MTH 9821 - L5

2017/10/05

Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)\cdot S\cdot\frac{\partial V}{\partial S} - rV = 0.$$

Properties of B-S PDE:

2° Parabolic PDE: $\frac{\partial u}{\partial t} - Lu = 0.$ (L: elliptic operater)*.

1° Linear PDE: $\left.\begin{array}{l} PDE(u)=0. \\ PDE(V)=0 \end{array}\right\} \Rightarrow PDE(C_1 u + C_2 V) = 0$ C_1, C_2: constant.

$L_{BS}(V_1) = 0 \Rightarrow L_{BS}(CV_1) = 0.$

$\left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} V\cdot\frac{\partial V}{\partial S} = 0\right.$: Not linear).

$L_{BS}(V_2) = 0 \Rightarrow L_{BS}(V_1 + V_2) = 0.$

*: Linear second order derivatives in $x_1, x_2, \cdots, x_n$.

3° Backward in time:

Boundary Conditions (for possibly unique solution).

$V(S,T)$ given (payoff at maturity). $\rightarrow$ Look for $V(S,0)$.

4°. Homogeneous Coefficient:

coeff. of $\frac{\partial^2 V}{\partial S^2}$ is $c\cdot S^2$. ; coeff of $\frac{\partial V}{\partial S}$ is $c\cdot S$.

Derivation of the BS PDE

Lognormal model for the evolution of the u.a.

$dS = (\mu - q)\cdot S\cdot dt + \sigma S\cdot dX$ dX: Wiener Process.

$V(S,t)$: value at t of a derivative security on an u.a with spot price S.

Set up a portfolio: Long 1 derivative security

short Δ units u.a.

$\Pi = V - \Delta S$: choose Δ such that the value of the portfolio does not depend on small changes in the value of the u.a.

$d\Pi = dV - \Delta\cdot dS - \Delta\cdot qS\,dt.$

$t \to t+dt$: 1 unit of u.a pays $qSdt$ dividents.

Itô's Lemma:

$$dV = \frac{\partial V}{\partial t}\cdot dt + \frac{\partial V}{\partial S}\cdot dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 , \qquad (dS)^2 = \sigma^2 S^2 (dX)^2 = \sigma^2 S^2 dt.$$

$$dV = \frac{\partial V}{\partial t} dt + \frac{\sigma^2 S^2}{2}\cdot\frac{\partial^2 V}{\partial S^2} dt + \frac{\partial V}{\partial S}\cdot dS.$$

Δ: The sensitivity of the derivative security's value w.r.t change in S.

[Not derivative!]. $\quad \Delta = \frac{\partial V}{\partial S}.$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\cdot dS - \Delta\cdot dS - \Delta q S dt.$$

$$= \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S}\right) dt.$$

For No-arbitrage:

$$d\Pi = r\Pi dt = r(V - \Delta S) dt = r\left(V - S\frac{\partial V}{\partial S}\right)\cdot dt.$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.$$

Financial Interpretation of the terms in BS formula.

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\Gamma + (r-q)S\Delta - rS = 0.$$

$$\frac{\partial V}{\partial t} = V(t+dt) - V(t) / dt. \quad = -\frac{\sigma^2 S^2}{2}\Gamma - (r-q)S\Delta + rV.$$

$$= r(V - \Delta S) + \Delta q S - \frac{\sigma^2 S^2}{2}\Gamma.$$

$$V(t+dt) - V(t) = r(V-\Delta S) dt + \Delta q S dt - \frac{\sigma^2 S^2}{2}\Gamma dt.$$

Replicate V by going long Δ units u.a., long $(V-\Delta S)$ cash position.

Change in value between t and $t+dt$:

For V: $V(t+dt) - V(t)$.

u.a: $\Delta S(t+dt) - \Delta S(t) + \Delta q S dt$.

cash: $r(V-\Delta S)dt$

$$V(t+dt) - V(t) = \underbrace{r(V-\Delta S)dt}_{A} + \underbrace{\Delta q S dt}_{B} \underbrace{- \frac{\sigma^2 S^2}{2}\Gamma dt}_{C}.$$

A: Interest on cash position in replicating portfolio.

B: dividerts on long Δ units u.a. position in replicating portfolio.

C: Slippage loss due to portfolio rebalancing

Solution of the Black-sholes PDE.

$V(S,T) = h(S)$ — Boundary condition. $\forall S>0$.

Find $V(S,0)$, for $S>0$.

Heat PDE: $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$. $\forall \tau>0$, $\forall x \in R$.

BC: $u(x,0) = f(x)$. $\forall x \in R$.

Heat kernal: pdf of $N(0,2\tau)$: $\frac{1}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}} = u_s(x,\tau)$.

Solution of Heat PDE is: $u(x,\tau) = u_s * f$ (convolution).

$(f*g)(x) = \int_R f(x-y) g(y) dy = \int_R f(y) g(x-y) dy$.

$u(x,\tau) = \int_R u_s(y,\tau) \cdot f(x-y) dy = \int_R u_s(x-y,\tau) f(y) dy$.

$= \frac{1}{\sqrt{4\pi\tau}} \int_R e^{-\frac{(x-y)^2}{4\tau}} f(y) dy$.

Change of variables:

$x = \ln(\frac{S}{K})$. $\tau = (T-t)\cdot\frac{\sigma^2}{2}$. $V(S,t) = w(x,\tau)$.

Given that $V(S,t)$ satisfies the BS PDE, what PDE does $w(x,\tau)$ satisfies.

$V(S,t) = w(x,\tau)$

$\frac{\partial V}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial w}{\partial \tau}$ (Chain Rule. & $\frac{\partial x}{\partial t}=0$).

$\frac{\partial V}{\partial S} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial S} + \frac{\partial w}{\partial \tau}\cdot\frac{\partial \tau}{\partial S} = \frac{1}{S}\frac{\partial w}{\partial x} = \frac{\partial V}{\partial S}$.

$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}[\frac{1}{S}\frac{\partial w}{\partial S}] = -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}(\frac{\partial w(x,\tau)}{\partial x})$

$= -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S}[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial S} + \frac{\partial^2 w}{\partial x \partial \tau}\cdot\overset{=0}{\frac{\partial \tau}{\partial S}}]$ (chain rule).

$= -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S^2}\frac{\partial^2 w}{\partial x^2}$.

$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)\cdot S\cdot\frac{\partial V}{\partial S} - r\cdot V = 0$.

$-\frac{\sigma^2}{2}\frac{\partial w}{\partial \tau} + \frac{\sigma^2 S^2}{2}\cdot\frac{1}{S^2}(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x}) + (r-q)S\cdot\frac{1}{S}\cdot\frac{\partial w}{\partial x} - rw = 0$.

$-\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + (\frac{2(r-q)}{\sigma^2} - 1)\frac{\partial w}{\partial x} - \frac{2r}{\sigma^2} w$. $\simeq (8.45)$.

Parabolic PDE: Constant coeff. & forward in time.

Then: Handout.

$a = \frac{(r-q)}{\sigma^2} - \frac{1}{2}$. $\quad b = \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}$.

$w(x,\tau) = \exp(-ax - b\tau)\, u(x,\tau)$.

$\Rightarrow \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0$. $\quad \forall x \in R, \quad \forall \tau > 0$.

$u(x,0) = f(x)$. $\forall x \in R$.

Know $V(S,T) = \max(K-S, 0)$. $\rightarrow$ find $f(x)$.

$V(S,t) = \exp(-ax - b\tau)\, u(x,\tau) \qquad t = T \Rightarrow \tau = 0$.

$V(S,T) = \exp(-ax)\, u(x,0)$, $\quad u(x,0) = (e^x)^a \max(K-S, 0)$.

$x = \ln\frac{S}{K} \Rightarrow e^x = \frac{S}{K}$, $S = K\cdot e^x$.

$u(x,0) = K\cdot e^{ax} \cdot \max(1-e^x, 0) = f(x)$.

From the general form of the solution of the heat PDE,

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_R e^{-\frac{(x-y)^2}{4\tau}} \cdot K \cdot e^{ay} \max(1-e^y, 0)\, dy$$

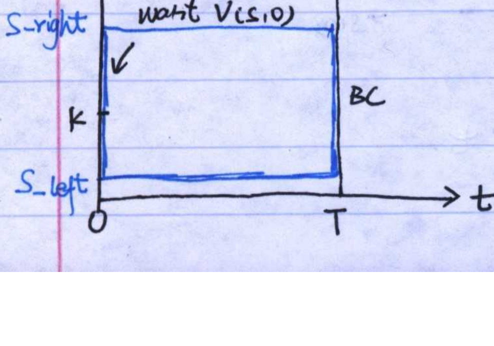
$$= \frac{K}{\sqrt{4\pi\tau}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\tau} + ay} (1-e^y)\, dy.$$

$$= \frac{K}{\sqrt{4\pi\tau}} \left[\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\tau} + ay} dy - \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\tau} + (a+1)y} dy \right].$$

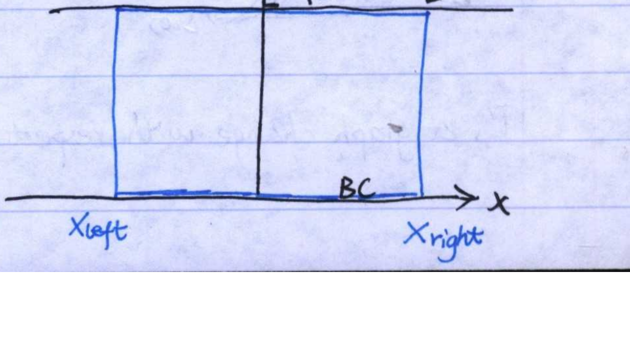
$$V(S,t) = \exp(-ax - b\tau) \cdot \frac{K}{\sqrt{4\pi\tau}} \cdot \left[\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\tau} + ay} dy - \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\tau} + (a+1)y} dy \right.$$

$x = \ln\left(\frac{S}{K}\right)$, $\tau = \frac{(T-t)\sigma^2}{2}$: $V(S,t) = Ke^{-r\tau} N(-d_2) - Se^{-q\tau} N(-d_1)$.

<Useful when pricing American options>.



$\rightarrow$



$t=0 \rightarrow \tau_{final} = \frac{\sigma^2}{2} T$

$V(S,T) = (K-S)^+$ for put $\qquad u(x,0) = Ke^{ax}(1-e^x)^+$

$\qquad (S-K)^+$ for call $\rightarrow \quad u(x,0) = Ke^{ax}(e^x-1)^+$

$S(T) = S(0)\exp((r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z)$

$S_{left} = S(0)\exp((r-q-\frac{\sigma^2}{2})T - 3\sigma\sqrt{T})$.

$S_{right} = S(0)\exp((r-q-\frac{\sigma^2}{2})T + 3\sigma\sqrt{T})$.

$X_{left} = \ln(\frac{S_{left}}{K}) = \ln(\frac{S(0)}{K}) + (r-q-\frac{\sigma^2}{2})T - 3\sigma\sqrt{T}$

$X_{right} = \ln(\frac{S_{right}}{K}) = \ln(\frac{S(0)}{K}) + (r-q-\frac{\sigma^2}{2})T + 3\sigma\sqrt{T}$.

Domain discretization in the (x,τ) space.

M nodes in the τ direction, $\delta\tau = \frac{\tau_{final}}{M}$; $\tau_i = i\cdot\delta\tau \quad i=0:M$.

N nodes in the x direction. $\alpha = \frac{\delta\tau}{(\delta x)^2}$ Courant Constant.

Start with α_{temp}, and M given.

$\delta\tau = \frac{\tau_{final}}{M}$. $\quad \delta x_{temp} = \sqrt{\frac{\delta\tau}{\alpha_{temp}}}$

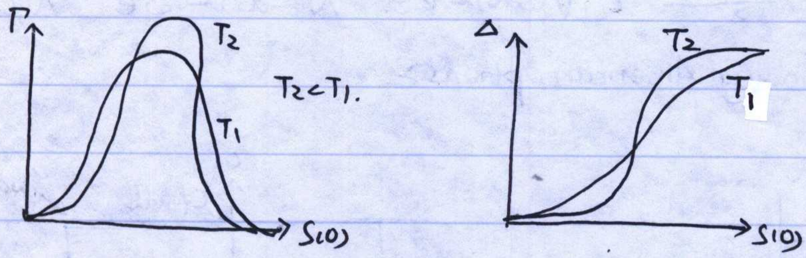
$N = \left(\frac{X_{right} - X_{left}}{\delta x_{temp}}\right) = 15.28$, Final α must be $\le \alpha_{temp}$.

floor

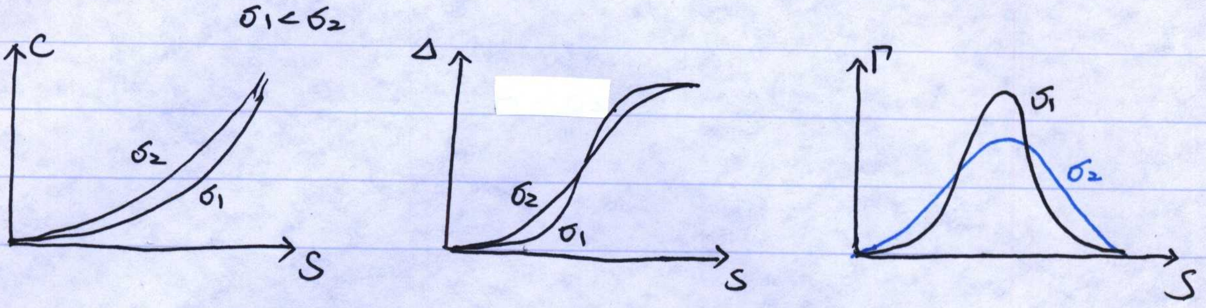
$(N=15)$ then $\delta x = \frac{X_{right} - X_{left}}{N}$.

$\alpha = \frac{\delta\tau}{(\delta x)^2}$, $\quad \alpha_{temp} = \frac{\delta\tau}{(\delta x_{temp})^2}$. $\quad \alpha \le \alpha_{temp}$

[$\delta x_{temp} < \delta x$, $N < N_{temp}$ (since floor). Sanity Check].



Γ, Δ graph change with respect to σ & t.



Higher σ: Higher Γ when OTM (deep), since change faster.
Lower Γ when ATM