MTH 9821 - L9

2017/11/02

Example: 1D parabolic ODE

$-u''(x) = f(x). \quad \forall x \in (0,1). \quad u(0) = u(1) = 0$

$h = \frac{1}{N+1}$

u_i = finite difference approximation to $u(x_i)$

FD Scheme:

$$-\left[\frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2} + O(h^2)\right] = f(x_i) \quad \forall\ i = 1:n.$$

$$-\left[\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + \cancel{O(h^2)}\right] = f(x_i).$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i) \quad \forall\ i = 1:n$$

$$T_N U_N = b_N: \quad T_N = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \quad U_N = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad b_N = h^2 \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} + \underbrace{\begin{pmatrix} u(0) \\ 0 \\ \vdots \\ u(1) \end{pmatrix}}_{=0}$$

Using Jacobi, GS, SOR:

<u>Jacobi</u>: $(A = L_A + D + U_A)$

$$x_{n+1} = -D^{-1}(L_A + U_A)x_n + D^{-1}b$$

$$x_{n+1} = R_J x_n + C_J. \quad \text{where } R_J = -D^{-1}(L_A + U_A)$$

<u>GS</u>: $A = L_A + D + U_A$

$$x_{n+1} = -(L_A + D)^{-1} U_A x_n + (L_A + D)^{-1} b$$

$$x_{n+1} = R_{GS} x_n + C_{GS}. \quad \text{where } R_{GS} = -(L_A + D)^{-1} U_A.$$

<u>SOR</u>: $x_{n+1} = R_{SOR} x_n + C_{SOR}$ where $R_{SOR} = -(D + wL_A)^{-1}((1-w)D + wU_A)$

$0 < w < 2$; In fact $1 < w < 2$ for SOR.

Convergence Properties:

- A spd $\Rightarrow$ SOR & GS convergent.
- A strictly diagonally dominate $\Rightarrow$ Jacobi & GS convergent

- A irreducable & weakly diagonal dominate $\Rightarrow$ Jacobi & GS convergent. GS is likely to need fewer iterations to converge than Jacobi for most matrix. (and $\rho(R_{GS}) \le \rho(R_J) < 1$) radius.

Solve $Ax = b$ $\Leftrightarrow$ Solve $\begin{cases} Ax_1 = b_1 \\ Ax_2 = b_2 \end{cases}$ This is reducible Matrix. $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$

Def: A is irreducible matrix if you can not find a permutation matrix P, s.t. $P^T A P$ is block diagonal

- A is consistently ordered (e.g: comes from a Red-Black ordering) then $\rho(R_{GS}) = \rho(R_J)^2$

Let $w_{opt} = 2/1+\sqrt{1-\rho(R_J)^2}$: then $\rho(R_{SOR}) = w_{opt} - 1$

Consistently ordered:

The graph associated to the matrix A look like this:

B	R	B	R
R	B	R	B
B	R	B	R
R	B	R	B

Evalues of T_N:

$\lambda_j = 2(1-\cos(\frac{\pi j}{N+1})) = 4\sin^2(\frac{\pi j}{2(N+1)})$

Note: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N < 4$; $\lambda_1 = 4\sin^2(\frac{\pi}{2(N+1)}) \approx 4(\frac{\pi}{2(N+1)})^2 = O(\frac{1}{N^2})$

$R_J = -D^{-1}(L_A + U_A)$.

$$= -\frac{1}{2} I \cdot \begin{pmatrix} 0 & -1 & \cdots & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{2} \\ & & \frac{1}{2} & 0 \end{pmatrix} = I - \frac{T_N}{2} \Rightarrow R_J = I - \frac{T_N}{2}$$

Then the eigenvalues of R_J are: $\mu_j = 1 - \frac{\lambda_j}{2}$

$\mu_j = \cos(\frac{\pi j}{N+1})$. $j = 1:N$. $-1 < \mu_N < \mu_{N-1} \cdots < \mu_1 < 1$.

$\rho(R_j) = \max(\mu_1, |\mu_N|) = \mu_1 = \cos(\frac{\pi}{N+1})$

$(\mu_N = \cos(\frac{N\pi}{N+1}) = -\cos(\frac{\pi}{N+1}) = -\mu_1)$

$$\rho(R_J) \approx 1-\frac{1}{2}\frac{\pi^2}{(N+1)^2} \approx 1-\frac{\pi^2}{2N^2}.$$

$$\rho(R_{GS}) = (\rho(R_J))^2 = (1-\frac{\pi^2}{2N^2})^2 \approx 1-\frac{\pi^2}{N^2}$$

$$w_{opt} = 2/1+\sqrt{1-(\rho(R_J))^2} = 2/1+\sqrt{1-(1-\frac{\pi^2}{N^2})} = \frac{2}{1+\frac{\pi}{N}}$$

$$\rho(R_{SOR}) = w_{opt}-1 = 1-\frac{\pi}{N}/1+\frac{\pi}{N} \approx 1-\frac{2\pi}{N}$$

$$\begin{cases} \rho(R_J) \simeq 1-\frac{\pi^2}{2N^2} & P \simeq -\ln 10/-\frac{\pi^2}{2N^2} = \frac{2\ln 10}{\pi^2}N^2 \\ \rho(R_{GS}) \simeq 1-\frac{\pi^2}{N^2} & P \simeq \frac{\ln 10}{\pi^2}N^2 \\ \rho(R_{SOR}) \simeq 1-\frac{2\pi}{N} & P \simeq \frac{\ln 10}{2\pi}N \end{cases}$$

Approximation Error Speed of Decrease in Iterative Method

$$X_{n+1} = RX_n + C, \ \forall n \geq 0$$

$$X^* = RX^*+C, \quad X_{n+1}-X^* = R(X_n-X^*), \ \forall n \geq 0.$$

$$\|X_{n+1}-X^*\| = \|R(X_n-X^*)\| \leq \|R\|\,\|X_n-X^*\|$$

- Norm of a matrix:

$$\|R\| = \sup_{V\in\mathbb{R}^n} \frac{\|RV\|}{\|V\|} = \sup_{w\in\mathbb{R}^n, \|w\|=1} \|Rw\|$$

$$\|R\|_2^2 = \sup_{\|w\|_2=1} \|Rw\|^2 = \sup_{w^tw=1} w^tR^tRw = \rho(R^tR) = \max\lambda_i \ ; \ (\lambda_i \text{ are evalues of } R^tR).$$

$$\|X_{n+p}-X^*\| \leq (\rho(R))^p \|X_n-X^*\|$$

In how many iteration does the approximation error decrease by a factor of 10?

Find p such that $p = -\frac{\ln 10}{\ln(\rho(R))}$. $(\rho(R))^p = \frac{1}{10}$.

$$= -\frac{\ln 10}{-0} \qquad \ln(1-x) \simeq -x(+O(x^2))$$

Crank-Nicolson for European/American Options

Solve linear systems using SOR: $A\,U^{m+1} = b^m$

$$A = \begin{pmatrix} 1+\alpha & -\frac{\alpha}{2} & & \\ -\frac{\alpha}{2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{\alpha}{2} \\ & & -\frac{\alpha}{2} & 1+\alpha \end{pmatrix}$$

A: spd tridiagonal

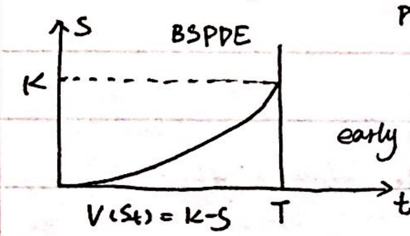
Until convergence:

for $j = 1:(N-1)$

$$X_{n+1}(j) = (1-w)X_n(j) + \frac{w\alpha}{2(1+\alpha)}\left(X_{n+1}(j-1) + X_n(j+1)\right) + \frac{w}{1+\alpha} b(j).$$

end.

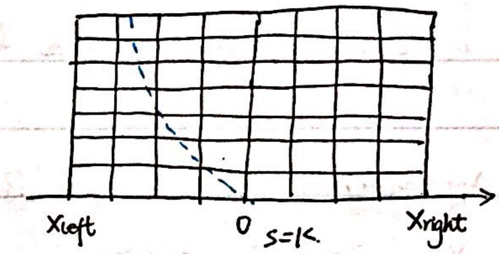
For American Option



put

$\Rightarrow$

early exercise



$S^{*''}(t) > 0.$ $\quad x = \ln\left(\frac{S}{K}\right)$ $\quad \tau = \frac{(T-t)\sigma^2}{2} \Rightarrow t = T - \frac{2\tau}{\sigma^2}.$

$x_0, x_1, \cdots, x_n, \cdots$ are approximation of U^{n+1}_{Amer}

Want $U^{n+1}_{Amer}(j) \geq$ early_ex_premium (j, m).

until convergence: $\|x_{n+1} - x_n\| \leq tol.$

for $j = 1:(N-1)$:

$$X_{n+1}(j) = \max\Big[(1-w)X_n(j) + \frac{w\alpha}{2(1+\alpha)}\left(X_{n+1}(j-1) + X_n(j+1)\right) + \frac{w}{1+w} b(j),$$
$$K - \text{early\_exercise\_premium}(j,m)\Big].$$