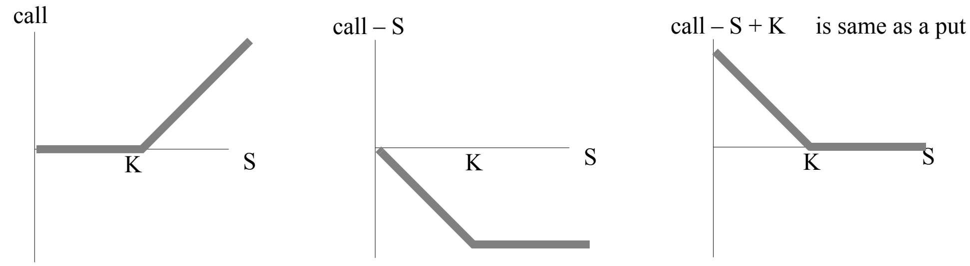
LECTURE 4

REPLICATION: VARIANCE SWAPS, MOSTLY

Recap: Static Replications of Various Kinds

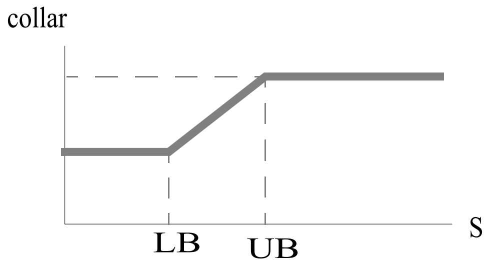
If you can create a static replicating portfolio for your payoff, you have very little model risk.

European put from a call: Put-Call Parity



Thus price of put = price of call - price of stock + PV(K).

A collar is a very popular instrument for portfolio managers who have made some gains during the year and now want to make sure they keep some upside but don't lose too much downside.



You can write the payoff as

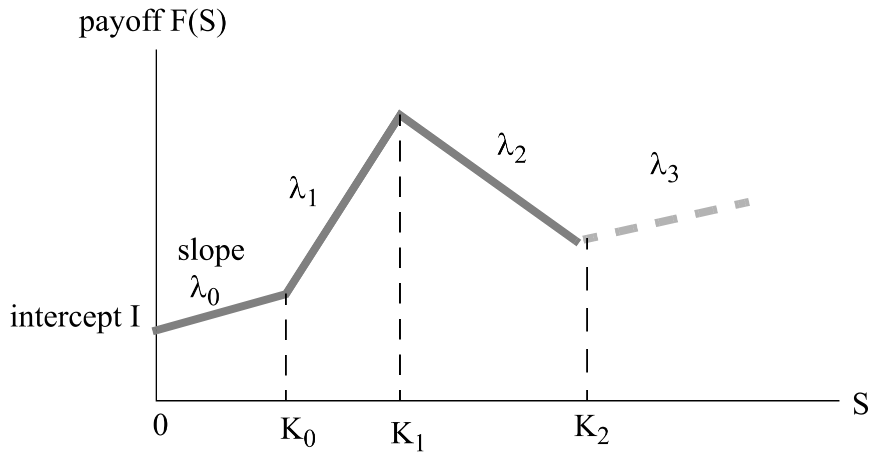
$$LB + call(S, LB) - call(S, UB)$$

Using put-call parity: $S + put(S, LB) - call(S, UB)$.

Its popularity forces dealers to be short puts and long calls.

Generalized European payoffs:

Piecewise-linear function of the terminal stock price S



Replicating portfolio consisting of a zero-coupon bond ZCB(I) plus a series of calls $C(K_i)$:

$$ZCB(I) + \lambda_0 S + (\lambda_1 - \lambda_0)C(K_0) + (\lambda_2 - \lambda_1)C(K_1) + \ldots$$

whose value can be determined from market prices.

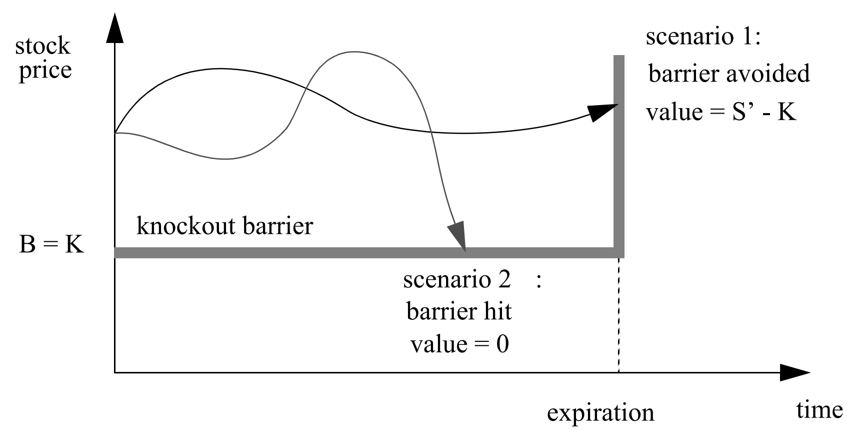
Static Hedge for a Down-and-Out Call with Strike = Barrier

This option has a time-dependent boundary.

Stock price S and dividend yield d, strike K and out-barrier $B = K$.

Scenario 1 in which the barrier is avoided and the option finishes in-the-money.

Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



In scenario 1 the call pays out $S'-K$, the payoff of a forward contract with delivery price K worth

$$F = Se^{-dt} - Ke^{-rt}$$

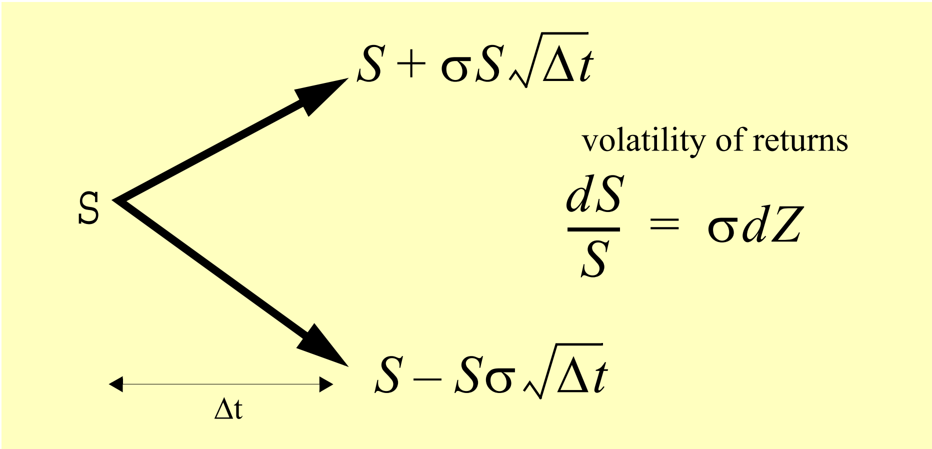
For paths in scenario 2, the down-and-out call immediately expires with zero value. In that case, the above forward F that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$. This is close to zero.

When the stock hits the barrier you must sell the forward to end the trade.

DYNAMIC REPLICATION

Quick Derivation of the Black-Scholes PDE

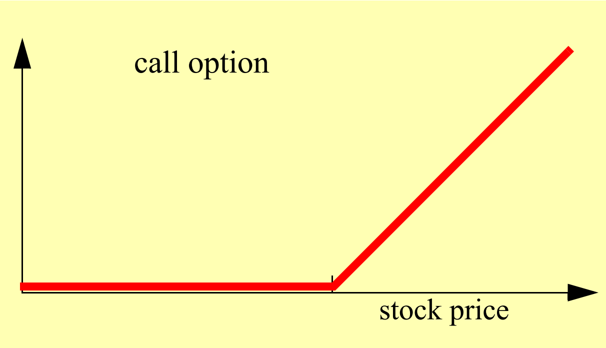
Assume GBM with zero rates for simplicity.



In time Δt, $\Delta S \approx \sigma S\sqrt{\Delta t}$.

The stock S is a primitive, linear underlying security that provides a linear position in ΔS.

If you are long an option, you profit whether the stock goes up or down! The call has curvature, or convexity. $\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$



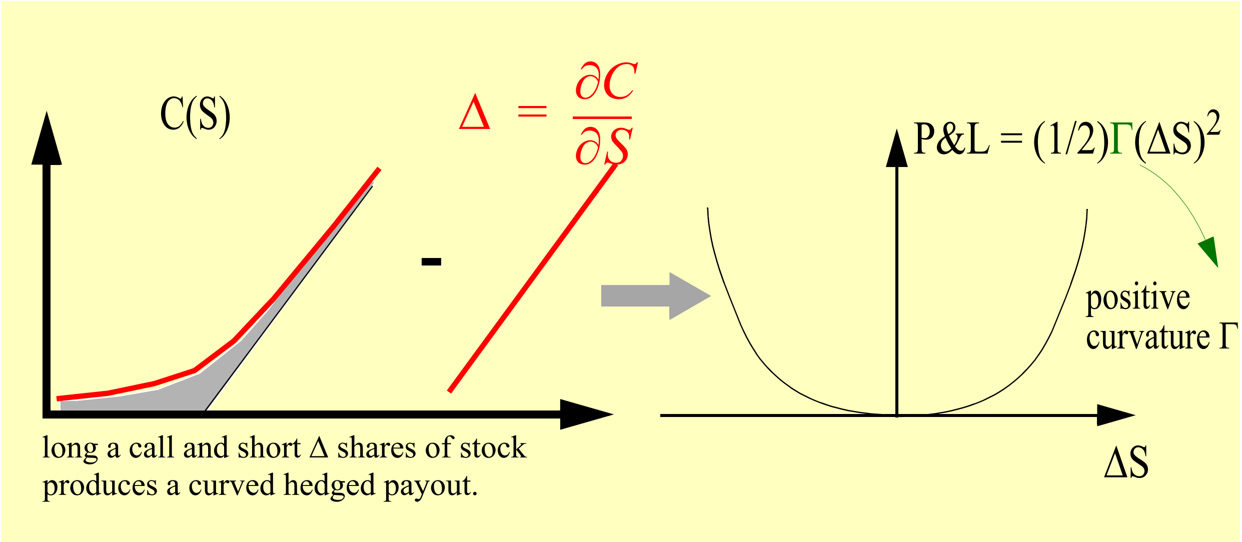
What is the fair price for *C(S,K,t,T)*?

We can do a Taylor series expansion on the unknown price *C( )* and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S+\Delta S, t+\Delta t) = C(S,t) + \left.\frac{\partial C}{\partial t}\right|_{S,t}\Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,t}\Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,t}\frac{(\Delta S)^2}{2} + \ldots$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

If you hedge away the linear term in ΔS by shorting $\Delta = \frac{\partial C}{\partial S}$ shares the profit and loss of the hedged option position looks like this:



Positive convexity generates a profit or loss that is quadratic in (ΔS).

What Should You Pay for Convexity?

Suppose we think we know the future volatility of the stock, Σ.

Over time Δt, the stock should move an amount $\Delta S = \pm \Sigma S Z(0, 1)\sqrt{\Delta t}$.

Binomially, this corresponds to $\Delta S = \pm \Sigma S \sqrt{\Delta t}$ with $(\Delta S)^2 = \Sigma^2 S^2 \Delta t$.

Change in value from the movement in stock price $= \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t)$

Change in value from passage of time $= \Theta(\Delta t)$ where $\Theta = \frac{\partial C}{\partial t}$

Total change in value of the hedged position is $d\text{P\&L} = d(C - \Delta S) = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t) + \Theta(\Delta t)$

If we know Σ, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it behaves like a riskless bond and must earn zero interest: $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

The Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

time decay and curvature are linked

$$C_{BS}(S, K, \Sigma, t, T) = SN(d_1) - KN(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

The option's Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate r is non-zero, we will show in a subsequent chapter that

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

Hedging an Option Means Betting On Volatility

Σ is the implied volatility that we inserted, our expectation of future volatility.

Suppose the stock actually evolves with a realized volatility σ Then the actual P&L is:

The gain from curvature is $\frac{1}{2}\Gamma\sigma^2 S^2 \Delta t$

The loss from time decay $\Theta\Delta t$ is $\frac{1}{2}\Gamma S^2 \Sigma^2 \Delta t$ because BS equation is $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

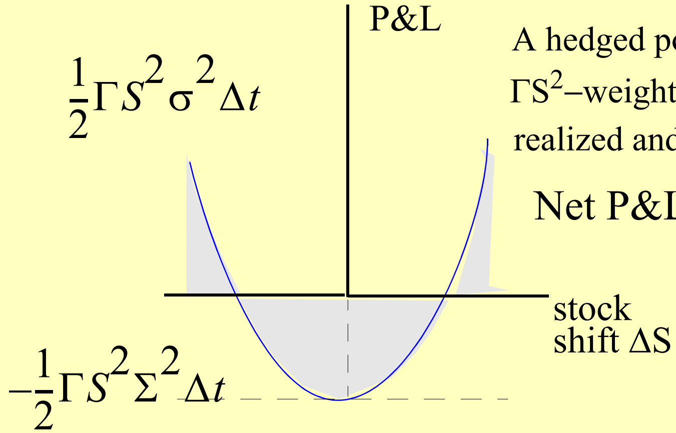
The net P&L during time Δt is $\left[\frac{1}{2}\Gamma(\sigma^2 - \Sigma^2)S^2 \Delta t\right]$.

This is path dependent unless ΓS^2 is independent of the stock price, which is not the usual case.

Here is an illustration of the contributions to the P&L:

To profit, you need the realized volatility to be greater than the implied volatility. A short position profits when the opposite is true.

Note: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S.



A hedged position earns the
ΓS^2–weighted difference between
realized and implied volatility

$$\text{Net P\&L} = \frac{1}{2}\int \Gamma S^2(\sigma^2 - \Sigma^2)\Delta t$$

Betting on Pure Volatility

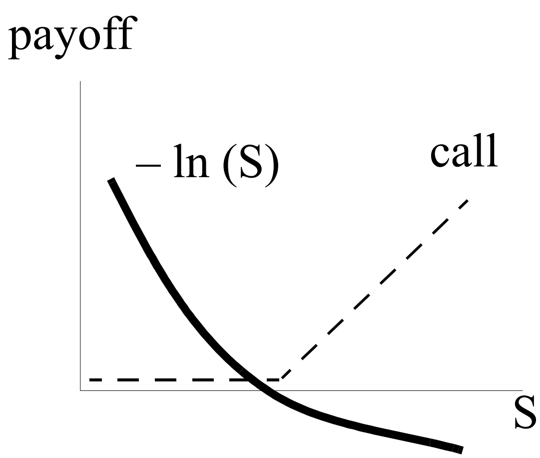
$$\text{Net P\&L} = \frac{1}{2}\int \Gamma S^2(\sigma^2 - \Sigma^2)\Delta t$$

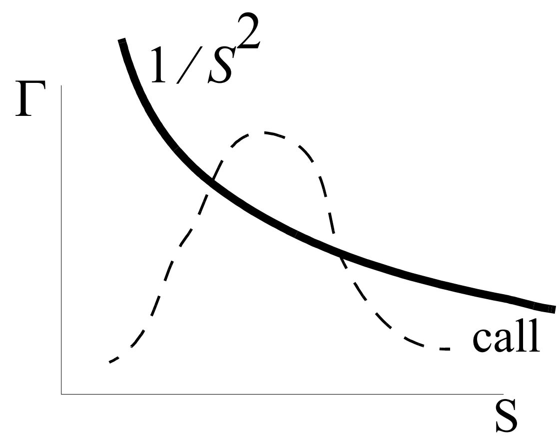
In a BS world, you can capture pure volatility if you own a derivative O whose curvature satisfies

$$\Gamma_O = 1/S^2 \quad \text{P\&L(O)} = \int \frac{1}{2}(\sigma^2 - \Sigma^2)\Delta t$$

The security with this gamma is the "log contract" with value $O = -\ln S$ and a hedge ratio $\Delta = -1/S$, **independent** of volatility! You hedge it by owning \$1 worth of stock always.

A log contract, hedged, will capture realized variance.





VARIANCE SWAPS
HOW TO TRADE VOLATILITY
(A LESSON IN REPLICATION)

Volatility and Variance Swap Contracts

A Volatility swap is a forward contract on realized volatility. At expiration it pays the difference in dollars between the actual return volatility realized by the index over the lifetime of the contract σ_R and some previously agreed upon "delivery" volatility K_{vol}:

$$(\sigma_R - K_{vol}) \times N \text{ where } N \text{ is the notional amount.}$$

Similarly, a **variance swap** is a forward contract on realized variance. It pays

$$\left(\sigma_R^2 - K_{var}\right) \times N$$

Note: $N\left(\sigma_R^2 - K_{var}\right) \approx 2N\sqrt{K_{var}}(\sigma_R - \sqrt{K_{var}})$, therefore notional vega is approximately $2\sqrt{K_{var}}$ times notional variance. But variance is a derivative of volatility.

The contract must also specify the precise method for calculating at expiration the realized volatility, including the source and observation frequency of prices, the annualization factor and whether the sample mean is subtracted from each return.

1.1 Intuitive Approach to Variance Replication in a BS World

Zero interest rates for simplicity, so $C = C(S, K, \nu)$ where $v = \sigma\sqrt{\tau}$.

$$C_{BS} = SN(d_1) - KN(d_2) \qquad d_{1,2} = \frac{\ln S/K \pm v^2/2}{v}$$

Then the exposure to volatility is given by

$$\kappa \equiv \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

You can see that the option has sensitivity to S and σ, and is therefore not a good way to make a clean bet on volatility. What we want is a portfolio whose exposure κ to volatility is independent of the stock price S, so that we can bet on volatility no matter what the stock price does.

Construct a portfolio $\pi(S) = \int_0^{\infty} \rho(K)C(S, K, v)dK$ such that $\kappa = \frac{\partial \pi}{\partial \sigma^2}$ is independent of S.

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(K)\frac{S\sqrt{\tau}}{2\sigma}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}dK \sim \int_0^{\infty} \rho(K)Sf\left(\frac{K}{S}, v\right)dK$$

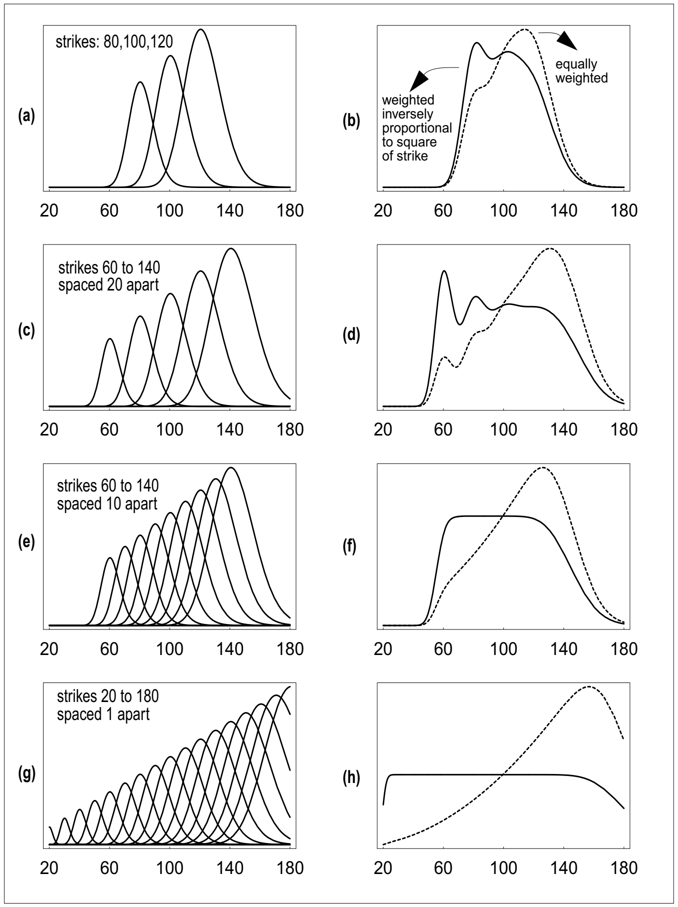
We can make the S-dependence of this explicit by changing variable to $x = K/S$ so that

$$\frac{\partial\pi}{\partial\sigma^2} = \int_0^{\infty} \rho(xS)S^2 f(x, v)dx$$

In order for this to be independent of S, we require that $\rho(K) \sim 1/K^2$

A density of options whose weights decrease as K^{-2} will give the correct volatility dependence.

FIGURE 1. The variance exposure, V, of portfolios of call options of different strikes as a function of stock price S. Each figure on the left shows the individual V_i contributions for each option of strike K_i. The corresponding figure on the right shows the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally-weighted sum of options; the solid line corresponds to weights inversely proportional to K^2, and becomes totally independent of stock price S inside the strike range



What Payoff Are We Replicating?

Use liquid puts below some strike S^* and use calls with strikes above S^*. The payoff at expiration is

$$\pi(S, S^*, v) = \int_{(K > S^*)} C(S, K, v)\frac{dK}{K^2} + \int_{(K < S^*)} P(S, K, v)\frac{dK}{K^2}$$

What does this payoff look like at expiration when $\tau = 0$: Call has S > K; Put has S < K

$$\pi(S, S^*, v) = \left(\int_{S^*}^{S} (S-K)\frac{dK}{K^2}\right) \text{ for S>S* and } \int_{S}^{S^*} (K-S)\frac{dK}{K^2} \text{ for S<S*}$$

$$= -\ln\frac{S}{S^*} + \left(\frac{S-S^*}{S^*}\right)$$

In order to be exposed purely to volatility, we need to short a log contract L and own a forward contract with delivery price S^*, which has no volatility dependence and can be replicated statically

1.2 Value of Log Contract in a Black-Scholes World

Solve the Black-Scholes equation $\frac{\sigma^2 S^2}{2}\frac{\partial L}{\partial S^2} + \frac{\partial L}{\partial t} = 0$ for r =0, with the boundary condition for the log payoff $L(S, S^*, 0) = -\ln\frac{S}{S^*}$.

Solution: $L(S, S^*, \tau) = -\ln S/S^* + (\sigma^2\tau)/2$. with volatility exposure $\kappa = \tau/2$.

The delta of the contract is $-1/S$.

Going long 1/S shares at any instant – i.e. by owning exactly \$1 worth of shares at any instant – you have exactly a $\Gamma = 1/S^2$ and the right vol. exposure.

At the start of the trade, when $t = 0$ and $\tau = T$, you need to buy 2/T contracts to have $\kappa = 1$, a variance exposure of \$1 for the whole trade.

$\Pi(S, S_*, t, T) = \frac{2}{T}\left[\frac{S-S_*}{S_*} - \ln\frac{S}{S_*}\right] + \frac{T-t}{T}\sigma^2$, after hedging, captures $\sigma_R^2 - \sigma_I^2$

If $S^* = S_0$ then $\Pi(S, S_0, t, T) = \frac{2}{T}\left[\frac{S-S_0}{S_0} - \ln\frac{S}{S_0}\right] + \frac{T-t}{T}\sigma^2$ and $\Pi(S_0, S_0, 0, T) = \sigma^2$

Proof that the fair value of a log contract with $S^*=S_0$ is actually the variance. (Assume r = 0)

Consider a log contract that pays out $\log(S_T/S_0)$ at expiration time T. Let its value today be denoted by L_0. Look at the trading strategy below that starts with a short position in one log contract and long \$1 worth of shares, and then maintains this dollar value of shares by rehedging as below.

time	stock	#shares	value	Dollars	log contracts	Total value
t_0	S_0	$1/S_0$	1	0	-1 worth L_0	$-L_0 + 1$
t_1	S_1	$1/S_0$	S_1/S_0	0	-1 worth L_1	$-L_1 + S_1/S_0$

Now rebalance to own \$1 worth of shares:. buy $(1/S_1 - 1/S_0)$ shares by borrowing $(1/S_1 - 1/S_0)S_1 =$ $(S_0 - S_1)/S_o$ dollars. You then own $1/S_1$ shares worth \$1, and you have borrowed (that is, you are short) $(S_0 - S_1)/S_o$ dollars. Then, after rebalancing,

time	stock	#shares	value	Dollars	log contracts	Total value
t_1	S_1	$1/S_1$	1	$-(S_0 - S_1) \div S_o$	-1 worth L_1	$-L_1 + 1 + (S_1 - S_0) \div S_0$

Now move to time t_2 and rebalance again, to get

time	stock	#shares	value	Dollars	log contracts	Total value
t_2	S_2	$1/S_2$	1	$-(S_0 - S_1)/S_o$ $-(S_1 - S_2)/S_1$	-1 worth L_2	$-L_2 + 1 + \begin{matrix} -(S_0 - S_1) \div S_o \\ -(S_1 - S_2) \div S_1 \end{matrix}$

Repeat rehedging N times to expiration:

$$1 - L_T + \frac{S_1 - S_0}{S_0} + \frac{S_2 - S_1}{S_1} + \ldots + \frac{S_N - S_{N-1}}{S_{N-1}} = 1 - L_T + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_{N-1}}\frac{S_{N-1}}{S_{N-2}}\cdots\frac{S_1}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \sum_{i=0}^{N-1}\left(\log\frac{S_{i+1}}{S_i}\right) + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$\approx 1 - \sum_{i=0}^{N-1}\left[\frac{\Delta S_i}{S_i} - \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2\right] + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$ in a Taylor expansion to second order

$$= 1 + \sum_{i=0}^{N-1}\frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 = 1 + \sum\frac{\sigma_i^2 \Delta t_i}{2}$$

Thus, if you assume zero interest rates, we've shown that an initial investment at time t = 0 of value $-L_0 + 1$, by dynamic rehedging, leads to a final value at time t = T of $1 + \sum \frac{\sigma_i^2 \Delta t_i}{2}$.

Therefore, the fair value of L_0 at the beginning must be $L_0 = -\sum_i \frac{\sigma_i^2 \Delta t_i}{2}$.

Being short a log contract with strike S_0 and being long \$1 worth of stock, dynamically rehedging as the stock moves, will guarantee you a final payoff equal to the realized volatility (assuming GBM with a variable volatility).

Problems with Replication

If you could buy a log contract you'd have exactly what you want. Instead you have to buy a continuum of calls and puts, which doesn't exist. You can only buy a discrete number in a discrete range, so you have no sensitivity to volatility outside the strike range.

1.3 More Rigorous Results

Everything so far has been in a Black-Scholes world, unlike the real one, which has skew, stochastic volatility and jumps. Does this still work?

As long as there is continuous diffusion (no jumps), the log contract still captures realized volatility.

$$\frac{dS}{S} = \mu dt + \sigma(t, \ldots)dZ_t$$

$$d\ln S = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t$$

$$\frac{dS}{S} - d\ln S = \frac{1}{2}\sigma^2 dt$$

$$\text{total variance} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\underbrace{\int_0^T \frac{dS_t}{S_t}}_{\text{rebalanced hedge}} - \underbrace{\ln\frac{S_T}{S_0}}_{\text{short log contract}}\right]$$

No expectations have been taken here; replication is enough to capture variance! Thus

$$-\ln\frac{S_T}{S_0} = \frac{1}{2}\int_0^T \sigma^2 dt - \int_0^T \frac{dS_t}{S_t} \text{ and } \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right]$$

You cannot buy/sell a log contract security $\ln(S/S_0)$, but you can replicate it out of hedging \$1 worth of stock, owning a forward, and owning calls and puts

$$-\ln \boldsymbol{S_T}/\boldsymbol{S_0} = -\ln S_*/S_0 - \ln \boldsymbol{S_T}/\boldsymbol{S_*} \qquad \textbf{and by payoff replication}$$

$$-\ln \boldsymbol{S_T}/\boldsymbol{S_*} = -\frac{(S_T - S_*)}{S_*} + \int_{(K > S^*)} C(S_T, K, v)\frac{dK}{K^2} + \int_{(K < S^*)} P(S_T, K, v)\frac{dK}{K^2}$$

Therefore

$$-\ln \boldsymbol{S_T}/\boldsymbol{S_0} = -\ln S_*/S_0 - \frac{(S_T - S_*)}{S_*} + \int_{(K > S^*)} C(S_T, K, v)\frac{dK}{K^2} + \int_{(K < S^*)} P(S_T, K, v)\frac{dK}{K^2}$$

Therefore, adding in the integral, we have **a prescription for generating realized volatility**:

$$\frac{1}{T}\int_0^T \sigma^2 dt =$$

"rehedging" \$1

$\ln(S_T/S_0)$

options at expiration

$$= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_*}{S_0} - \frac{(S_T - S_*)}{S_*} + \int_{(K > S^*)} C(S_T, K, 0)\frac{dK}{K^2} + \int_{(K < S^*)} P(S_T, K, 0)\frac{dK}{K^2}\right]$$

The RHS consists of a strategy trade and some payoffs of forwards and options that tell us how to replicate the variance.

Its fair value is given by the risk-neutral expectation. If in the risk-neutral world $\frac{dS_t}{S_t} = r\,dt + \sigma_t dZ_t$

- $\frac{dS_t}{S_t} = \frac{S_{n+1} - S_n}{S_n}$ is the adjustment in the position that always holds \$1 worth of stock.
-
- $E\left[\int_0^T \frac{dS_t}{S_t}\right] = rT$ and $E[S_T] = S_0 e^{rT}$
- The current call value $C(S, K) = e^{-rT} E[C(S_T, K, 0)]$ so $E[C(S_T, K, 0)] = e^{rT} C(S, K)$

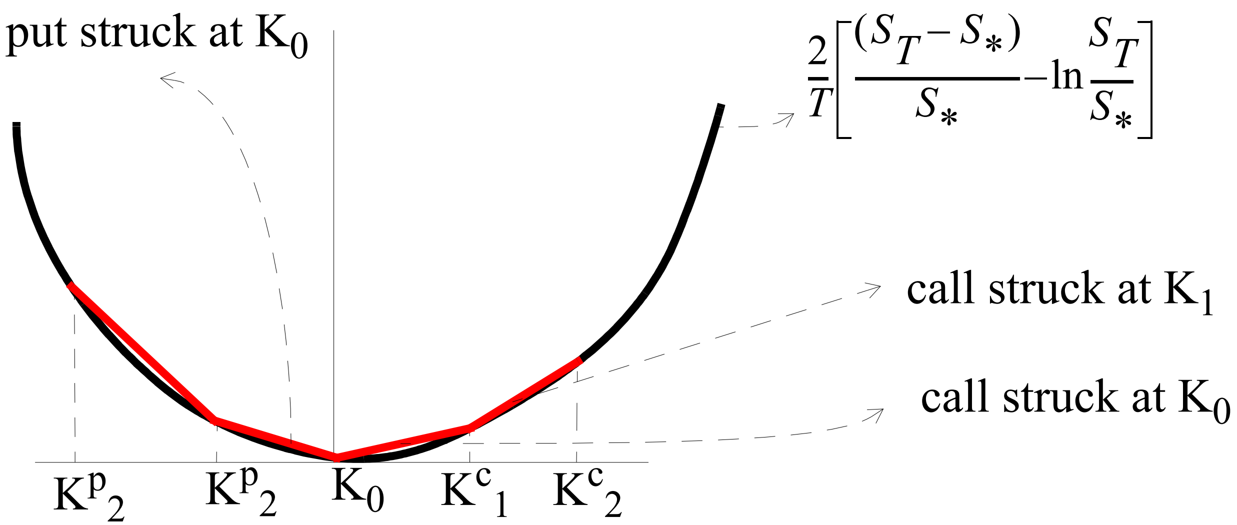
Thus the fair value of the total variance is

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{rT}}{S_*} - 1\right) + e^{rT} \int_{(K > S^*)} C(S, K)\frac{dK}{K^2} + e^{rT} \int_{(K < S^*)} P(S, K)\frac{dK}{K^2}\right]$$

Every option's price can be taken from the marketplace, even with a skew, and we can value the variance almost independent of theory.

Fair variance in a skew

Replicate $\frac{2}{T}\left[\frac{(S_T - S_*)}{S_*} - \ln\frac{S_T}{S_*}\right]$ by linear-payoff calls and puts that dominate it, with strikes $K_i^{c,p}$ for the calls and puts.



1.4 Imperfections in Valuation by Replication

- Discrete strikes with a limited range capture less variance than the true variance. You gamble by omitting some strikes because when/if the stock price gets to those strikes, you have no options to capture the variance.
- Effect of jumps
  The log contract doesn't capture the true variance if jumps occur, for two reasons.

  1. Jumps can move the stock price out of the range of replication.
  2. Jumps contribute to the realized variance proportional to J^2, but jumps contribute to the log contract with a J^3 term too.

  The log contract captures

$$\sum \frac{\Delta S_i}{Si} - \log\frac{S_T}{S_0} = \sum\left[\frac{\Delta S_i}{Si} - \log\frac{S_{i+1}}{S_i}\right] = \sum\left[\frac{\Delta S_i}{Si} - \log\left(1 + \frac{\Delta S_i}{S_i}\right)\right]$$

$$\approx \sum \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 - \frac{1}{3}\left(\frac{\Delta S_i}{S_i}\right)^3 + \ldots$$

The first term is the true variance contribution; the second is normally negligible, but for a large jump $(\Delta S_i)/S_i = J$ will add an asymmetric term to the P&L that is absent from the true variance

.

1.5 Valuing Volatility Swaps

Volatility is the square root of variance, a derivative. You can replicate it with the continuous dynamic trading of portfolios of variance swaps, just as you can replicate $\sqrt{S}$ by trading S.

Expand about V_E, the expected variance.

$$\sigma = \sqrt{\sigma^2} = \sqrt{V} \equiv \sqrt{V_E + \{V - V_E\}}$$

$$= \sqrt{V_E}\left(1 + \frac{V - V_E}{V_E}\right)^{1/2}$$

$$\approx \sqrt{V_E}\left[1 + \frac{V - V_E}{2V_e} - \frac{1}{8}\left(\frac{V - V_E}{V_E}\right)^2 + \ldots\right]$$ The square root has negative convexity therefore worth less.

$$\approx \sqrt{V_E} + \frac{V - V_E}{2\sqrt{V_E}} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$$

Taking risk-neutral expectations: $E(\sigma) \approx \sqrt{V_E} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$

Thus the fair volatility is smaller than the square root of the variance, and depends on the volatility of variance, like an option on variance.

1.6 The VIX Volatility Index

The VIX, from 1993 - 2003, used to defined as the weighted average of various atm and otm implied volatilities. This was rather arbitrary. In 2003 the CBOE changed the definition of the VIX to be the square root of the fair delivery price of variance as captured by a variance swap, using the formula from this paper with stock dividends.

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[(r-d)T - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{(r-d)T}}{S_*} - 1\right) + e^{rT}\int_{(K > S^*)} C(S, K, 0)\frac{dK}{K^2} + e^{rT}\int_{(K < S^*)} P(S, K, 0)\frac{dK}{K^2}\right]$$

The RHS is

$$\frac{2}{T}\left\{\ln\frac{F}{S_0} - \ln\frac{S_*}{S_0} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls above S* plus puts below S*}]\right\}$$

$$= \frac{2}{T}\left\{\ln\frac{F}{S^*} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$= \frac{2}{T}\left\{\ln\left(1 + \frac{F}{S^*} - 1\right) - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$\approx \frac{2}{T}\left\{e^{rT}[\text{sum of calls and puts}] - \frac{1}{2}\left(\frac{F}{S_*} - 1\right)^2\right\}$$

The CBOE uses a finite sum over traded options at two expirations near 30 days, and then interpolates/extrapolates to thirty day volatility.

Some advantages of the new VIX

- The VIX is an estimate of one-month future realized volatility based on listed options prices. The value of the VIX depends on implied volatility.
- The estimate is independent of market level because it involves the sum of different options prices.
- It is relatively insensitive to model issues, because it assumes only continuous underlier movement, but doesn't assume Black-Scholes.
- It is hedgeable because it involves a portfolio of listed options.

You can therefore in principle price futures, forwards and options on the VIX.

Future Extensions

Many variance swaps are capped and implicitly contain embedded volatility options.

Valuing options on volatility is the big challenge.

Modeling the VIX and VIX futures because it's the most liquid measure of volatility.

1.7 Aside: The Black-Scholes Equation and Sharpe Ratios

Valuation by perfect replication. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

$$
\begin{aligned}
dS_t &= \mu_S S_t dt + \sigma_t S_t dZ_t \\
dB_t &= B_t r_t dt
\end{aligned}
\qquad \text{Eq.1.1}
$$

The option price $C(S_t, t)$ whose evolution is given by

$$
\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S^2} (\sigma_t S_t)^2 dt \\
&= \left\{ \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S} \mu_S S_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S^2} (\sigma_t S_t)^2 \right\} dt + \frac{\partial C_t}{\partial S} \sigma_t S_t dZ_t \\
&\equiv \mu_C C_t dt + \sigma_C C_t dZ_t
\end{aligned}
$$

where by definition

$$\mu_C = \frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2\right\} \tag{Eq.1.2}$$

$$\sigma_C = \frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_t S_t\right)$$

Riskless portfolio $\pi = \alpha S + C$

Then

$$\begin{aligned} d\pi &= \alpha\{\mu_S S_t dt + \sigma_t S_t dZ_t\} + \{\mu_C C_t dt + \sigma_C C_t dZ_t\} \\ &= (\alpha\mu_S S_t + \mu_C C_t)dt + (\alpha\sigma_t S_t + \sigma_C C_t)dZ_t \end{aligned} \tag{Eq.1.3}$$

Riskless necessitates

$$\alpha = -\frac{\sigma_C C}{\sigma_S S} \tag{Eq.1.4}$$

That no riskless arbitrage: $d\pi = \pi r dt$.

Requires $\alpha\mu_S S + \mu_C C = (\alpha S + C)r$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for α from Equation 4.4 leads to the relation

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S} \qquad \text{Eq.1.5}$$

This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 4.2 into Equation 4.5 for μ_C and σ_C we obtain

$$\frac{\frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2\right\} - r}{\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_t S_t\right)} = \frac{\mu_S - r}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \qquad \text{Black-Scholes equation, no drift} \qquad \text{Eq.1.6}$$

It's good to get very familiar with manipulating this solution and its derivatives.

The solution, the Black-Scholes formula and its implied volatility, is the quoting currency for trading prices of vanilla options.

You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$C(S, K, t, T, r, \sigma) = e^{-r(T-t)} \times [S_F N(d_1) - KN(d_2)]$$

$$S_F = e^{r(T-t)} S$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.1.7}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma^2(T-t)$. If you rewrite the formula in terms of the prices of traded securities – the present value of the bond K_{PV} and the stock price S – then indeed time and volatility always appear together:

$$C(S, K, t, T, \sigma) = [SN(d_1) - K_{PV}N(d_2)]$$

$$K_{PV} = e^{-r(T-t)}K$$

$$d_{1,2} = \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.1.8}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

Note that the time to expiration appears in the formulas in two different combinations, $r(T-t)$ the discount factor and $\sigma^2(T-t)$ the total variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.