MTH 9821 - L10

2017/11/16

Finite Difference Schemes for Parabolic PDEs

Steps & Notation

Definition: Consistency, Convergence, Stability, Discretization Order.

Lax Equivalence Theorem.

Von Neuman Analysis of stability of FD Schemes.

Forward Euler - Convergence Analysis - Von Neuman Analysis.

Von Neuman Analysis - Backward Euler - Crank-Nicolson.

Parabolic PDEs: $u_t = Lu$. L: second-order elliptic differential operator in x. $Lu = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

+ Boundary conditions.

FD Discretization: $U^{m+1} = AU^m + b^m$. $\forall m = 0:(M-1)$.

- M intervals of size Δt in the t-direction
- N intervals of size Δx in the x-direction

$$U^m = \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix} \quad \|U^m\|_{\Delta x} = \sqrt{\Delta x \sum_{n=1}^{N-1} (u_n^m)^2} = \sqrt{\Delta x}\, \|U^m\|_2.$$

$$\|g\|_{L^2([a,b])} = \sqrt{\int_a^b g^2(x)\,dx} = \sqrt{\lim_{N\to\infty} \sum_{i=1}^{N} \Delta x\, g^2(\xi_i)} \quad x_{i-1} < \xi_i < x_i$$

$$= \sqrt{\Delta x \cdot \lim_{N\to\infty} \sum_{i=1}^{N} g^2(\xi_i)} = \lim_{N\to\infty} \sqrt{\Delta x \cdot \sum_{i=1}^{N} |u_i^m|^2}$$

Let $\bar{u}(x,t)$ = exact solution of the PDE

$\delta_n^m = u_n^m - \bar{u}(x_n, t_m)$, where $x_n = x_{\text{left}} + n\Delta x$, $t_m = m\Delta t$. $\Delta^m = \begin{pmatrix} \delta_1^m \\ \vdots \\ \delta_{N-1}^m \end{pmatrix}$

FD Scheme is: **Consistent** $\lim_{\Delta x\to 0} \left(\max_{m=0:M,\, n=0:N} |\delta_n^m| \right) = 0$

↳ in synch with $\Delta t = O((\Delta x)^2)$.

Convergent $\lim_{\Delta x\to 0} \left(\max_{m=0:M} \|\Delta^m\|_{\Delta x} \right) = 0$.

Stable $\exists\, C > 0$ s.t. for any M $\|U^m\|_{\Delta x} \le C$, $\forall m = 0:M$.

$$\underline{\text{Order } P}: \quad \bar{U}^{m+1} = A\bar{U}^m + b^m + O((\Delta x)^{2+P}) \ ; \quad \bar{U}^m = \begin{pmatrix} \bar{u}(x_1, t_m) \\ \bar{u}(x_2, t_m) \\ \vdots \\ \bar{u}(x_{N-1}, t_m) \end{pmatrix} = U^m_{exact}$$

Lax Equivalence Theorem

FD Scheme Convergent $\Leftrightarrow$ the FD Scheme is stable and of order $p \geq 1$.

Convergence Analysis - Forward Euler.

$u_\tau = u_{xx}$. FE: $\dfrac{u_n^{m+1} - u_n^m}{\delta\tau} = \dfrac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}$

$$u_n^{m+1} = \alpha u_{n+1}^m + (1-2\alpha) u_n^m + \alpha u_{n-1}^m \quad ①$$

$$U^{m+1} = AU^m + b^m, \quad A = \begin{pmatrix} 1-2\alpha & \alpha & & 0 \\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \alpha \\ 0 & & \alpha & 1-2\alpha \end{pmatrix} \qquad b^m = \begin{pmatrix} \alpha u_0^m \\ 0 \\ \vdots \\ 0 \\ \alpha u_N^m \end{pmatrix}$$

$\bar{u}_\tau(\tau, x) = \bar{u}_{xx}(\tau, x)$. $\quad \bar{u}_\tau(\tau_m, x_n) = \bar{u}_{xx}(\tau_m, x_n)$

$$\frac{\bar{u}_\tau(\tau_{m+1}, x_n) - \bar{u}(\tau_m, x_n)}{\delta\tau} + O(\delta\tau) = \frac{\bar{u}(\tau_m, x_{n+1}) - 2\bar{u}(\tau_m, x_n) + \bar{u}(\tau_m, x_{n-1})}{(\delta x)^2} + O((\delta x)^2)$$

from ①: $\bar{u}(\tau_{m+1}, x_n) = \alpha\bar{u}(\tau_m, x_{n+1}) + (1-2\alpha)\bar{u}(\tau_m, x_n) + \alpha\bar{u}(\tau_m, x_{n-1}) + O((\delta x)^4)$. ②

$$\cancel{+ O((\delta x)^4) + \delta\tau \cdot O((\delta x)^2) - \delta\tau \cdot O((\delta x))}.$$

$\bar{U}^{m+1} = A\bar{U}^m + b^m + O((\delta x)^4)$. FX is discretization is of order 2.

①-②: $\delta_n^{m+1} = \alpha\delta_{n+1}^m + (1-2\alpha)\delta_n^m + \alpha\delta_{n-1}^m + V_n^m$, where $|V_n^m| \leq C(\Delta x)^4$

$$\Delta^{m+1} = A \cdot \Delta^m + \cancel{\begin{pmatrix} \alpha\delta_0^m \\ 0 \\ 0 \\ \alpha\delta_N^m \end{pmatrix}} + V^{m+1} \quad \text{where } V^{m+1} = \begin{pmatrix} V_1^{m+1} \\ \vdots \\ V_{N-1}^{m+1} \end{pmatrix}.$$

Goal: $\lim\limits_{M,N\to\infty} \left(\max\limits_{m=0:M} \|\Delta^m\|_{\Delta x} \right) = 0$.

$$\Delta^0 = \begin{pmatrix} \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \end{pmatrix} = 0. \qquad \Delta^1 = A\Delta^0 + V^1 = V^1 \ ; \quad \Delta^2 = A\Delta^1 + V^2 = AV^1 + V^2.$$

$$\Delta^3 = A\Delta^2 + V^3 = A^2V^1 + AV^2 + V^3 \quad \cdots$$

$$\Delta^M = A^{M-1}V^1 + A^{M-2}V^2 + \cdots + AV^{M-1} + V^M \quad \forall m = 1:M.$$

$$\|\Delta^m\|_{\Delta x} \leq \|A^{m-1}V^1\|_{\Delta x} + \|A^{m-2}V^2\|_{\Delta x} + \cdots + \|AV^{m-1}\|_{\Delta x} + \|V^m\|_{\Delta x}$$

$$\|A^{m-i}V^i\|_{\Delta x} = \sqrt{\Delta x}\,\|A^{m-i}V^i\|_2 \leq \sqrt{\Delta x} \cdot \|A^{m-i}\|_2 \cdot \|V^i\|_2 \leq \cancel{\sqrt{\Delta x}}$$

power notation

$$v^i = \begin{pmatrix} v_1^i \\ \vdots \\ v_{N-1}^i \end{pmatrix}. \text{ where } |v_j^i| \le C \cdot |\Delta x|^4. \quad \forall j = 1:N-1. \qquad \Delta x = \frac{x_{right} - x_{left}}{N}.$$

$$N-1 \simeq \frac{C_1}{\Delta x}.$$

$$\|v^i\|_2 = \sqrt{\sum_{j=1}^{N-1} |v_j^i|^2} \le \sqrt{(N-1)\cdot C^2 \cdot |\Delta x|^8} \simeq \sqrt{C_1 \cdot C^2 \cdot |\Delta x|^7} = \tilde{C}\cdot(\Delta x)^{\frac{7}{2}}$$

$$\Rightarrow \|A^{m-i} v^i\|_{\Delta x} \le \sqrt{\Delta x} \cdot \|A\|_2^{m-i} \cdot \tilde{C}(\delta x)^{7/2} = \|A\|_2^{m-i} \cdot \tilde{C}(\delta x)^4. \qquad \delta x = \Delta x.$$

$$\Rightarrow \|\Delta^m\|_{\Delta x} \le \sum_{i=1}^{m} \|A\|_2^{m-i} \cdot \tilde{C} \cdot (\delta x)^4 = \tilde{C}(\delta x)^4 \cdot \frac{1 - \|A\|_2^m}{1 - \|A\|_2}$$

$$\|\Delta^m\|_{\Delta x} \le \tilde{C}(\delta x)^4 \cdot \frac{1 - \|A\|_2^m}{1 - \|A\|_2}, \qquad \frac{(\|A\|_2)^M}{M^2} \xrightarrow[(\|A\|_2 > 1)]{M\to\infty} \infty$$

If $\|A\|_2 < 1$, then $\max_{m=0:M} \|\Delta^m\|_{\Delta x} \le \tilde{C}(\delta x)^4 \cdot \frac{1}{1-\|A\|_2} \xrightarrow{N\to\infty\,(\delta x\to 0)} 0$

FE is convergent.

If $\|A\|_2 > 1$, the upper bound is on the order of $\frac{\|A\|_2^M}{M^2} \xrightarrow{M\to\infty} \infty$ (useless bound).

FE is not convergent.

If $\|A\|_2 = 1$, then $1 + \|A\|_2 + \cdots + \|A\|_2^{m-1} = m$.

$$\max_{m=0:M}(\|\Delta^m\|_{\Delta x}) \le \tilde{C} m \cdot (\delta x)^4 \le \tilde{C} \cdot M \cdot (\delta x)^4 \simeq \tilde{C} N^2 \cdot (\delta x)^4 = \tilde{C}(N\delta x)^2 \cdot \delta x^2$$

$$\simeq C_3 \cdot \delta x^2 \xrightarrow{\delta x \to 0} 0 \qquad (N\delta x = x_{right} - x_{left}).$$

FE Convergent $\Leftrightarrow \|A\|_2 \le 1$.

A symmetric $\Rightarrow \|A\|_2 = \max|\lambda|$. ($\lambda$: evalue of A).

$$A = I - \alpha \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \quad \text{evalue } \lambda_j = 2\left(1 - \cos\frac{\pi j}{N}\right), \ j = 1:(N-1).$$

evalues of A: $\mu_j = 1 - 4\alpha \sin^2\left(\frac{\pi j}{2N}\right)$. $j = 1:(N-1)$.

$$\mu_{N-1} < \cdots < \mu_1 = 1 - 4\alpha\sin^2\left(\frac{\pi}{2N}\right) < 1.$$

$$\mu_{N-1} = 1 - 4\alpha\sin^2\left(\frac{\pi(N-1)}{2N}\right) > 1 - 4\alpha$$

$$\|A\|_2 = \max(|1-4\alpha|, \mu_1) \le 1 \Leftrightarrow 1 - 4\alpha \ge -1, \ \alpha \le \frac{1}{2}.$$

Fourien / Von Neuman Analysis

Substitute $(g(\theta))^m e^{in\theta}$ for u_n^m in the FD Scheme

FD scheme is stable $\Leftrightarrow |g(\theta)| \le 1$

$$u_n^{m+1} = \alpha u_{n+1}^m + (1-2\alpha) u_n^m + \alpha u_{n-1}^m$$

$$(g(\theta))^{m+1} e^{im\theta} = \alpha (g(\theta))^m e^{i(n+1)\theta} + (1-2\alpha)(g(\theta))^m e^{in\theta} + \alpha (g(\theta))^m e^{i(n-1)\theta}$$

$$e^{i\theta} - e^{-i\theta} = 2\cos\theta$$

$$g(\theta) = \alpha e^{i\theta} + 1 - 2\alpha + \alpha e^{-i\theta} = 1 - 2\alpha + \alpha (e^{i\theta} - e^{-i\theta}) = 1 - 2\alpha(1-\cos\theta).$$

$1 - 4\alpha \le g(\theta) \le 1$. $\quad |g(\theta)| \le 1 \iff 1 - 4\alpha \ge -1$. $\alpha \le \frac{1}{2}$.

FE stable $\iff \alpha \le \frac{1}{2}$. FE order 2. [LE Thm: FE convergent $\iff \alpha \le \frac{1}{2}$].

Backward Euler $\Rightarrow$ Order 2 Method.

$$\frac{u_n^{m+1} - u_n^m}{\delta t} = \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2} \qquad g(\theta)^m e^{i\theta n} \leftrightarrow u_n^m.$$

$$-\alpha u_{n+1}^{m+1} + (2\alpha+1) u_n^{m+1} - \alpha u_{n-1}^{m+1} = u_n^m$$

$$-\alpha (g(\theta))^{m+1} e^{i(n+1)\theta} + (2\alpha+1)(g(\theta))^{m+1} e^{in\theta} - \alpha (g(\theta))^{m+1} e^{i(n-1)\theta} = (g(\theta))^m e^{in\theta}.$$

$g(\theta)[-\alpha e^{i\theta} + 2\alpha + 1 - \alpha e^{-i\theta}] = 1$. $\quad g(\theta)(1 + 2\alpha - 2\alpha\cos\theta) = 1$.

$$g(\theta) = \frac{1}{1 + 2\alpha(1-\cos\theta)} \le 1.$$

$\Rightarrow$ B.E is stable for all $\alpha > 0$

LE Thm: B.E is convergent for all $\alpha > 0$.

Crank-Nicolson

$$\frac{u_n^{m+1} - u_n^m}{\delta t} = \frac{1}{2}\left[\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}\right]$$

$$-\frac{\alpha}{2} u_{n+1}^{m+1} + (1+\alpha) u_n^{m+1} - \frac{\alpha}{2} u_{n-1}^{m+1} = \frac{\alpha}{2} u_{n+1}^m + (1-\alpha) u_n^m + \frac{\alpha}{2} u_{n-1}^m \quad \Big|: (g(\theta))^m e^{in\theta}$$

$$-\frac{\alpha}{2} g(\theta) e^{i\theta} + (1+\alpha) g(\theta) - \frac{\alpha}{2} g(\theta) e^{-i\theta} = \frac{\alpha}{2} e^{i\theta} + 1 - \alpha + \frac{\alpha}{2} e^{-i\theta}.$$

$$g(\theta)\left[1 + \alpha - \frac{\alpha}{2}(e^{i\theta} + e^{-i\theta})\right] = 1 - \alpha + \frac{\alpha}{2}(e^{i\theta} + e^{-i\theta})].$$

$$g(\theta) = \frac{1 - \alpha + \alpha\cos\theta}{1 + \alpha - \alpha\cos\theta} = \frac{1 - \alpha(1-\cos\theta)}{1 + \alpha(1-\cos\theta)} \le 1.$$

$|g(\theta)| \le 1$ for all α. $\Rightarrow$ C-N stable

LE Thm $\Rightarrow$ CN convergent. (CN order 4 scheme).

Black-Scholes PDE for American Options.

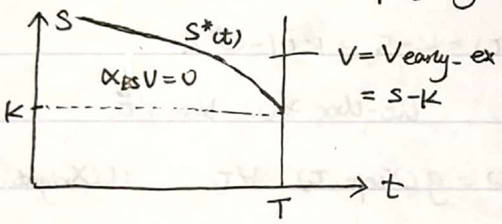
~~European~~ American Option. $\pi = V - \Delta S$. → If do not exercise the option over dt time period.

$d\pi = (\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S})\, dt$. $\cancel{d\pi = r\pi dt}$.

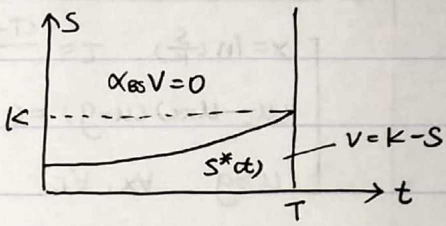
$\leq r\pi dt$.

$\alpha_{BS} V = \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV$. $\alpha_{BS} V \leq 0$. $V(S,t) \geq V_{early\text{-}ex}(S,t)$

Am Call on divident-paying asset.



Am Put



$\frac{\partial V}{\partial S}(S^*(t), t) = 1$.

$\frac{\partial V}{\partial S}(S^*(t), t) = -1$.

Taylor Approximation:

$P(S^*(t)+dS, t) = P(S^*(t), t) + dS \cdot \frac{\partial P}{\partial S}(S^*(t), t) + O((dS)^2)$.

$P(S^*(t)+dS, t) = K - S^*(t) + dS \cdot \frac{\partial P}{\partial S}(S^*(t), t) + O((dS)^2)$. $(P(S^*(t)+dS, t) > K - S^*(t) - dS)$

$-dS < dS \cdot \frac{\partial P}{\partial S}(S^*(t), t) + O((dS)^2)$.

$\Rightarrow \frac{\partial P}{\partial S}(S^*(t), t) \geq -1$.

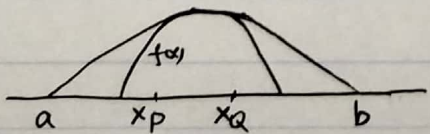
Free Boundary PDE Formulation. (For Put).

$\mathcal{L}_{BS} V = 0$, $\forall\, 0 < t < T$. $\forall\, S^*(t) < S$.

$V(S,t) = K - S$. $\forall\, 0 < t < T$. $\forall\, 0 < S \leq S^*(t)$.

$\frac{\partial V}{\partial S}(S^*(t), t) = -1$. $\forall\, 0 < t < T$.

Obstacle Pattern.



$u \geq f$. $u'' \leq 0$. $u(a) = u(b) = 0$.

$u(x) = f(x)$. $\forall\, x_P \leq x \leq x_Q$, $u''(x) = 0$. $\forall\, a < x < x_P$.

$u''(x) = 0$. $\forall\, x_Q < x < b$. $u'(x_P) = f''(x_P)$.

$u'(x_Q) = f''(x_Q)$.

Linear Complementarity Formulation.

$\mathcal{L}_{BS} V \leq 0. \quad \forall\ 0<t<T, \ \forall\ 0<S.$

$V(S,t) \geq K-S. \ \forall\ 0<t<T. \ \forall\ 0<S.$

$\mathcal{L}_{BS} V \cdot (V(S,t) - (K-S)) = 0. \ \forall\ 0<t<T. \ \forall\ 0<S.$

$\mathcal{L}_{BS}$: BS Operator.

$\mathcal{L}_{BS} V = \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S \cdot \frac{\partial V}{\partial S} - rV.$

Variational Formulation.

$$\left[\begin{array}{lll} x = \ln(\frac{S}{K}). \ \tau = \frac{(T-t)\sigma^2}{2}, & g(x,\tau) = K-S = K(1-e^x). & \\ (u_\tau - u_{xx})(u-g) = 0. \ \forall x, \forall \tau. & u_\tau - u_{xx} \geq 0. \ \forall x, \forall \tau. & \\ u \geq g. \ \forall x, \forall \tau. & u(x_{left},\tau) = g(x_{left},\tau), \forall \tau. & u(x_{right},\tau) = 0. \ \forall \tau. \end{array}\right]$$

u, u_x continuous

Let $\mathcal{S} = \{ Q : [x_{left}, x_{right}] \times [0, \tau_{final}] \to \mathbb{R}, \ Q_\tau$ cont. Q_x piecewise cont.$\}$.

such that $Q \geq g$. $Q(x,0) = g(x,0)$. $Q(x_{left},\tau) = g(x_{left},\tau)$.

$Q(x_{right},\tau) = 0.$

Let $a : \mathcal{S} \times \mathcal{S} \to \mathbb{R}.$

$$a(Q_1, Q_2) = \int_0^{\tau_{final}} \int_{x_{left}}^{x_{right}} (Q_1)_\tau \cdot Q_2 + (Q_1)_x (Q_2)_x \, dx d\tau.$$

Find $u \in \mathcal{S}$ solution for $a(u,Q) \geq a(u,u)$. $\forall Q \in \mathcal{S}$.

If u sol of free Boundary formulation $\Rightarrow$ u sol to Linear Comp $\Rightarrow$ u sol var form.

Theorem: There exists an unique solution to the Var form problem

This solution is the unique sol to the linear comp problem.

and is the unique sol to the free boundary PDE.