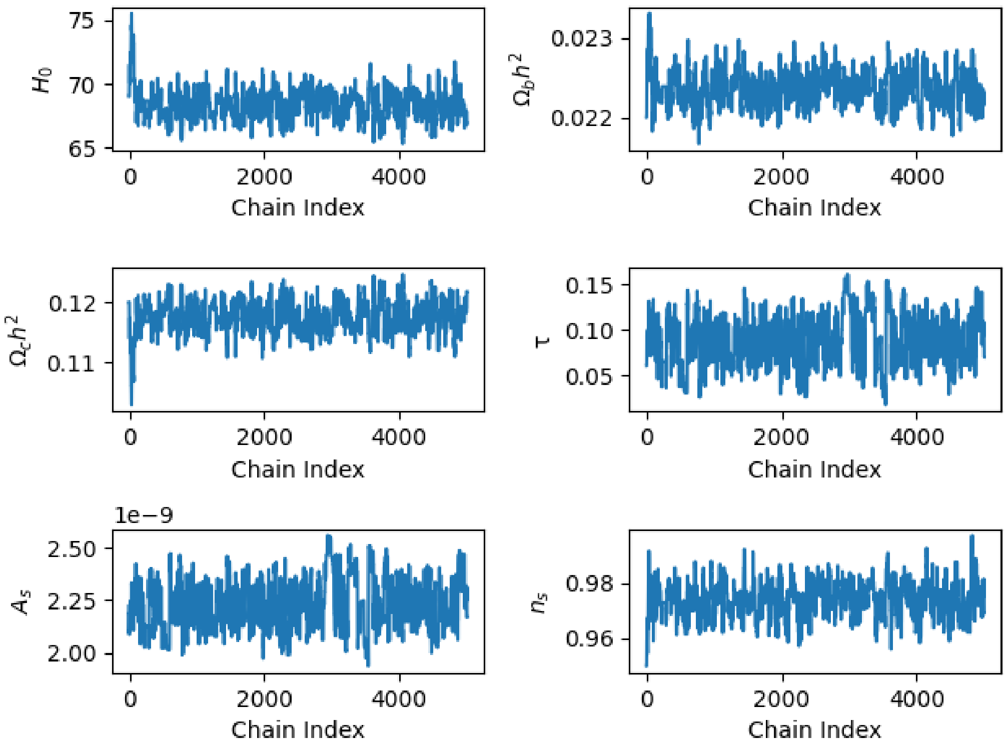

Figure 4. Parameter convergence with respect to chain index – MCMC Part 3

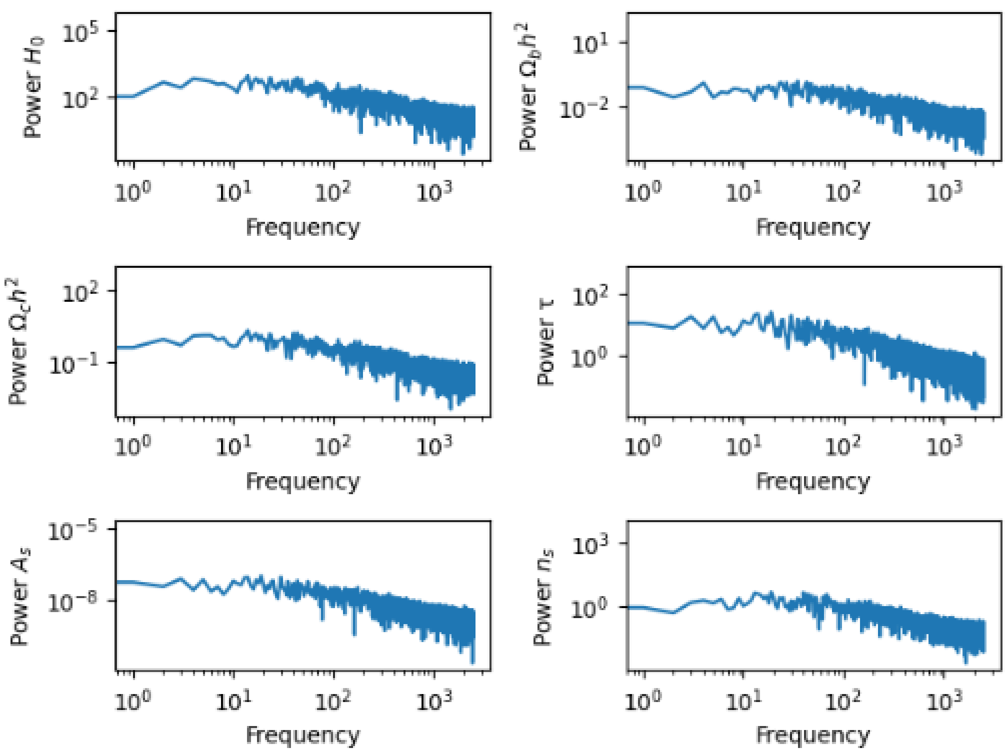

Figure 5. Parameter trends in the frequency domain – MCMC Part 3

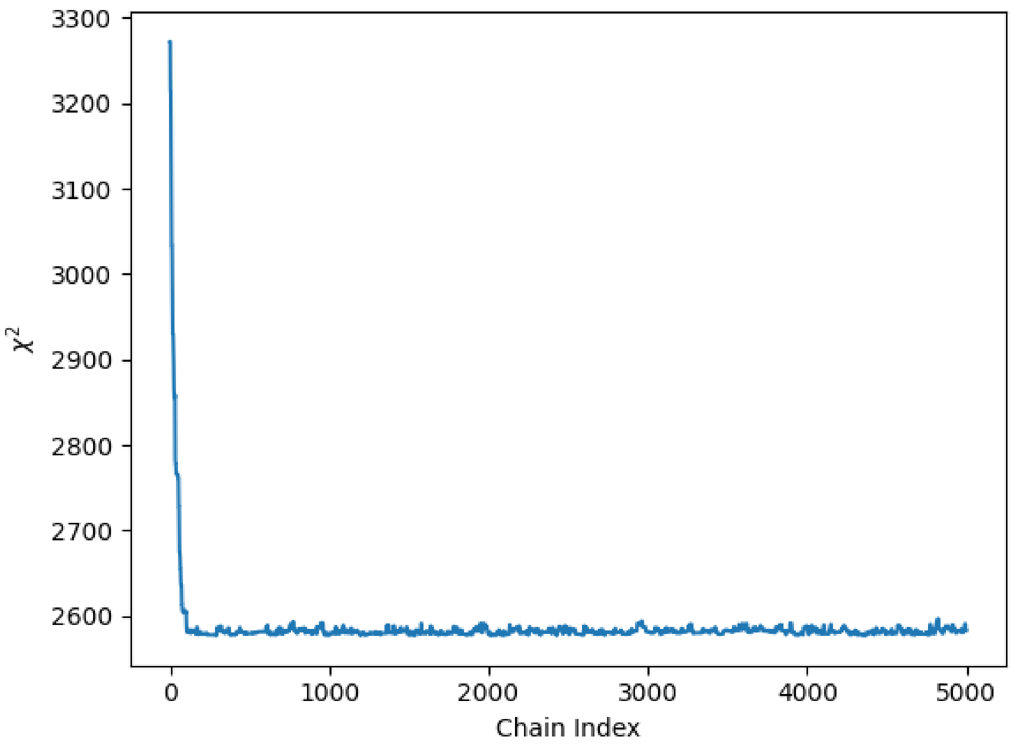

Figure 6. χ^2 variation with respect to chain index – MCMC Part 3

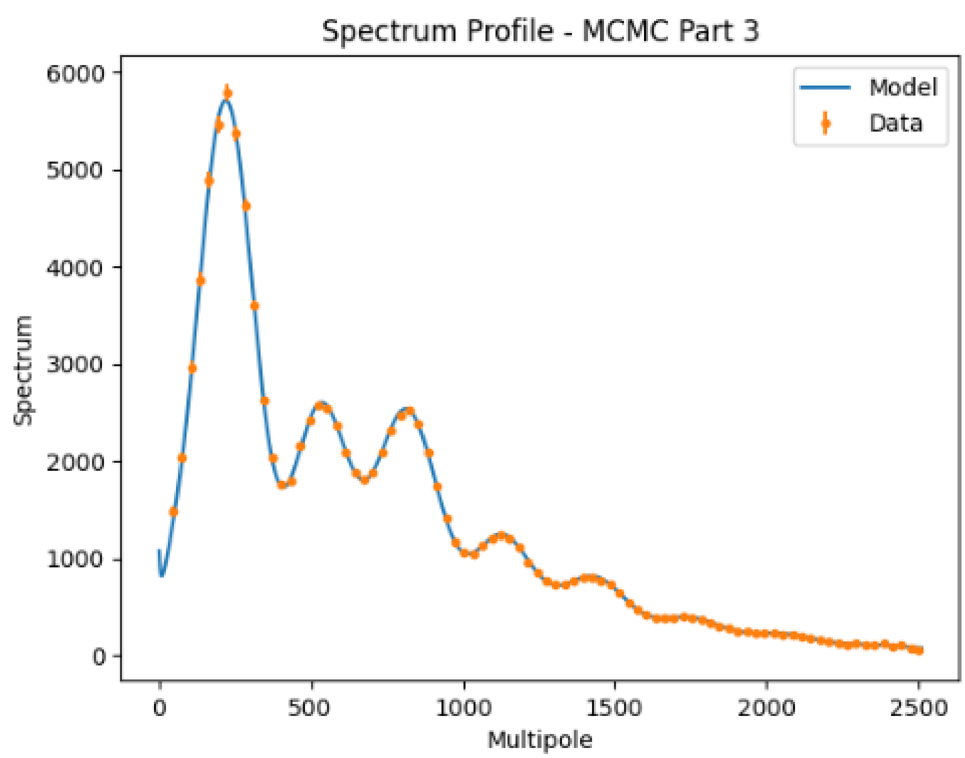

Figure 7. Spectrum profile based on mean parameters – MCMC Part 3

4)

Polarization data was expected to impose a much better constraint on reionization, with the specified $\tau = 0.0540 \pm 0.0074$. The additional term $\Delta\chi^2$ was defined as:

$\Delta\chi^2 = (\tau - \tau_m)^2\sigma_\tau^{-2}$

where τ is the estimated value at each step of MCMC analysis, $\tau_m = 0.0540$, and $\sigma_\tau = 0.0074$. Accordingly, the probability, accept_prob, utilized in the accept step of the algorithm would be:

accept_prob = $\exp[-0.5\ (\chi^2 + \Delta\chi^2 - \chi_o^2)]$

where χ^2 was newly calculated for each MCMC iteration, and χ_o^2 was the referenced value from previous step in iteration.

A re-estimation of the parameter covariance matrix was carried out based on the weight w:

$w = \exp[-\Delta\chi^2/2]$

Subsequently, a new chain was run where this constraint was implemented; results were saved to planck_chain_tauprior.txt. Convergence of chains was apparent as the chain progressed for each parameter with the datapoints constituting a near Gaussian distribution about the mean (Figure 8). The Fourier transform of each parameter within the frequency domain (Figure 9) showed a relatively stable lead zone, further supporting convergence.

First 1000 burn-in datapoints were removed; a spectrum fit based on the mean parameters resulted in a χ^2 of **2577.04685** whose progression along the MCMC chain is shown in Figure 10, whereas the model fit based on the mean parameters is given in Figure 11. Moreover, corner plots allowed for visualization of how the parameters converged. Figure 13 shows the inter-parametric agreements, providing additional support for convergence. After removal of the burn-in, the resultant mean parameters and associated errors calculated were:

H_0	=	$67.8583289 \pm 0.980005551$
$\Omega_b h^2$	=	$0.0222991368 \pm 0.000214489231$
$\Omega_c h^2$	=	$0.118509802 \pm 0.00219938345$
τ	=	$0.0562836325 \pm 0.00774573895$
A_s	=	$2.09895309\text{x}10^{-9} \pm 3.32210404\text{x}10^{-11}$
n_s	=	$0.970793820 \pm 0.00570252597$

Importance Sampling Analysis:

An alternative importance sampling analysis could be conducted based on the chain results in Part (3). The weighted average of parameters would provide insight into an improved estimate thereof, based on the aforementioned weighting factor w. The resultant parameters and associated errors calculated were:

H_0 = $67.7181558 \pm 1.04687113$
$\Omega_b h^2$ = $0.0222961236 \pm 0.000210802010$
$\Omega_c h^2$ = $0.118918032 \pm 0.00238169122$
τ = $0.0555899311 \pm 0.00708192135$
A_s = $2.09810736 \mathrm{x} 10^{-9} \pm 3.07207848 \mathrm{x} 10^{-11}$
n_s = $0.969892676 \pm 0.00606400427$

Computed $\tau = 0.0559 \pm 0.0071$ is well within 1-σ of the constraint mean τ. Furthermore, all parameters calculated based on the importance sampling analysis trial lay within 1-σ of those generated by the MCMC analysis constrained by the τ. A reasonable parametric estimate is thus possible with the importance sampling approach without a rerun of the MCMC based on the set constraint.

Bonus:

5σ-equivalent error bars estimated for the parameters were calculated based on the MCMC constrained by τ, ensuring that none of the parameters could be negative. The estimated $\pm 5\sigma$ range of each parameter follows:

H_0 = [62.9583012, 72.7583567]
$\Omega_b h^2$ = [0.0212266906, 0.0233715829]
$\Omega_c h^2$ = [0.107512885, 0.129506720]
τ = [0.0175549377, 0.0950123272]
A_s = [$1.93284789 \mathrm{x} 10^{-9}$, $2.26505829 \mathrm{x} 10^{-9}$]
n_s = [0.942281191, 0.999306450]

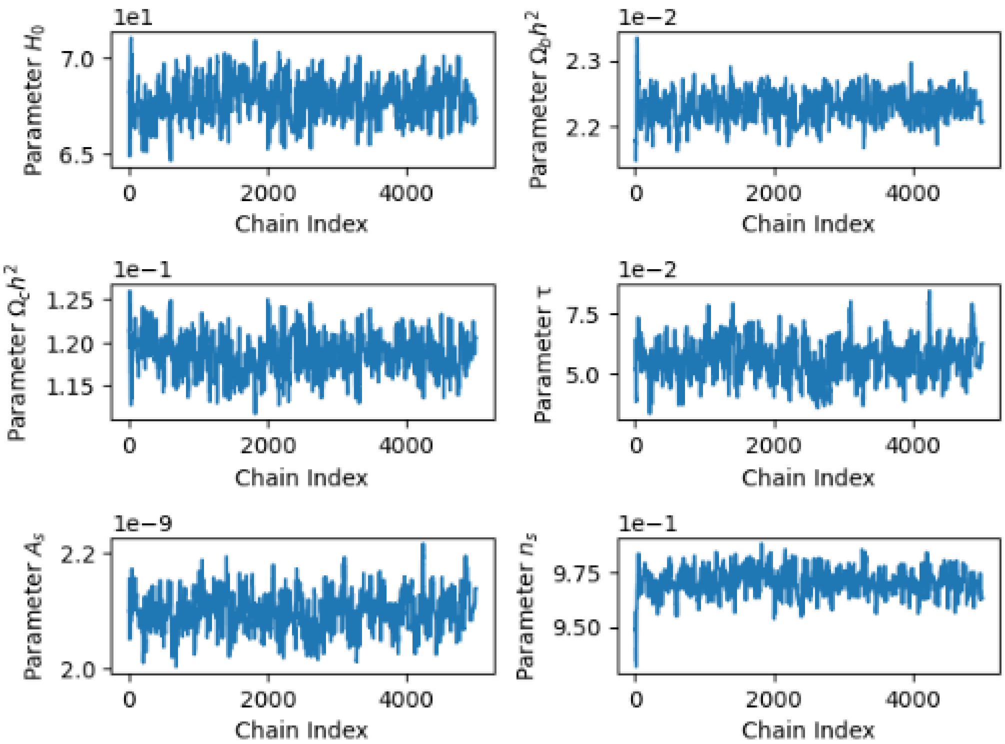

Figure 8. Parameter convergence with respect to chain index – MCMC Part 4

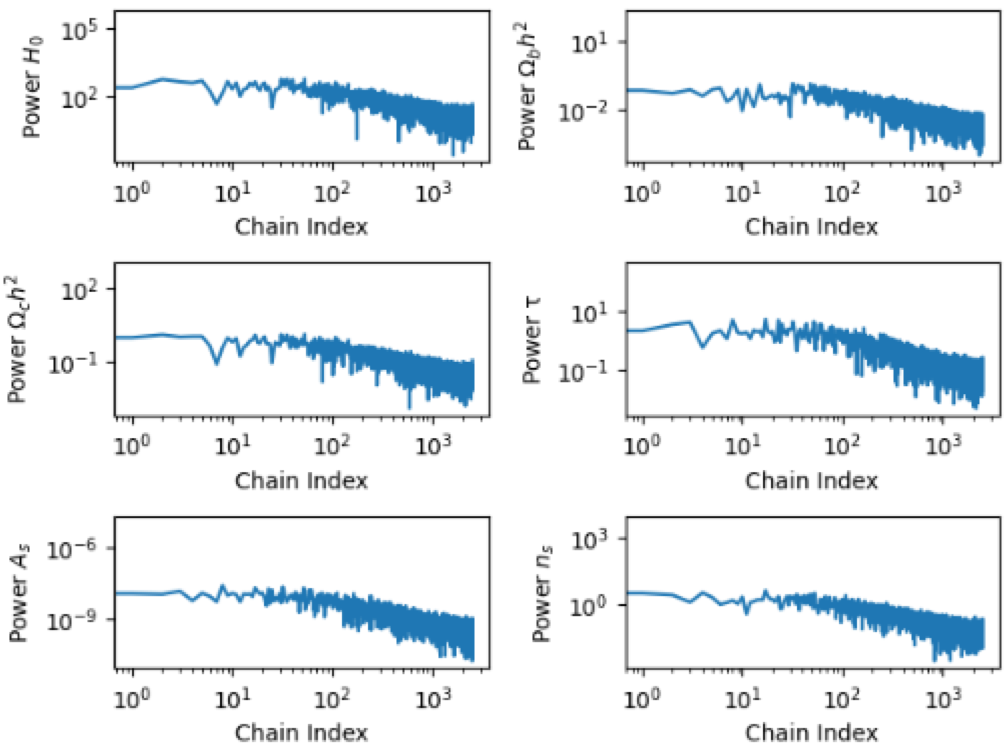

Figure 9. Parameter trends in the frequency domain – MCMC Part 4

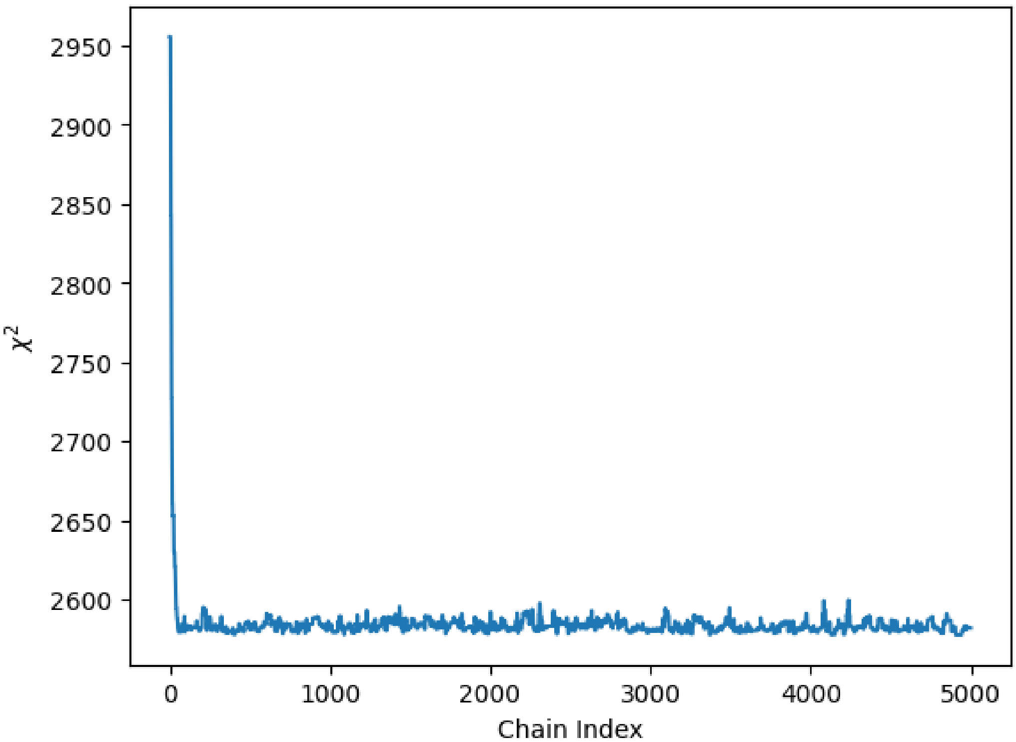

Figure 10. χ^2 variation with respect to chain index – MCMC Part 4

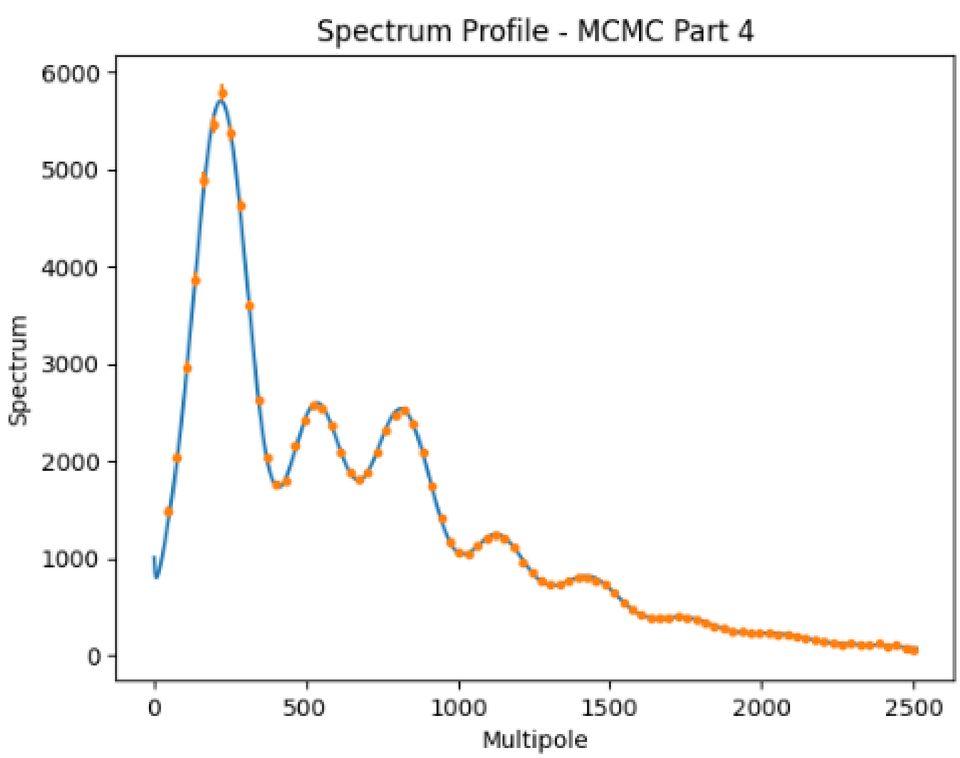

Figure 11. Spectrum profile based on mean parameters – MCMC Part 4

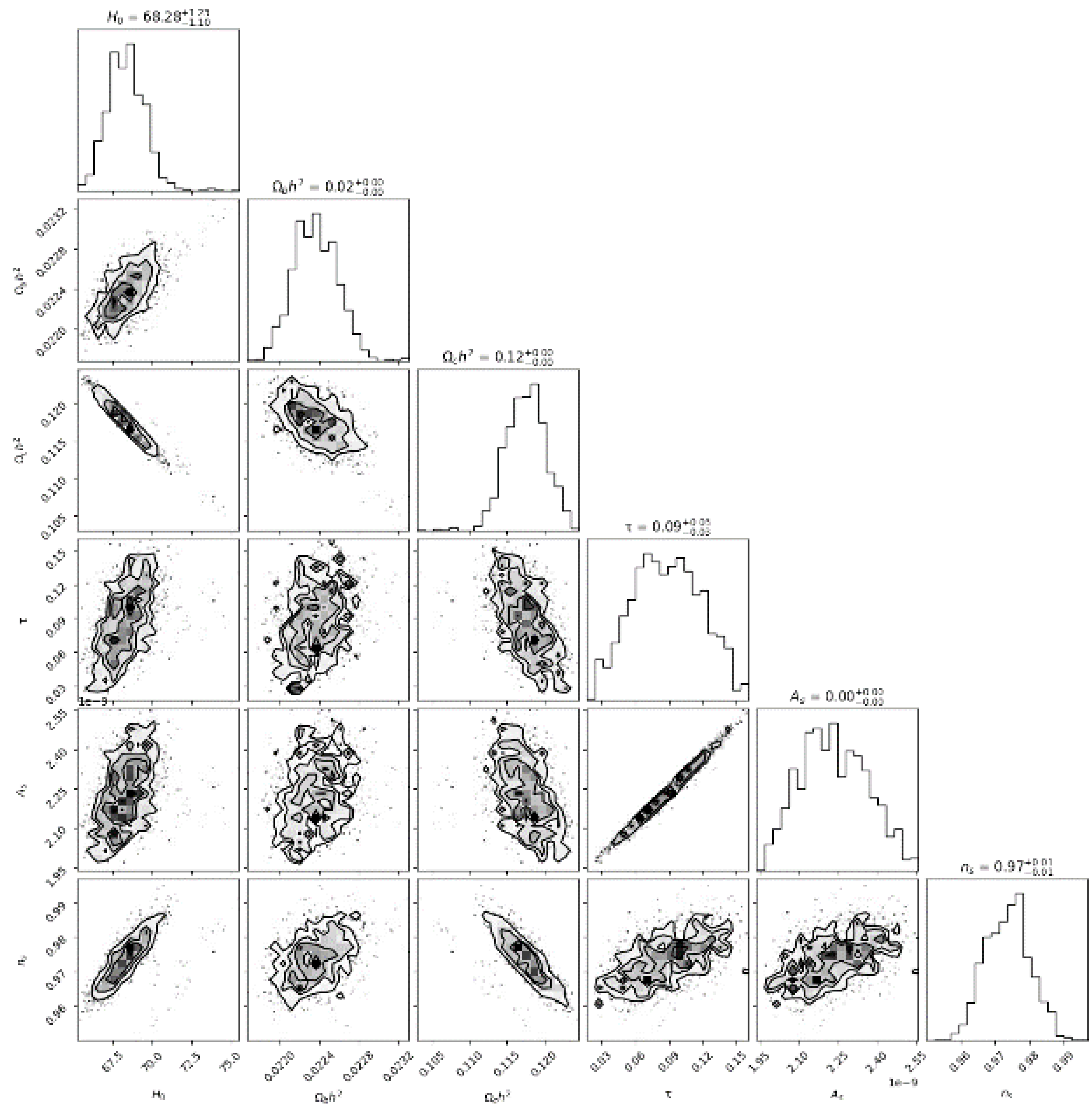

Figure 12. Corner plot of inter-parametric relationships for MCMC Part 3

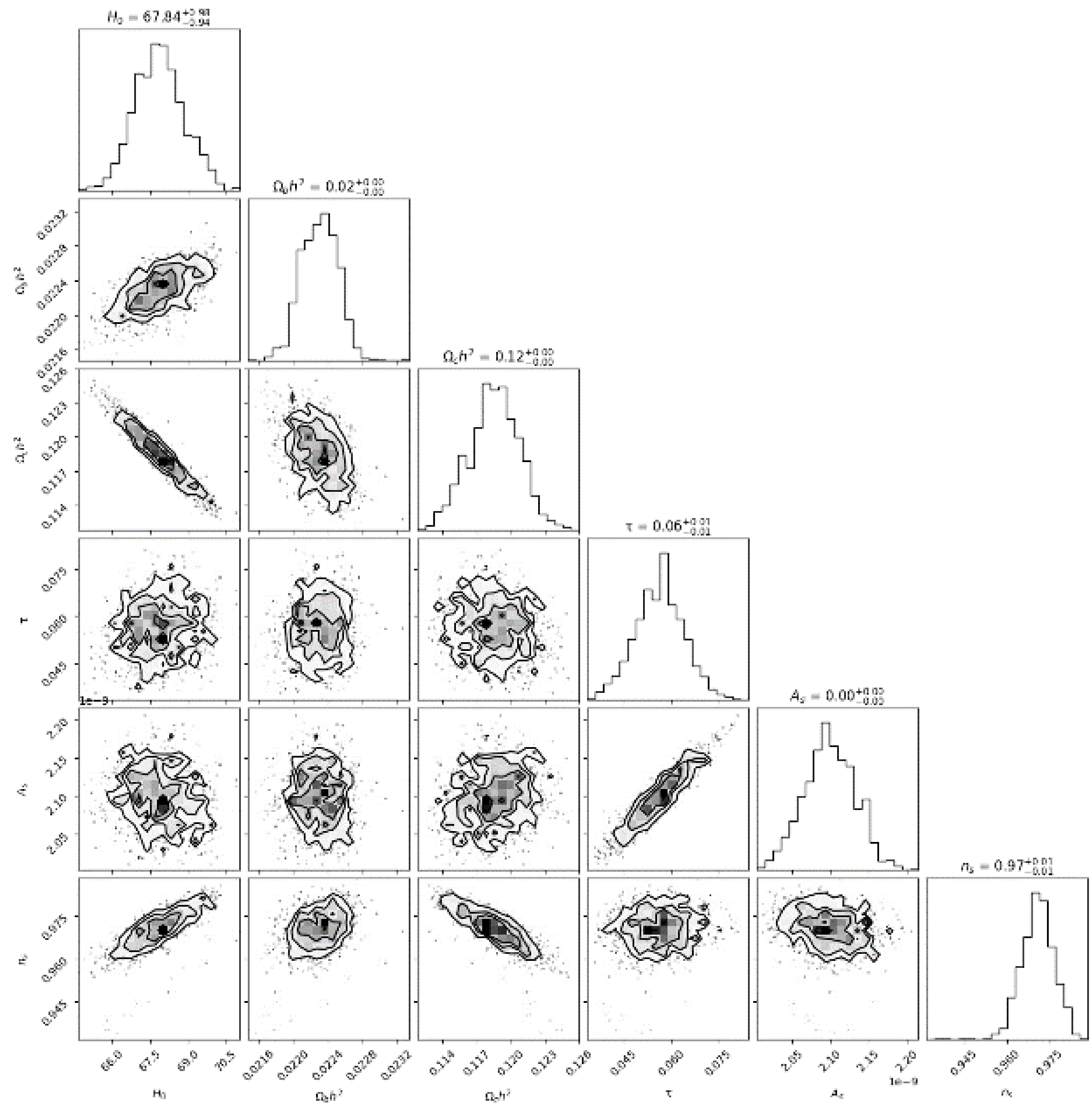

Figure 13. Corner plot of inter-parametric relationships for MCMC Part 4

Appendix A: Python Code

Jupyter notebook with relevant Python code and outputs is available at:

https://github.com/ck22512/comp_phys/tree/main/Assignment5