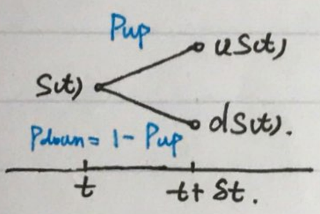
MTH 9821 - L1

2017/08/31

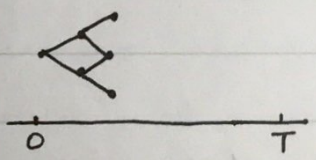
Binomial tree: model the evolution of underlying asset.

Tree (Binomial) Models

- Model for the evolution of the price of an asset.
- Once we have the evolution of the asset, we can ~~derive~~ price derivative securities on that asset.



u, d are functions of δt.



N time intervals $\delta t = T/N$

$S(T)$ has a discrete probability distribution.

WANT: Prob distribution of $S^{(N)}(T)$ to converge to the prob. distribution of a lognormal $S(T)$ with drift μ and volatility σ.

Necessary and Sufficient condition for this to happen: CALIBRATION.

$E_{continuous}[S(t+\delta t) \mid S(t)] = E_{discrete}[S(t+\delta t) \mid S(t)]$.

$Var_{continuous}[S(t+\delta t) \mid S(t)] = Var_{continuous}[S(t+\delta t) \mid S(t)]$.

CLT. pd of $S^{(N)}(T) \xrightarrow{N\to\infty}$ pdf of lognormal $S(T)$.

$E_{cont.}[S^2(t+\delta t) \mid S(t)] = E_{discrete}[S^2(t+\delta t) \mid S(t)]$.

$$e^{(\mu-q)\delta t} S(t) = S(t)(up + d(1-p)).$$

$$e^{2(\mu-q)\sigma^2\delta t} S^2(t) = S^2(t)(u^2 p + d^2(1-p)).$$

Two constraints. 3 unknowns: u, d, p

Add the constraint $ud = 1$ (CRR) Model.

or Add the constraint $p = \frac{1}{2}$.

"Binomial Trees" → Model for the evolution of the price of an asset.

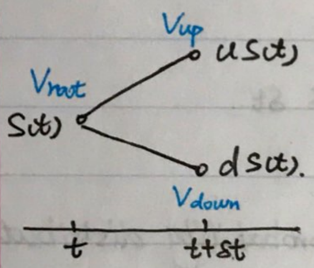
$u = e^{\sigma\sqrt{\delta t}} + O(\delta t)$. $d = e^{-\sigma\sqrt{\delta t}} + O(\delta t)$.

Do Not Care about p.

Calibration of tree model to lognormal distribution ONLY care about u and d (function of σ and δt only), not depend on μ. up to order $\sqrt{\delta t}$.

(Bigger order than 1 could depends on μ).

Valuation of Derivative Securities on assets with binomial tree distribution.



<u>Hedging for derivative valuation</u>

Set up portfolio: Long 1 derivative security, short Δ units ua.

$$\Pi_r \begin{cases} \Pi_{up} \\ \Pi_{dow.} \end{cases} \qquad \Pi_r = V_r - \Delta S(t). \qquad \left.\begin{array}{l} \Pi_{up} = V_{up} - \Delta u S(t) \\ \Pi_{down} = V_{down} - \Delta d S(t). \end{array}\right\} \text{Choose } \Delta \text{ to make them equal.}$$

$$\Delta = \frac{V_{up} - V_{down}}{(u-d)S(t)} . \quad \Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}$$

$$\Pi_{up} = \frac{uV_{down} - dV_{up}}{u-d} = \Pi_{down}.$$

$$\text{No arbitrage: } \Pi_r = e^{-r\cdot\delta t} \cdot \frac{uV_{down} - d\cdot V_{up}}{u-d} \leftarrow \text{Solve for } V_r$$

$$V_r - \Delta\cdot S(t) = V_r - \frac{V_{up} - V_{down}}{u-d}$$

$$V_r = V_{up} \cdot \frac{1 - de^{-r\delta t}}{u-d} + V_{down} \cdot \frac{u\cdot e^{-r\delta t} - 1}{u-d}$$

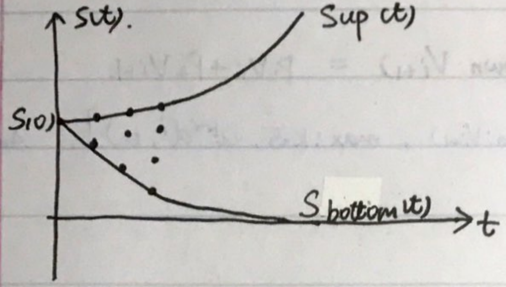
$$V_r = e^{-r\delta t} \cdot \left[V_{up} \cdot \frac{e^{+r\delta t} - d}{u-d} + V_{down} \cdot \frac{u - e^{+r\delta t}}{u-d} \right]$$

$$= e^{-r\delta t} \cdot [V_{up} P_{RN,up} + V_{down} \cdot P_{RN,down}].$$

$$P_{RN,up} + P_{RN,down} = 1. \quad P_{RN,up} > 0. \quad P_{RN,down} > 0$$

$$\Leftrightarrow d < e^{r\delta t} < u \quad \text{arbitrage free condition.}$$

The values of non-path dependent derivative securities on assets with binomial tree model for price evolution, only depends on u and d, and not on the P_{up}, P_{down} probability for the price of the asset on the tree.



$t_n = n\cdot\delta t$. $\quad S_{top}(t_n) = S(0)\cdot u^n = S(0)\cdot e^{n\sigma\sqrt{\delta t}}$

$$= S(0)\cdot e^{\sigma\sqrt{n t_n}}$$

$$S_{bottom}(t_n) = S(0)\cdot e^{-\sigma\sqrt{n t_n}}$$

If the asset pays divident, continuous rate q. The change:

short $\Delta e^{-q\delta t}$ units u.a → become Δ units of u.a at time $t+\delta t$.

$$\Pi_r = V_n - \Delta S(t)\, e^{-q\delta t} = V_n - e^{-q\delta t}\cdot\frac{V_{up}-V_{down}}{u-d}$$

$$V_r = V_{up}\cdot\frac{e^{-q\delta t} - d e^{-r\delta t}}{u-d} + V_{down}\,\frac{u\cdot e^{-r\delta t} - e^{-q\delta t}}{u-d}$$

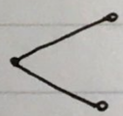
$$P_{RN,up} = \frac{e^{(r-q)\delta t} - d}{u-d} \qquad P_{RN,down} = \frac{u - e^{(r-q)\delta t}}{u-d}$$

$d < e^{(r-q)\delta t} < u$. $\quad$ (Δ: Not change) $\quad$ [$S(t)$ won't change].

Replication for derivative valuation.

Build a portfolio made of cash and u.a such that $\Pi_{up} = V_{up}$. $\Pi_{down} = V_{down}$.

$\Rightarrow \Pi_r = V_r$ (No Arb).



$\Pi_r = B + \Delta S(t)$.

$$\left.\begin{aligned}\Pi_{up} &= Be^{r\delta t} + \Delta\cdot u S(t) = V_{up}\\ \Pi_{down} &= Be^{r\delta t} + \Delta\cdot d S(t) = V_{down}\end{aligned}\right\}\quad \Delta = \frac{V_{up}-V_{down}}{uS(t) - dS(t)} = \frac{V_{up}-V_{down}}{S_{up}-S_{down}}$$

Pseudocode Binomial Tree Pricer:

Input: $S_0, K, T, \sigma, n, r, q$. N = number of time steps.

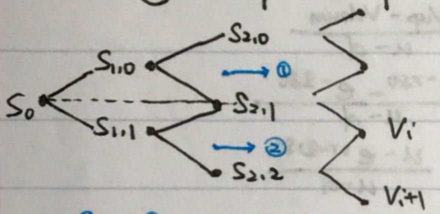
Output: V_0 (Think it as $V_{numerical}(N)$).

$\delta t = T/N$; $u = e^{\sigma\sqrt{\delta t}}$; $d = \frac{1}{u}$; $P_1 = \frac{e^{-q\delta t} - de^{-r\delta t}}{u-d}$. $\quad P_2 = \frac{ue^{-r\delta t} - e^{-q\delta t}}{u-d}$.

```
for i = 0:N
    V_i = V(S_0 u^(N-i) d^i, T)
end
```

$$\text{for } k = (N-1):0$$
$$\quad \text{for } i = 0:k$$
$$\quad\quad V_i = e^{-r\delta t}\left(P_{RN,up} V_i + P_{RN,down} \cdot V_{i+1}\right) = p_1 V_i + p_2 V_{i+1}.$$
$$\quad \text{end} \quad \hookrightarrow \max\left[e^{-r\delta t}\left(P_{RN,up} V_i + P_{RN,down} \cdot V_{i+1}\right),\ \max(K-S,\ u^{k-i} d^{i},\ 0)\right]. \quad \text{American.}$$
$$\text{end}$$

Numerical Calculation of the Greeks is equally if not more important than calculating the price of the derivatives.



$$\Delta_0 = \frac{V_{1,0} - V_{1,1}}{S_{1,0} - S_{1,1}} \qquad \Gamma_0 = \frac{\Delta_{1,0} - \Delta_{0,0}}{(S_{2,0} - S_{2,2})/2}$$

$$\Gamma_0 = \frac{\dfrac{V_{2,0} - V_{2,1}}{S_{2,0} - S_{2,1}} - \dfrac{V_{2,1} - V_{2,2}}{S_{2,1} - S_{2,2}}}{\dfrac{S_{2,0} - S_{2,2}}{2}}$$

①: $\dfrac{S_{2,0} + S_{2,1}}{2}$. ②: $\dfrac{S_{2,1} + S_{2,2}}{2}$. ① − ② $= \dfrac{S_{2,0} - S_{2,2}}{2}$

$S_{2,1} = S_{0,0}\ (u = d)$. $\qquad \theta_0 = \dfrac{V_{2,1} - V_{0,0}}{2\delta t}$

Vega: $\dfrac{\underset{tol=10^{-6}}{V^{\sigma+d\sigma}_{binomial}(N)} - \underset{tol=10^{-6}}{V^{\sigma}_{binomial}(N)}}{d\sigma}$

$d\sigma = 0.001, \quad \sigma = 0.1 = 10\%.$

V_0 is known. how to calculate implied volatility of binomial tree.

$V_{exact} = \$7.53. \quad f(\sigma) = V^{N}_{numerical}(\sigma) - V_{exact}$

Newton: $\sigma_{n+1} = \sigma_n - f(\sigma_n)/f'(\sigma_n) \rightarrow$ Vega.

Secant: $\sigma_{n+1} = \sigma_n - \dfrac{f(\sigma_n)}{f(\sigma_n) - f(\sigma_{n-1})} \cdot (\sigma_n - \sigma_{n-1})$. [Correct Method].

[If calculate vega, need to ~~cod~~ compute the tree twice].

N is given by the tolerance.

Ex: $\sigma_0 = 10\% \quad f(\sigma_0) = V^{(N)}_{numerical}(10\%) - V_{exact}$.

where N is determined by: $tol = 10^{-6}$

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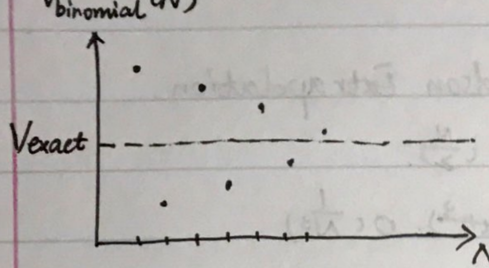
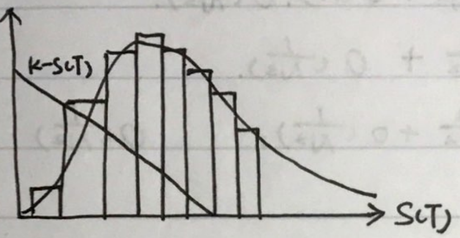
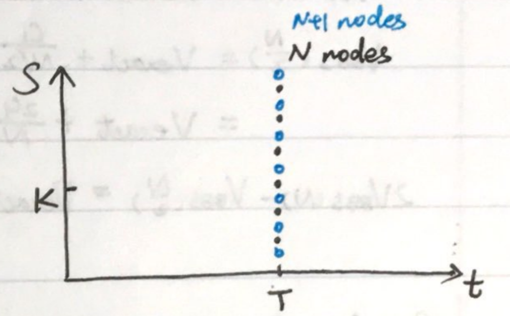
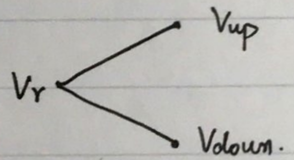


Overshooting. under shooting, ...
(The pattern of convergence). <?>





Binomial Tree Pricer for American options,



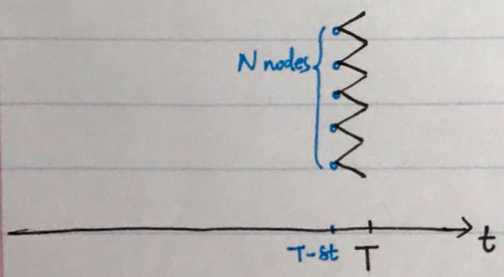
$V_r = \max(V_{root,\,binomial},\ V_{early\ exercise})$.

Change of the code ...

Average Binomial Tree method.

$$V_{Average,\,Bin}(N) = \frac{V_{binomial}(N) + V_{binomial}(N+1)}{2}$$

Binomial Black-Scholes.



for $i = 0 : (N-1)$

$V_i = \max[V_{BS}(S_0 u^{N-1-i} d^i, K, \delta t, \sigma, q, r),\ K - S_0 u^{N-1-i} d^i]$.

end.

For American Options:

Binomial Black-Scholes with Richardson Extrapolation.

$$V_{BBSR}(N) = 2V_{BBS}(N) - V_{BBS}\left(\frac{N}{2}\right).$$

$$V_{BBS}(N) = V_{exact} + \frac{C_1}{N} + \frac{C_2}{N^2} + \cancel{O(N^3)}.\ O\left(\frac{1}{N^3}\right)$$

$$V_{BBS}\left(\frac{N}{2}\right) = V_{exact} + \frac{C_1}{N/2} + \frac{C_2}{(N/2)^2} + \cancel{O(N^3)}.\ O\left(\frac{1}{N^3}\right).$$

$$= V_{exact} + \frac{2C_1}{N} + \frac{4C_2}{N^2} + O\left(\frac{1}{N^3}\right).$$

$$2V_{BBS}(N) - V_{BBS}\left(\frac{N}{2}\right) = V_{exact} - \frac{2C_2}{N^2} + O\left(\frac{1}{N^3}\right). \qquad O\left(\frac{1}{N^2}\right)$$

Greeks of Average Binomial Tree.

$$\Delta_{Average, Bin}(N) = \frac{1}{2}\left[\Delta_{Binomial}(N) + \Delta_{Binomial}(N+1)\right].$$

Greeks Calculation of BBSR

Variance Reduction for American Options.

$V_{numerical, European}(N)$

$V_{numerical, American}(N)$

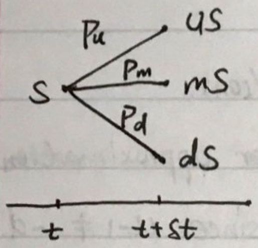
$$V^{VarRed}_{American}(N) = V_{numerical, American}(N) -$$

$$- \left(V_{numerical, European}(N) - V_{BS}\right).$$

MTH 9821 - L2

2017/09/07

Trinomial Model for the evolution of the price of a asset.



Calibrate to a lognormal model with drift μ and volatility σ.

$\mathbb{E}_{cont}[S(t+\delta t)\mid S(t)] = \mathbb{E}_{trinomial}[S(t+\delta t)\mid S(t)]$.

$Var_{cont}[S(t+\delta t)\mid S(t)] = \mathbb{E}_{trinomial}[S(t+\delta t)\mid S(t)]$.

$\Leftrightarrow \mathbb{E}_{cont}[S^2(t+\delta t)] = \mathbb{E}_{tri}[S^2(t+\delta t)]$.

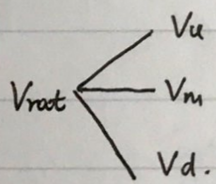
6 unknowns: u, m, d, p_u, p_m, p_d.

$p_u + p_m + p_d = 1$. 3 constraints.

$u = e^{\sigma\sqrt{3\delta t}}$, $d = e^{-\sigma\sqrt{3\delta t}}$, $m = 1$.

$p_{u,RN} = \frac{1}{6} + (r-q-\frac{\sigma^2}{2})\sqrt{\frac{\delta t}{12\sigma^2}}$. $p_{m,RN} = \frac{2}{3}$. $p_{d,RN} = \frac{1}{6} - (r-q-\frac{\sigma^2}{2})\sqrt{\frac{\delta t}{12\sigma^2}}$.

Used in the valuation of the derivative securities.



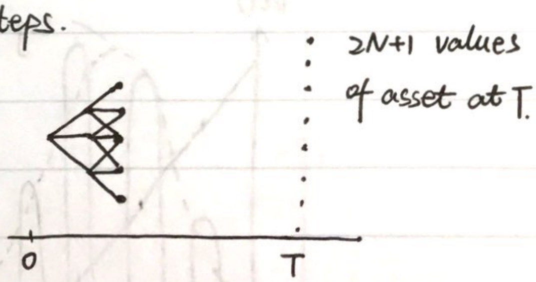
$V_{root} = e^{-r\delta t}(p_{u,RN}\cdot V_u + p_{m,RN}\cdot V_m + p_{d,RN}\cdot V_d)$.

(To choose u and d, should cover more probability space).

(Many solutions ...).

Trinomial Tree Pseudocode for American Put Valuation.

Input: S_0, K, T, σ, r, q. $N \rightarrow$ number of tree steps.



$\delta t = T/N$; $p_u = \cdots$, $p_m = \frac{2}{3}$, $p_d = \cdots$;

for $i = 0 : 2N$

$V_i = \max(K - S_0 \cancel{u^{2N-i} d^i}, 0)$.

end. $S_0 u^{N-i}$ ($\because ud = 1, m = 1$).

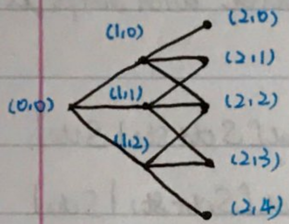
for $j = (N-1) : 0$

for $i = 0 : 2j$ $\quad V_i = \max[e^{-r\delta t}[V_i p_u + V_{i+1} p_m + V_{i+2} p_d], \max(K - S_0 u^{j-i}, 0)]$. American.

$V_i = e^{-r\delta t}[V_i p_u + V_{i+1} p_m + V_{i+2} p_d]$. Return V_0.

end end

Greeks approximation using trinomial trees.



$$\Delta_{approx} = \frac{V_{1,0} - V_{1,2}}{S_{1,0} - S_{1,2}}$$

Assume have nodes. $V(0,-1)$, $V(0,1)$.

$$\Delta \simeq \frac{V(0,-1) - V(0,1)}{S(0,-1) - S(0,1)}$$ second order approximation.

But not always work well, since $u-1 \neq 1-d$. $(u+d \approx 2)$.

The distance between nodes are not equal.

$$\Theta_{approx} = \frac{V_{11} - V_{00}}{\delta t}$$

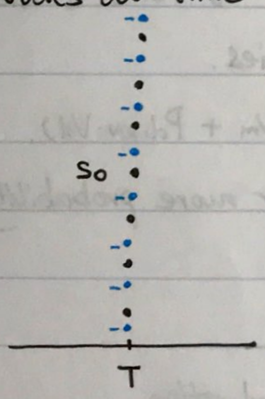
$$\Gamma_{approx} = \frac{\Delta_{10} - \Delta_{12}}{S_{10} - S_{12}} = \frac{\frac{V_{2,0} - V_{2,2}}{S_{2,0} - S_{2,2}} - \frac{V_{2,2} - V_{2,4}}{S_{2,2} - S_{2,4}}}{S_{1,0} - S_{2,2}}$$

$\hookrightarrow \simeq \frac{S_{2,0} - S_{2,4}}{2}$

Binomial Model

N time steps, N+1 time steps.

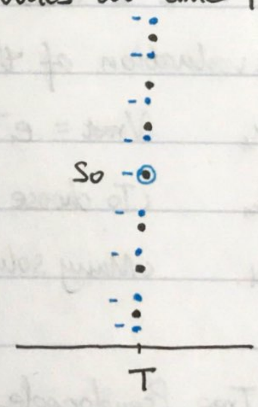
Nodes at time T.

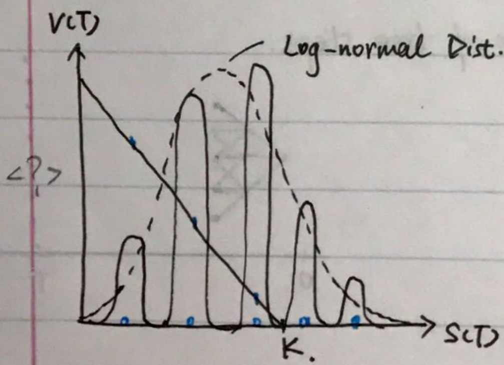


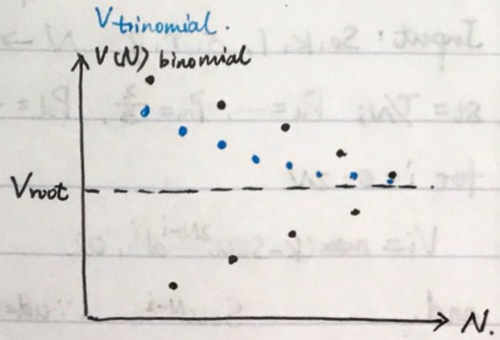
Trinomial Model.

N time steps, N+1 time steps.

Nodes at time T







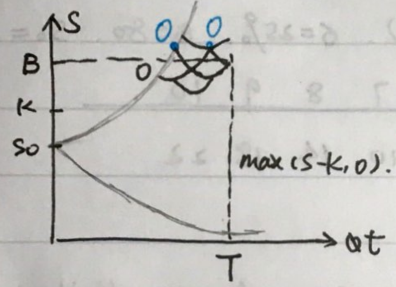
∴ Bi: Oscilate Convergence ⇒ Take average

Tri: Monototic Convergence.

Oscilations:

Tree Methods for Barrier Option Pricing

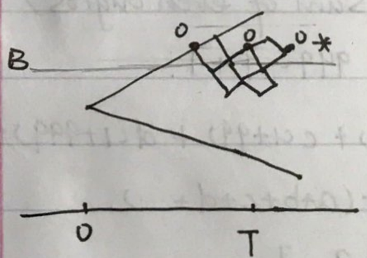
Up and ~~Fall~~ Out Calls.

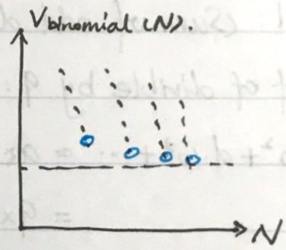


Hard to hedge such options. when S approaches Barrier, usually close the position, sell options.

*. Slightly higher than the barrier.

Since nodes on the barrier do not exist.





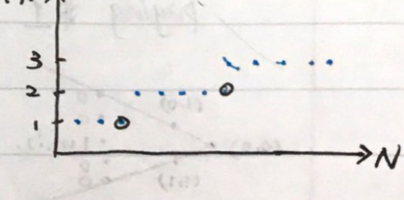
As N larger, nodes move slightly left and lower.

Convergence Pattern.

∴ Choose the ∘ points give best approximation.

Assume $B > S_0$, Given N, denote by K_N the number of spot price levels between S_0 and B. $S_0 \cdot u_N^{K_N} < B < S_0 u_N^{K_N+1}$, where $u_N = e^{\sigma\sqrt{\frac{T}{N}}}$

Goal: Find N such that $K_{N+1} = K_N + 1$.



$u_N^{K_N} \leq \frac{B}{S_0} < u_N^{K_N+1}$, take ln.

$K_N \cdot \sigma\sqrt{\frac{T}{N}} \leq \ln\left(\frac{B}{S_0}\right) < (K_N+1)\sigma\sqrt{\frac{T}{N}}$.

$K_N \leq \ln\left(\frac{B}{S_0}\right) / \sigma\sqrt{\frac{T}{N}} < K_N + 1$

$K_{N+1} \leq \ln\left(\frac{B}{S_0}\right) / \sigma\sqrt{\frac{T}{N+1}} < K_{N+1} + 1$.

Set $k_N = p$. Look for N such that.

$p \le \ln(\frac{B}{S_0})/\sigma\sqrt{\frac{T}{N}} < p+1$. $p+1 \le \ln(\frac{B}{S_0})/\sigma\sqrt{\frac{T}{N+1}} < p+2$.

$\ln(\frac{B}{S_0})/\sigma\sqrt{\frac{T}{N}} < p+1 \le \ln(\frac{B}{S_0})/\sigma\sqrt{\frac{T}{N+1}}$.

$\sqrt{N} < \frac{(p+1)\sigma\sqrt{T}}{\ln(B/S_0)} \le \sqrt{N+1}$. $N < \frac{(p+1)^2\sigma^2 T}{[\ln(\frac{B}{S_0})]^2} \le N+1$.

$N = \text{ceil}\left(\frac{(p+1)^2\sigma^2 T}{(\ln(\frac{B}{S_0}))^2}\right) - 1$. $N = \text{floor}\left(\frac{(p+1)^2\sigma^2 T}{(\ln(\frac{B}{S_0}))^2}\right)$.

$\boxed{\text{floor}\left(\frac{p^2\sigma^2 T}{(\ln(\frac{B}{S_0}))^2}\right)}$

floor $(\frac{p^2}{\cancel{169}})$. $\sigma = 25\%$. $B = 80$. $S_0 = 50$. $T = 4m$.

p	6	7	8	9	10
N	8	10	14	18	22

[P.S: Divisible by 11: (Sum of odd digits - Sum of even digits) divisble by 11;. Proof of divible by 9: $10^k = 99\cdots9 + 1$.

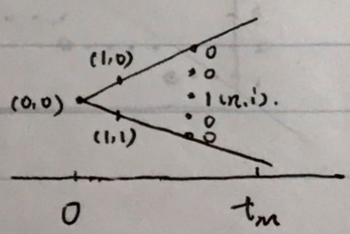
$\therefore a + b\cdot 10 + c\cdot 10^2 + d\cdot 10^3 + \cdots = a + b(1+9) + c(1+99) + d(1+999) + \cdots$

$= 9\times[\cdots] + (a+b+c+d+\cdots)$.

$\Rightarrow$ Sum of digits are divisible by 9.].

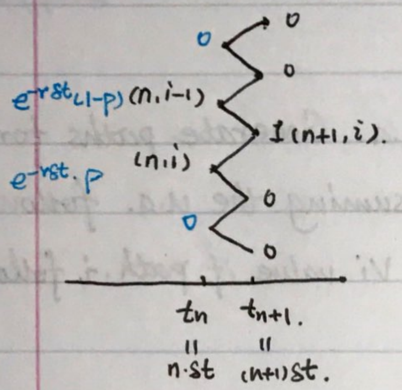
Arrow - Debreu Prices.

Def: $\lambda_{m,i}$ = value at time 0 of a derivative security paying \$1 if $S(t_m) = S_{m,i}$ / 0 if $S(t_m) \neq S_{m,i}$.



$\lambda_{0,0} = 1$. $\lambda_{1,0} = e^{-r\delta t} p$.

$p = \frac{e^{(r-q)\delta t} - d}{u - d}$.



knows $\lambda_{n,i}$ for all $i = 0:n$.

find $\lambda_{n+1,i}$ for $i = 0:n+1$.

$$\begin{cases} \lambda_{n+1,i} = e^{-r\delta t}(1-p)\lambda_{n,i-1} + e^{-r\delta t}p\lambda_{n,i} & \forall i = 1:n \\ \lambda_{n+1,0} = e^{-r\delta t}\lambda_{n,0}\cdot p \\ \lambda_{n+1,n+1} = e^{-r\delta t}(1-p)\cdot\lambda_{n,n} \end{cases}$$

Useful when pricing multiple derivative securities on the same underlying asset.

The value $V(0)$ of a derivative security with payoff at maturity given by:

$V_{N,i}$ if $S(T) = S_{N,i}$ for $i = 0:N$. $V(0) = \sum_{i=1}^{N} \lambda_{N,i} V_{N,i}$

Use Arrow-Debreu pricer to price multiple derivative securities on the same underlying asset with multiple maturities.

Problem: Calculate the Greeks ?

MTH 9821 - L3

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Monte Carlo Method

Goal: Value derivative security on an u.a. Generate paths for the u.a.
Value the derivative security assuming the u.a. follows that path.
n paths, m time steps per path. V_i value if path i followed.
Compute Average $\hat{V}(n) = \frac{1}{n}\sum_{i=1}^{n} V_i$

Convergence pattern for MC:

$E[\hat{V}(n)] = V$ → unbiased estimator

CLT ⇒ $\hat{V}(n) - V / \frac{\hat{\sigma}_V(n)}{\sqrt{n}} \to N(0,1)$. (in distribution).

$\hat{\sigma}_V(n) = \left[\frac{1}{n-1}\sum_{i=1}^{n}(V_i - \hat{V}(n))^2\right]^{\frac{1}{2}}$

Confidence Intervals: As $n \to \infty$, there is a 95% probability that

$-\frac{\sigma_V}{\sqrt{n}} \cdot 1.96 \le \hat{V}(n) - V \le \frac{\sigma_V}{\sqrt{n}} 1.96$ $(\mathbb{P}(|Z| \le 1.96) = 0.95)$.

Monte Carlo approximation error. $|\hat{V}(n) - V| = O(\frac{1}{\sqrt{n}})$.

2 types of approximations when doing Monte Carlo:

Sampling error (n value) $O(\frac{1}{\sqrt{n}})$. → Finite Difference Method?

Time step discretization error $O(\delta t)$. $\delta t = \frac{T}{m}$.

⇒ $|\hat{V}(n) - V| = O(\max(\frac{1}{\sqrt{n}}, \frac{T}{m}))$.

N independent samples of Standard Normal, Choose m, n such that

$m \simeq \sqrt{n} \cdot \overset{(x)}{T}$. $N \simeq m \cdot n \simeq m^3$. $m = \sqrt[3]{N}$ $n = (\sqrt[3]{N})^2$

Advantages:

Easy to implement

Scales well for multi-asset derivatives.

Easy to parallelize

Easy to apply for path-dependent options.

Disadvantages:

Slow convergence;

Greeks computation are not straightforward;

[ex: PDE method $O(1/n^{\frac{d}{2}})$. MC: $O(\frac{1}{\sqrt{n}})$]

American option Monte Carlo not straightforward

<u>MC for non path-dependent securities.</u>

Given $S(T)$, the value of the derivative security at T only depends of $S(T)$ (and not on the paths $S(t)$, for $0<t<T$).

[Risk neutral pricing. $V(0) = e^{-rT} E_{RN}[V(T)] = E_{RN}[e^{-rT}V(T)]$.

Lognormal Model:

$$S(T) = S(0) \cdot \exp[(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z]$$

If z_i is sample of Z.

$$S_i = S(0) \cdot \exp[(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}z_i] \quad . \quad V_i = e^{-rT}V(S_i).$$

MC estimation: $\hat{V}(N) = \frac{1}{N}\sum_{i=1}^{N} V_i$.

[Generate N independent samples $z_1, z_2, \cdots, z_N$ of Z].

<u>MC for path-dependent securities</u>

$dS = (r-q)\cancel{S}dt + \sigma S\,dX$. $\quad d(\ln S) = (r-q-\frac{\sigma^2}{2})dt + \sigma\,dX$.

Discrete path corresponding to $0=t_0<t_1<\cdots<t_m=T$.

$\delta t = t_i - t_{i-1} = \frac{T}{m}$. $\forall\, i=1:m$.

$S_{j+1} = (r-q)S_j\cancel{\delta t} + \sigma S_j\sqrt{\delta t}\,z_j$. [Don't use this one, small p that <0].

$S_{j+1} = S_j \cdot \exp[(r-q-\frac{\sigma^2}{2})\delta t + \sigma\sqrt{\delta t}\,z_j)$. $j=0:(m-1)$.

On each path, compute V_i, $i=1:n$.

n paths $\quad \hat{V}(n) = \frac{1}{n}\cdot\sum_{i=1}^{n} V_i$.

MC Error: $O(\frac{1}{\sqrt[3]{N}})$. $\quad (\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{N^{\frac{2}{3}}}})$.

Issues:

1. Generation of independent normal samples. (*).
   - Inverse - Transform Method.
   - Acceptance - Rejection Method.
   - Box-Muller Method
2. Improve convergence speed $O(\frac{\sigma_V}{\sqrt{n}})$ by reducing σ_V using Variance Reduction Method.

(*) → Based on generating independent samples from uniform dist. $U[0,1]$.

Goal: Obtain $V_1, V_2, \dots, V_N$ indep samples of $U[0,1]$.

<u>Linear Congruential Generation</u>

Choose x_0, a, c, m position integers $(x \equiv y \bmod (t)$ iff $t \mid (x-y))$.

$x_{i+1} = ax_i + c \pmod m$ [x_{i+1} is the residual of the division of ax_i+c by m]

$u_{i+1} = \frac{x_{i+1}}{m} \quad (0 \le u_{n+1} < 1)$.

Want $(a, m) = 1$.

(p.s. a be prime number).

<u>(Good) Features of Linear Congruential Generation</u>

1. Period has maximal length → generate $m-1$ samples without repetition
2. Fast → fewer operation.
3. Portable → obtain same sequence of numbers on different platforms.
4. Good randomness properties.

Inverse Transform Method

Let X be r.v. with $F(x)$ strictly increasing cumulative density. ($F: \mathbb{R} \to [0,1]$)

If $U \sim U([0,1])$. then $F^{-1}(U) \sim X$ (has the same distribution as).

$F^{-1}(x)$ is strictly increasing.

$P(F^{-1}(U) \le a) = P(U \le F(a)) = F(a) = P(X \le a)$.

Given $u_1, u_2, \cdots, u_N$ independent of $U([0,1])$.

Then $N^{-1}(u_1), N^{-1}(u_2), \cdots, N^{-1}(u_N)$ indep sample of Z.

where $N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{s^2}{2}} ds$. (Handout P68).

Acceptance - Rejection Method

Goal: Generate indep samples for a r.v. with pdf f.

Assume we know how to generate indep samples for r.v. with pdf g where there exists $c > 0$ such that $f(x) \le c \cdot g(x)$. $\forall x \in \mathbb{R}$.

Generate Normal Distribution from Double Exponential.

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. $g(x) = \frac{1}{2} e^{-|x|}$. $\left(\int_{-\infty}^{+\infty} \frac{1}{2} e^{-|x|} = 1\right)$.

$f(x)/g(x) = \sqrt{\frac{2}{\pi}} e^{|x| - \frac{x^2}{2}} \le \sqrt{\frac{2e}{\pi}}$.

$\sup_{x \in \mathbb{R}} e^{|x| - \frac{x^2}{2}} \underset{x=1}{=} e^{\frac{1}{2}} = \sqrt{e}$.

$G^{-1}(x) = \begin{cases} \ln(2u), & \text{if } 0 < u < \frac{1}{2} \\ -\ln(2(1-u)), & \text{if } \frac{1}{2} \le u < 1 \end{cases}$.

Generate a sample X from g; accept it with probability $\frac{f(x)}{cg(x)}$.

(implement by generating $U \sim U([0,1])$. and accept X if $U \le \frac{f(x)}{cg(x)}$).

Step 0: Generate $U_1, U_2 \sim U[0,1]$.

Step 1: Generate X from g. [$X = G^{-1}(U_1)$ Inverse Transform].

Step 2: if $U_2 \le \frac{f(x)}{cg(x)}$: return $Y = X$; else: go to step 0.

pdf of y is f

Another Version of Generating Normal Distribution.

Step 1: Generate $U_1, U_2, U_3 \sim U[0,1]$.

Step 2: $X = -\ln(U_1)$. $\quad$ (In $G^{-1}(x)$: $2(1-u) \sim U(0,1)$).

Step 3: if $U_2 > \exp(-\frac{1}{2}(x-1)^2)$, go to step 1.

else if $U_3 \le 0.5$, $X = -X$.

Return X.

$$\frac{f(x)}{cg(x)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big/ \sqrt{\frac{2e}{\pi}} \cdot \frac{1}{2} \cdot e^{-|x|} = \frac{1}{\sqrt{e}} e^{|x| - \frac{x^2}{2}} = e^{-\frac{1}{2} + |x| - \frac{x^2}{2}} = \exp(-\frac{1}{2}(|x|-1)^2).$$

Generate only positive samples of X (on Z).

Randomize the sign.

Box-Muller Method.

Goal: Generate a sample from the bivariate standard normal (2 indep standard normals).

Note: if Z_1, Z_2 are indep standard normals

(i) $R = Z_1^2 + Z_2^2$ is exponentially distributed with mean 2 $[\mathbb{P}(R \le x) = 1 - e^{-\frac{x}{2}}]$.

(ii) Given R, the point (Z_1, Z_2) is uniformly distributed on the circle $C(0, \sqrt{R})$.

Step 0: $U_1, U_2 \sim U[0,1]$.

($G(x) = 1 - e^{-\frac{x}{2}} = u$. $\quad G^{-1}(u) = 2\ln(1-u)$).

Step 1: $R = -2\ln(U_1)$.

Step 2: $\theta = 2\pi U_2 \Rightarrow Z_1 = \sqrt{R}\cos(\theta)$. $Z_2 = \sqrt{R}\sin(\theta)$.

Marsaglia - Bray Algorithm

$U_1, U_2 \sim U[-1,1]$. Select U_1, U_2 only if (U_1, U_2) are inside $D(0,1)$.

Then $X = U_1^2 + U_2^2 \sim U([0,1])$. $\quad$ (Then $(U_1, U_2) \sim$ Uniform $(D(0,1))$).

$$\underbrace{\mathbb{P}(a \le X \le b)}_{\substack{\parallel \leftarrow \text{Want} \\ b-a}} = \mathbb{P}(a \le U_1^2 + U_2^2 \le b).$$

$$= \iint_{D(0,1)} 1_{(a \le s^2+t^2 \le b)} \cdot \frac{ds\,dt}{\pi}. \quad = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} 1_{(a \le r^2 \le b)}\, r\, dr\, d\theta.$$

$s = r\cos\theta$, $t = r\sin\theta$. $ds\,dt = r\,dr\,d\theta$.

$$= \frac{1}{\pi}\cdot 2\pi \int_0^1 r \cdot 1_{(a \le r^2 \le b)}\, dr = 2\int_{\sqrt{a}}^{\sqrt{b}} r\,dr = b-a.$$

$R = -2\ln X$ (R exponential distrib mean 2).

$$\left[\begin{array}{ll} Z_1 = U_1 Y & \text{such that } Z_1^2 + Z_2^2 = R \Rightarrow Y^2 = \frac{R}{U_1^2+U_2^2} = \frac{-2\ln X}{X}. \\ Z_2 = U_2 Y. & \end{array}\right]$$

$Y = \sqrt{\frac{-2\ln X}{X}}$; $Z_1 = U_1 Y$; $Z_2 = U_2 Y$.

Greeks with MC for non-path dependent derivatives.

$S(T) = S(0)\exp\left((r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right)$.

Sample $Z_i \to S_i \to V_i = e^{-rT}V(S_i, T)$ $[= e^{-rT}\max(K-S_i, 0)]$.

Delta : $\Delta = \frac{\partial V}{\partial S(0)} = \frac{\partial V}{\partial S(T)} \cdot \frac{\partial S(T)}{\partial S(0)}$

$\frac{\partial S(T)}{\partial S(0)} = \exp[(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z]$.

$e^{-rT}\frac{\partial V(T)}{\partial S(T)}$: Put Option : $e^{-rT}(-1_{(S(T)<K)}) = -e^{-rT}\cdot 1_{(S_i<K)}$.

$\Rightarrow \Delta_i = -e^{-rT} 1_{\{S_i<K\}} \cdot \frac{S_i}{S_0}$

$\hat{\Delta}(n) = \frac{1}{n}\sum_{i=1}^{n} \Delta_i$.

Vega : $= \frac{\partial V}{\partial \sigma} = \frac{\partial V}{\partial S(T)} \cdot \frac{\partial S(T)}{\partial \sigma}$.

$\frac{\partial S(T)}{\partial \sigma} = (-\sigma T + \sqrt{T}Z)\cdot S(T)$.

$\text{vega}_i = -e^{-rT} 1_{(S_i<K)} \cdot (-\sigma T + \sqrt{T}Z_i)\, S_i$.

$\text{Vega}(N) = \frac{1}{N}\sum_{i=1}^{N} \text{vega}_i$.