Answers

1.

The claim that the leapfrog scheme preserves energy was investigated. Let the leapfrog scheme be represented as:

$$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\frac{f(t,x+dt)-f(t,x-dx)}{2dx}$$

Substitution of a general solution f(x,t) = ξ^texp(ikx) where ξ would be complex and a function of k, in the preceding generates the following, for $\alpha = \frac{vdt}{dx}$

$$\xi^{t+dt}\,exp(ikx) - \xi^{t-dt}\,exp(ikx) = -\alpha \exp(\mathrm{ikx})\,[\xi^t\,exp(ikdx) - \xi^t\,exp(-ikdx)]$$

$$\xi^t[\xi^{dt} - \xi^{-dt}] = -\alpha\xi^t[exp(ikdx) - exp(-ikdx)]$$

$$[\xi^{dt} - \xi^{-dt}] = -\alpha[exp(ikdx) - exp(-ikdx)]$$

where exp(ikdx-ikx) = isin(kdx)+cos(kdx) and exp(-ikdx) = -isin(-kdx) +cos(kdx) according to Euler's identity.

$$[\xi^{dt} - \xi^{-dt}] = -\alpha[isin(kdx) + \cos(kdx) + isin(kdx) - \cos(kdx)]$$

$$[\xi^{dt} - \xi^{-dt}] = -2\alpha isin(kdx)$$

$$[\xi^{dt}]^2 + [\xi^{dt}]2\alpha isin(kdx) - 1 = 0$$

The quadratic expression can be solved:

$\xi^{dt} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where a=1, b=$2\alpha isin(kdx)$, and c=-1.

$$\xi^{dt} = \frac{-2\alpha isin(kdx)\pm\sqrt{-4\alpha^2 sin^2(kdx)+4}}{2} = -\alpha isin(kdx) \pm \sqrt{-\alpha^2 sin^2(kdx) + 1}$$

For the CFL condition $|\alpha| < 1$ at dt>0:

$$|\xi^{dt}| = \alpha^2 sin^2(kdx) - \alpha^2 sin^2(kdx) + 1 = 1$$

|ξ|=1, evidencing that the energy is conserved.

2.

a)

A 2D point charge should look like a 3D line charge with the following voltage dependence on the coordinates:

$V(x,y) = m\ln[x^2+y^2]^{1/2} \sim= \rho(x,y)+(1/4)[V(x-1,y)+V(x+1,y)+V(x,y+1)+V(x,y-1)$

where m could be assigned a value of 1, and ρ was the charge density at a given point, noting the four neighboring potentials would need to be averaged for iterative calculation of the potential map. The constraints of $\rho(0,0)=1$ and $V(0,0) = 1$ were implemented. The resultant potential map $V_c(x,y)$ is shown in Figure 1. The computed Green's function was subseqeuntly utilized in parts (b) and (c).

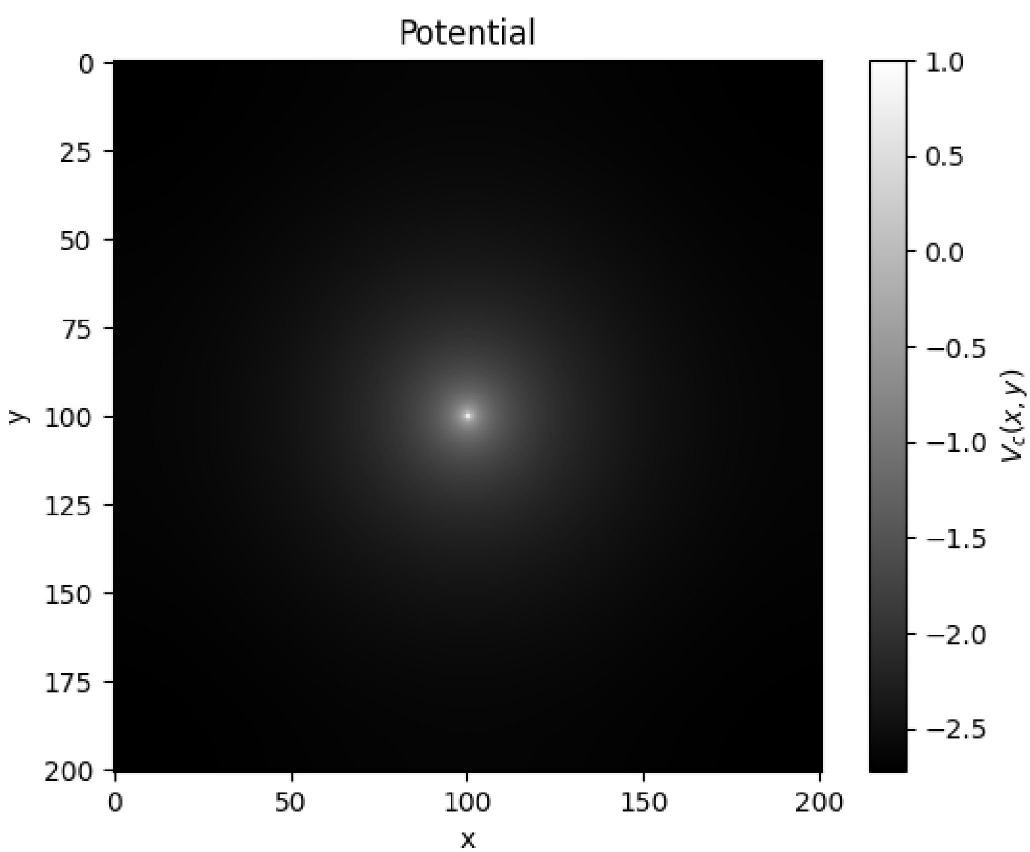

Figure 1. Potential map due to point charge at [0,0] located at point (100,100) on grid

Potentials were computed at select locations:

V[0,0] = 1.00 V

V[1,0] = 0.00 V

V[2,0] = -0.45 V

V[5,0] = -1.05 V

b)

Calculation of the potential V everywhere in space from an arbitrary charge distribution is possible via convolution of the charge by the computed Green's function, G:

$V = G*\rho$

The conjugate gradient approach was utilized in line with the course discussions. Initially, the potential on select surfaces were set with the aim to determine the charge distribution on those same surfaces. A conjugate-gradient solver was adopted to compute for charge density on a mask given V on that mask. A square box of 10 edge length held at a potential of 1, positioned symmetrically about the central charge was analyzed. The corresponding charge density plot is shown in Figure 2. V(x,y) was equal to zero over the plane, except along the box boundaries where V(x,y)=1. Charge density along one side of the box is given in Figure 3.

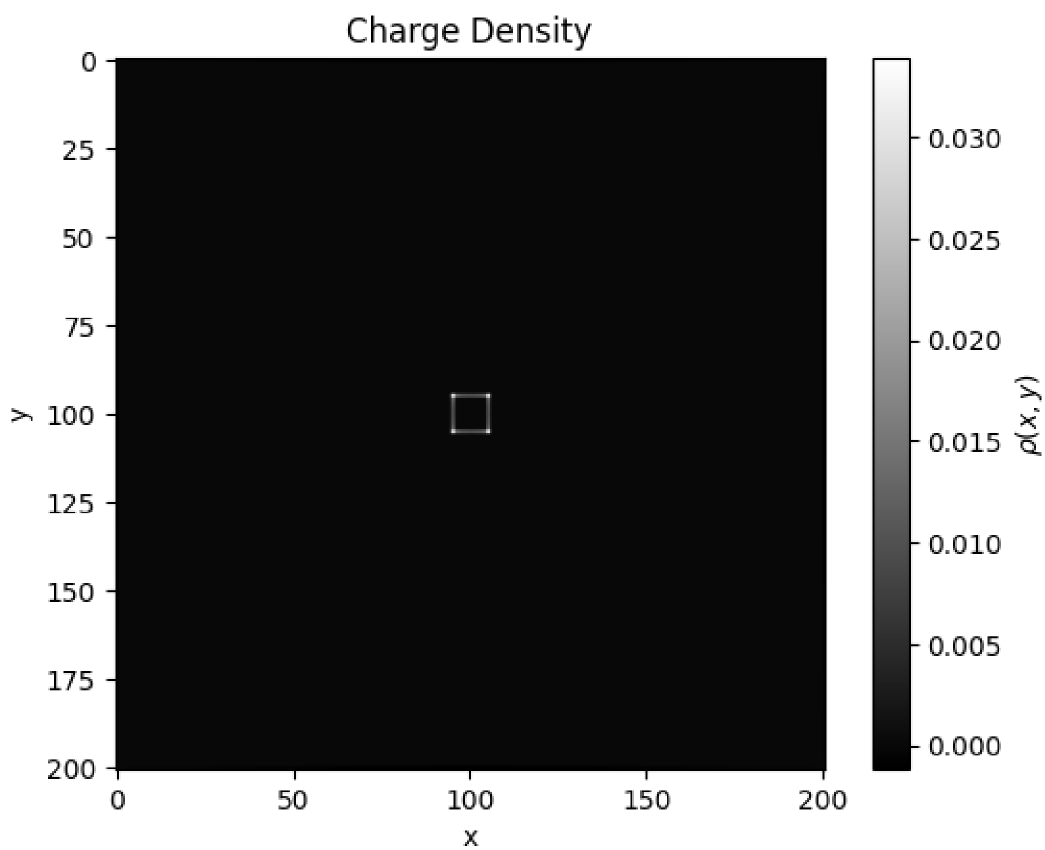

Figure 2. 2D Charge density profile subsquent to a square box of edge 10 held at V=1

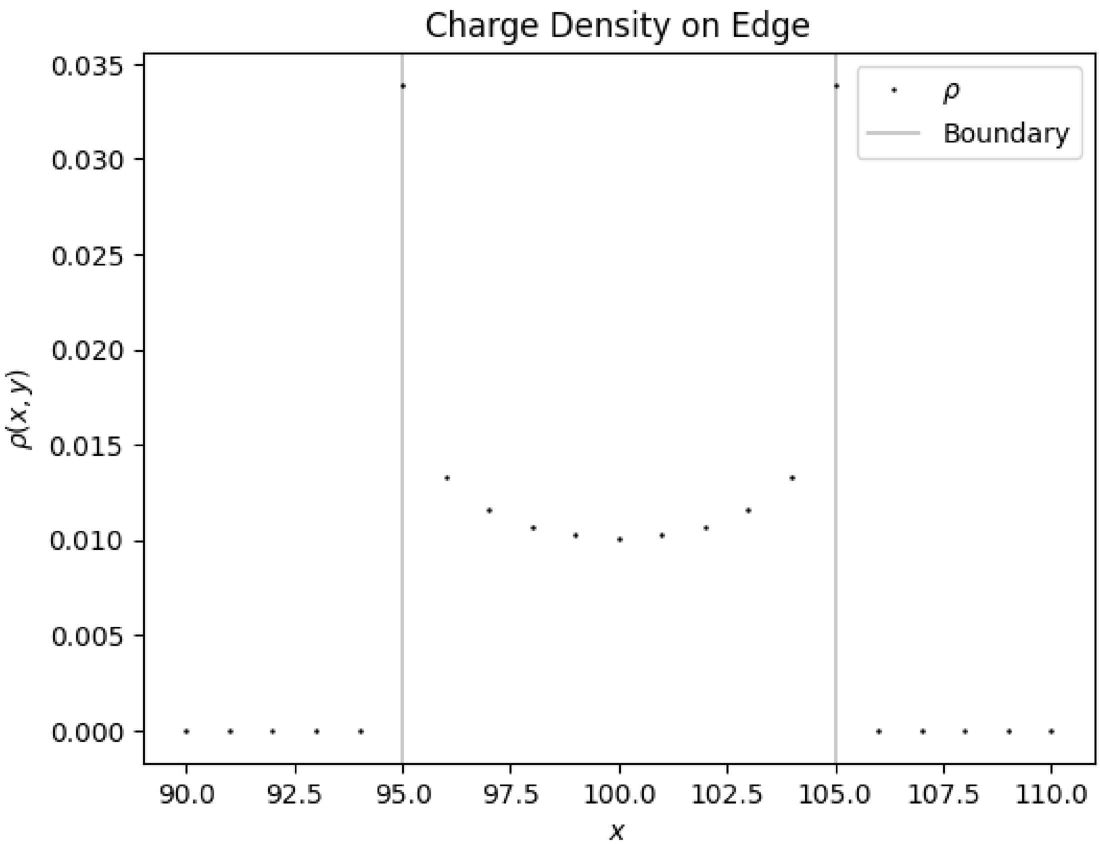

Figure 3. 1D Charge density profile along one side of the square box of edge 10 held at V=1

c)

Given the charge density profile, a convolution of the original Green's function with the stipulated charge density of the square box would facilitate computation of the potential V_f everywhere in space (Figure 4). Mean and standard deviation of the potential inside the box where 0.8117 and 0.0203, respectively. Observed ~2% standard deviation of inner potential evidenced that the potential was relatively constant wihin the boundaries of the box. Electric field $E_{x,y}$ just outside the box was calculated based on the gradient - ∇V. Figure 5 demonstrates the electric field vector map. Strongest field was observed at the corners of box, which also corresponded to greatest charge density along the boundary. Strength of electric field was diminished inside the box, as per the behavior of the enclosure of a conductor. Electric field lines were observed to be perpendicular to the equipotential box surface, in agreement with theory.

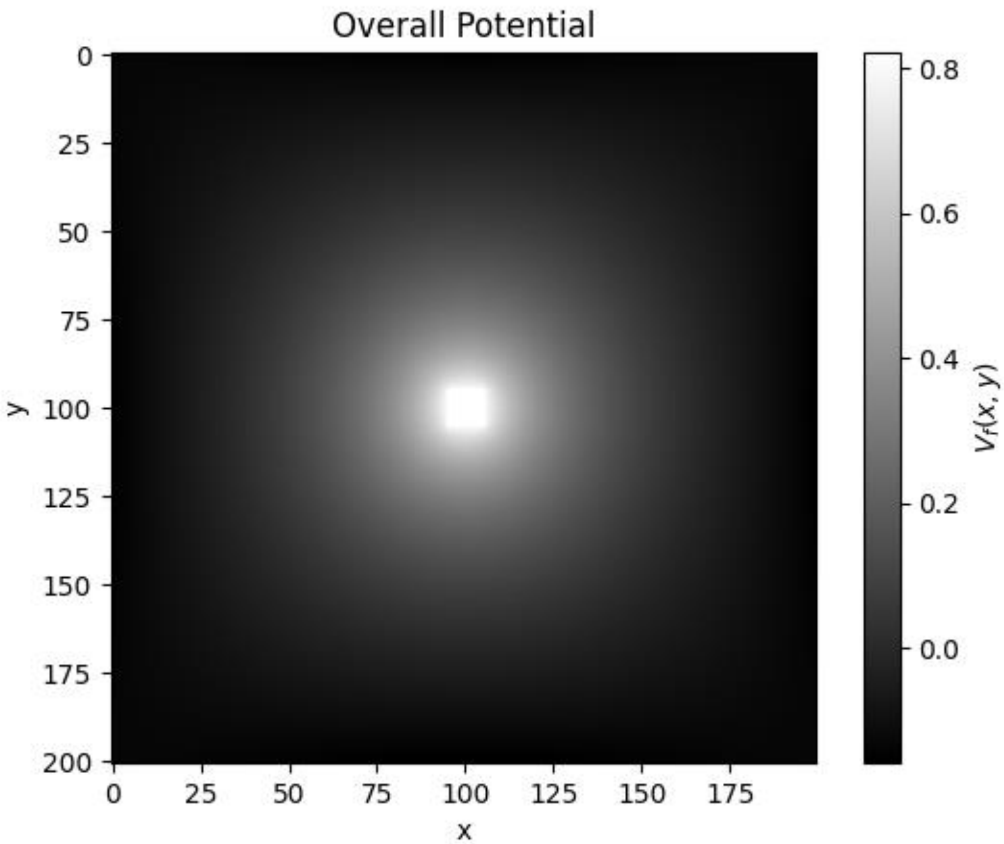

Figure 4. 2D Potential profile resultant from the square box of edge 10 held at V=1

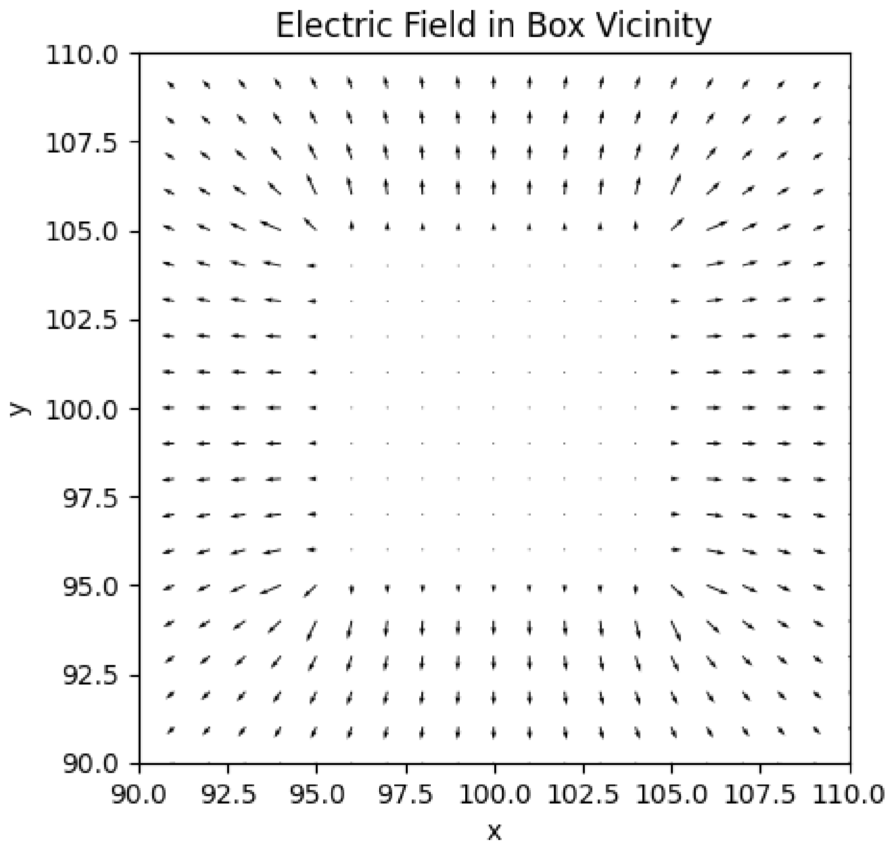

Figure 5. Electric field lines around the square box of edge 10 held at V=1

Appendix A: Python Code

Jupyter notebook with relevant Python code and outputs is available at:

https://github.com/ck22512/comp_phys/tree/main/Assignment8