MTH 9821 - L6

2017/10/12

Interview: ① Tell ~~you~~ sth. about yourself: Sth. relevant to the job requirement

"... make me prepared for ...".

② Tell me what you're proud of: Sth. that makes you really good.

For EY: Not much brain teasers.

Finite Difference Hedging and Valuation for European & American Options.

$V(S,t)$ Satisfies the B-S PDE.

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)\cdot S\cdot\frac{\partial V}{\partial S} - rV = 0.\quad \forall S>0,\ \forall\, 0<t<T.$$

Boundary condition: $V(S,T)$ = option payoff : puts $\max(K-S,0)$. call: $\max(S-K,0)$

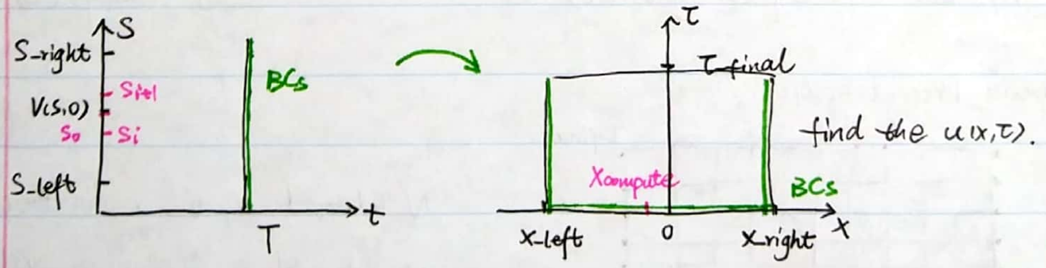
Change of variable:

$$x = \ln\left(\frac{S}{K}\right),\quad \tau = \frac{(T-t)\sigma^2}{2},\quad V(S,t) = \exp(-ax-b\tau)\, u(x,\tau).$$

$$a = \frac{r-q}{\sigma^2} - \frac{1}{2},\quad b = \left(\frac{r-q}{\sigma^2}+\frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}.$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},\quad \forall x\in\mathbb{R},\quad \forall\, 0<\tau<\tau_{final},\quad \tau_{final} = \frac{\sigma^2 T}{2}$$

Boundary conditions: $\begin{cases} Ke^{ax}\cdot\max(1-e^x,0), & \text{puts} \\ Ke^{ax}\cdot\max(e^x-1,0), & \text{calls} \end{cases} = u(x,0).$



$$S(T) = S(0)\cdot\exp\left[\left(r-q-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right].$$

$$S\_left = S(0)\cdot\exp\left[\left(r-q-\frac{\sigma^2}{2}\right)T - 3\sigma\sqrt{T}\right] \Rightarrow x_{left} = \ln\left[\frac{S(0)}{K}\right] + \left(r-q-\frac{\sigma^2}{2}\right)T - 3\sigma\sqrt{T}$$

$$S\_right = S(0)\cdot\exp\left[\left(r-q-\frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}\right] \Rightarrow x_{right} = \ln\left[\frac{S(0)}{K}\right] + \left(r-q-\frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}.$$

Put options Boundary Condition:

$u(x,0) = Ke^{ax}\max(1-e^{x},0)$, for $x \in \mathbb{R}$.

As $s \to \infty$: $P(S,t) \to 0$.

As $s \to 0$: $P(S,t) \to Ke^{-r(T-t)} - Se^{-q(T-t)}$.

[Put-Call parity: $C(t) - P(t) = Se^{-q(T-t)} - Ke^{-r(T-t)}$.

$u(X_{left},\tau) = g_{left}(\tau)$ $\quad u(X_{right},\tau) = g_{right}(\tau)$. [Want].

$u(X_{left},\tau) = [Ke^{-r\frac{2\tau}{\sigma^2}} - Ke^{X_{left}} \cdot e^{-q\frac{2\tau}{\sigma^2}}]\exp(aX_{left} + b\tau)$.

$(T-t = \frac{2\tau}{\sigma^2},\quad S_{left} = Ke^{X_{left}})$.

$g_{left}(\tau) = K\cdot\exp(ax_{left}+b\tau)\,[e^{-\frac{2r\tau}{\sigma^2}} - e^{X_{left}-\frac{2q\tau}{\sigma^2}}]$; $g_{right}(\tau) = 0$.

Call options Boundary Condition.

$u(x,0) = Ke^{ax}\max(e^{x}-1,0)$, for $x \in \mathbb{R}$.

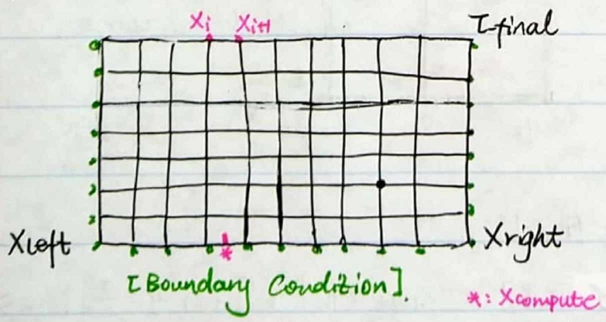
As $s \to 0$: $C(S,t) \to 0 \Rightarrow C(S_{left},t) = 0 \Rightarrow u(X_{left},\tau) = 0 \Rightarrow g_{left}(\tau) = 0$.

As $s \to \infty$: $C(S,t) \simeq S_{right}\,e^{-q(T-t)} - Ke^{-r(T-t)}$

$= \exp(-aX_{right} - b\tau)\,u(X_{right},\tau)$.

$g_{right}(\tau) = K\cdot\exp(aX_{right}+b\tau)\,[e^{X_{right}-\frac{2q\tau}{\sigma^2}} - e^{-\frac{2r\tau}{\sigma^2}}]$; $g_{left}(\tau) = 0$.

Domain Discretization.



N intervals in the x direction.

M intervals in the y direction.

$\delta x = \frac{X_{right}-X_{left}}{N}$. $\quad \delta\tau = \frac{T_{final}}{M}$.

$\alpha = \delta\tau/(\delta x)^2$ constant.

Start with α_{temp} and M, $\delta\tau = \frac{T_{final}}{M}$. $\quad \delta X_{temp} = \sqrt{\frac{\delta\tau}{\alpha_{temp}}}$.

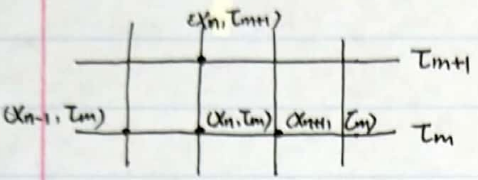
$N = \text{floor}\left(\frac{X_{right} - X_{left}}{\delta X_{temp}}\right)$. $\quad \delta X = \frac{X_{right}-X_{left}}{N}$

$\alpha = \frac{\delta\tau}{(\delta X)^2}$. $\quad$ Note $\alpha \le \alpha_{temp}$.

(X_n, τ_m): $X_n = X_{left} + n\delta X$ $(n = 0{:}N)$. $\tau_m = m\cdot\delta\tau$ (m in $0{:}M$).

Finite Difference approximation of the heat PDE.

$$\frac{\partial u}{\partial \tau}(x,\tau_m) = \frac{\partial^2 u}{\partial x^2}(x_n, \tau_m), \quad \forall\ x_{left} < x < x_{right}, \quad \forall\ 0 < \tau < \tau_{final}$$



[Forward Euler. explicit FD method].

Forward FD approx: $\frac{\partial u}{\partial \tau}(x_n, \tau_m) = \frac{u(x_n, \tau_{m+1}) - u(x_n, \tau_m)}{\delta\tau} + O(\delta\tau)$.

Central FD approx: $\frac{\partial^2 u}{\partial x^2}(x_n, \tau_m) = \frac{u(x_{n+1}, \tau_m) - 2u(x_n, \tau_m) + u(x_{n-1}, \tau_m)}{(\delta x)^2} + O(\delta x)^2$.

$$\frac{u(x_n, \tau_{m+1}) - u(x_n, \tau_m)}{\delta\tau} + O(\delta\tau) = \frac{u(x_{n+1}, \tau_m) - 2u(x_n, \tau_m) + u(x_{n-1}, \tau_m)}{(\delta x)^2} + O((\delta x)^2).$$

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}$$ keep $\alpha = \frac{\delta\tau}{(\delta x)^2}$ constant | courant

$$u_n^{m+1} - u_n^m = \alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m).$$

$$u_n^{m+1} = \alpha u_{n-1}^m + (1-2\alpha)u_n^m + \alpha u_{n+1}^m \quad \forall n = 1:(N-1).$$

For $m = 0:(M-1)$, $\downarrow$, end.

For $m = 0:(M-1)$:

$$U^{m+1} = A \cdot U^m + \underbrace{\begin{pmatrix} \alpha u_0^m \\ 0 \\ \vdots \\ 0 \\ \alpha U_N^m \end{pmatrix}}_{N-3} \qquad A = \begin{bmatrix} 1-2\alpha & \alpha & & 0 \\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \alpha \\ 0 & & \alpha & 1-2\alpha \end{bmatrix}$$

End.

$$U^m = \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix} \qquad u_0^m = g_{left}(m\delta\tau). \qquad U^{m+1} = A \cdot U^m + \alpha \begin{pmatrix} g_{left}(m\delta\tau) \\ 0 \\ \vdots \\ 0 \\ g_{right}(m\delta\tau) \end{pmatrix}$$

$$u_N^m = g_{right}(m\delta\tau).$$

Then, go back to the original domain. $x_{compute} = \ln(\frac{S_0}{K})$.

Let x_i and x_{i+1} such that $x_i \le x_{compute} \le x_{i+1}$ $[S_i = Ke^{x_i};\ S_{i+1} = Ke^{x_{i+1}}]$.

Found u_i and u_{i+1} finite difference approximate solution for

$u(x_i, \tau_{final})$ and $u(x_{i+1}, \tau_{final})$.

approx of $V(S_i, 0) \leftarrow V_i = \exp(-ax_i - b\tau_{final})\, u_i^M$.

approx of $V(S_{i+1}, 0) \leftarrow V_{i+1} = \exp(-ax_{i+1} - b\tau_{final})\, u_{i+1}^M$.

$$V_{approx}(S_0, 0) = \frac{V_i(S_{i+1} - S_0) + V_{i+1}(S_0 - S_i)}{S_{i+1} - S_i}$$

[Linear Interpolation $f(x) = \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a)$].

For Greeks Computation:

$$\Delta_{approx}(S_0, 0) = \frac{V_{i+1} - V_i}{S_{i+1} - S_i}. \quad (\Delta_{fd}).$$

p.s.: The domain discretization is uniform in x, but not in S.

$$\Gamma_{fd} = \frac{\frac{V_{i+2} - V_{i+1}}{S_{i+2} - S_{i+1}} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}}{\frac{S_{i+2} + S_{i+1}}{2} - \frac{S_i + S_{i-1}}{2}}$$

For American Put Option:

$V(S,t) \geq K - S, \quad \forall S > 0, \quad \forall\, 0 < t < T. \qquad [K - S = K(1 - e^x)]$.

$V(S,t) = \exp(-ax - b\tau)\, u(x,\tau). \qquad u(x,\tau) \geq K \exp(ax + b\tau)(1 - e^x)$.

for $m = 0 : M-1$;

$$U^{m+1} = \max\left(AU^m + \begin{pmatrix} \alpha U_0^m \\ 0 \\ \vdots \\ 0 \\ \alpha U_N^m \end{pmatrix}, \; (\text{early-ex-premium}(x_n, \tau_{m+1}))_{m=0:N-1} \right)$$

End.

Implied Volatility:

Given V_{market}, find σ such that $V_{numerical}(\sigma) = V_{market}$.

$f(\sigma) = V_{numerical}(\sigma) - V_{market}$.

Start with σ_0, σ_1 until convergence

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)(\sigma_n - \sigma_{n-1})}{f(\sigma_n) - f(\sigma_{n-1})}$$

End