LECTURE 6

1. VARIANCE SWAPS REPLICATION CONT.
2. P&L OF TRADING STRATEGIES

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Valuing Volatility Swaps

Volatility is the square root of variance, a derivative. You can replicate it with the continuous dynamic trading of portfolios of variance swaps, just as you can replicate $\sqrt{S}$ by trading S.

Expand about V_E, the expected variance.

$$\sigma = \sqrt{\sigma^2} = \sqrt{V} \equiv \sqrt{V_E + \{V - V_E\}}$$

$$= \sqrt{V_E}\left(1 + \frac{V - V_E}{V_E}\right)^{1/2}$$

$$\approx \sqrt{V_E}\left[1 + \frac{V - V_E}{2V_e} - \frac{1}{8}\left(\frac{V - V_E}{V_E}\right)^2 + \dots\right]$$ The square root has negative convexity therefore worth less.

$$\approx \sqrt{V_E} + \frac{V - V_E}{2\sqrt{V_E}} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$$

Taking risk-neutral expectations: $E(\sigma) \approx \sqrt{V_E} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$

Thus the fair volatility is smaller than the square root of the variance, and depends on the volatility of variance, like an option on variance.

The VIX Volatility Index

The VIX, from 1993 - 2003, used to defined as the weighted average of various atm and otm implied volatilities. This was rather arbitrary. In 2003 the CBOE changed the definition of the VIX to be the square root of the fair delivery price of variance as captured by a variance swap, using the formula from this paper with stock
dividends.

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[(r-d)T - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{(r-d)T}}{S_*} - 1\right) + e^{rT}\int_{(K > S^*)} C(S, K, 0)\frac{dK}{K^2} + e^{rT}\int_{(K < S^*)} P(S, K, 0)\frac{dK}{K^2}\right]$$

The RHS is

$$\frac{2}{T}\left\{\ln\frac{F}{S_0} - \ln\frac{S_*}{S_0} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls above S* plus puts below S*}]\right\}$$

$$= \frac{2}{T}\left\{\ln\frac{F}{S^*} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$= \frac{2}{T}\left\{\ln\left(1 + \frac{F}{S^*} - 1\right) - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$\approx \frac{2}{T}\left\{e^{rT}[\text{sum of calls and puts}] - \frac{1}{2}\left(\frac{F}{S_*} - 1\right)^2\right\}$$

The CBOE uses a finite sum over traded options at two expirations near 30 days, and then interpolates/extrapolates to thirty day volatility.

Some advantages of the new VIX

- The VIX is an estimate of one-month future realized volatility based on listed options prices. The value of the VIX depends on implied volatility.
- The estimate is independent of market level because it involves the sum of different options prices.
- It is relatively insensitive to model issues, because it assumes only continuous underlier movement, but doesn't assume Black-Scholes.
- It is hedgeable because it involves a portfolio of listed options.

You can therefore in principle price futures, forwards and options on the VIX.

Future Extensions

Many variance swaps are capped and implicitly contain embedded volatility options.

Valuing options on volatility is the big challenge.

Modeling the VIX and VIX futures because it's the most liquid measure of volatility.

Quick Aside: The Black-Scholes Equation and Sharpe Ratios

Valuation by perfect replication. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

$$
\begin{aligned}
dS_t &= \mu_S S_t dt + \sigma_S S_t dZ_t \\
dB_t &= B_t r_t dt
\end{aligned}
\qquad \text{Eq.6.1}
$$

The option price $C(S_t, t)$ whose evolution is given by

$$dC_t = \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2 dt$$

$$= \left\{ \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2 \right\} dt + \frac{\partial C_t}{\partial S}\sigma_S S_t d$$

$$\equiv \mu_C C_t dt + \sigma_C C_t dZ_t$$

where by definition

$$\mu_C = \frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2\right\} \qquad \text{Eq.6.2}$$

$$\sigma_C = \frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_S S_t\right)$$

Riskless portfolio $\pi = \alpha S + C$

Then

$$\begin{aligned} d\pi &= \alpha\{\mu_S S_t dt + \sigma_S S_t dZ_t\} + \{\mu_C C_t dt + \sigma_C C_t dZ_t\} \\ &= (\alpha\mu_S S_t + \mu_C C_t)dt + (\alpha\sigma_t S_t + \sigma_C C_t)dZ_t \end{aligned} \qquad \text{Eq.6.3}$$

Riskless necessitates

$$\alpha = -\frac{\sigma_C C}{\sigma_S S} \qquad \text{Eq.6.4}$$

That no riskless arbitrage: $d\pi = \pi r dt$.

Requires $\alpha\mu_S S + \mu_C C = (\alpha S + C)r$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for α from Equation 6.4 leads to the relation

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S} \qquad \text{Eq.6.5}$$

This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 4.2 into Equation 4.5 for μ_C and σ_C we obtain

$$\frac{\frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2\right\} - r}{\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_S S_t\right)} = \frac{\mu_S - r}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \qquad \text{Black-Scholes equation, no drift} \qquad \text{Eq.6.6}$$

It's good to get very familiar with manipulating this solution and its derivatives.

The solution, the Black-Scholes formula and its implied volatility, is the quoting currency for trading prices of vanilla options.

You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$C(S, K, t, T, r, \sigma) = e^{-r(T-t)} \times [S_F N(d_1) - K N(d_2)]$$

$$S_F = e^{r(T-t)} S$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm 0.5\sigma_S^2 (T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.6.7}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma_S^2(T-t)$. If you rewrite the formula in terms of the prices of traded securities – the present value of the bond K_{PV} and the stock price S – then indeed time and volatility always appear together:

$$C(S, K, t, T, \sigma) = [SN(d_1) - K_{PV}N(d_2)]$$

$$K_{PV} = e^{-r(T-t)}K$$

$$d_{1,2} = \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.6.8}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

Note that the time to expiration appears in the formulas in two different combinations, $r(T-t)$ the discount factor and $\sigma^2(T-t)$ the total variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.

NEW TOPIC:
P&L FROM HEDGED TRADING STRATEGIES

The P&L of Hedged Trading Strategies

Consider an initial position at time t_0 in an option C that is Δ-hedged with borrowed money which earns interest r, and then rehedged using in discrete steps at times t_i and stock prices S_i.

Notation: $C_n = C(S_n, t_n)$ $\quad \Delta_n = \Delta(S_n, t_n)$.

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares Traded	Net Value of Position: Option + Stock + Cash
t_0, S_0	Buy C_0, short Δ_0 shares	$-\Delta_0$	$-\Delta_0 S_0$	$\Delta_0 S_0$	C_0
t_1, S_1	none	$-\Delta_0$	$-\Delta_0 S_1$	$\Delta_0 S_0 e^{r\Delta t}$	$C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\Delta t}$
	get short Δ_1 shares by shorting $\Delta_1 - \Delta_0$ shares	$-\Delta_1$	$-\Delta_1 S_1$	$\Delta_0 S_0 e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1$	$C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1$
t_2, S_2	none	$-\Delta_1$	$-\Delta_1 S_2$	$\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$	$C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares Traded	Net Value of Position: Option + Stock + Cash
t_2, S_2	get short Δ_2 shares by shorting $\Delta_2 - \Delta_1$ shares	$-\Delta_2$	$-\Delta_2 S_2$	$\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$	$C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$
etc.					
t_n, S_n	get short Δ_n shares by shorting $\Delta_n - \Delta_{n-1}$ shares	$-\Delta_n$	$-\Delta_n S_n$	$\Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$	$C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$

The initial value of the positions was C_0 and would have generated $C_0 e^{r(T-t)}$

The final value is $C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$

where the subscript b at the end of the formula denotes a backwards Ito integral.

Therefore, the fair value of C_0 is given by equating these two quantities:

$$e^{r(T-t)}C0 = C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$$

or

$$(C_0 - \Delta_0 S_0)e^{r(T-t)} = (C_T - \Delta_T S_T) + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b \qquad \text{[A]}$$

You can integrate by parts using the relation

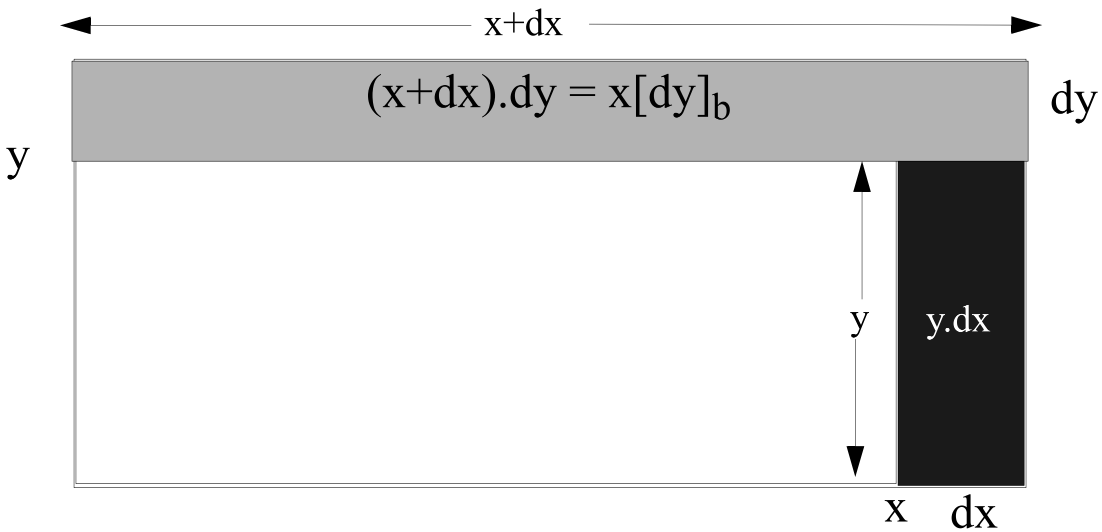
$$e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b = d\left[e^{r(T-\tau)} S_\tau \Delta_\tau\right] + re^{r(T-\tau)} \Delta_\tau S_\tau d\tau - e^{r(T-\tau)} \Delta_\tau dS_\tau$$

to obtain

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[dS_\tau - S_\tau r d\tau] e^{-r(\tau - t)} \qquad \text{[B]}$$

[A] and [B] provide a way to calculate the value of the call in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration.

Backward Ito Integral



$$d[xy] = ydx + x[dy]_b$$

The Effect of Different Hedging Strategies
How do the return profiles depend on the hedging strategy?

Realized volatility is noisy. Implied volatility is a parameter reflecting fear, hedging costs, inability to hedge perfectly, uncertainty of future volatility, the chance to make a profit, etc., usually greater than realized volatility.

Hedging with Realized (Known) Volatility

We buy the option at its implied volatility and then hedge it at the realized volatility to replicate the option perfectly. The P&L is value gained from hedging MINUS Black-Scholes implied value:

$$\text{Total P\&L} = V(S, \tau, \sigma) - V(S, \tau, \Sigma)$$

Recap: How Do We Capture $V_r - V_i$:

Table 1: Position Values when Hedging with Realized Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_r\vec{S}, -\Delta_r S$	$\Delta_r S - V_i$	0
t + dt	$\vec{V}_i(t+dt, S+dS)$, $V_i + dV_i$	$-\Delta_r\vec{S}$, $-\Delta_r(S+dS)$	$(\Delta_r S - V_i)(1+rdt)$ interest received; $-\Delta_r DSdt$ dividends paid	$(V_i + dV_i - \Delta_r(S+dS))$ $(\Delta_r S - V_i)(1+rdt)$ $-\Delta_r DSdt$

$$dP\&L = dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt \qquad \text{Eq.1.9}$$

This was bought at implied, hedged at realized.

But dP&L hedged *and bought* at realized volatility is zero which is statement of Black-Scholes:

$$0 = d\boldsymbol{V_r} - \Delta_r dS - rdt(\boldsymbol{V_r} - \Delta_r S) - \Delta_r DSdt$$

So substituting in 5.1

$$\begin{aligned} dP\&L &= dV_i - dV_r - rdt(V_i - V_r) \\ &= e^{rt} d\left[e^{-rt}(V_i - V_r)\right] \end{aligned}$$

$$d\mathrm{PV}(\mathrm{P\&L}) = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_i - V_r)\right] = e^{rt_0} d\left[e^{-rt}(V_i - V_r)\right]$$

$$\mathrm{PV}(\mathrm{P\&L}) = e^{rt_0}\int_{t_0}^{T} d\left[e^{-rt}(V_i - V_r)\right]$$

$$= 0-(V_i - V_r) = V_r - V_i \qquad \text{if T is expiration}$$

The final P&L at the expiration of the option is known and deterministic, provided that we know the realized volatility and that we can hedge continuously.

How is this deterministic P&L realized over time? Stochastically -- like a zero coupon bond whose final principal is known but whose present value varies with interest rates.

$$d\mathrm{P\&L} = dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

Use Ito

$$d\mathrm{P\&L} = \left\{\Theta_i dt + \Delta_i dS + \frac{1}{2}\Gamma_i S^2\sigma^2 dt\right\} - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

$$= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2\sigma^2\right\}dt + (\Delta_i - \Delta_r)dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

But Black-Scholes with option bought at implied vol and with realized volatility set to implied volatility gives you $\Theta_i = -\frac{1}{2}\Gamma_i S^2 \Sigma^2 + rV_i - (r-D)S\Delta_i$

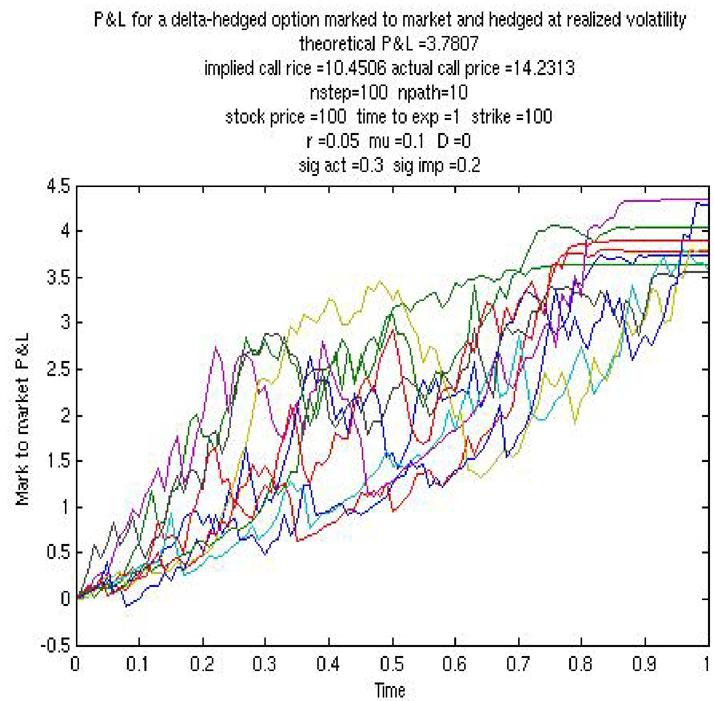
and so

$$dP\&L = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt + (\Delta_i - \Delta_r)\{(\mu - r + D)Sdt + \sigma SdZ\}$$

The total integrated P&L is deterministic but the increments have a random component dZ.

To illustrate this, plot cumulative **P&L** along ten random stock paths, 100 steps

P&L starts at zero because initial position is totally finance,

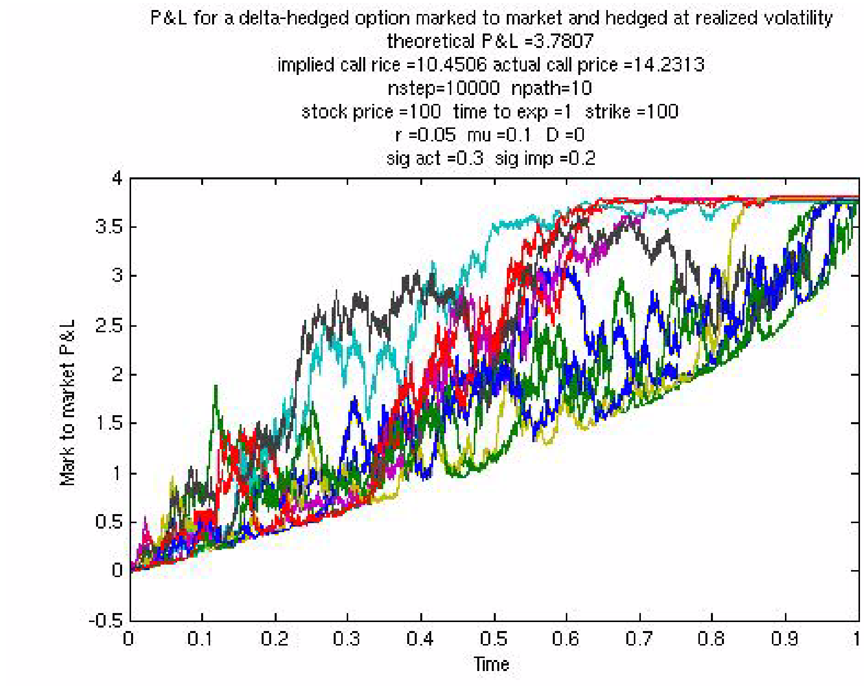


$$\sigma_r = 0.3$$

$$\sigma_i = 0.2$$

The final P&L is almost path-independent – almost, because 100 rehedgings per year is not quite the same continuous hedging.

Rehedge 10,000 times, almost independent **P&L**:



Bounds on the P&L When Hedging at the Realized Volatility

We had $\sigma > \Sigma$. Notice the upper and lower bounds. Why? We had

$$d\text{PV(P\&L)} = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_i - V_r)\right] = e^{rt_0} d\left[e^{-rt}(V_i - V_r)\right]$$

Integrate from $t_0 \equiv 0$, S_0 to t, S, to obtain

$$PV(\text{P\&L(t)}) = -[V(\sigma, S, t) - V(\Sigma, S, t)]e^{-rt} + [V(\sigma, S_0, 0) - V(\Sigma, S, 0)] \qquad \text{Eq.1.10}$$

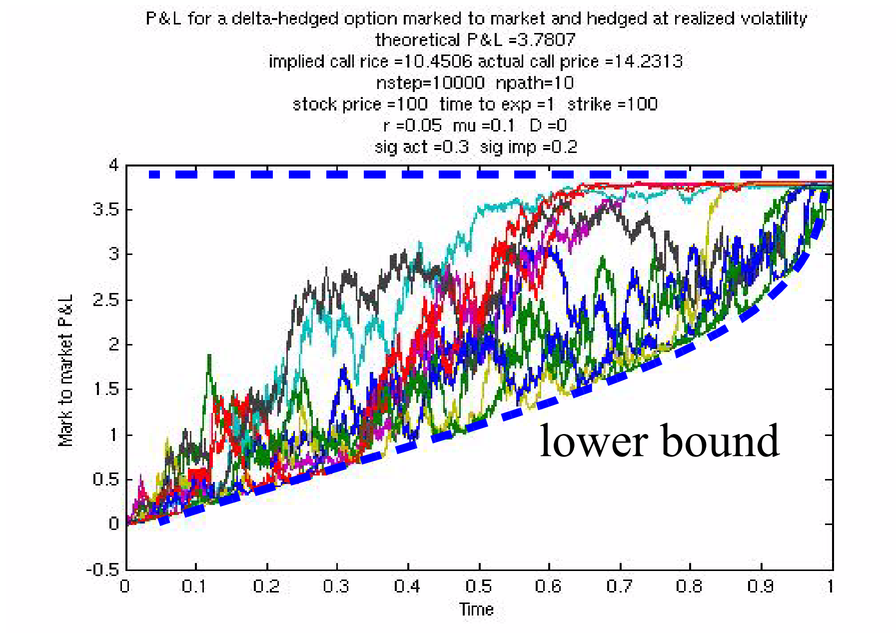
value along way> 0 value at inception > 0

Both terms in the square brackets in are positive.

Upper bound $[V(\sigma, S_0, 0) - V(\Sigma, S, 0)]$ occurs when the first term is zero, which occurs at $S = 0$ or $S = \infty$, and is $[V(\sigma, S_0, 0) - V(\Sigma, S, 0)]$.

The lower bound to the P&L occurs when the first term $[V(\sigma, S, t) - V(\Sigma, S, t)] \sim \frac{\partial V}{\partial \sigma}(\Sigma - \sigma)$ is a maximum, i.e. when vega is largest, close to at-the money, which turns out to be at

$$S = Ke^{-(r - 0.5\sigma\Sigma)(T-t)}$$



P&L for a delta-hedged option marked to market and hedged at realized volatility
theoretical P&L =3.7807
implied call rice =10.4506 actual call price =14.2313
nstep=10000 npath=10
stock price =100 time to exp =1 strike =100
r =0.05 mu =0.1 D =0
sig act =0.3 sig imp =0.2
Mark to market P&L
Time
lower bound

Hedging with Implied Volatility, Evolving at Realized

When you hedge with implied, the final value of the P&L depends on the path taken, and is not deterministic, but there is **no random mishedging component at each instant**.

Table 2: Position Values when Hedging with Implied Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_i\vec{S}$	$\Delta_i S - V_i$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_i\vec{S}, -\Delta_i(S + dS)$	$(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$	$(V_i + dV_i - \Delta_i(S + dS))$ $(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$

$$
\begin{aligned}
d\text{P\&L} &= [V_i + dV_i - \Delta_i(S + dS)] + (\Delta_i S - V_i)(1 + rdt) - \Delta_i DSdt \\
&= dV_i - \Delta_i dS - r(V_i - \Delta_i S)dt - \Delta_i DSdt
\end{aligned}
$$

Using Ito:

$$
\begin{aligned}
dP\&L &= \left[\Theta_i dt + \cancel{\Delta_i dS} + \frac{1}{2}\Gamma_i S^2 \sigma^2 dt\right] \cancel{- \Delta_i dS} - r(V_i - \Delta_i S)dt - \Delta_i DSdt \\
&= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2 \sigma^2 + (r - D)\Delta_i S - rV_i\right\}dt
\end{aligned}
$$

Black-Scholes equation for all volatilities, hedging and realized, equal to σ_i, is

$$\Theta_i + \frac{1}{2}\Gamma_i S^2 \Sigma^2 + (r - D)\Delta_i S - rV_i = 0$$

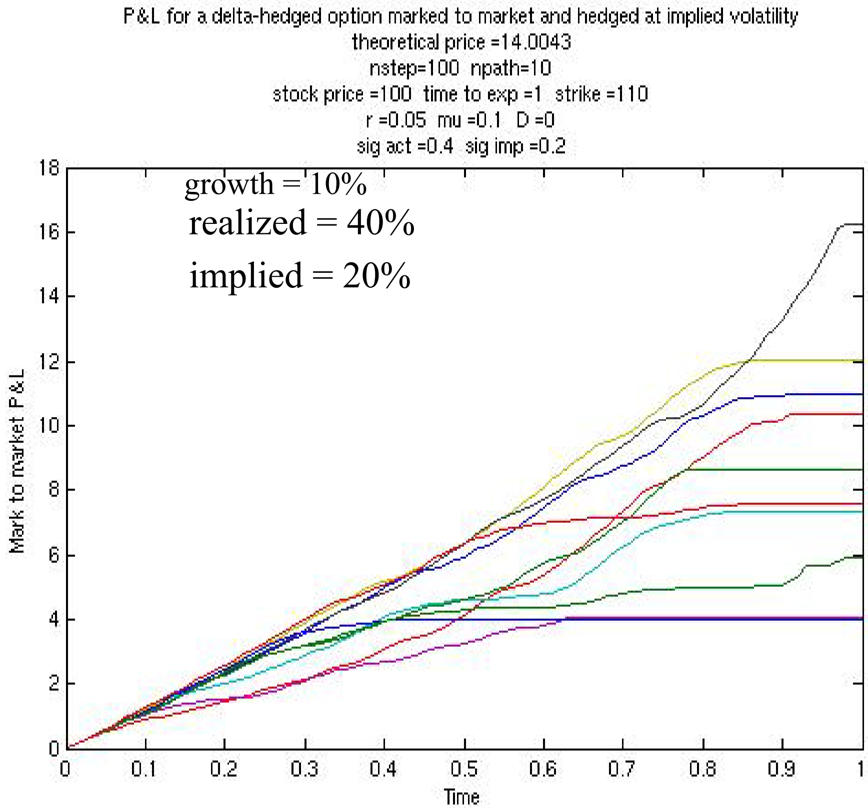
So

$$dP\&L = \frac{1}{2}\Gamma_i S^2 (\sigma^2 - \Sigma^2)dt \qquad \text{Eq.1.11}$$

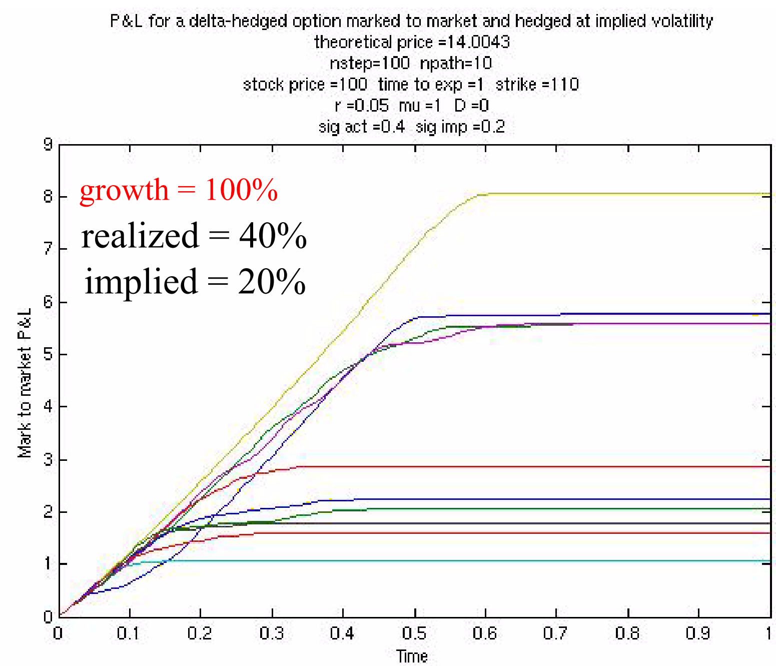
$$PV(P\&L) = \frac{1}{2}\int_{t_0}^{T} \Gamma_i S^2 (\sigma^2 - \Sigma^2) e^{-r(t - t_0)} dt \qquad \text{Eq.1.12}$$

The P&L is highly path-dependent. Although the hedging strategy captures a value proportional to $(\sigma^2 - \Sigma^2)$, it depends strongly on moneyness.

Cumulative P&L along 10 random stock paths, 100 hedging steps to expiration



worth less because more often out of the money



In practice, realized volatility isn't known in advance. A trading desk would most likely hedge at the constantly varying implied volatility which would move in synchronization with the recent realized volatility.