LECTURE 8

P&L OF TRADING STRATEGIES
HEDGING DISCRETELY
TRANSACTIONS COSTS

BACK TO THE SMILE

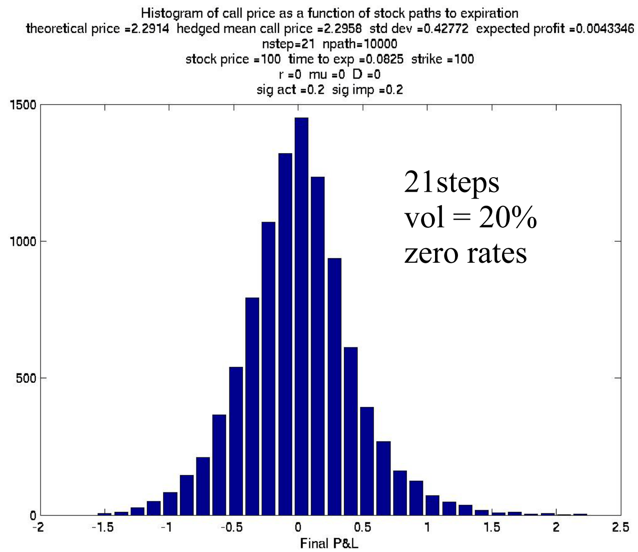
Hedging Errors from Discrete Hedging

We cannot hedge continuously:

A Simulation Approach

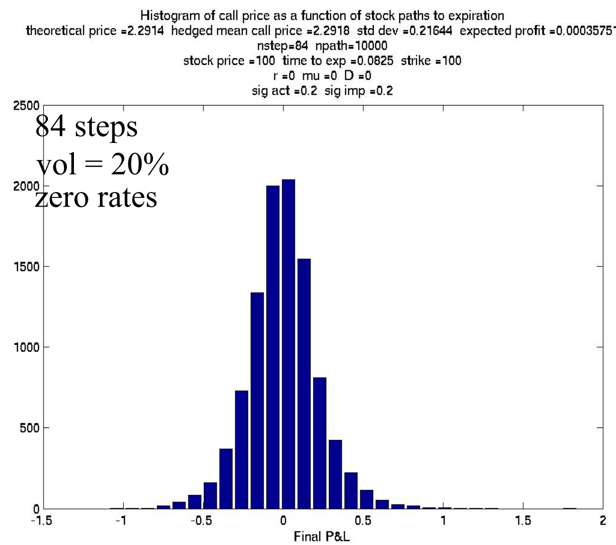
You cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss hedging at regular time intervals.

Monte Carlo: ATM option, expiration 1 month, the realized volatility is 20%, $\mu = r = 0.05$, hedged at an implied volatility of 20% equal to the realized volatility.



21 Rehedgings, Std. deviation. = 0.42

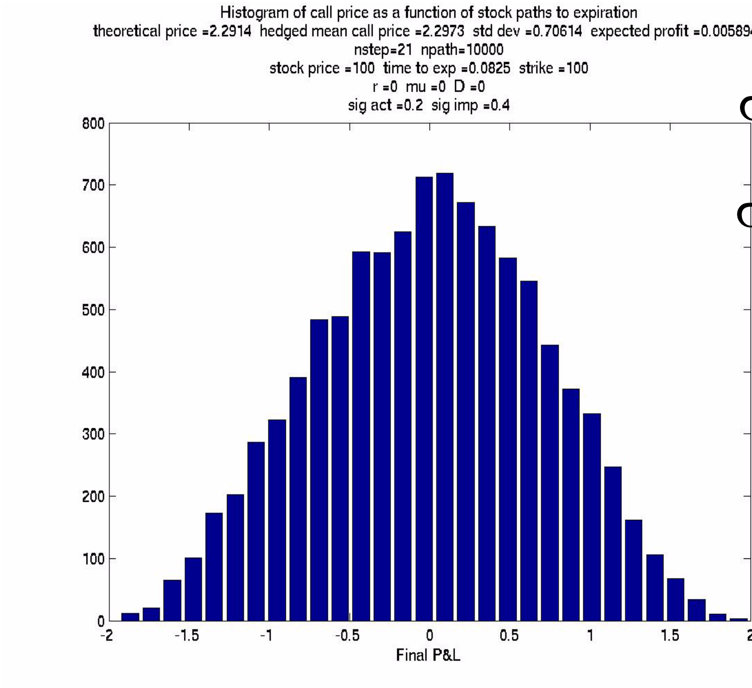
$$\sigma_i = \sigma_r$$



84 Rehedgings, Std. deviation. = 0.21

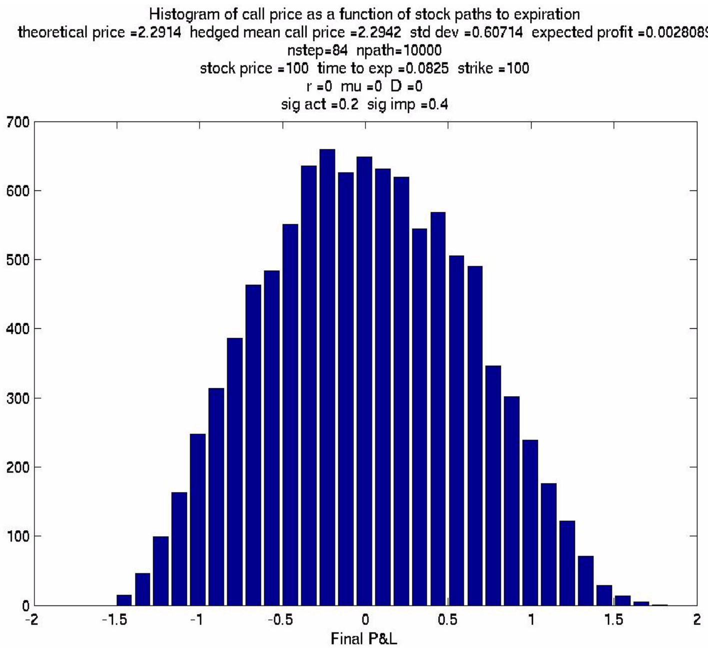
The mean P&L is zero; When we quadruple the number of hedgings, the standard deviation of the P&L halves.

Now let's see what happens $\sigma_i \neq \sigma_r$. Choose an implied volatility of 40% as the hedging volatility, that is, as the volatility used to calculate the value of Δ.



$$\sigma_i = 40\%$$

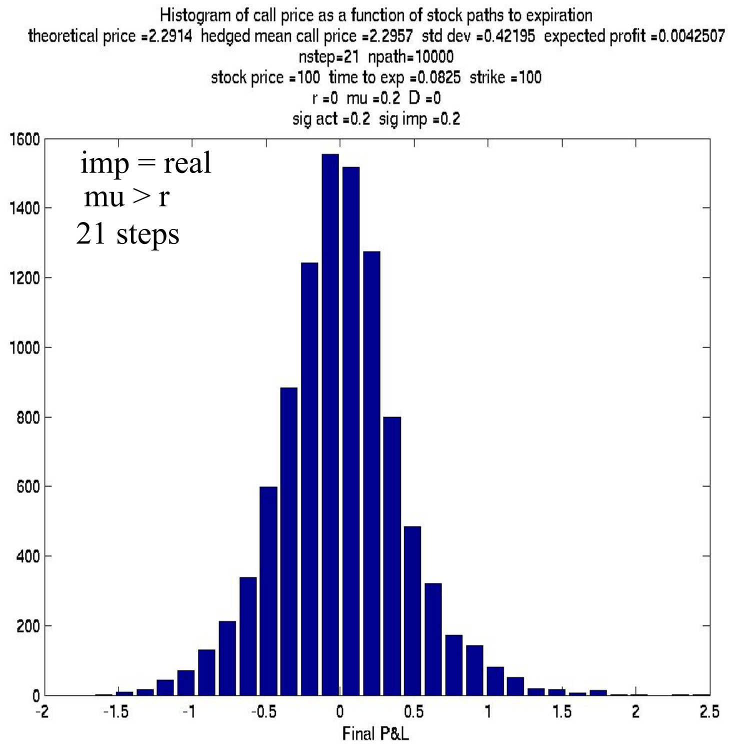
$$\sigma_r = 20\%$$



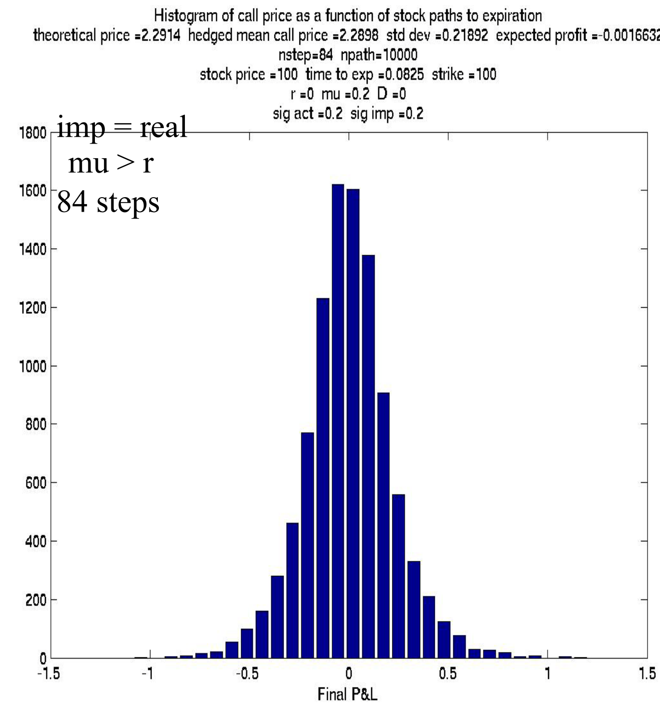
no reduction in variance with increasing rehedges unless hedge vol = realized vol

No longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

Finally let's see what happens when the drift μ is not the same as the riskless rate, even though implied/hedging) and the realized volatility are both set equal to 0.2.

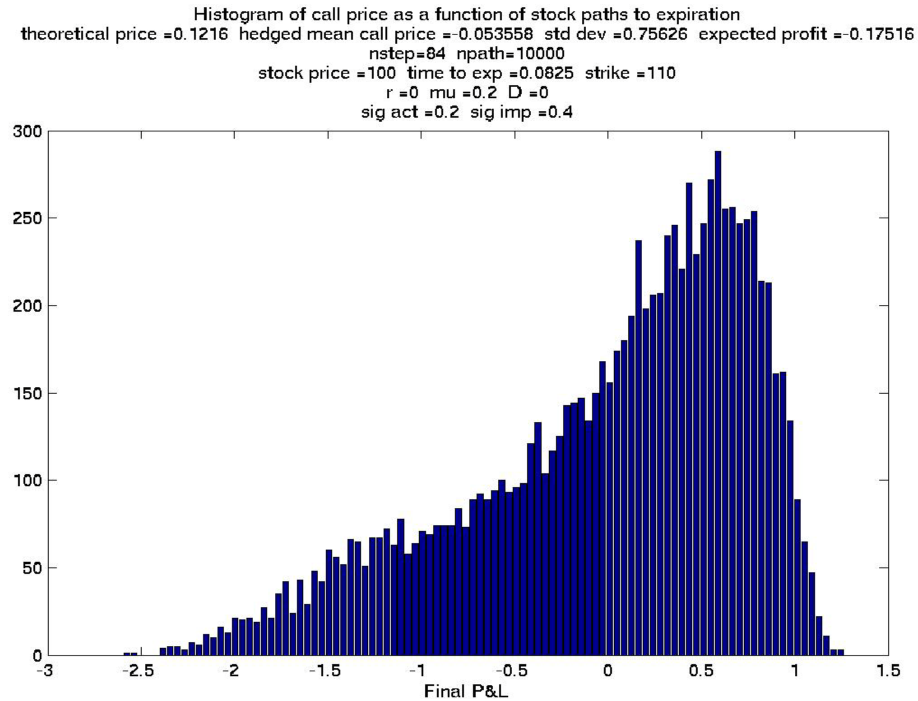


$$\sigma_i = \sigma_r$$



Std deviation again decreases by a factor of two.

Finally, for completeness, we look at the where $\sigma_i \neq \sigma_r$ *and* $\mu \neq r$. In this case the distribution is very asymmetric.



Understanding Discrete Hedging Error Analytically when $\sigma_i = \sigma_r \equiv \sigma$.

(Assuming we know the future volatility)

Discrete time Δt is larger than infinitesimal $\dfrac{\Delta S}{S} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}$

$$\varepsilon \in N(0, 1)$$

Hedged portfolio $\pi = C - \left(\dfrac{\partial C}{\partial S}\right)S$; Initial long π bought with borrowed money. If we hedged continuously the P&L would be zero.

Hedging error owing to mismatch between **continuous** hedge ratio and **discrete** time step

$$HE = \pi + \Delta\pi - \pi e^{r\Delta t} \approx \Delta\pi - r\pi dt$$

$$\approx \left[C_t\Delta t + C_S\Delta S + C_{SS}\frac{\sigma^2 S^2 \varepsilon^2}{2}\Delta t - C_S\Delta S\right] - r\Delta t\left[C - \left(\frac{\partial C}{\partial S}\right)S\right]$$

$$\approx \left(\mathbf{C_t} + \underbrace{C_{SS}\frac{\sigma^2 S^2 \varepsilon^2}{2}}_{\text{discrete}} - \underbrace{\mathbf{r}\left[\mathbf{C} - \left(\frac{\partial \mathbf{C}}{\partial \mathbf{S}}\right)\mathbf{S}\right]}_{\text{continuous}}\right)\Delta t$$

Now from Black-Scholes $r\left[C - \left(\dfrac{\partial C}{\partial S}\right)S\right] = \mathbf{C_t} + C_{SS}\dfrac{\sigma^2 S^2}{2}$

$$HE = \frac{1}{2}C_{SS}\sigma^2 S^2(\varepsilon^2 - 1)\Delta t \text{ Gamma error} \qquad \text{Eq.8.1}$$

$\varepsilon \in Z(0, 1)$ is normal with $E(\varepsilon^2) = 1$ so $E[HE] = 0$ with a χ^2 distribution.

Over n steps to expiration, the total HE is

$$HE = \sum_{i=1}^{n} \frac{1}{2}\Gamma_i S_i^2 \sigma_i^2 \left(\varepsilon_i^2 - 1\right)\Delta t \qquad \text{Eq.8.2}$$

The variance of the hedging error can be approximately calculated and shown to be

$$\sigma_{HE}^2 = E\left[\sum_{i=1}^{n} \frac{1}{2}\left[\Gamma_i S_i^2\right]^2 \left(\sigma_i^2 \Delta t\right)^2\right] \text{ over all paths} \qquad \text{Eq.8.3}$$

Integrating over all paths starting from S_0 for an atm option

$$E\left[\Gamma_i S_i^2\right]^2 = S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Thus for constant volatility

$$\sigma_{HE}^2 = \frac{1}{2}\sum_{i=1}^{n} S_0^4\Gamma_0^2\sqrt{\frac{T^2}{T^2 - t_i^2}}(\sigma^2\Delta t)^2$$

$$= \left(S_0^4\Gamma_0^2\right)(\sigma^2\Delta t)^2 \frac{1}{2\Delta t}\int_0^T \sqrt{\frac{T^2}{T^2-\tau^2}}d\tau$$

$$= \left(S_0^4\Gamma_0^2\right)(\sigma^2\Delta t)^2 \frac{\pi}{4} \times \frac{T}{\Delta t} \quad n$$

$$= \frac{\pi}{4}n\left(S_0^2\Gamma_0\sigma^2\Delta t\right)^2$$

From BS we can interpret $S_0^2\Gamma_0 = \frac{1}{\sigma(T-t)}\frac{\partial C}{\partial \sigma}$

$$\sigma_{HE}^2 = \frac{\pi}{4}n\left(\frac{1}{\sigma}\frac{\partial C}{\partial \sigma}\sigma^2\frac{\Delta t}{T-t}\right)^2 = \frac{\pi}{4}n\left(\frac{1}{n}\frac{\partial C}{\partial \sigma}\sigma\right)^2 = \frac{\pi}{4n}\left(\frac{\partial C}{\partial \sigma}\sigma\right)^2$$

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial \sigma}\frac{\sigma}{\sqrt{n}} \qquad \text{Eq.8.4}$$

Thus, the hedging error is approximately $\frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$. What does this mean?

Understanding The Results Intuitively

Hedging discretely introduces uncertainty in the hedging outcome but no bias: $E[HE] = 0$

Simple analytic rule

$$\sigma_{HE} \sim \frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$$

For $S \sim K$, more simply

$$\frac{\sigma_{\mathrm{P\&L}}}{\text{fair option value}} \sim \sqrt{\frac{\pi}{4n}} \; .$$

Think of this as statistical sampling error: discrete hedging samples volatility discretely and is therefore subject to error.

The standard deviation of a constant volatility σ measured discretely is $\frac{\sigma}{\sqrt{2n}}$.

This is quite a large error even assuming we know the future volatility with certainty.

In real life your hedge ratio is incorrect not just because hedging is discrete, but because you don't know the appropriate volatility to use.

The Effect of Transactions Costs

It costs money to hedge: **Simulation**

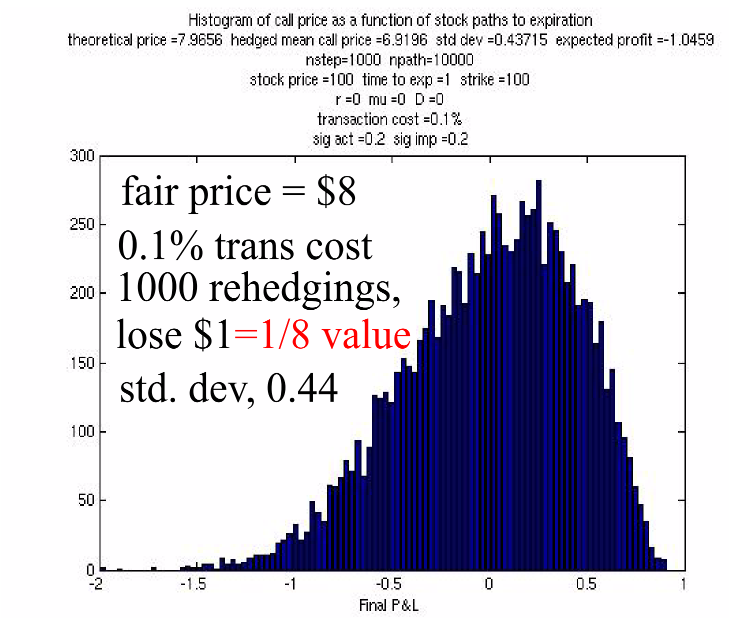
Suppose there is a fee to buy and sell the stock each time you rehedge. Then, not only is the P&L uncertain because of the discrete hedging, but the cost of hedging also lowers the fair value of the option if you buy it, and raises the cost to you if you sell it.

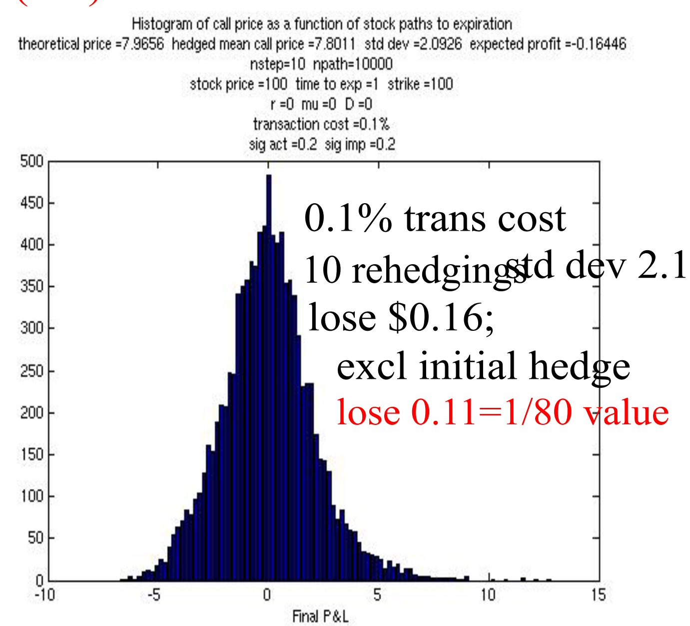
Assume a simple transactions cost proportional to the cost of the shares traded, and hedge at the realized volatility.

Rehedging at regular intervals

Rebalance hedge at every step, whether necessary or not.

cost of initial hedge is $k\Delta S = 0.001(0.5)100 = \$0.05$



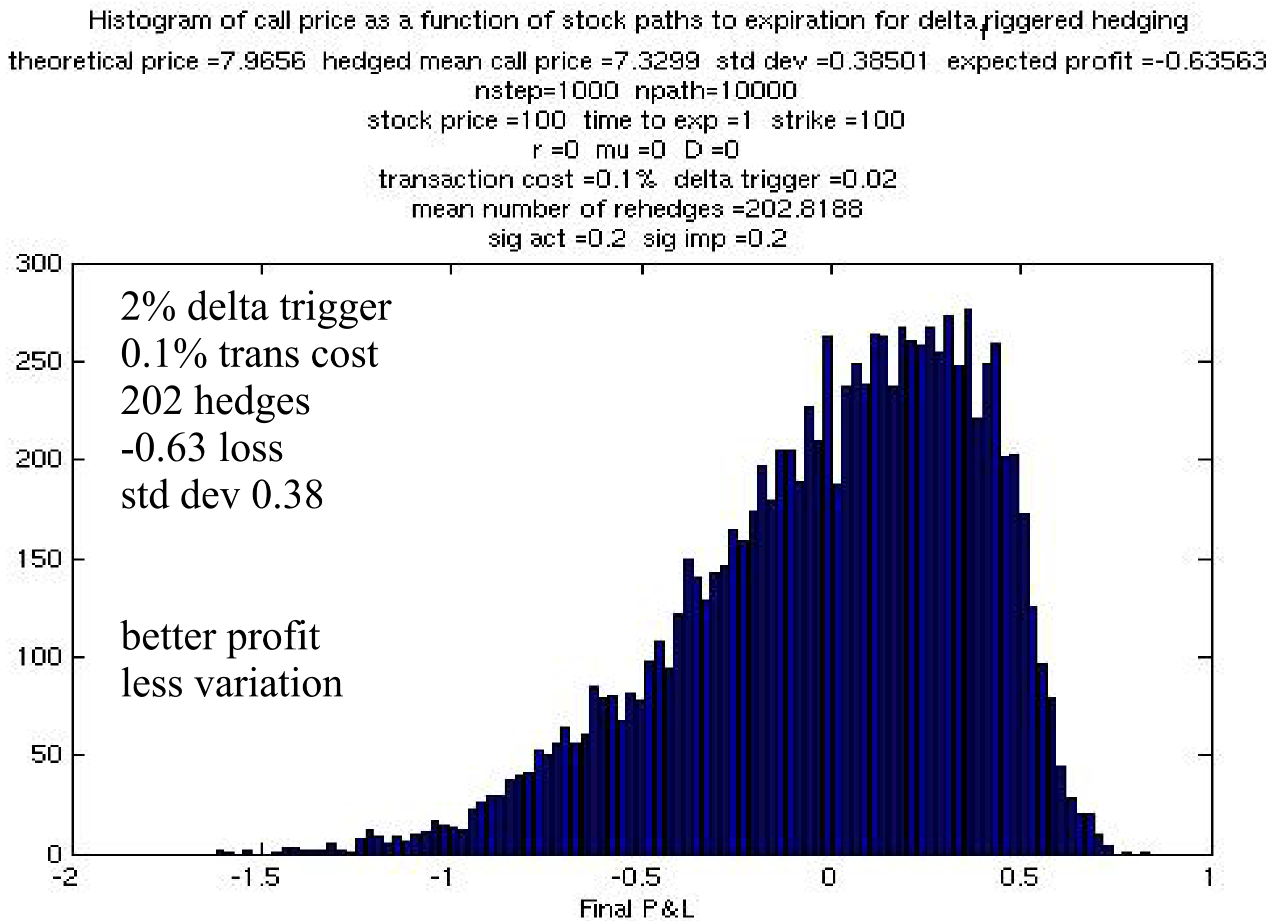


Tension between diminishing hedging error and reducing cost! What is optimal rebalancing

Rehedging triggered by changes in the hedge ratio

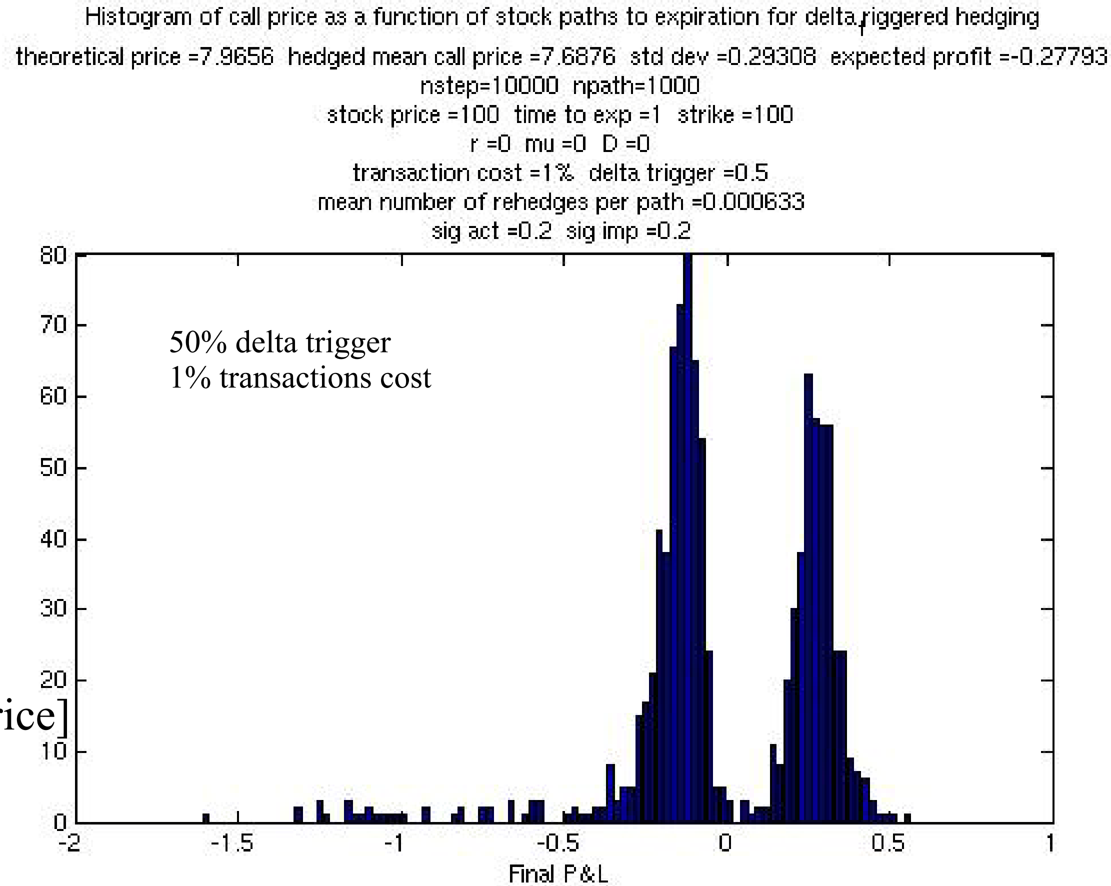
More efficiently rehedge when necessary, on a substantial change in delta.

Hedging an at-the-money call with a delta trigger of 0.02 or 2% and a transactions cost of 0.1%.



The loss owing to the transactions cost is smaller; the standard deviation of the P&L is smaller too.

Extreme case: rehedge only when the delta changes by 50 percentage points and with a transactions cost of 1%.



Some moves lead to no rehedging and high value; some moves lead to one rehedge and loss in value below the mean:
If you rehedge once, half the time, then *expected* loss in value is probability x cost = (1/2)[k x shares traded x price] =(0.5)(0.01)(0.5)100 =0.25

The distribution is bimodal. The reason is that if you rehedge only when the delta of the option changes by 50 points, then rehedges only occur when the stock makes a substantial move up or down in order to achieve such a large change in the delta. Hence one set of final call prices involve no rehedging transactions costs (over the paths where delta changed by less than 50 points) and hence lie above the mean; the other set of call final call prices involve one rehedging and its cost (over the paths where delta did change by 50bp or more) and hence lie below the mean.

Analytical Approximations to Transactions Cost

$$HE = \sum_{i=1}^{n} \frac{1}{2}\Gamma_i S_i^2 \sigma_i^2 \left(\varepsilon_i^2 - 1\right)\Delta t$$

Eq.8.5

$$E[HE] = 0$$

$$\sigma_{HE}^2 \sim O([\Delta t]^2)$$

The total number of rehedgings is $T/(\Delta t)$

$$\sigma_{HE}^2 \sim O\left(\frac{T}{\Delta t}[\Delta t]^2\right) \sim O(T\Delta t) \to 0 \qquad \text{as } \Delta t \to 0$$

Hedging continuously captures exactly the value of the option.

Now include transactions costs. Assume that you rehedge an option C with value C every time Δt passes. Every time you trade the stock (buying *or* selling), you pay a fraction k of the cost of the shares traded.

Then, every time you rehedge, you have to trade a number of shares equal to

$$N \approx \Delta(S + \delta S, t + \Delta t) - \Delta(S, t) \approx \frac{\partial^2 C}{\partial S^2}\delta S$$

Cost is value of number of shares traded times the fraction k, that is

$$\left|\frac{\partial^2 C}{\partial S^2}\delta S \times (kS)\right|$$

where the absolute value reflects the fact that you pay a positive transaction cost irrespective of whether you buy or sell shares. (Therefore nonlinear!)

$$\delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}\varepsilon$$

To order $(\Delta t)^{1/2}$ the expected transactions cost in time Δt is

$$E\left[\left|\frac{\partial^2 C}{\partial S^2}\sigma S^2 k \varepsilon \sqrt{\Delta t}\right|\right]$$

$$E[|\varepsilon|] \neq 0$$

There are $T/(\Delta t)$ rehedgings to expiration.

Total cost of order $\frac{T}{\Delta t}\sqrt{\Delta t} \sim \frac{1}{\sqrt{\Delta t}} \to \infty$ as the time between rehedgings goes to zero.

A PDE Model of Transactions Costs

One can approach transactions costs even more analytically in the framework of Hoggard, Whaley & Wilmott. (There are many other treatments.)

Assume zero rates and dividend yield, and

$$dS = \mu S dt + \sigma S Z \sqrt{dt}$$

where ε is drawn from a standard normal distribution.

$$\begin{aligned}
d\text{P\&L} &= dV - \Delta dS - \text{cash spent on transactions costs} \\
&\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \Delta dS - \kappa S|N| \\
&= \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S Z \sqrt{dt}) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \kappa S|N| \\
&= \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)}\sigma S Z \sqrt{dt} + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)} + \frac{\partial V}{\partial t}\right)dt - \kappa S|N|
\end{aligned}$$

Choose the continuous hedge ratio $\Delta = \frac{\partial}{\partial S}V(S, t)$ to eliminate the first term.

After time δt we have to rehedge, so that the change in the hedge is

$$
\begin{aligned}
N(S, t) &= \frac{\partial}{\partial S}V(S+\delta S, t+\delta t) - \frac{\partial}{\partial S}V(S, t) \\
&\approx \frac{\partial^2 V}{\partial S^2}\delta S \\
&\approx \frac{\partial^2 V}{\partial S^2}\sigma S Z \sqrt{\delta t}
\end{aligned}
$$

N itself is stochastic and related to Γ of course. The P&L is not riskless.

The average number of shares traded is

$$
E[N] \approx \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S E|Z| \sqrt{\delta t} = \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S \sqrt{\delta t}
$$

The **average** transactions cost obtained by multiplying the above by the cost κS per share is

$$
\sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa \sigma S^2 \sqrt{\delta t} = \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa \sigma S^2 \frac{\delta t}{\sqrt{\delta t}}
$$

The expected value of the change in the P&L is therefore given by

$$E[d\mathrm{P\&L}] = E\left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa\sigma S^2\right) dt\right]$$

$$\approx \left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa\sigma S^2\right) dt\right]$$

This isn't riskless.Nevertheless assume it expects to earn the riskless rate on the hedge, on average.

$$E[d\mathrm{P\&L}] = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

Combining, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa\sigma S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

Modified BS equation with nonlinear extra term proportional to the value of $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

The sum of two solutions to the equation is not necessarily a solution too; you cannot assume that the transactions costs for a portfolio of options is the sum of the transactions costs for hedging each option in isolation.

For a single long position in a call or a put, $\frac{\partial^2 V}{\partial S^2} \geq 0$, so we can drop the modulus sign.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.8.6}$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} \qquad \hat{\sigma} \approx \sigma - \kappa\sqrt{\frac{2}{\pi\delta t}}$$

This is the Black-Scholes equation with a modified reduced volatility, first derived by Leland, and the option is worth less. If you are long, you must pay less than the fair BS value since the hedging will cost you. For a short position, the effective volatility is enhanced, given by

$$\hat{\sigma} \approx \sigma + \kappa\sqrt{\frac{2}{\pi\delta t}}$$

When you sell the option you must ask for money because hedging it is going to cost you. For very small δt this expression diverges and the approximation becomes invalid.

Compare with our Simulations

Percentage change in ATM option is $\frac{\hat{\sigma} - \sigma}{\sigma} = \frac{\kappa}{\sigma}\sqrt{\frac{2}{\pi\delta t}}$

Case 1: $\frac{0.001}{0.2}\sqrt{\frac{2}{\pi\frac{1}{1000}}} = 0.005(25) = 0.125 = \frac{1}{8}$

Case 2: $\frac{0.001}{0.2}\sqrt{\frac{2}{\pi\frac{1}{10}}} = 0.005(2.6) = 0.013 = \frac{1}{80}$

excluding initial transactions cost of setting up hedge.

That's the number we found in our simulations.

More About The Smile

The Columbia Smile Generated by a Truck with Stochastic Volatility in 2004

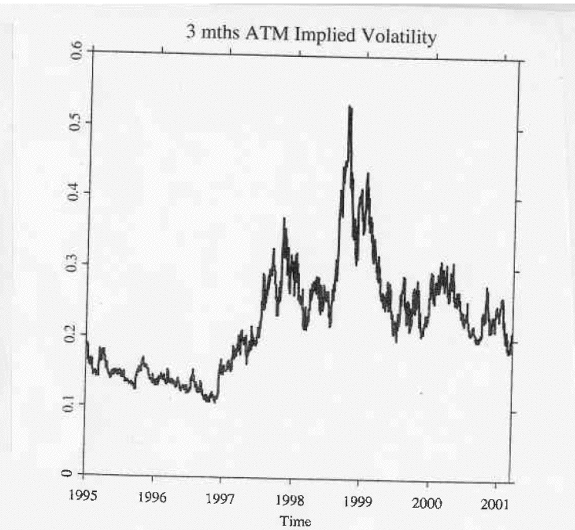


Fig. 2.13. The three-months ATM IV levels of DAX index options

short-term implieds
move more
than long-term

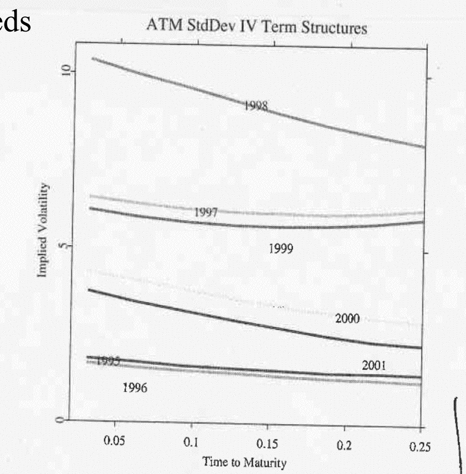


Fig. 2.12. ATM standard deviation of IV term structures between 1995 and 2001, computed from smoothed surfaces

negative correlation
during crisis

From Fengler's book

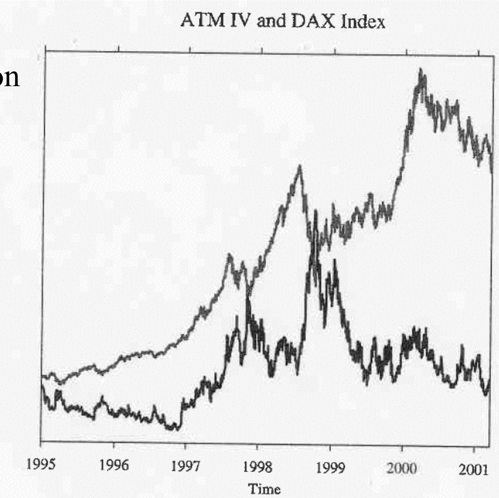


Fig. 2.14. German DAX $\times 10^{-4}$ (*upper line*) and three-months ATM IV levels (*lower line*), also given in Fig. 2.12

DAX Implied Volatility Surface 2008

2 Matthias R. Fengler

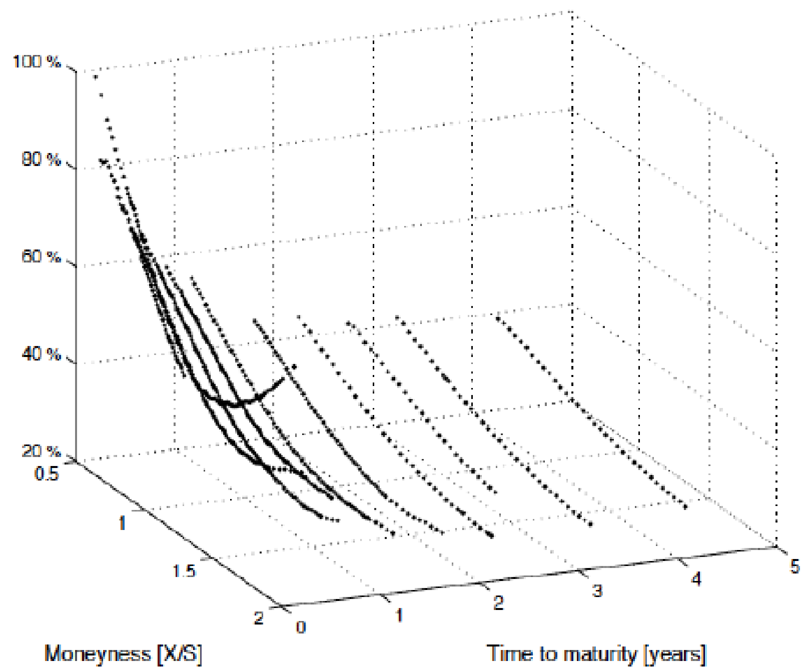


Fig. 1. IV surface of DAX index options from 28 Oct. 2008, traded at the EUREX. IV given in percent across a spot moneyness metric, time to expiry in years.

Implied Volatility as a Function of Strike/Spot for Different Expirations. (Crash-o-phobia: A Domestic Fear Or A Worldwide Concern? Foresi & Wu JOD Winter 05

The quoting convention is the Black-Scholes implied volatility

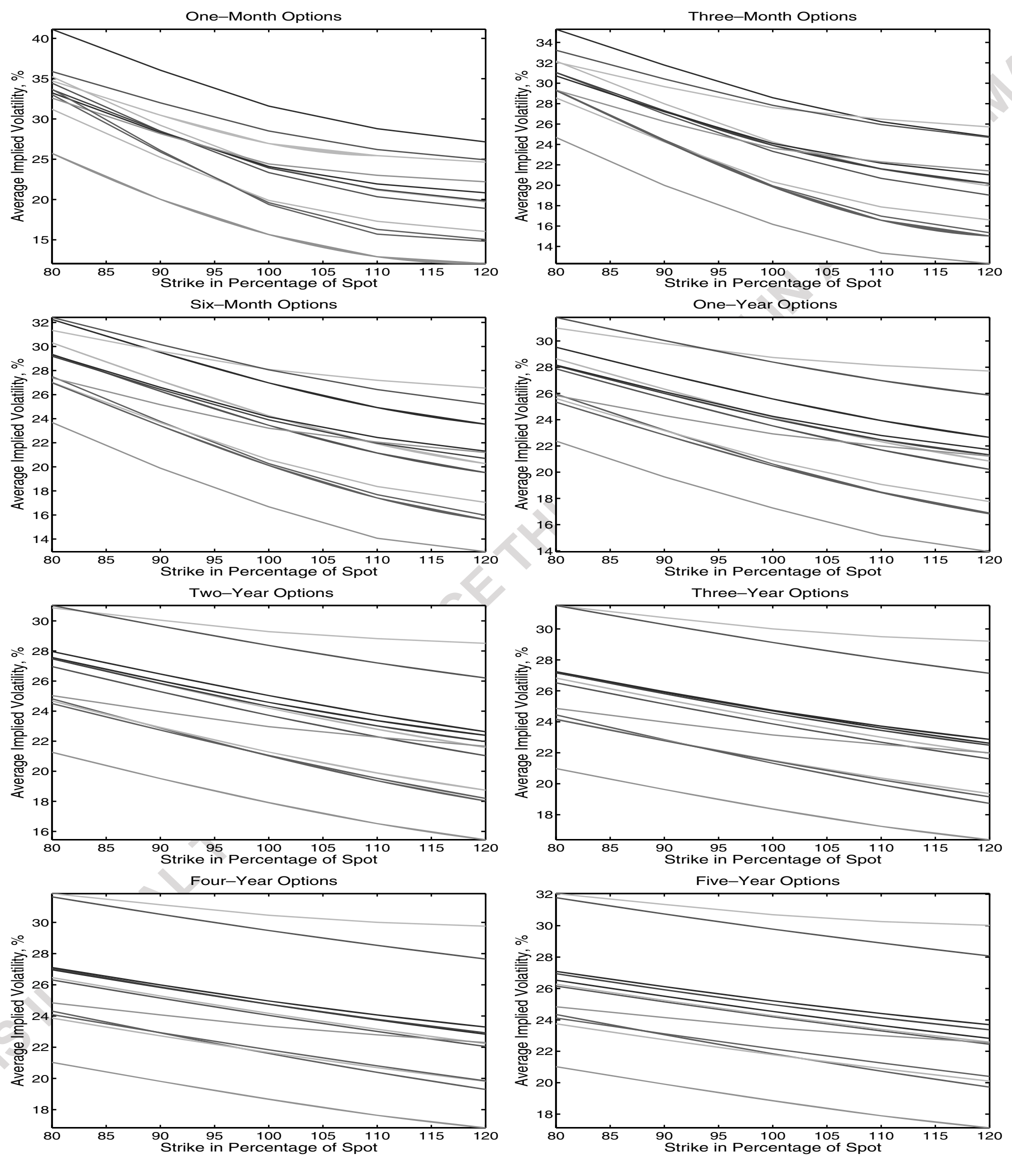
EXHIBIT 2

Implied Volatility Smirk on Major Equity Indexes

Notice the patterns that persist across all indexes:

- out-of-the-money puts have higher implied Black-Scholes volatilities than out-of-the-money calls. (Why?)

- The slope of implied volatil-

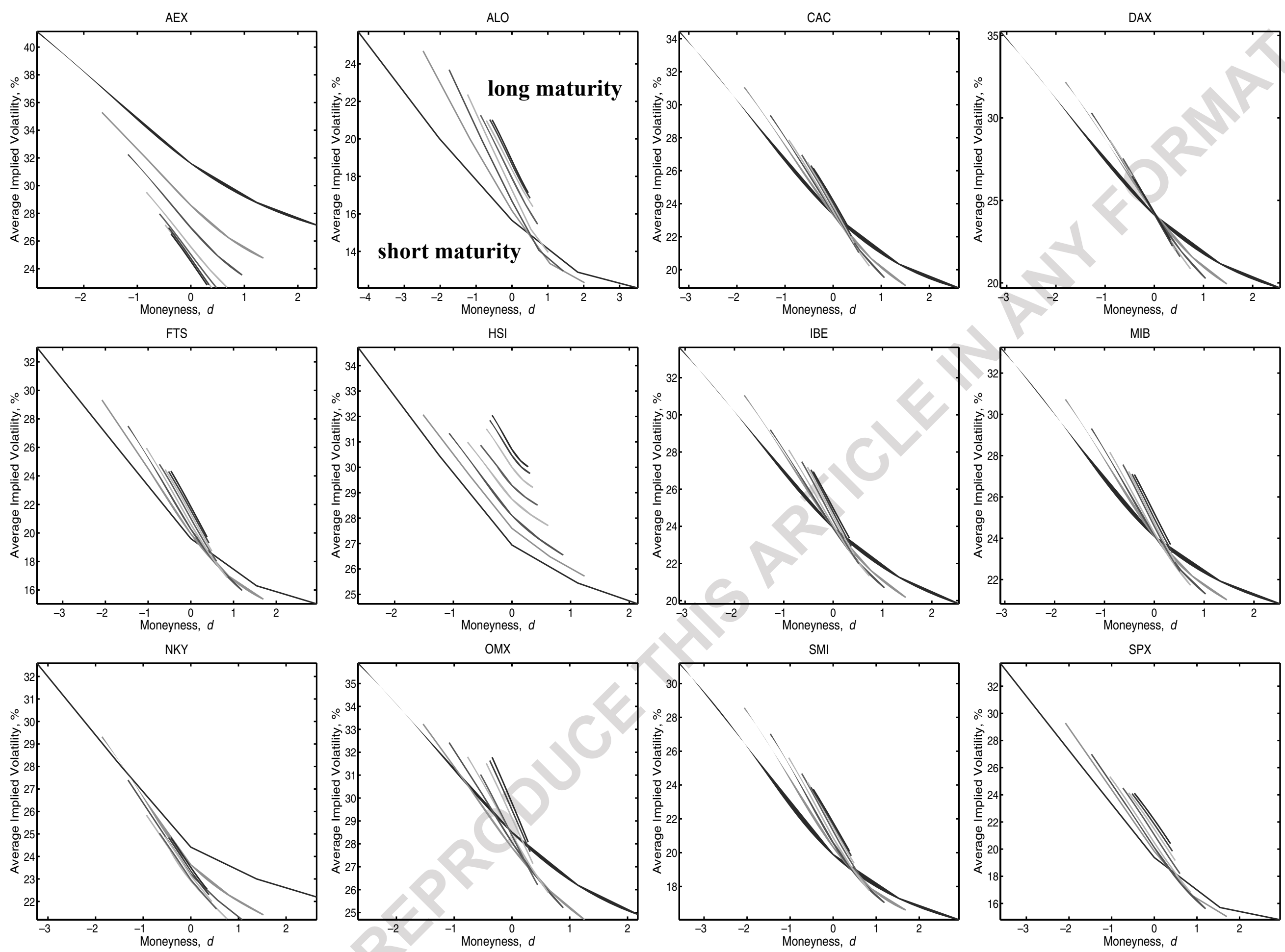


Implied Volatility as a Function of $\left(\log\dfrac{\text{Strike}}{\text{Spot}}\right)/(\sigma\sqrt{\tau})$

related to d_1

EXHIBIT 3

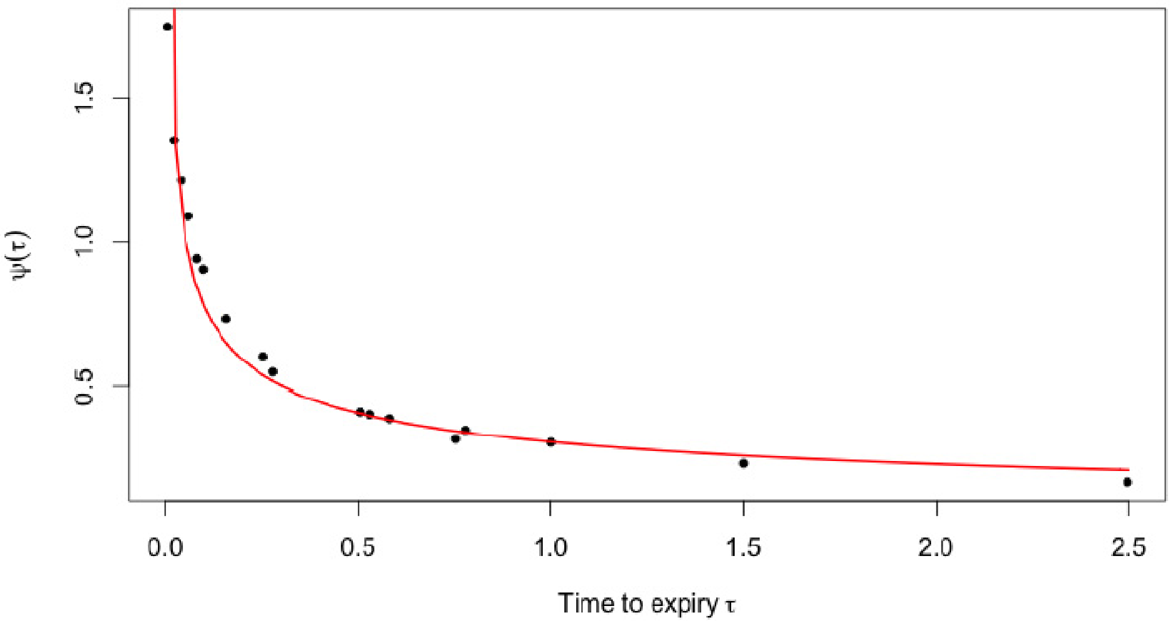
Maturity Pattern of Implied Volatility Smirks



O REPRODUCE THIS ARTICLE IN ANY FORMAT

When plotted against the number of standard deviations between the log of the strike and the log of the spot price for a lognormal process, the slope of the skew actually increases with expiration. Whatever is happening to cause this doesn't fade away with future time.

But plotted just against log moneyness, we see the slope decreases.

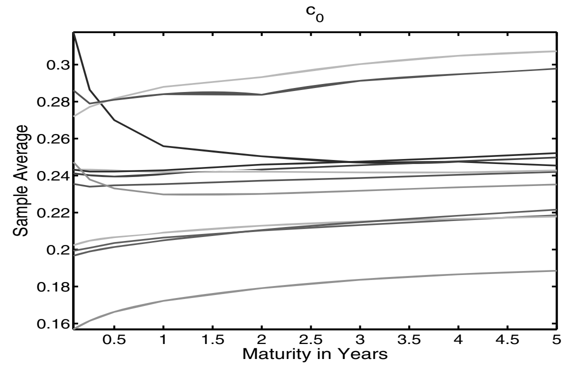


The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013 ; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.4}$.

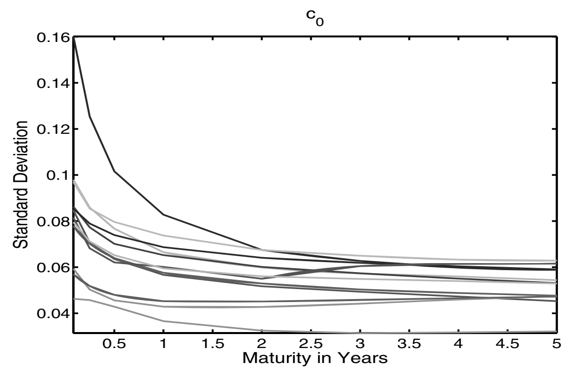
which is consistent by change of variable.

$$\frac{d\sigma}{d\left(\frac{m}{\sqrt{\tau}}\right)} = \sqrt{\tau}\frac{d\sigma}{d(m)} \sim \sqrt{\tau}\frac{1}{\tau^{0.4}} \sim \tau^{0.1}$$

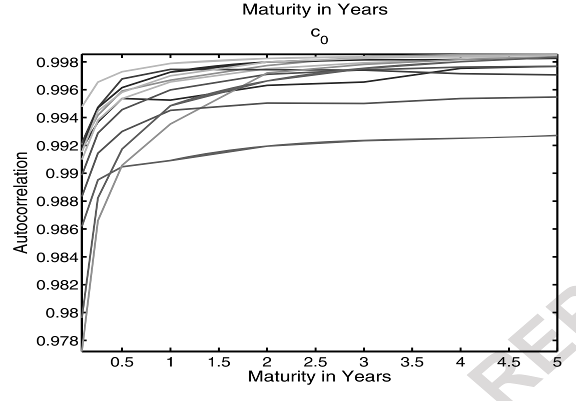
Behavior of implied volatility level c_0 as a function of option expiration.



term structure of
implied volatility is
roughly flat



volatility of volatility
decreases with expiration,
suggesting mean reversion
or stationarity for the
instantaneous volatility
evolution

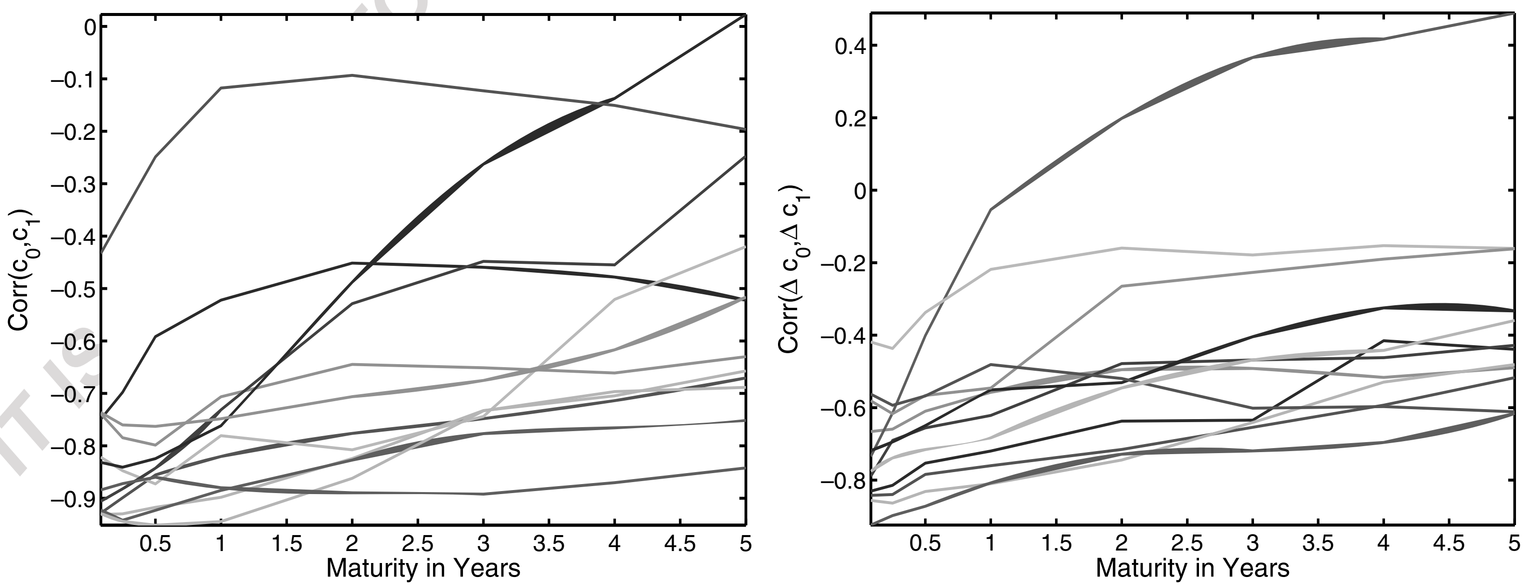


daily autocorrelation of
implied volatility is large,
and larger for longer
maturities
[excitement or depression
tends to continue]

The cross-correlation between volatility level and slope of the skew is large.

EXHIBIT 5

Cross Correlations between Volatility Level and Smirk Slope



Lines denote the cross-correlation estimates between the volatility level proxy (c_0) and the volatility smirk slope proxy (c_1). The left panel measures the correlation based on daily estimates, the right panel measures the correlation based on daily changes of the estimates.

short-term slope tends to get more negative as volatility increases

Some characteristics of the equity implied volatility smile

- Volatilities are steepest for small expirations as a function of strike, shallower for longer expirations.

- The minimum volatility as a function of strike occurs near atm strikes or strikes corresponding to slightly otm call options.

- Low strike volatilities are usually higher than high-strike volatilities but high strike volatilities can rise a little too.

- The term structure can be up or down.

- The volatility of implied volatility is greatest for short maturities, as with Treasury rates.

- There is a negative correlation between changes in implied atm volatility and changes in the underlying asset itself. [Fengler: $\rho = -0.7$ for the DAX in the 2000's]

- Implied volatility appears to be mean reverting.

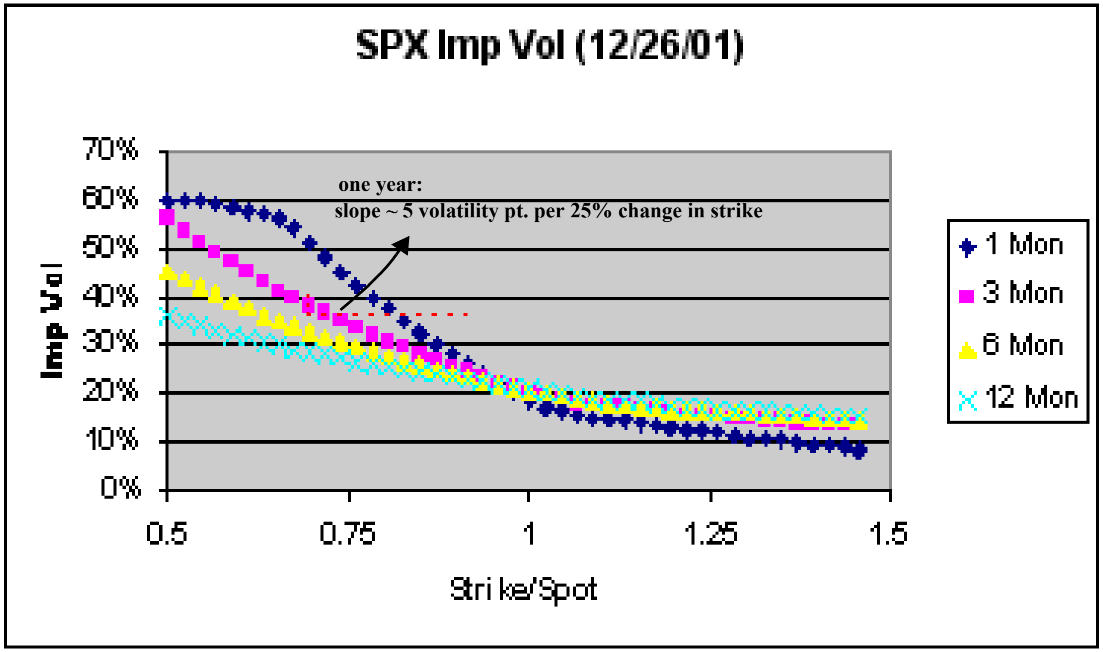
- Implied volatility tends to rise fast and decline slowly.

- Shocks across the surface are highly correlated. There are a small number of principal components or driving factors. We'll study these effects more closely later in the course.

- Implied volatility is usually greater than recent historical volatility.

Different Smiles in Different Markets

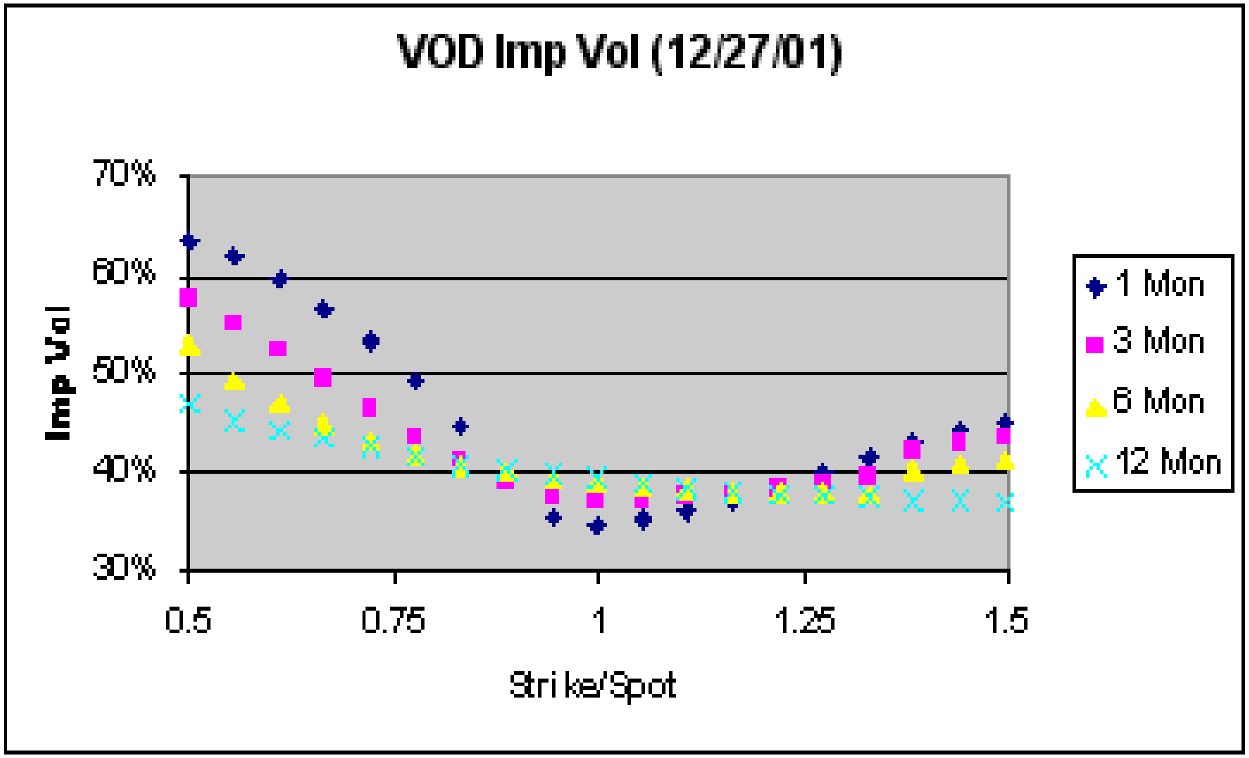
Here are some old smiles for the S&P 500, plotted a little differently:



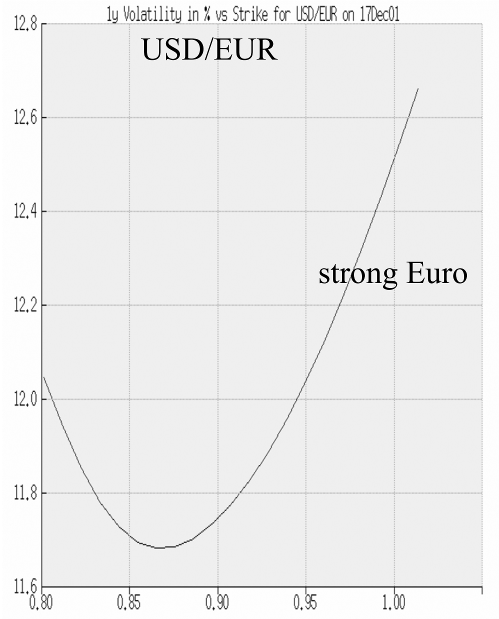
Indexes generally have a negative skew. The slope here for a one-year option is of order 5 volatility points per 250 S&P points, or about $\frac{0.05}{250} = 0.0002$. Note that the slope for a 3-month option is about twice as much, which roughly confirms the idea that the smile depends on $\frac{(\ln K/S)}{(\sigma\sqrt{\tau})}$, because a four-fold decrease in time to expiration then implied a doubling of the slope of the smile. The magnitude of the slope of the one-month option volatility is about 23 volatility points per 250 S&P points, or about 0.001.

Single stock smiles

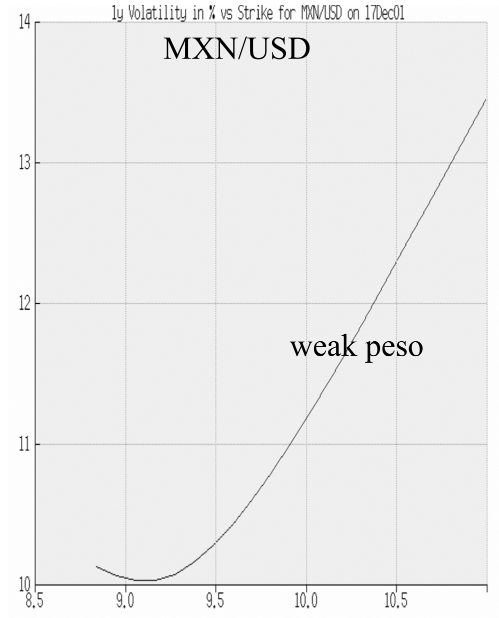
A single stock smile is more of an actual smile with both sides turning up.



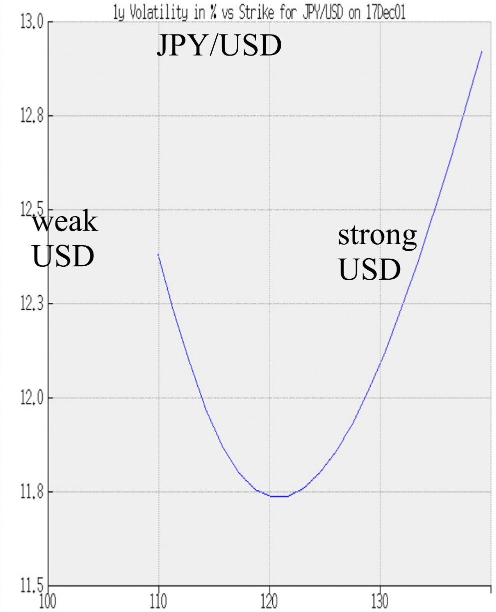
Some currency smiles....



ATM Strike = 0.90



9.85



123.67

The smiles are more symmetric for “equally powerful” currencies, less so for “unequal” ones. Equally powerful currencies are likely to move up or down.

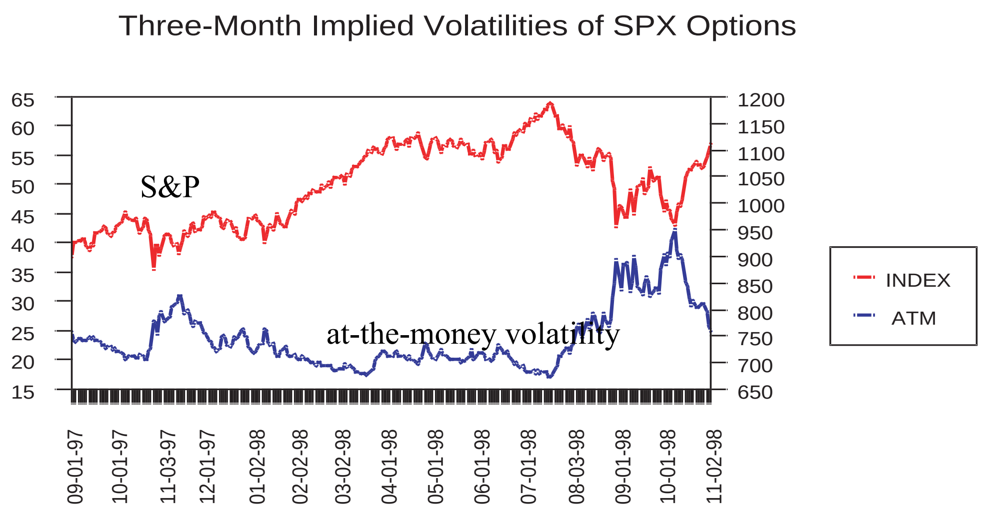
Equity index smiles tend to be skewed to the downside. The big painful move for an index is a downward move, and needs the most protection. Upward moves hurt almost no-one. An option on index vs. cash is very different and much more asymmetric than an option on JPY vs. USD.

Single-stock smiles tend to be more symmetric than index smiles. Single stock prices can move dramatically up or down. Indexes like the S&P when they move dramatically, move down.

Interest-rate or swaption volatility, which we will not consider much in this course, tend to be more skewed and less symmetric, with higher implied volatilities corresponding to lower interest rate strikes. This can be partially understood by the tendency of interest rates to move normally rather than lognormally as rates get low.

Variation of implied volatility and the smile over time

Example: here is the behavior of "volatility" itself as time passes.



Why do traders talk most about atm volatility?

ATM volatility is therefore not the volatility of a particular option you own.

Some of the apparent correlation in the figure above would occur even if $\Sigma(S, t, K, T)$ didn't change with S at all. How much of the correlation is true co-movement and not incidental?

People in the market often talk about how "volatility changed." One must be very careful in speaking about volatility

realized volatility, at-the-money implied volatility, and implied volatility for a *definite strike K and tenor T - t*, $\Sigma = \Sigma(S, t; K, T)$

When you talk about the change in Σ, what are you keeping fixed?

Consequences of the Smile for Trading

- Liquid standard call and put options prices are simply *quoted* via the Black-Scholes formula, so the model doesn't really matter for pricing.

- The model does matter if you wanted to generate your own idea of fair options values and then arbitrage them against market prices, but that is a very risky long-term business. This is a buy-side view.

- It does matter for hedge ratios.

- The model matters for pricing illiquid OTC exotic options.

- The question in both of these cases is of course: which model?

How to Graph the Smile

You see the yield curve at one instant and wonder what will happen to it later.

The snapshot $\Sigma(S_0, t_0, K, T)$. What is $\Sigma(S, t; K, T)$?

Plot $\Sigma(\ )$ vs. strike K, moneyness K/S, forward moneyness K/S_F, $(\ln K/S_F)/(\sigma\sqrt{\tau})$, or even more generally against $\Delta = N(d_1)$, which depends on S, K, t and *implied volatility*, itself.

Traders like to plot the smile against Δ because they believe it's more invariant. Also:

- Plotting implied volatilities against Δ immediately indicates the hedge.

- Since Δ depends on both strike and expiration, you can compare the implied volatilities of differing expirations and strikes as a function of single variable.

- Finally, d_2 is roughly the number of standard deviations the stock price must move to expire in the money and $N(d_2)$ is the risk-neutral probability of this happening. An "actuarial" measure.

Using the wrong quoting convention can distort the simplicity of the underlying dynamics. Perhaps the Black-Scholes model uses the wrong dynamics for stocks and therefore the smile looks peculiar in that quoting convention: ABM vs. GBM: constant arithmetic volatility corresponds to variable lognormal volatility. Plotting lognormal volatility against stock price would obscure the simplicity of the underlying evolution.

Δ and the Smile

The meaning of delta

Suppose that

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

$$d\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ$$

$$\ln\frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\varepsilon$$

The risk-neutral probability of $S_t > K$ is $P(S_t > K)$ given by

$$P(\ln S_t > \ln K) = P\left(\ln\frac{S_t}{S_0} > \ln\frac{K}{S_0}\right) = P\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\varepsilon > \ln\frac{K}{S_0}\right]$$

$$= P\left[\varepsilon > \frac{\ln K/S_0 - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right] = P[\varepsilon > -d_2] = P[\varepsilon < d_2] = N(d_2) \approx N(d_1)$$

$$= \Delta$$

Delta is approximately the risk-neutral probability of the option finishing in the money at expiration.

The Relationship between Δ and Strike

- The most popular and liquid option is $\Delta \sim 0.5$. Why?

- Far out-of-the-money options are also popular for buyers. Trading desks don't like to sell them. Why?

A standard measure of the skew is the *risk reversal*: difference in volatility between an out-of-the-money call option with a 25% Δ and an out-of-the-money put with a –25% Δ.

Moneyness rather than strike because relative rather than absolute.

What percentage of moneyness corresponds to a given Δ?

For simplicity set $r = 0$.

$$\Delta = N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} \exp\left(-\frac{y^2}{2}\right)dy + \int_0^{d_1} \exp\left(-\frac{y^2}{2}\right)dy\right] \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}$$

$$d_{1,2} = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} \pm \frac{\Sigma\sqrt{\tau}}{2} \text{ and } \tau = T - t$$

At the money $S = K$ $$\Delta \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\Sigma\sqrt{\tau}}{2} \approx 0.5 + (0.4)(0.5)\Sigma\sqrt{\tau}$$

For 20% volatility 1 year expiration $\Delta \approx 0.5 + 0.04 = 0.54$.

Slightly out of the money: $K = S + \delta S$ $\ln\left(\frac{S}{S+\delta S}\right) = -\ln(1 + \delta S/S) \approx -\frac{\delta S}{S}$

$$d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} \approx -\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

Then for a slightly out-of-the-money option, a fraction J away from the at-the-money level,

percent move

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{\Sigma\sqrt{\tau}}{2} - \frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

total variance

Suppose $(\delta S)/S = 0.01$, $T = 1$ year $\Sigma = 0.2$.

Then $\Delta \approx 0.54 - \frac{(0.4)(0.01)}{0.2} = 0.54 - 0.02 = 0.52$

Thus, Δ decreases by two basis points for every 1% that the strike moves out of the money.

The difference between a 50-delta and a 25-delta option therefore corresponds to about a 12% or 13% move in the strike price.

The move δS to decrease the delta from atm 0.54 to 0.25 is approximately given by

$$\frac{1}{\sqrt{2\pi}}\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} \approx 0.29 \text{ or } (\delta S)/S = 0.29\sqrt{2\pi}\Sigma\sqrt{\tau} \approx 0.29 \times 2.5 \times 0.2 \approx 0.15$$

Thus the strike of the 25-delta call is about 115. Actually it's about 117 if you use the exact Black-Scholes formula to compute deltas.

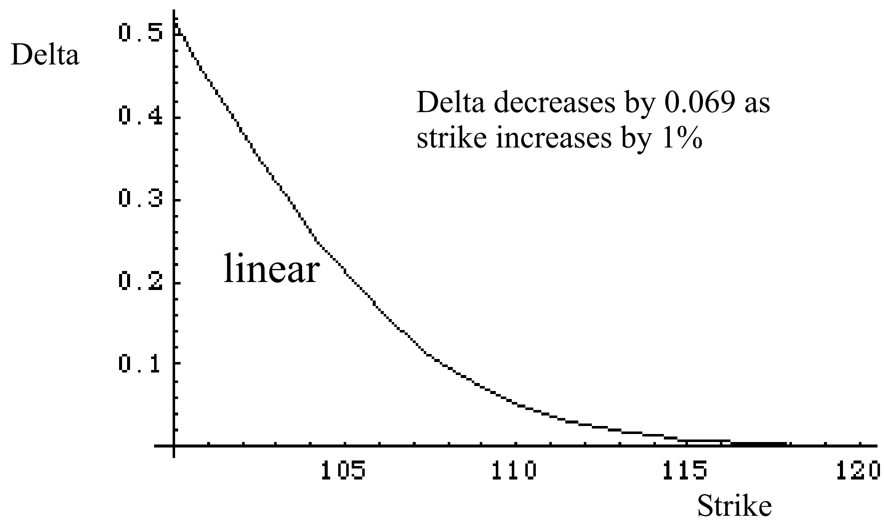
More generally

$$\text{change in Delta} \approx \frac{1}{\sqrt{2\pi}}\left(-\frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

and a one-basis point change in Δ corresponds to a change in $(\delta S)/S$ of about $0.025\Sigma\sqrt{\tau}$.

Key is the percent move in stock price divided by the square root of the annual variance. For a greater volatility or time to expiration and you need a bigger move in the strike to get to the same Δ.

A 1-month call with zero interest rates, 20% volatility



No-Arbitrage Bounds on the Smile

Yield to maturity: the parameter that determines bond prices: $B_T = 100\exp(-y_T T)$

Σ: the parameter that determines options prices in Black-Scholes.

There are no-arbitrage bounds on bond yields.

For example suppose $B_1 = 90$ and $B_2 = 91$.

$\pi = \frac{91}{90}B_1 - B_2$ has zero cost.

After one year the long position is worth more than \$100, so if you wait for B_2 to mature and pay off the face value you have a riskless profit so there is something wrong with these yields.

Similar constraints on options implied volatilities

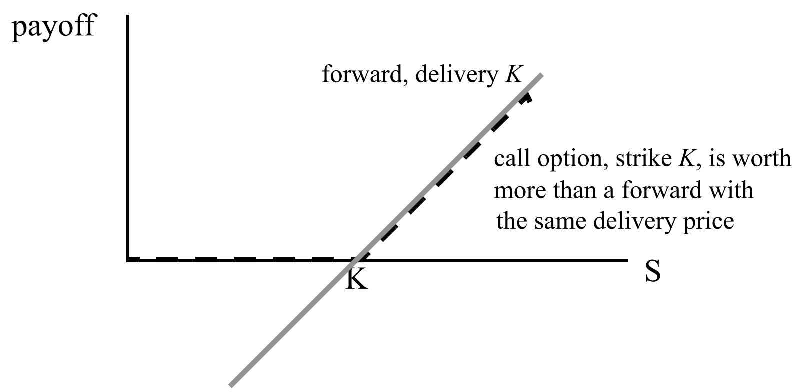
1.0.1 Some of the Merton Inequalities for Strike

Assume zero dividends, European calls.

1. A call is always worth more than a forward: $C \geq S - Ke^{-r(T-t)}$:

   Proof: An option is always worth more than a forward, because it has the same payoff when $_T > K$, and is worth more when $_T < K$.

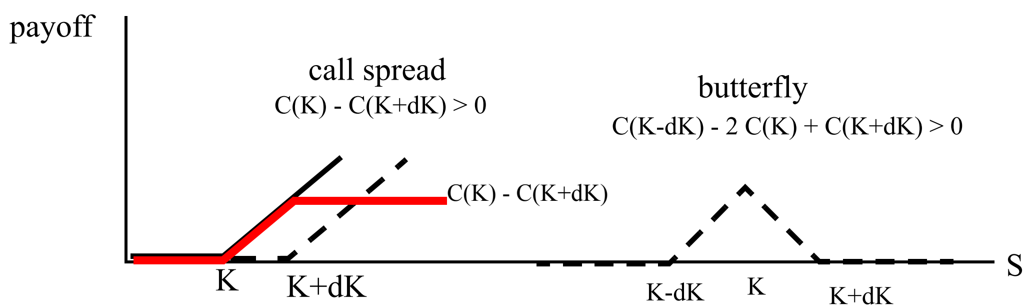
   Diagrammatically:



2. For the same expiration, options prices satisfy two constraints on their derivatives:

$$\frac{\partial C}{\partial K} < 0 \text{ and } \frac{\partial^2 C}{\partial K^2} > 0,$$

Proof: Look at payoff of a *call spread* and a *butterfly*.



There are similar constraints on European put prices:

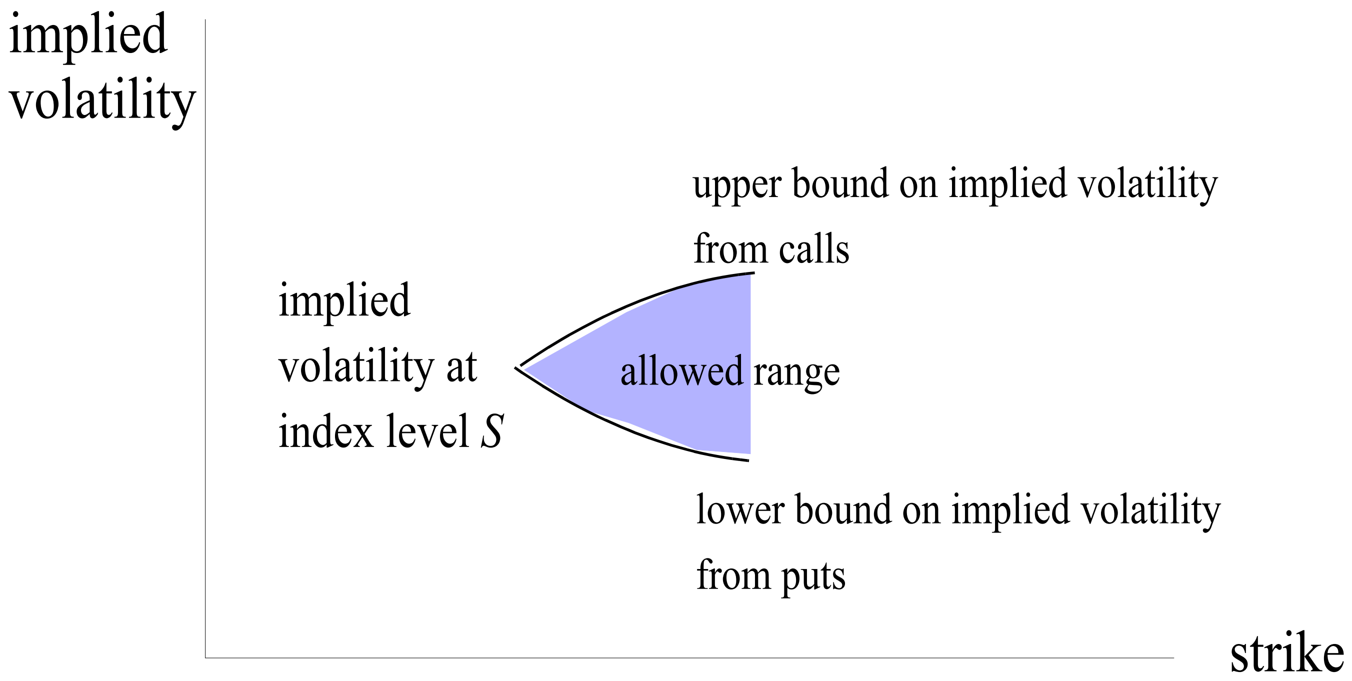
$$\frac{\partial P}{\partial K} > 0 \text{ and } \frac{\partial^2 P}{\partial K^2} > 0$$

Not just partials.

Inequalities for the slope of the smile

The constraints on $\frac{\partial C}{\partial K} < 0$ and $\frac{\partial P}{\partial K} > 0$ put limits on the slope of the smile independent of model.

Therefore they put constraints on the implied volatility parameters as a function of strike.



These constraint are true in the Black-Scholes formula with strike-independent volatility.

Now let's develop this idea more quantitatively.

$$C = C_{BS}(S, t, K, T, r, \Sigma)$$

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} < 0$$

$$\frac{\partial C_{BS}}{\partial \Sigma} = S\sqrt{\tau}N'(d_1) \equiv Ke^{-r\tau}\sqrt{\tau}N'(d_2) \qquad \text{Eq.1.7}$$

$$\frac{\partial \Sigma}{\partial K} \le -\frac{\left(\frac{\partial C_{BS}}{\partial K}\right)}{\left(\frac{\partial C_{BS}}{\partial \Sigma}\right)} = \frac{e^{-r\tau}N(d_2)}{Ke^{-r\tau}\sqrt{\tau}N'(d_2)} = \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

For small volatility, at the money: $d_2 \approx 0$, $N(d_2) \approx 0.5$ and $N'(d_2) \approx \frac{1}{\sqrt{2\pi}}$:

$$\frac{\partial \Sigma}{\partial K} \le \sqrt{\frac{\pi}{2}}\frac{1}{K\sqrt{\tau}} \approx \frac{1.25}{K\sqrt{\tau}} \qquad \text{Eq.1.8}$$

For 1-month options on the S&P $\quad \frac{\partial \Sigma}{\partial K} < 0.0043$.

For a 1% change in strike ($\Delta K \approx 10$) volatility must change less than 4.3 volatility points.

Recall: the S&P skew slope for one-month options was ~ 0.001, or 1 volatility point for a 1% change in the strike, only a factor of 4 below the arbitrage limit.

Asymptotically short expiration

$$\frac{\partial \Sigma}{\partial K} \le \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

At-the-money forward, as $\tau \to 0$

$$d_2 \to -\frac{\Sigma\sqrt{\tau}}{2} \to 0$$

$$N(d_2) \to \frac{1}{2}$$

$$N'(d_2) \to \frac{1}{\sqrt{2\pi}}$$

and so

$$\frac{\partial \Sigma}{\partial K} \le O(\tau^{-1/2}) \qquad \text{as } \tau \to 0.$$

As the time to expiration $\tau \to 0$, the slope steepness can increase no faster than $O(\tau^{-1/2})$.

Asymptotically short expiration

At the other extreme, as $\tau \to \infty$, $d_2 \to -\infty$, and therefore

$$\frac{\partial \Sigma}{\partial K} \le \frac{1}{K\sqrt{\tau}} \frac{N(d_2)}{N'(d_2)} \sim O\left(\frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{\tau}}\right) \sim O\left(\frac{1}{\tau}\right)$$

To prove the line above we have made use of the asymptotic relation

$N(d_2)/N'(d_2) \sim O(\tau^{-0.5})$ as $\tau \to \infty$.

The area under the tail gets smaller faster than the height of the tail.

Thus, the slope of the smile can decrease with time to expiration no more slowly than $O(\tau^{-1})$.

Reference: *Arbitrage Bounds on the Implied Volatility Strike and Term Structures of European-Style Options*. Hardy M. Hodges, Journal of Derivatives, Summer 1996, pp. 23-35.