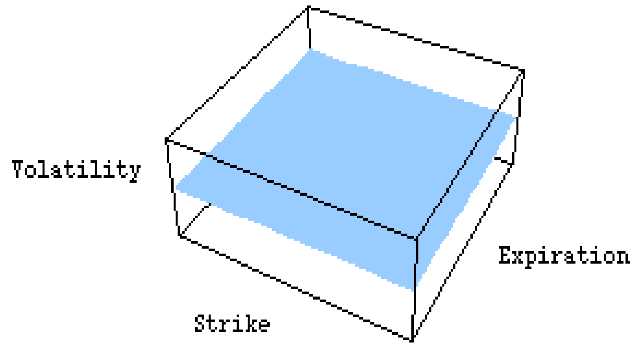
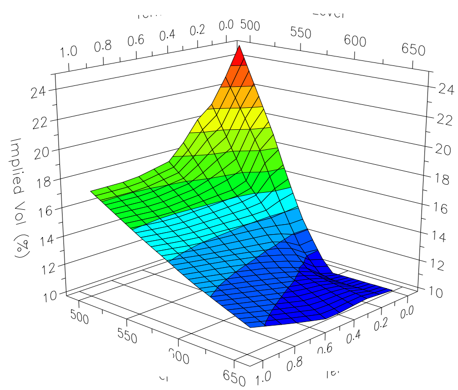
Lecture 1: Principles of Valuation; Introduction to the Smile

According to classic theory, the Black-Scholes implied volatility of an option should be independent of its strike and expiration. Plotted as a surface, it should be flat, as shown at right.

The volatility surface according to Black-Scholes



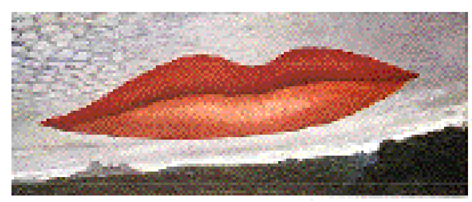
The volatility surface according to S&P options markets



Prior to the stock market crash of October 1987, the volatility surface of index options was indeed fairly flat.

Since the crash, the volatility surface of index options has become skewed. Referred to as the volatility smile, the surface changes over time. Its level at any instant is a varying function of strike and expiration, as shown at left.

The smile phenomenon has spread to stock options, interest-rate options, currency options, and almost ever other volatility market. Since the Black-Scholes model cannot account for the smile, trading desks have begun to use more complex models to value and hedge their options.



After 25 years, there is still no overwhelming consensus as to the correct model. Each market has its own favorite (or two). Despite initial optimism about finding **the** model to replace Black-Scholes, we are still searching in the dark.

Introduction

- The Great Financial Crisis and models
- *I am not interested in proofs, but only in what nature does* -- Paul Dirac
- Solutions, behavior, intuition about their behavior.

Aim of the Course

1. The nature of financial modeling
2. Understanding volatility as a quality, a quantity, and an asset
3. Understanding the practical use of the Black-Scholes-Merton model. There's more to it than just knowing the equation and its solution.
4. Understanding the successes and limitations of the Black-Scholes model.
5. Coming to grips with the volatility smile
6. The extensions of the Black-Scholes model to accommodate/explain the volatility smile.
7. Understanding the consequences of these extensions. It's easy to make up new and richer models but we want to understand whether they are realistic, whether they are advantageous, and what they lead to.
8. Learn how to use, when to use, how much to rely, some sense of efficacy and fallibility of models
9. How to make your own models …

Preamble

"Options theory is the most successful theory not only in finance, but in all of economics" Ross

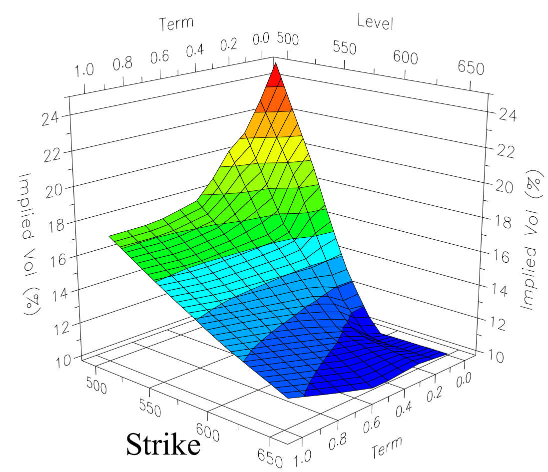
Why? What does that mean?

Assumptions?

Violations?

Robust, used as quoting mechanism, but it's not a solved problem.

FIGURE 1.1. The implied volatility surface for S&P 500 index options as a function of strike level and term to expiration on September 27, 1995.



What's the right replacement?

Why is it important?

Because of: hedging, valuation, model uncertainty

- This is a course about learning how to model and think about models, not about efficient implementation.
- So, I'll put a lot of effort into deriving *simple or approximate proofs* of the key model formulas and ideas.

References

Mostly my course is self-contained.

I don't require you to buy any textbook for the course but I can recommend the following for some additional material.

- *The Volatility Surface: A Practitioner's Guide* by Jim Gatheral, Wiley 2006.
  This book probably contains material most relevant to what I'm doing, though it approaches it somewhat differently and much more formally.

Some other useful books on the volatility smile:

- *Derivatives in Financial Markets with Stochastic Volatility* by JP Fouque, G. Papanicolaou and R. Sircar, Cambridge U. Press. This book is devoted to a particular perturbative treatment of rapidly mean-reverting stochastic volatility models. It has a very good introductory chapter, then gets pretty technical.
- There is also a qualitative chapter (Chapter 14) on the smile in my book, *My Life as a Quant: Reflections on Physics and Finance.* I will post an electronic version of that chapter on Courseworks.

Some more useful books of a more general nature are listed below:

- *Paul Wilmott on Quantitative Finance*, Wiley, by Paul Wilmott (who else?) is a very good general book on options theory. He's not afraid to tell you what he thinks is important and what isn't, which is valuable.
- *The Concepts and Practice of Mathematical Finance*, CUP 2004, by Mark S. Joshi.
  Good book, particularly on static hedging.

- *Options, Futures and Other Derivatives,* Prentice Hall, by John Hull. The standard comprehensive teaching book.
- *Introduction to Quantitative Finance*, Stephen Blyth, 2013, is a very elegant and compact book with a straightforward approach that tries to make things simple and clear rather than complicated and obscure.

Journals

- Risk Magazine
- Wilmott Magazine
- Journal of Derivatives
- Quantitative Finance, lots of econophysics and quant papers
- www.ssrn.com has many papers in the FEN (Financial Economics Network) section, and most of the latest papers get posted there before publication.

Contacting me

ed2114@columbia.edu.

If you have questions, please come to my office hours, which I will post. I cannot answer very long questions before or after class, or by email.

Grades

Homework assignments weekly. Occasional unannounced pop quizzes in class for about 10 minutes at a time, unannounced. These, together with the homework sets, will count for 10% of the final grade. A further 40% will depend on the midterm and 50% on the final examination.

Homework is due once a week, on a Monday **at the beginning of each class,** to be left in a pile at the front of the classroom.

Ethics

I don’t mind if you discuss homework among yourselves, but then I expect you to think about it and work it out and write it up by yourself afterwards.

Please don’t be part of several people handing in identical copies of solutions; don’t hand in xeroxes of someone else’s solution. If you do, I will consider it cheating.

1. Everyone is an adult in this course, here to learn. Responsible behavior is assumed.

2. Please don’t read email or browse the web in class. I have a policy of

NO PHONES, NO LAPTOPS OPEN DURING CLASS.

If you have some emergency that requires you to keep your phone on, please let me know.

3. Please don’t come late to class.

4. In the past couple of years there has been some cheating on homework as well as on exams, in many FE courses. I don’t want to be a judge or jury. If someone is found cheating in any way, I will simply send their name to the dean’s office and let them take appropriate action.

Course Outline

This is roughly what I would like to cover in the course, but we will have to play it by ear and see exactly how things progress.

- **The Principles of Financial Modeling**
  Aim of the course.
  A quick look at the smile.
  Viewpoints: relative rather than absolute valuation
  The foundations of financial theory
  Valuation: Static hedging, Dynamic hedging, Utility-based
  The theory of dynamic hedging.

- **Option Valuation: Realities and Myths**
  The theory of dynamic replication
  Option replication
  The Black-Scholes equation
  P&L (profit and loss) of options trading
  The difficulties of dynamic hedging; which hedge ratio to use
  The approximations and assumptions involved
  Simulations of discrete hedging
  Reserves for illiquid securities

- **Introduction to the Implied Volatility Smile**
  The smile in various markets
  The difficulties the smile presents for trading desks and for theorists
  Pricing and hedging
  Different kinds of volatility
  Parametrizing options prices: delta, strike and their relationship
  Estimating the effects of the smile on delta and on exotic options

Reasons for a smile
No-riskless-arbitrage bounds on the size of the smile
Fitting the smile
Some simple models and a look at their smiles

- **Implied Distributions Extracted from the Smile**
  Arrow-Debreu state prices
  Breeden Litzenberger formula
  Black-Scholes implied density and its use
  Static replication of path-independent exotic options with vanilla options

- **Static Hedging**
  Static replication of path-dependent exotic options with vanilla options

- **Extending Black-Scholes beyond constant-volatility lognormal stock price evolution**
  Binomial trees
  Time-dependent deterministic rates
  Time-dependent deterministic volatility
  Calibration to rates and volatility
  Changes of numeraire to simplify problems
  Alternative stochastic processes that could account for the smile

- **Local Volatility Models/ Implied Trees**
  Derman-Kani binomial local volatility trees
  Difficulties encountered
  Trinomial local volatility trees

- **Fitting Implied Binomial Trees to the Volatility Smile**
  Dupire equation
  Fokker-Planck/ forward Kolmogorov equation.

Calibration of implied binomial trees
How to build an implied tree from options prices.
The relation between local and implied volatilities

- **The Consequences of Local Volatility Models**
  The local volatility surface
  The relationship between local and implied volatility
  Estimating the deltas of vanilla options in the presence of the smile
  Estimating the values of exotic options
  Static hedging of barrier options
  Some specific local volatility models: displaced diffusion, CEV, mixed distributions

- **Model classification**
  Empirical behavior of implied volatility with time and market level
  Sticky strike, sticky delta, sticky implied tree

- **Stochastic Volatility Models**
  The effect of changes in volatility in the Black-Scholes formula
  The Vanna-Volga way of looking at things
  Mean reversion of volatility
  The SABR model
  The PDE for option value under stochastic volatility
  The mixing formula for option value under stochastic volatility
  Estimating the smile in stochastic volatility models
  Simulations of the smile in these models
  The relationship between local and stochastic volatility

- **Jump-Diffusion Models**
  Are they reasonable, and if so, when?
  Poisson jumps
  The Merton jump-diffusion model and its solution

Estimating the smile in jump-diffusion models
Simulations

- **Other Models**

...

Some Guest Speakers

Today and Next Class:

A Quick Look at the Implied Volatility Smile

Principles of Financial Valuation

Modeling Markets

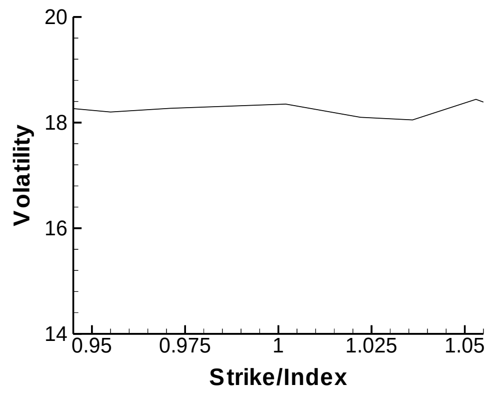
Methods of Replication

THE SMILE

A Quick Look at the Implied Volatility Smile

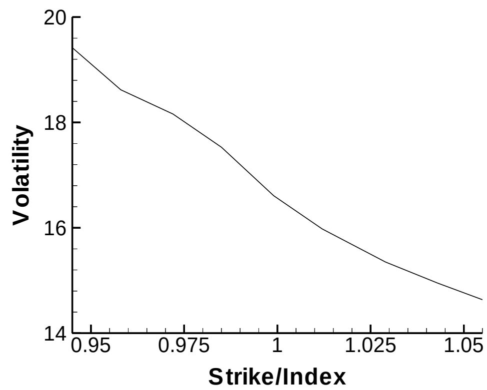
What is the definition of implied volatility? Which are the most liquid options?

• Representative S&P 500 implied volatilities prior to 1987. Data taken from M. Rubinstein,



Similar shapes with different levels of volatility at different times.

• Representative S&P 500 implied volatilities after 1987.



• The volatility of a *stock* itself cannot depend upon the option with which you choose to view it.

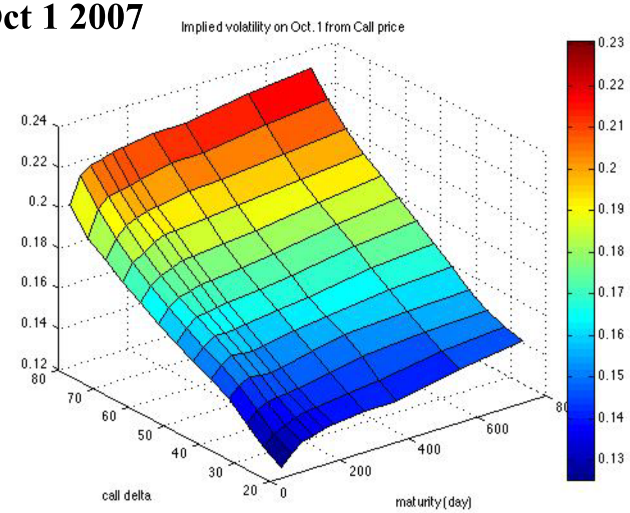
Development of the Smile

- There was always a bit of a smile in currency options markets.
- The equity "smile" is really more of a skew or a smirk.
- 1987 crash: a giant market could drop by 20% or more in a day. Low-strike puts more likely to end up in the money than high-strike. (What if you hedge?)
- Volatility smile has spread to most other options markets.
- Traders and quants in every product area have had to model the smile.
- No area where model risk/uncertainty is more of an issue than in the modeling of the volatility smile.

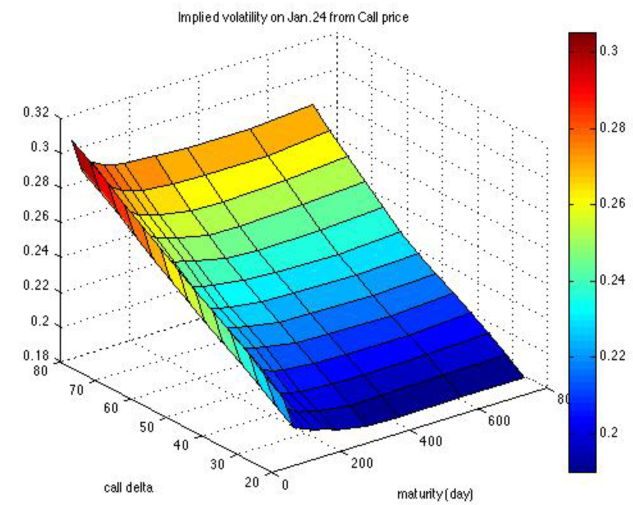
A Brief Look at the Equity Index Smile

1. Recent S&P Smiles

Oct 1 2007



Jan 24 2008



DAX Implied Volatility Surface 2008

2 Matthias R. Fengler

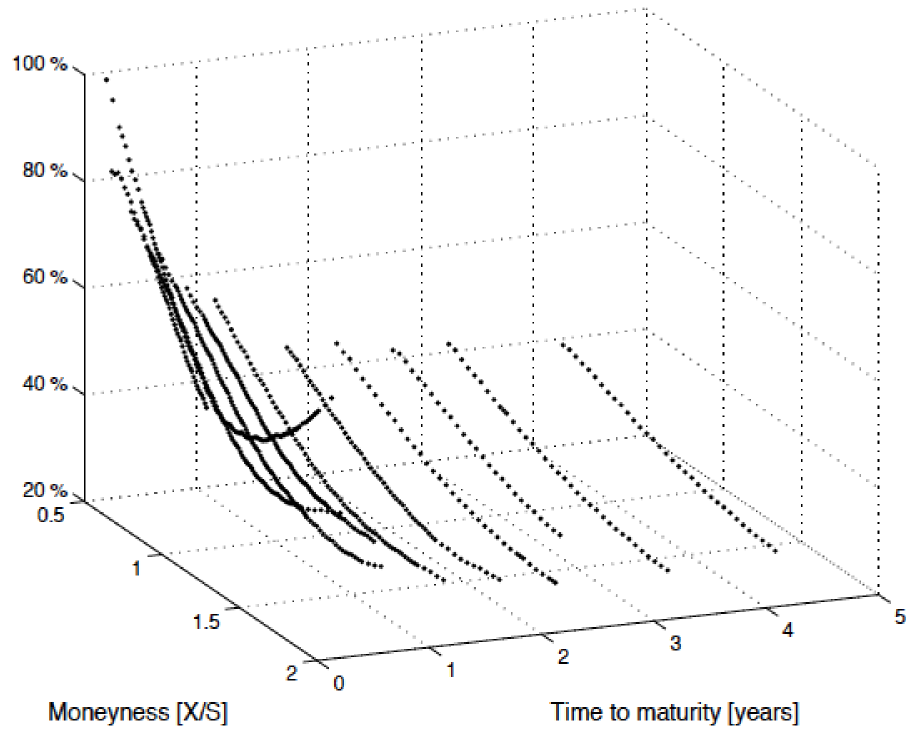


Fig. 1. IV surface of DAX index options from 28 Oct. 2008, traded at the EUREX. IV given in percent across a spot moneyness metric, time to expiry in years.

SVI Parameterization

- A commonly used parameterization of the implied volatility smile for a fixed expiration T is given by

$$\sigma_I^2(m, T) = a + b[\rho(m - c) + \sqrt{(m-c)^2 + \theta^2}]$$

where $m = \ln\frac{K}{F}$ is the forward moneyness, and $F = Se^{rT}$ is the forward price of the underlying stock, and T is the time to expiration of the option.

- But we must worry about arbitrage violations, since this is a parameter, not a price. (Cf. interest rates)

Stylized Facts about the Smile for Equity Indexes

- Steep for short expiration, flatter for longer ones.
- Wasn't that way before 1987.
- Negatively skewed almost always
- Term structure is upward sloping except in crises, when skew also steepens.

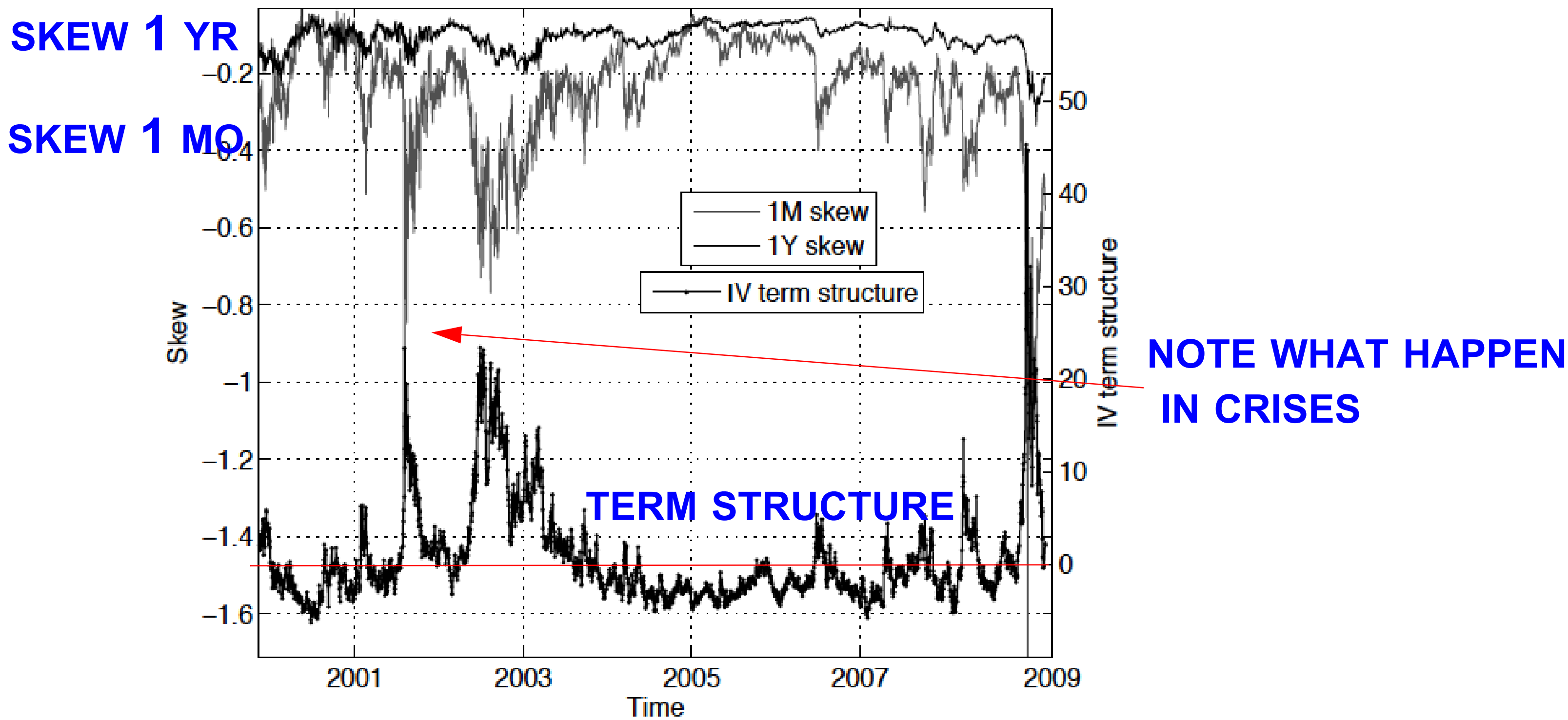


Fig. 3. Time series of 1M and 1Y IV skew (left axis, gray line and black line respectively) and time series of the IV term structure (right axis, black dotted line). Skew is defined as $\left.\frac{\partial \bar{\sigma}^2}{\partial m}\right|_{m=0}$, where m is log-forward moneyness. The derivative is approximated by a finite difference quotient. IV term structure is the difference between 1M ATM and 1Y ATM in terms of percentage points. Negative values indicate an upward sloping term structure.

• Changes in the at-the-money implied volatility and the index are negatively correlated.

corr = -0.69

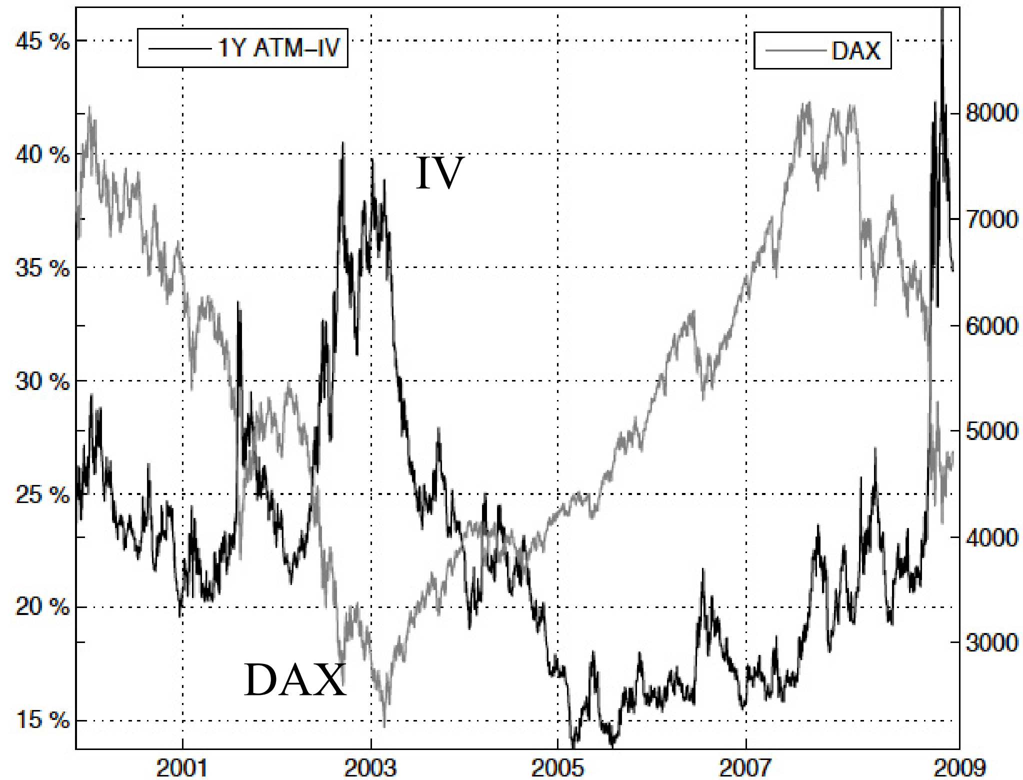
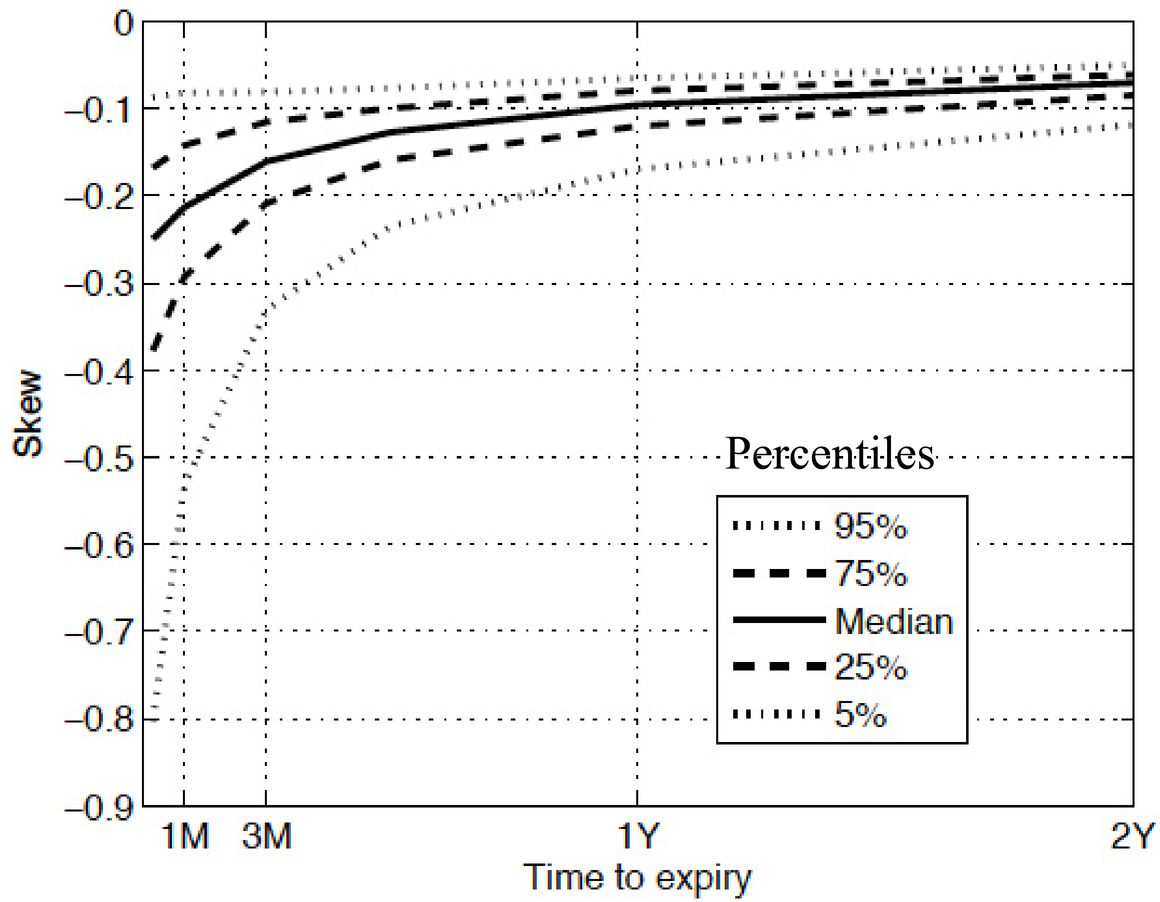


Fig. 2. Time series of 1Y ATM IV (left axis, black line) and DAX index closing prices (right axis, gray line) from 2000 to 2008.

• Implied volatility seems to be mean-reverting.

• Fluctuations in the short term skew are much larger (cf interest rates)



PRINCIPLES OF VALUATION

The Principles of Financial Valuation

Fundamental Theorem of Finance. *Security prices exclude arbitrage if and only if there exists a strictly positive value functional, under the technical restrictions that the space of portfolios and the space of contingent claims are locally convex topological vector spaces and the positive cone of the space of contingent claims is compactly generated, that is, there exists a compact set K of X (not containing the null element of X) such that*

$$C = \{x \in X : x \geq 0\} = \bigcup_{\lambda \geq 0} \lambda K.$$

What other fields have theorems? Why can you value options?

So formal and axiomatic. But we are studying financial "**engineering**".

Don't get misled by mathematics, theorems, lemmas. Understand them, but we're dealing with the real world.

- What is financial engineering?
  Cf. Mechanical engineering, Electrical engineering, Bio-engineering.

  Science seeks to discover the fundamental principles that describe the world, and is usually reductive.
  Engineering is about using those principles, constructively, for a purpose.

  Financial engineering, layered above financial science, would be the study of how to create functional financial devices – convertible bonds, warrants, default swaps, etc. – that perform in desired ways.
- What is financial science?
  Our theories don't describe the behavior of assets very well.
  Stock evolution isn't Brownian.

Price & Value

- Price = what you have to pay to acquire a security.
  Value is what it is worth. The price is fair when it is equal to the value.
- Judging value, in even the simplest way, involves the construction of a model or theory.
- Black: markets are efficient when prices are 1/2 to 2 times value.

The Purpose of Models

- Example: Valuing a Park Ave apartment.
- Models are used to rank securities by value.
- Models are used to interpolate or extrapolate from liquid prices to illiquid prices.
- Models transform linear quantities you can have intuition about into nonlinear dollar values.

Styles of Modeling

- Absolute vs. Relative Value.
- Absolute Value: Quantum mechanics is a theory of the world, absolute.
  Geometric Brownian motion is a *model* of absolute valuation, but not correct.
- Relative Value: illiquid --> liquid
  Derivatives are like molecules made out of simpler atoms.
  Relative valuation is less ambitious
  Black-Scholes tells you the price of an option in terms of the price of a stock and a bond.
- We are taking the view point of an options trading desk, as manufacturers or arbitrageurs
  Derivatives can be constructed or deconstructed
  Stocks to Options, Exotics to Vanilla
  Fruit salad: What is *the implied price* of pears

The One Commandment of Quantitative Finance

If you want to know the value of a security, use the price of another security that's as similar to it as possible.

The ***law of one price***, or ***the principle of no riskless arbitrage:***

Any two securities with identical future payoffs, no matter how the future turns out, should have identical current prices.

Valuation by Replication

Target security
Replicating portfolio, a collection of more *liquid* securities that, collectively, has the same future payoffs as the target *no matter how the future turns out.*

The target's value is then simply that value of the replicating portfolio.

No matter how the future turns out: the science of markets

Replication: engineering, a model

Styles of Replication

Static: rarely possible: put-call parity and variance swaps

Dynamic: BSM

Dynamic replication is very elegant but makes many assumptions.

Neftci's *Principles of Financial Engineering*
"Real life complications make dynamic replication a much more fragile exercise than static replication. The problems that are encountered in static replication are well known. There are operational problems, counterparty risk, and so the theoretically exact synthetics may not be identical to the original asset. There are liquidity problems and other transactions costs. But all these are relatively minor and in the end, static replicating portfolios used in practice generally provide good synthetics.

With dynamic replication, these problems are magnified, because the underlying positions needs to be readjusted many times. For example, the effect of transaction costs is much more serious if dynamic adjustments are required frequently. Similarly, the implications of liquidity problems will also be more serious. But more importantly, the real-life use of dynamic replication methods brings forth *new* problems that would not exist with static synthetics."

> We have to worry not just about current liquidity and bid-ask spreads, but about how they vary in the future.
>
> Dynamic replication is imperfect; it depends upon models, which imply assumptions and the approximations involved in working in discrete time steps.
>
> Even if the theory is easy, "the strategy needs to be implemented using appropriate position-keeping and risk-management tools. The necessary software and human skills required for these tasks may lead to significant new costs, but also to many jobs producing and taking care of these tools.
>
> Finally, dynamic replication is often used to replicate securities with nonlinear payoffs. This leads to exposure to the level of volatility, and who knows what the future level of volatility will be. Managing exposure to volatility can be much more difficult than managing exposure to interest rates or currencies, because there are (almost) no underliers to trade."

- So, in this course, *first try use static replication* for valuing new securities. If we cannot, *then we will use dynamic replication*.
- Models are unreliable guides to the world of finance, and because you don't know which is the right one, *it's best to use as little modeling as possible. And, if you have to use a model, it's always good to use more than one so you understand the model-dependence of your result.*

Implied Variables and Realized Variables

Physics models start from today and **predict the future**.

Financial models think about the future and **predict values today**.

What matters is not only what will happen, but what people *think* will happen.

What people think will happen affects what happens today.

Realized variables describe what actually happens.

Financial models calibrate the future to current known prices to produce implied variables about the future that match known prices today. One then has to compare these implied values to the future values that are actually realized as time passes.

Implied variables describe what people think will happen filtered through a model.

Should you hedge with implied volatility or realized volatility? Later lectures.

Testing Models

- Here are some counter-intuitive and interesting remarks about quantitative finance by Fischer Black.
- “It's better to 'estimate' a model than to test it. I take 'calibration' to be a form of estimation, so I'm sympathetic with it, so long as we don't take seriously the structure of a model we calibrate. Best of all, though, is to 'explore' a model.”
- “My job, I believe, is to persuade others that my conclusions are sound. I will use an array of devices to do this: theory, stylized facts, time-series data, surveys, appeals to introspection and so on.”
- “In the real world of research, conventional tests of [statistical] significance seem almost worthless.”
- No-one could make these remarks about models or theories in physics, chemistry or engineering. Think about the differences between these fields and quantitative finance, and why that should be the case.

MODELING MARKETS:

The Efficient Market Hypothesis/Model

Experience shows that it is difficult or impossible to successfully and consistently predict what's going to happen to the stock market tomorrow based on all the information you have today.

The EMH formalizes this experience by stating that it is impossible to beat the market, because current prices reflect all current economic and market information.

Jiu-jitsu approach: *I can't figure out how things work, so I'll make the inability to do that a principle.*

Uncertainty, Risk & Return

Quantifiable Uncertainty or Risk

What do you mean when you say there's a 1/8 chance of throwing 3 heads in succession?

Frequentist Probabilities

Tossing a coin: history doesn't matter

Unquantifiable Uncertainty:

What do you mean when you say that there's a small probability of a revolution overthrowing the United States government?

The likelihood of a revolution in some country or the probability of a terrorist attack. The chance that an earthquake of magnitude 6.7 or greater will occur before the year 2030 in the San Francisco Bay Area.

No way of honestly estimating probabilities.

In human affairs frequentist probabilities are not known and history matters.

Most financial models assume that unquantifiable uncertainties are actually frequentist. We are mostly going to do this too, but need to remember the assumptions we make.

Agent-based models perhaps go one step closer to reality, but are still models.

First, Modeling A Share of Stock

- A company is a tremendously complex and structured endeavor. Consider Apple.

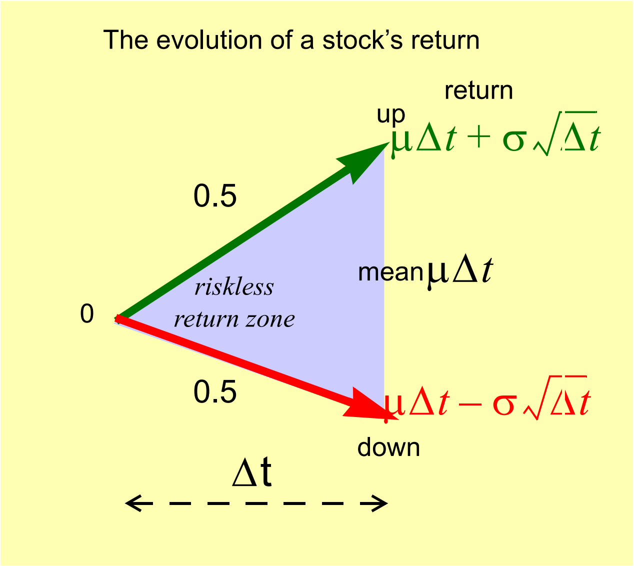
The value of the organization is reflected in the price to buy or sell just *one* incremental share of the company. Financial modeling aims to tell you what you should pay today for a share of its future performance.

The EMH model treats a stock as a simple kind of atom that undergoes a simple kind of quantifiable uncertainty.

A stock's most important feature is the uncertainty of its return.

No Arbitrage means the riskless return is a convex combination of up and down.

This naive either-up-or-down model captures much of the inherent risk of owning a stock and many other securities. But not all.



The Efficient Market Models often uses Geometric Brownian Motion to describe return distributions. This means that all we care about is volatility and return; these parameters specify the entire distribution of returns. Obviously not strictly true, but let's see where it takes us.

What return should we expect for a given volatility?

The law of one price will give us the answer.

The Law of One Price Relates Risk to Return

We can extend the law of one price (identical payoffs have identical prices*)* to demand that *identical expected risks have identical expected returns*.

But some risks can be avoided. Therefore the principle: *identical* ***unavoidable*** *expected risks have identical expected returns*.

How can you avoid risk?

There are three ways to reduce or avoid risk:

(i) dilution with a riskless bond

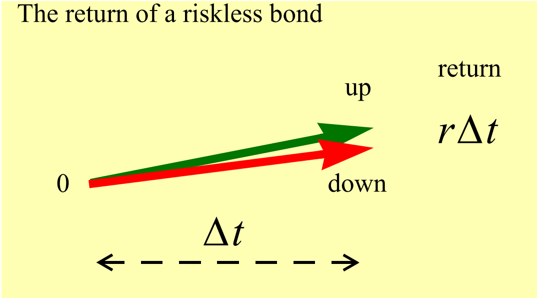
(ii) diversification

(iii) hedging

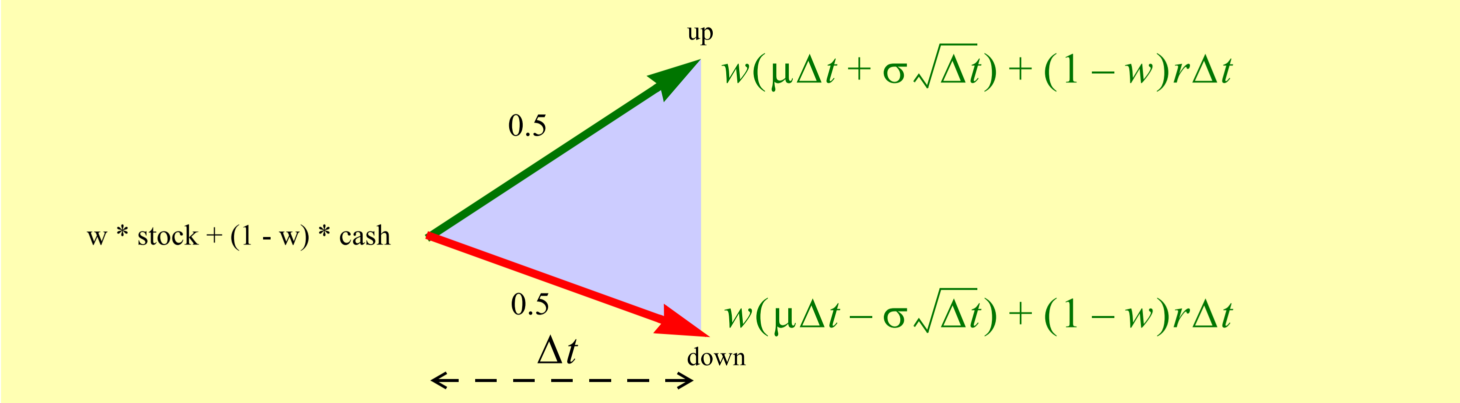
We'll combine these with the law of one price to derive everything we can use.

(i) Risk Reduction by Dilution Means Risk and Return are Proportional

By adding a riskless bond with zero volatility to the stock of volatility σ and expected return μ, you reduce both the risk and return of your investment.



Consider a mixture of w% risky stock with volatility σ and (1 - w)% riskless bond.



The expected return for this mixture is $w\mu\Delta t + (1-w)r\Delta t \; = \; r\Delta t + w(\mu - r)\Delta t$.
The volatility of returns is $w\sigma$.

Thus extra risk of magnitude $w\sigma$ must generate extra return $w(\mu - r)\Delta t$

$$\frac{\mu - r}{\sigma} = \lambda \qquad\qquad \mu - r = \lambda\sigma$$

Excess return is proportional to risk. By law of one price, must be true for all securities. A similar result holds for options values, and is equivalent to the Black-Scholes equation.

(ii) Risk Reduction by Diversification Means λ is Zero

If you can accumulate a portfolio of so many uncorrelated unavoidable risks that they cancel in the limit as the number of stocks become large, the portfolio's net volatility σ approaches zero.

Then, by the law of one price, it must produce an excess return of zero for the entire portfolio.

But the excess return of the entire portfolio is the weighted sum of the excess returns of each individual member of the portfolio, each of which is proportional to their individual non-zero volatility via the Sharpe ratio.

Hence the Sharpe ratio in the equations above for this portfolio must be zero. But the Sharpe ratio is the same for all portfolios of stocks, so that $\lambda = 0$ in general. Thus,

$$\mu = r$$

All stocks must be expected to earn the riskless rate if you can diversify.

(iii) Risk Reduction by Hedging

You can't always diversify because stocks are sensitive to the entire market M.

Let ρ be the correlation of the returns between stock S with volatility σ and the market M with volatility σ_M

$$\frac{dS}{S} = \mu dt + \sigma\left(\sqrt{1-\rho^2}\,dZ + \rho dZ_M\right)$$

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

You can *hedge away* the M-related risk of any stock to create an M-neutral portfolio:

$dS_M = dS - \Delta dM$ has no market risk if $\Delta = \rho(\sigma/\sigma_M)\frac{S}{M} \equiv \beta\frac{S}{M}$.

This M-neutral stock has expected return $\frac{\mu - \beta\mu_M}{1-\beta}$ sensitive only to the volatility of the stock.

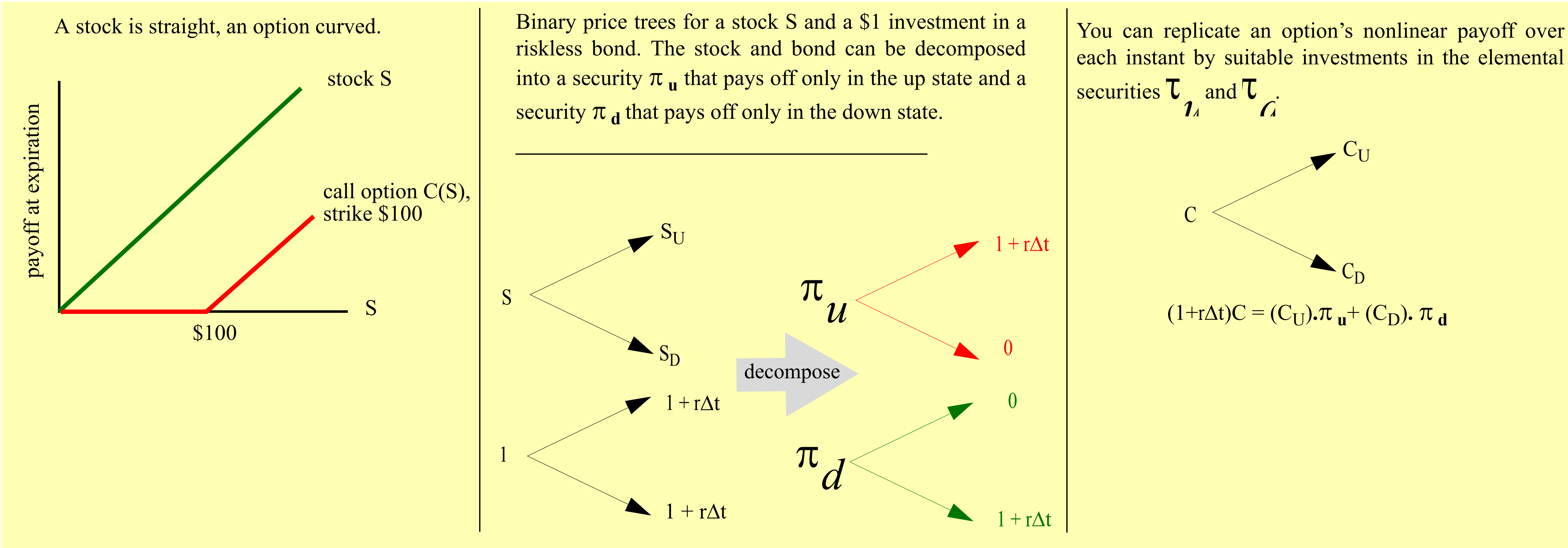
By diluting it, we can show that its excess return of the M-neutral stock must then be proportional to its residual volatility. Furthermore, by diversifying over many M-neutral stocks we can show that the M-neutral stock can expect only the riskless rate r, so that[1]

$$(\mu - r) = \beta(\mu_M - r) \qquad \text{CAPM in "Efficient Markets"}$$

Do you believe this?

1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://www.ederman.com/new/docs/qf-market_bubbles.pdf

Derivative Valuation by Replication



A derivative is a contract whose payoff depends on the price of a "simpler" *underlier*. The most relevant characteristic is the *curvature* of its payoff *C(S)*, as illustrated for a simple call option. **What is the value of curvature?**

You can use linear algebra to decompose the stock and bond into a basis of two more elemental securities π_u and π_d, each respectively paying \$(1+rΔt) in only one of the final states.

Then you can replicate the payoff of any non-linear function *C(S)* over the next instant of time Δt, no matter into which state the stock evolves. Note that the portfolio consisting of both π_u and π_d is riskless and is therefore worth \$1.

The value of the option is the price of the mixture of stock and bond that replicates it. The coefficients depend on the difference between the up-return and the down-return at each node, that is, on the stock's volatility σ.

The choice-of-currency/numeraire trick

You can use any currency to value a security if markets are efficient.

A convenient choice of currency can greatly simplify thinking about a problem, and sometimes reduce its dimensionality.

Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modeled by choosing a bond itself as the natural valuation currency.