MTH 9821 - L7

2017/10/19

- forward Euler:

for m = 1 : M-1

$u^m = \max(A u^{m-1} + b^m, \text{early\_ex}^m)$.

end.

- Backward Euler:

Implicit method : faster in pricipal but have to solve linear systems
$\Rightarrow$ Not as good as forward Euler.

for m = 0 : M-1

$A u^m = B \cdot u^{m-1} + b^m$

end. (A: spd; tridiagonal matrix).

~~u^m = linear_solve_cholesky (A, $Bu^{m-1}+b^m$)~~

V = cholesky (A).

y = forward_subst (V^t, $Bu^m + b^m$).

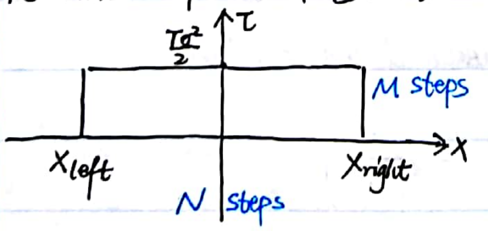
u^m = backward_subst (V, y).

Find u^m such that : $A u^m = B u^{m-1} + b^m$

and : $u^m \geq \text{early\_ex\_premium}^m$

Implicit Finite Difference Methods.

Reminder (forward Euler).



$x = \ln\left(\frac{S}{K}\right)$

$-3 < \frac{\ln\left(\frac{S}{S_0}\right)}{\sigma\sqrt{T}} < 3.$

$u_n^m = u(x_{left} + n\delta x, m\delta\tau) = u(x_n, \tau_m)$.

In Forward Euler:

$$\frac{\partial u}{\partial \tau}(x_n, \tau_m) = \frac{u(x_n, \tau_{m+1}) - u(x_n, \tau_m)}{\delta\tau} + O((\delta t)^2).$$

$$\frac{\partial^2 u}{\partial x^2}(x_n, \tau_m) = O((\delta x)^2) + \frac{u(x_{n+1}, \tau_m) - 2u(x_n, \tau_m) + u(x_{n-1}, \tau_m)}{(\delta x)^2}$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \qquad \alpha = \frac{\delta \tau}{(\delta x)^2}$$

By Trancation: $$\frac{u_n^{m+1} - u_n^m}{\delta t} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}$$

$$u_n^{m+1} = \alpha u_{n+1}^m + (1-2\alpha) u_n^m + \alpha \cdot u_{n-1}^m$$

Backward Euler

Same scheme, but different discretization of δx and δt.

$$\frac{u_n^m - u_n^{m-1}}{\delta t} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} \qquad \begin{cases} f'(x) = \dfrac{f(x+h) - f(x)}{h} \\ f'(x) = \dfrac{f(x) - f(x-h)}{h} \end{cases}$$

$$-\alpha u_{n+1}^m + (1+2\alpha) u_n^m - \alpha u_{n-1}^m = u_n^{m-1}$$

$(\forall n = 1:N-1 ; \quad m = 1:M)$.

$A u^m = b^m \quad (*)$.

$$u^m = \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix} \quad b^m = u^{m-1} + \begin{pmatrix} \alpha u_0^m \\ 0 \\ \vdots \\ 0 \\ \alpha u_N^m \end{pmatrix} \quad A = \begin{pmatrix} 1+2\alpha & -\alpha & \cdots & \\ -\alpha & 1+2\alpha & -\alpha & 0 \\ & \ddots & \ddots & \ddots \\ 0 & & & -\alpha \\ & & -\alpha & 1+2\alpha \end{pmatrix}$$

A : ① Symmetric. ② tridiagonal ③ positive definite
④ Strictly diagonally dominated.

$$(*) \Rightarrow \begin{pmatrix} 1+2\alpha & -\alpha & & 0 \\ -\alpha & 1+2\alpha & \ddots & \\ & \ddots & \ddots & -\alpha \\ 0 & & -\alpha & 1+2\alpha \end{pmatrix} \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix} = \begin{pmatrix} u_1^{m-1} + \alpha u_0^m \\ \vdots \\ u_{N-1}^{m-1} + \alpha u_N^m \end{pmatrix}$$

Solve using LU decomposition or Cholesky decomposition.

$$A = \left(\begin{array}{c|c} a & b^T \\ \hline b & c \end{array}\right) = \left(\begin{array}{c|c} U_{11} & 0 \\ \hline U_{21} & U_{22} \end{array}\right) \left(\begin{array}{c|c} U_{11} & U_{21}^T \\ \hline 0 & U_{22} \end{array}\right).$$

$a = U_{11}^2 \qquad b^T = U_{11} U_{21}^T \qquad b = U_{21} \cdot U_{11} \qquad C = U_{21} U_{21}^T + U_{22} U_{22}^T$

$U_{11} = \sqrt{a} \qquad U_{21} = b \cdot \frac{1}{U_{11}} \qquad C - U_{21} U_{21}^T = U_{22} U_{22}^T$

$$U_{21} U_{21}^T = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix} (* \ 0 \cdots 0) = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Solve the linear equation using Cholesky decomposition.

Since A is tridiagonal, C is also tridiagonal,

and by the above argument, so is $C - U_{21}U_{21}^T$

<u>Psuedo Code for Cholesky decomposition</u>

```
def cholesky
    ''' A is spd tridiagonal matrix out U is upper triangular
        bidiagonal s.t. U^tU=A '''
  for i = 1: (n-1)
      u(i,i) = sqrt(A(i,i))
      u(1,i+1) = A(1,i+1) / u(i,i)
      A(i+1, i+1) = A(i+1, i+1) - u(i, i+1)^2.
  end.
  u(n,n) = sqrt(A(n,n))
```

$Ax=b$. $\quad U^TUx=b$. $\quad U^Ty=b$ (forward substitution)

$\qquad\qquad\qquad\qquad\qquad Ux=y$ (backward substitution).

$u_{11}=a \quad U_{12}=b \quad l_{21}=\frac{1}{a}c \quad 0-l_{21}U_{12}=l_{22}U_{22}$

<u>Psuedo Code for LU decomposition.</u>

```
def lu_non_pivoting_tridiag (A):
    '''A is nonsingular tridiagonal matrix of size n with LU decomp returns (L,U) '''
   where L = lower triangular bidiagonal with is one the diagonal.
         U = upper triangular bidiagonal such that A= LU
```

for $i = 1 : (n-1)$:

$L(i,i) = 1$; $L(i+1,i) = A(i+1,i) / A(i,i)$

$U(i,i) = A(i,i)$; $U(i,i+1) = A(i,i+1)$.

$A(i+1,i+1) = A(i+1,i+1) - L(i+1,i)\, U(i,i+1)$.

end

$L(n,n) = 1$. $U(n,n) = A(n,n)$

operation count: $3n-3$

Crank-Nicolson [Average the forward euler, backward euler].

$$\frac{\partial u}{\partial \tau}(x_n, \tau_{m+\frac{1}{2}}) = \frac{u(x_n,\tau_{m+1}) - u(x_n,\tau_m)}{\delta\tau} + O((\delta\tau)^2).$$

$$\frac{\partial^2 u}{\partial x^2}(x_n, \tau_{m+\frac{1}{2}}) = \frac{1}{2}\Bigg[\underbrace{\frac{u(x_{n+1},\tau_{m+1}) - 2u(x_n,\tau_{m+1}) + u(x_{n-1},\tau_{m+1})}{(\delta x)^2}}_{\to \text{ Backward}} +$$

$$\underbrace{\frac{u(x_{n+1},\tau_m) - 2u(x_n,\tau_m) + u(x_{n-1},\tau_m)}{(\delta x)^2}}_{\to \text{ Forward}}\Bigg] + O((\delta x)^3).$$

$\Rightarrow \forall n = 1 : N-1,\ \forall m = 1 : M$

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{1}{2(\delta x)^2}\left[u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n+1}^m - 2u_{n+1}^m + u_{n-1}^m + u_{n-1}^{m+1}\right].$$

$$-\frac{\alpha}{2}u_{n+1}^{m+1} + (1+\alpha)u_n^{m+1} - \frac{\alpha}{2}u_{n-1}^{m+1} = \frac{\alpha}{2}u_{n+1}^m + (1-\alpha)u_n^m + \frac{\alpha}{2}u_{n-1}^m$$

$$\begin{pmatrix} 1+\alpha & -\frac{\alpha}{2} & \cdots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & \ddots & \\ & \ddots & \ddots & -\frac{\alpha}{2} \\ 0 & & -\frac{\alpha}{2} & 1+\alpha \end{pmatrix} \begin{pmatrix} u_1^{m+1} \\ u_2^{m+1} \\ \vdots \\ u_{n-1}^{m+1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2}u_2^m + (1-\alpha)u_1^m + \frac{\alpha}{2}u_0^m + \frac{\alpha}{2}u_0^{m+1} \\ \frac{\alpha}{2}u_3^m + (1-\alpha)u_2^m + \frac{\alpha}{2}u_1^m + \frac{\alpha}{2}u_2 \\ \vdots \\ \frac{\alpha}{2}u_N^m + (1-\alpha)u_{N-1}^m + \frac{\alpha}{2}u_{N-2}^m + \frac{\alpha}{2}u_N^{m+1} \end{pmatrix}$$

Black-Sholes inequality for American Options $\Rightarrow$ Transfered into heat equation.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S \cdot \frac{\partial C}{\partial S} \le rC. \qquad C \ge (S-K)^+$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \ge 0. \quad u(x,\tau) - g(x,\tau) \ge 0$$

$\to$ Early Exercise Value