MTH 9821 - L4

2017/09/19

Variance Reduction.

$\hat{V}(n) = \frac{1}{n}\sum_{i=1}^{n} V_i$. $Var(\hat{V}(n)) = \frac{1}{n} Var(V)$

Confidence Interval $\alpha \frac{1}{\sqrt{n}} std(V)$. ($\alpha$: some number, could be 1.96).

$O(\frac{1}{\sqrt{n}})$ Convergence.

Reference: Paul Glasserman. Monte Carlo Method in Financial Engineering
— Chapter 4.

Control Variate

$y_1, \dots, y_n$ i.i.d: outputs. $x_1, \dots, x_n$ i.i.d. different output.

$\tilde{y}_i = y_i - b(X_i - \mathbb{E}[X])$ for some fixed b.

$\hat{y}(n) = \frac{1}{n}\sum_{i=1}^{n} y_i$. $\hat{y}_{cv}(n) = \frac{1}{n}\sum_{i=1}^{n}\tilde{y}_i = \hat{y}(n) - b(\hat{X}(n) - \mathbb{E}[X])$.

Unbiased Estimator: $\mathbb{E}[\tilde{y}_{cv}(n)] = \mathbb{E}[y]$.

Example: $y_i = V_i = (K - S_i(T))^+$, $X_i = S_i(T)$.

$$V_{cv}(n) = \frac{1}{n}\sum_{i=1}^{n} V_i - b\Big(\frac{1}{n}\sum_{i=1}^{n} S_i(T) - \underbrace{e^{rT}S(0)}_{\mathbb{E}[S(T)]}\Big).$$

$y_{cv}(n) - \mathbb{E}[y] = \hat{y}(n) - \mathbb{E}[y] - b(\hat{X}(n) - \mathbb{E}[X])$.

Pick b, that minimize $var(y_{cv}(n))$.

$Var(y_{cv}(n)) = \frac{1}{n} Var(\tilde{y}_i)$.

$Var(\tilde{y}_i) = Var(y_i) - 2b\,cov(y_i, X_i - \mathbb{E}[X]) + b^2 Var(X_i)$.

$$b^* = \frac{cov(y_i, X_i)}{Var(X_i)} = \frac{\sigma_Y}{\sigma_X}\rho_{XY}.$$

(We try to estimate the mean of y).

$Var(\tilde{y}_i) = \sigma_Y^2(1 - \rho_{XY}^2)$.

Look for X very strongly positively / negatively correlated with Y.

$\rho_{XY} = 0.95 \Rightarrow Var(y_{cv}(n)) = \frac{1}{10} Var(\hat{y}(n))$.

Confidence Interval. $\alpha \frac{1}{\sqrt{n_{cv}}}\sqrt{Var(\tilde{y}_i)} = \alpha\frac{1}{\sqrt{n}}\sqrt{Var(y)}$.

$$n \cdot \frac{Var(\tilde{y}_i)}{Var(y_i)} = n_{cv}$$

In practice, we don't know b^*. Instead,

$$b(n) = \frac{\sum_{i=1}^{n} (x_i - \hat{x}_i)(y_i - \hat{y}_i)}{\sum_{i=1}^{n} (x_i - \hat{x}(n))^2}, \quad \tilde{y}_i = y_i - b(n)(X_i - \mathbb{E}[X]).$$

P.S.: $Var(V-bS) = Var(V) - 2b\,Cov(V,S) + b^2 Var(S)$.

$$b^* = \frac{cov(V,S)}{Var(S)} \quad (\text{Min variance}).$$

$$Y_{cv}(n) = \frac{1}{n}\sum_{i=1}^{n} \tilde{y}_i = \sum_{i=1}^{n} w_i y_i \qquad \sum_{i=1}^{n} w_i = 1. \qquad [\text{Weighted Monte Carlo}]$$

Tractable Option.

$$\tilde{V}_{barriar,i} = V_{barriar,i} - b(V_{Eur,i} - V_{BS})$$

Antithetic Variables

Reduce variance by introducing negative dependence between points of replication.

Generate: $U_1, U_2, \cdots, U_n$, i.i.d. $U([0,1])$.

$1-U_1, 1-U_2, \cdots, 1-U_n$.

$; \quad Z_{1,i} = F^{-1}(U_i). \quad F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$

$Z_{2,i} = F^{-1}(1-U_i) = -Z_{1,i}.$ identical distribution

$(X_1, Y_1), \cdots, (X_n, Y_n)$ are i.i.d. X_i, Y_i are i.d but not i.i.d.

$$Y_{AV}(n) = \frac{1}{n}\sum_{i=1}^{n} \frac{X_i + Y_i}{2} \qquad \text{Compare to } \hat{X}(2n) = \frac{1}{2n}\sum_{i=1}^{n} X_i.$$

$(X_i = f(Z_{1,i}). \quad Y_i = f(Z_{2,i})).$

$$Var(Y_{AV}(n)) = \frac{1}{4n} Var(X_i + Y_i) = \frac{1}{2n}(Var(X_i) + Cov(X_i, Y_i))$$

$$\frac{Var(Y_{AV}(n))}{Var(\hat{X}(2n))} = \frac{Var(X) + Var(X)\cdot \rho_{X_i,Y_i}}{Var(X)} = 1 + \rho_{X,Y}.$$

$$S_{i,1}(T) = S(0)\cdot e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T} Z_{1,i}}. \qquad S_{i,2}(T) = S(0)\cdot e^{(r-\frac{\sigma^2}{2})T - \sigma\sqrt{T} Z_{1,i}}.$$

$X_i = e^{-rT} V(S_{i,1}(T)) = e^{-rT}(S_{i,1}(T) - K)^+.$ (Example).

$Y_i = e^{-rT} V(S_{i,2}(T))$. Want $cov(X_i, Y_i) < 0$.

Thm: If $x_1, \cdots, x_n$ i.i.d. and $y = f(x_1, \cdots, x_n)$. $\tilde{y} = g(x_1, \cdots, x_n)$.

s.t. f increasing, g decreasing. Then $E[y\tilde{y}] \le E[y]E[\tilde{y}]$.

Moment Matching

$\hat{S}(n) = \frac{1}{n} \sum_{i=1}^{n} S_i(T)$. $\hat{S}(n) \neq E[S]$ (unless you're really lucky).

$\tilde{S}_i(T) = S_i(T) \cdot \frac{E[S(T)]}{\hat{S}(n)}$. or $\tilde{S}_i(T) = S_i(T) + E[S(T)] - \hat{S}(n)$.

$\hat{C}(n) = \frac{1}{n} \sum_{i=1}^{n} e^{-rT} \max(S_i - K, 0)$.

$\hat{P}(n) = \frac{1}{n} \sum_{i=1}^{n} e^{-rT} \max(K - S_i, 0)$

$C(0) - P(0) = S(0) - Ke^{-rT}$ (put-Call Parity) $= e^{-rT}(E[S(T)] - K)$.

$\hat{C}(n) - \hat{P}(n) = \frac{1}{n} e^{-rT} \sum_{i=1}^{n} (S_i - K) = e^{-rT}(\hat{S}(n) - K)$.

Put-Call Parity is satisfied iff $\hat{S}(n) = E[S(T)] = S(0)e^{rT}$.

Basket Call

$V(S_1(T), S_2(T)) = \max(S_1(T) + S_2(T) - K, 0)$ (S_1 and S_2: Two different stocks).

$dS_1 = (r - q_1)S_1 dt + \sigma_1 S_1 dX_1$. $\langle dX_1, dX_2 \rangle = \rho dt$.

$dS_2 = (r - q_2)S_2 dt + \sigma_2 S_2 dX_2$.

$S_1(T) = S(0) \cdot \exp((r - q_1 - \frac{\sigma_1^2}{2})T + \sigma_1 \sqrt{T} X_1)$. $S_2(T) = S(0) \exp((r - q_2 - \frac{\sigma_2^2}{2})T + \sigma_2 \sqrt{T} Z_2)$.

$\mathrm{corr}(Z_1, Z_2) = \rho$. $\Sigma_Z = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = U^T U$

$$\begin{pmatrix} \mathrm{Var}(Z_1) & \mathrm{Cov}(Z_1, Z_2) \\ \mathrm{Cov}(Z_1, Z_2) & \mathrm{Var}(Z_2) \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Rightarrow U^T = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = U^T \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \quad \begin{cases} Z_1 = \sigma_1 W_1 \\ Z_2 = \sigma_2(\rho W_1 + \sqrt{1-\rho^2} W_2). \end{cases}$$

W_1 and W_2: i.i.d standard normal.

$\Sigma_Z = U^T \Sigma_W U = U^T U$.

$S_1(t_{j+1}) = S_1(t_j)\exp[(r-q_1-\frac{\sigma_1^2}{2})\delta t + \sigma_1\sqrt{\delta t}\, Z_{2j+1}]$.

$S_2(t_{j+1}) = S_2(t_j)\exp[(r-q_2-\frac{\sigma_2^2}{2})\delta t + \sigma_2\sqrt{\delta t}\,(\rho Z_{2j+1} + \sqrt{1-\rho^2}\, Z_{2j+2})]$ for all $j \in [0, n-1]$.

Z_i : i.i.d : $N(0,1)$.

Heston Model

$dS(t) = \mu(t)S(t)dt + \sqrt{V(t)}\,S(t)\,dX_1$,

$dV(t) = -\lambda(V(t)-\bar{V})dt + \eta\sqrt{V(t)}\,dX_2$. CIR : Cox-Ingersoll-Ross Process

$corr(dX_1, dX_2) = \rho\, dt$.

$S(t_{j+1}) = S(t_j)\exp[(r-\frac{V(t_j)}{2})\delta t + \sqrt{V(t_j)}\sqrt{\delta t}\, Z_j^{(1)}]$.

$V(t_{j+1}) = V(t_j) - \lambda(V(t_j)-\bar{V})\delta t + \eta\sqrt{V(t_j)}\sqrt{\delta t}\,(\rho Z_j^{(1)} + \sqrt{1-\rho^2}\, Z_j^{(2)})$. $j=1:m$.

In the finite difference approximation (recursion) for $V(t)$, it's possible for $V(t)$ to become negative. To get around that, substitute:

$$V(t_j) \text{ by } \begin{cases} 0 & \text{(absorbing)} \\ -V(t_j) & \text{(reflecting)} \end{cases}$$

Greeks for Path-dependent options

$$V(T) = \max(\underbrace{\tfrac{1}{n}\sum_{i=1}^{n} S(t_j)}_{\bar{S}} - k, 0).$$

$$\frac{\partial V(0)}{\partial S(0)} = \frac{\partial V(0)}{\partial \bar{S}}\cdot\frac{\partial \bar{S}}{\partial S(0)} = \mathbb{1}_{\bar{S}>K}\, e^{-rT}\cdot\frac{\partial \bar{S}}{\partial S(0)} = e^{-rT}\,\mathbb{1}_{\bar{S}>K}\cdot\frac{\bar{S}}{S(0)} \quad \text{as } \bar{S} \text{ linear in } S(0).$$

$$\hat{V} = \frac{1}{n}\sum_{i=1}^{n}\max(\frac{1}{m}\sum_{j=1}^{m} S(t_j) - k, 0)\, e^{-rT}.$$

$$\frac{\partial V}{\partial \sigma} = \frac{\partial V}{\partial \bar{S}}\cdot\frac{\partial \bar{S}}{\partial \sigma}.$$

$$S(t_j) = S(0)\cdot\exp[(r-\frac{\sigma^2}{2})t_j + \sigma\sqrt{\delta t}\sum_{l=1}^{j} Z_l].$$

$$= S(t_{j-1})\cdot\exp[(r-\frac{\sigma^2}{2})\delta t + \sigma\sqrt{\delta t}\, Z_j].$$

$$\frac{\partial S(t_j)}{\partial \sigma} = S(t_j)\cdot(-\sigma t_j + \sqrt{\delta t}\sum_{l=1}^{j} Z_l)$$

$$= S(t_j)\cdot[-\sigma t_j + \frac{1}{\sigma}\ln(\frac{S(t_j)}{S(0)}) - (r-\frac{\sigma^2}{2})t_j].$$