LECTURE 20

Stochastic Volatility Models Continued Continued Continued:

Solutions to the PDE

Looking Ahead

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Stochastic Volatility Models Wrap Up

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3 Lectures on Jump Diffusion Models

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Guest Speakers

Michael Kamal - April 15
Jackie Rosner - April 20
Scott Weiner - April 27

If you have questions come to my office hours or see me some other time by appointment.

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Mixing Theorem

The price of an option in a stochastic volatility model with zero correlation is the weighted integral/sum over BS prices over the distribution of path volatilities.

$$V = \sum_{\sigma_T} p(\sigma_T) \times BS(S, K, r, \sigma_T, T)$$

The Smile That Results From Stochastic Volatility

The zero-correlation smile depends on moneyness

Mixing: average BS solutions over the volatility distribution to get the stochastic volatility solution.
Example: path volatility can be one of two values, either high or low, with equal probability:

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)] \qquad \text{Eq.20.1}$$

Homogeneity: $$C_{SV} = \frac{1}{2}\left[SC_{BS}\left(1, \frac{K}{S}, \sigma_H\right) + SC_{BS}\left(1, \frac{K}{S}, \sigma_L\right)\right] = Sf\left(\frac{K}{S}\right)$$

Now, defining BS Σ by $$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}\left(1, \frac{K}{S}, \Sigma\right)$$

and so $$\Sigma = g\left(\frac{K}{S}\right)$$

IThe zero correlation smile is symmetric

The mixing theorem: $$C_{SV} = \int_0^{\infty} C_{BS}(\sigma_T)\phi(\sigma_T)d\sigma_T$$

Taylor expansion about the average value $\overline{\sigma_T}$ of the path volatility, dropping subscript *T*.

$$C_{SV} = \int_0^{\infty} C_{BS}(\bar{\sigma} + \sigma - \bar{\sigma})\phi(\sigma)d\sigma$$

$$= \int\left\{C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma}C_{BS}(\bar{\sigma})\right](\sigma - \bar{\sigma}) + \frac{1}{2}\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})(\sigma - \bar{\sigma})^2 + \dots\right\}\phi(\sigma)d\sigma$$

$$= C_{BS}(\bar{\sigma}) + 0 + \frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})\right]var[\sigma] + \dots$$

where $var[\sigma]$ is the *variance of the path volatility of the stock over the life* τ *of the option*.

Define BS implied volatility Σ by

$$C_{SV} \equiv C_{BS}(\Sigma) = C_{BS}(\bar{\sigma} + \Sigma - \bar{\sigma})$$

$$= C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma} C_{BS}(\bar{\sigma})\right](\Sigma - \bar{\sigma}) + \ldots \qquad \text{Eq.20.2}$$

Then equating two expressions

$$\Sigma \approx \bar{\sigma} + \frac{\frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2} C_{BS}(\bar{\sigma})\right] var[\sigma]}{\frac{\partial}{\partial\sigma} C_{BS}(\bar{\sigma})} \qquad \text{Eq.20.3}$$

Use BS derivatives to find the functional form of $\Sigma(S, K)$.

$$\frac{\partial C}{\partial\sigma} = \frac{Se^{-\frac{1}{2}d_1^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}} \quad \text{vega is always positive} \qquad \text{Eq.20.4}$$

$$\frac{\partial^2 C}{\partial\sigma^2} = \frac{S\sqrt{\tau}N'(d_1)}{\sqrt{2\pi}\sigma}(d_1 d_2) = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) = \frac{\partial C}{\partial\sigma}\frac{d_1 d_2}{\sigma} \quad \text{volga is mostly positive}$$

except atm

$$\frac{C_{\sigma\sigma}}{C_{\sigma}} = \frac{1}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) \qquad \text{Eq.20.5}$$

So

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}var[\sigma]\left[\frac{(\ln S_F/K)^2 - \left(\bar{\sigma}^4\tau^2\right)/4}{\bar{\sigma}^3\tau}\right] \qquad \text{Eq.20.6}$$

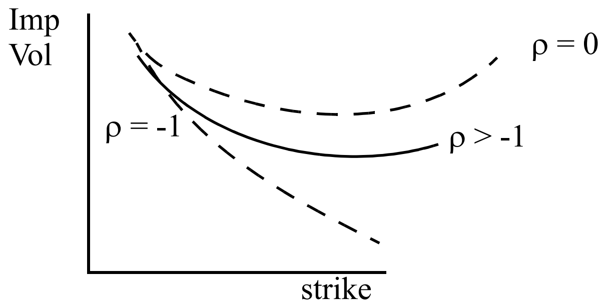
and note that $\bar{\sigma}$ is the average of the path volatility over the life of the option and $var[\sigma]$ is the *variance of the path volatility of the stock over the life of the option.*

Quadratic function of $\ln S_F/K$, parabolically shaped smile that varies as $(\ln S_F/K)^2$ or $\left(K - S_F^{\,2}\right)$ close to atm. Sticky moneyness smile, no scale, a function of K/S_F.

Non-zero correlation ρ in stochastic volatility models

No correlation lead to a symmetric smile.

With correlation the smile still depends on (K/S_F) but the dependence is not quadratic.:

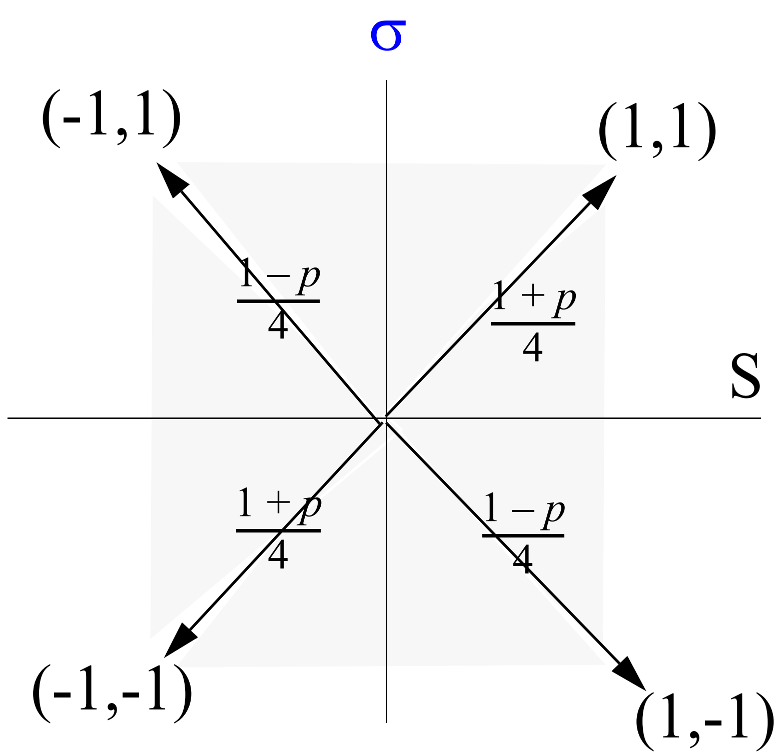


A very steep short-term skew is difficult in these models; since volatility diffuses continuously in these models, at short expirations volatility cannot have diffused too far. A very high volatility of volatility and very high mean reversion are needed to account for steep short-expiration smiles. (There is more on this in Fouque, Papanicolaou and Sircar's book.)

Approximate Imperfect Analytic Intuitive Approximation $\rho \neq 0$

Use the convexity approach: Estimate value of the option with stochastic volatility as an average over the four states with different stock prices and volatilities:

Correlated moves in stock price and volatility with correlation p



$$E(S) = \frac{1}{4}[(1+p)1 + (1-p)(-1) + (1+p)(-1) + (1-p)1] = 0$$

$$E(\sigma) = \frac{1}{4}[(1+p)1 + (1-p)(1) + (1+p)(-1) + (1-p)(-1)] = 0$$

$$\mathrm{var}(S) = \frac{1}{4}[(1+p)1^2 + (1-p)1^2 + (1+p)(-1)^2 + (1-p)(-1)^2] = 1$$

$$\mathrm{var}(\sigma) = 1$$

$$\mathrm{cov}(S, \sigma) = \frac{1}{4}[(1+p)1 \times 1 + (1-p)(-1) \times 1 + (1+p)(-1) \times (-1) + (1-p)$$

$$= \frac{1}{4}[(1+p) - (1-p) + (1+p) - (1-p)] = p$$

Let the four nodes correspond to $S \pm \sigma S$, $\sigma \pm \xi\sigma$ and average over BS prices at these ranges of stock prices and path volatilities.

$$
\begin{aligned}
C_{SV} \approx\; & \frac{(1+\rho)}{4} C_{BS}(S+\sigma S, \sigma+\xi\sigma) \\
& + \frac{(1-\rho)}{4} C_{BS}(S+\sigma S, \sigma-\xi\sigma) \\
& + \frac{(1+\rho)}{4} C_{BS}(S-\sigma S, \sigma-\xi\sigma) \\
& + \frac{(1-\rho)}{4} C_{BS}(S-\sigma S, \sigma+\xi\sigma)
\end{aligned}
$$

Do Taylor expansion for small vol and small vol of vol.

We then find that all additional terms cancel out except for the volga and vanna terms, to second order in Taylor series:

$$C_{SV} = C_{BS}(S, \sigma) + C_{\sigma\sigma}\frac{var[\sigma]}{2} + \rho C_{s\sigma} S var^{\frac{1}{2}}[\sigma]$$

But implied volatility is defined in terms of

$$C_{SV} \equiv C_{BS}(S, \sigma + \Sigma - \sigma) \approx C_{BS}(S, \sigma) + C_{\sigma}(\Sigma - \sigma)$$

Thus $\Sigma \approx \sigma + \frac{C_{\sigma\sigma}}{C_\sigma}\frac{(\xi\sigma)^2}{2} + \frac{C_{s\sigma}}{C_\sigma}\rho S\sigma^2\xi$

Now from what we know about vanna and volga in a Black Scholes world,

$$\frac{C_{\sigma\sigma}}{C_\sigma} = \frac{1}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) \qquad \frac{C_{s\sigma}}{C_\sigma} = \frac{1}{S}\left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]$$

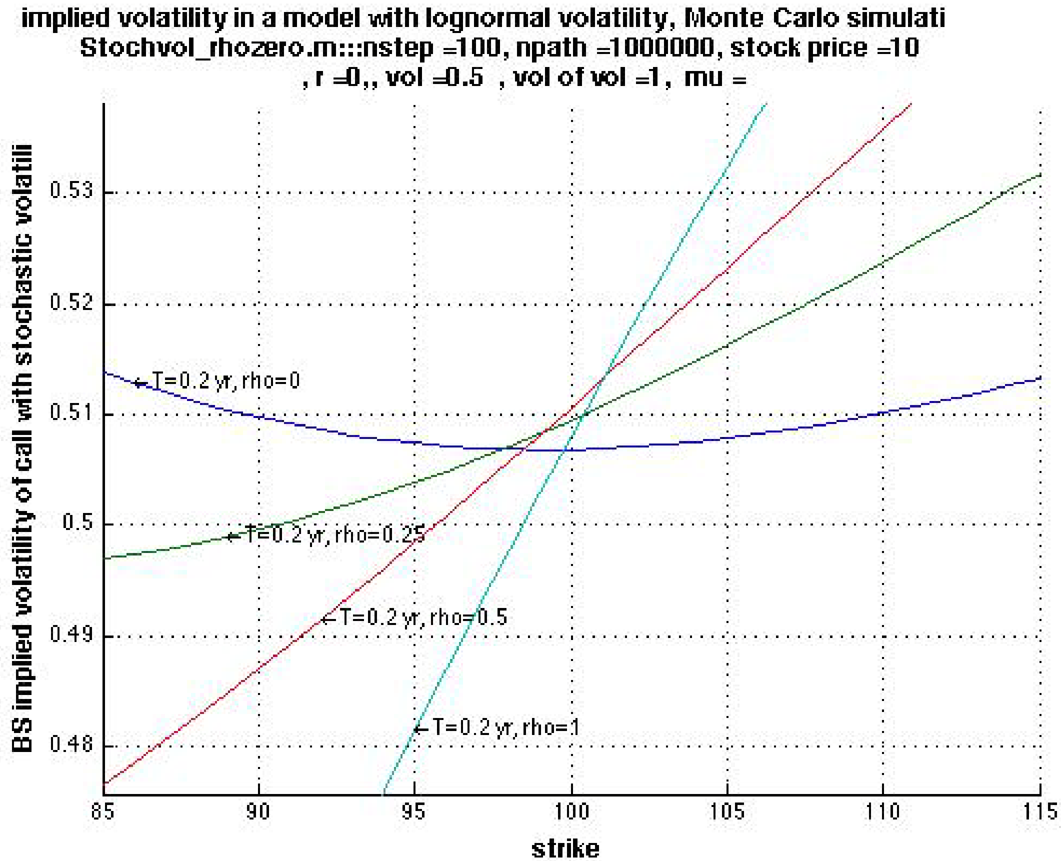
So

$$\Sigma \approx \sigma + \left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right)\frac{\xi^2\sigma}{2} + \left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]\rho\sigma^2\xi$$

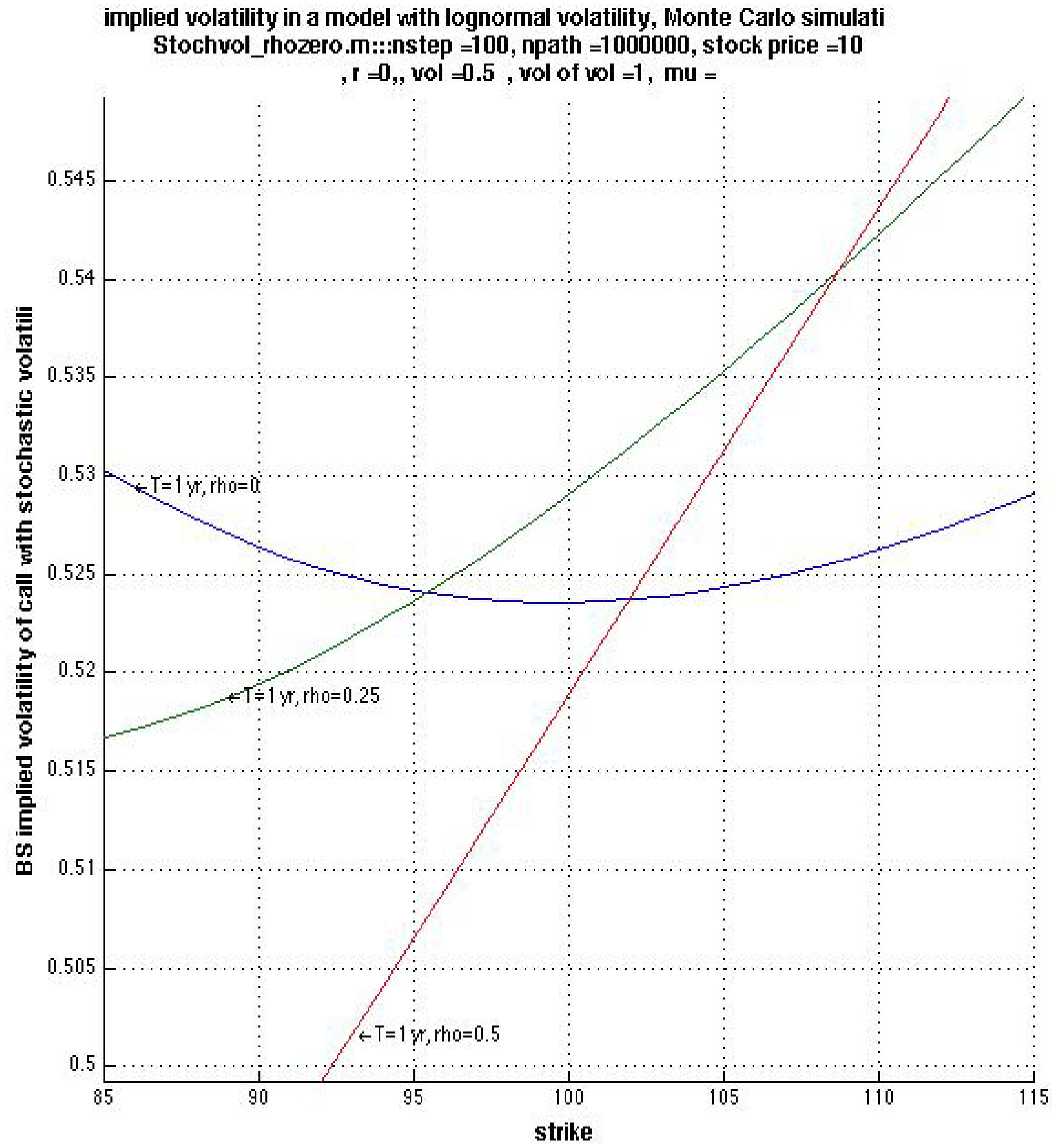
We see a quadratic and a linear term, depending on correlation.

Simulation

Monte Carlo simulation for $\tau = 0.2\text{yrs}$ with non-zero ρ. You can see that increasing the value of the correlation steepens the slope of the smile.



$\tau = 1yr.$


implied volatility in a model with lognormal volatility, Monte Carlo simulati
Stochvol_rhozero.m:::nstep =100, npath =1000000, stock price =10
, r =0,, vol =0.5 , vol of vol =1, mu =
BS implied volatility of call with stochastic volatili
strike
0.545
0.54
0.535
0.53
0.525
0.52
0.515
0.51
0.505
0.5
85
90
95
100
105
110
115
←T=1 yr, rho=0
←T=1 yr, rho=0.25
←T=1 yr, rho=0.5

The Smile in Mean-Reverting Stochastic Volatility Models

Finally, we explore the smile when volatility mean reverts:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad d\sigma = \alpha(m - \sigma)dt + \beta\sigma dW \qquad dZdW = \rho dt$$

The following pages show the results of a Monte Carlo for BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation.

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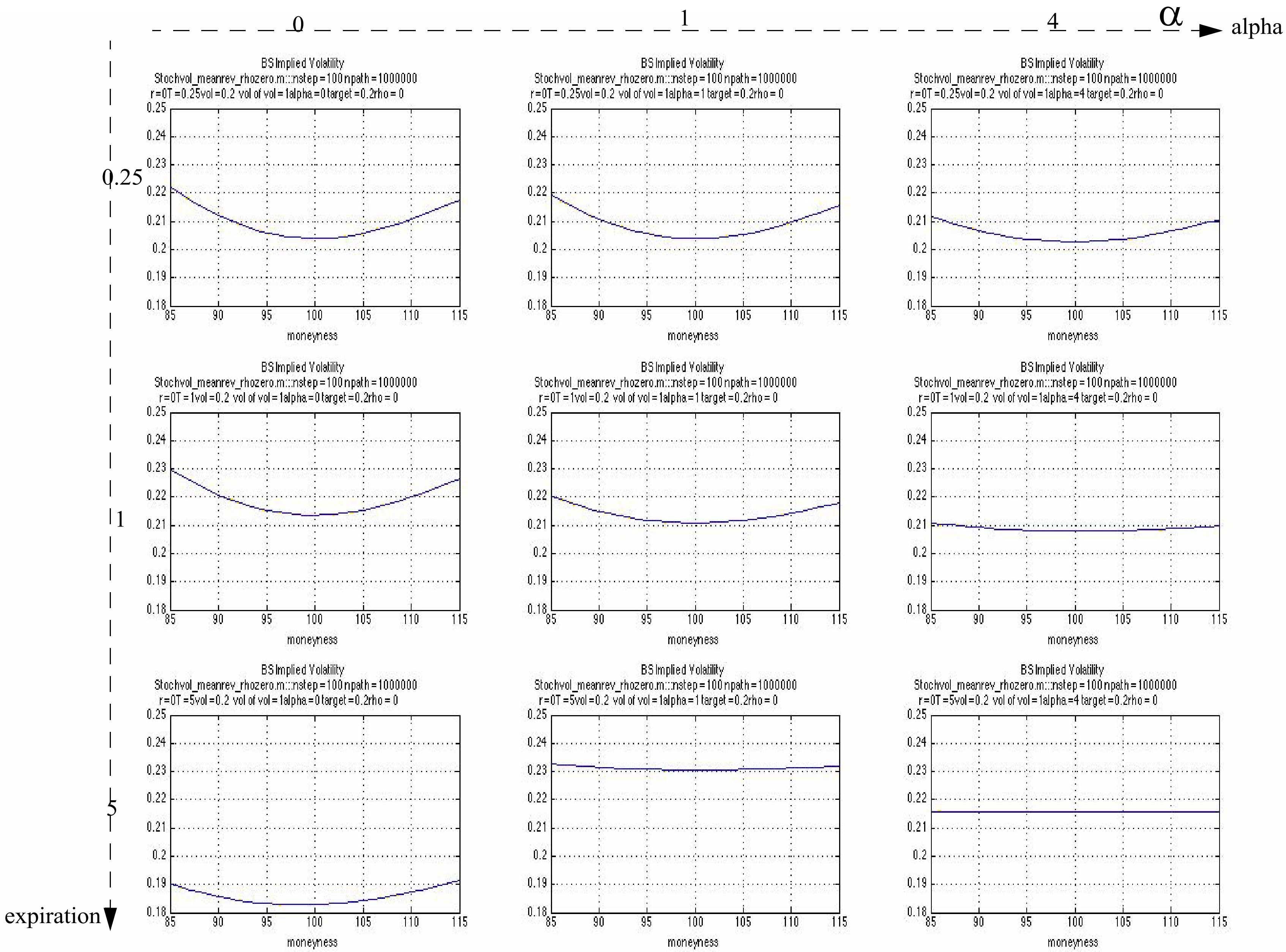
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.The target volatility and the initial volatility are both 0.2



BS Implied Volatility as a function of mean reversion s and expiration for correlation -0.1. The target and the initial volatility are both 0.2

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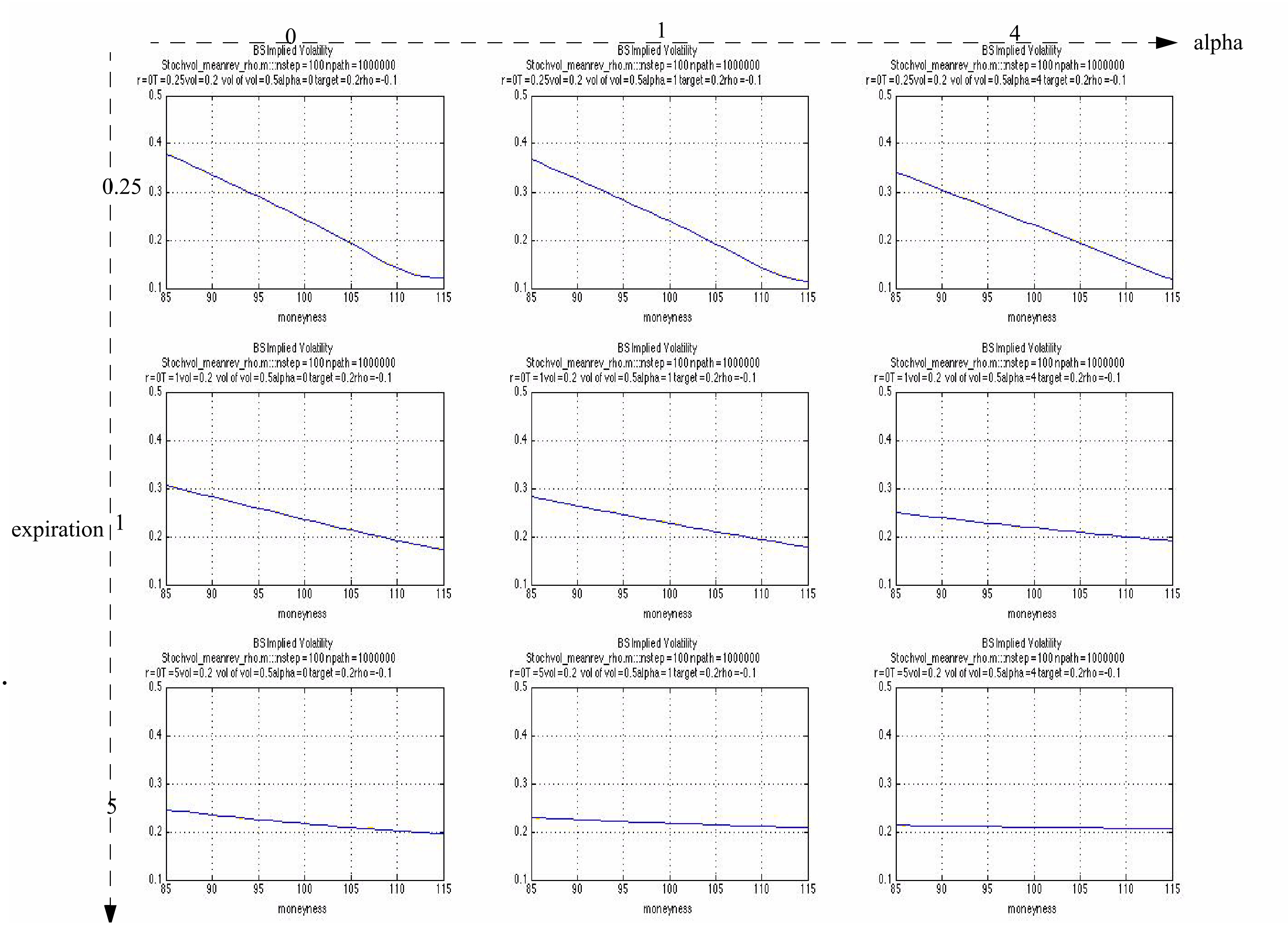
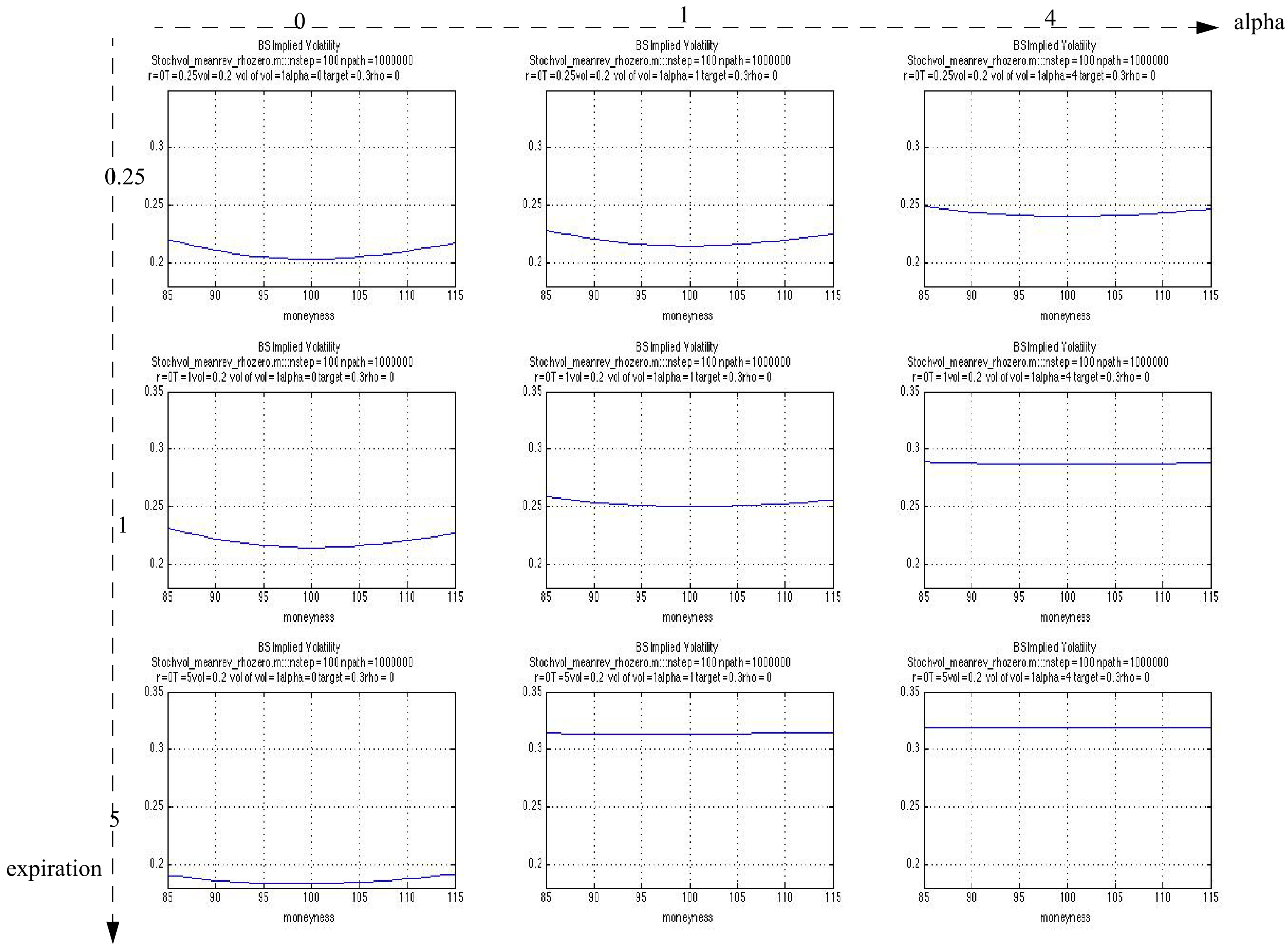


Note the flattening of the smile with both expiration and mean-reversion strength α

BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.



Mean-Reverting Stochastic Volatility and the Asymptotic Behavior of the Smile.

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma] \left[\frac{(\ln S_F/K)^2 - \left(\bar{\sigma}^4 \tau^2\right)/4}{\bar{\sigma}^3 \tau} \right] \qquad \text{approximately for small vol of vol} \qquad \text{Eq.20.7}$$

and insert intuition about mean reversion for σ.

Short Expirations, Zero Correlation

In the limit that $\tau \to 0$

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma] \left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3 \tau} \right]$$

$var[\bar{\sigma}] = \beta\tau$. Substituting this relation into Equation leads to the expression

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} \beta \left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3} \right] \qquad \tau \to 0 \text{ limit} \qquad \text{Eq.20.8}$$

Smile is quadratic and finite as $\tau \to 0$ for short expirations.

Long Expirations

As $\tau \to \infty$ $$\Sigma_{SV} \approx \bar{\sigma} - \frac{1}{2} var[\sigma]\left[\frac{\bar{\sigma}\tau}{4}\right]$$

where $\bar{\sigma}$ is the path volatility over the life of the option and is itself a function of the time to expiration due to the stochastic nature of the instantaneous volatility.

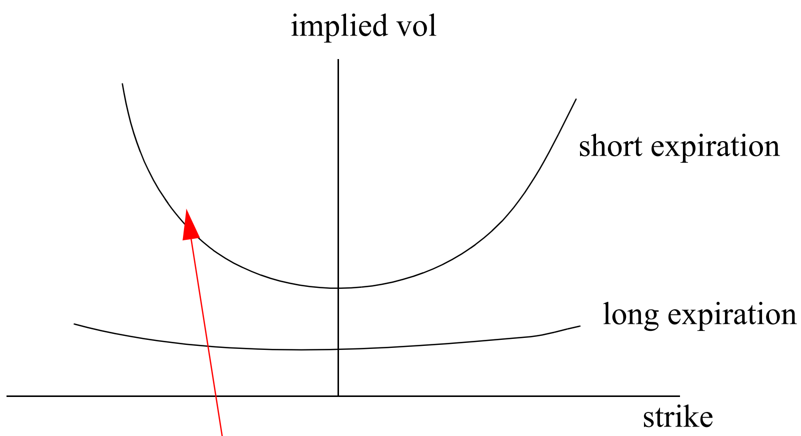
For Ornstein-Uhlenbeck the path volatility to expiration $\bar{\sigma}$ converges to a constant along all paths as $\tau \to \infty$, and so $\bar{\sigma}$ has zero variance as $\tau \to \infty$, $var[\bar{\sigma}] \to \mathrm{const}/\tau$.

$$\Sigma_{SV} \approx \bar{\sigma} - \frac{\mathrm{const}}{8}\bar{\sigma} \qquad \text{Eq.20.9}$$

NO smile at large expirations.

Why is the correction term negative? The option price $C_{BS}(\sigma)$ has negative convexity, and for a concave function $f(x)$, the average of the function $\overline{f(x)}$ is less than the function $f(\bar{x})$ of the average.

Thus, for zero correlation, we expect to see stochastic volatility smiles that look like this:



We can understand this intuitively as follows. In the long run, all paths will have the same volatility if it mean reverts, and so the long-term skew is flat. In the short run, bursts of high volatility act almost like jumps, and induce fat tails

Comparison of vanilla hedge ratios under Black-Scholes, local volatility and stochastic volatility models when all are calibrated to the same negative skew

Calibrated to the same current negative skew for the S&P, different models have different evolutions of volatility, different hedge ratios, different deltas, different forward skews.

Black-Scholes: Implied volatility is independent of stock price. The correct delta is the Black-Scholes delta.

Local Volatility: Local volatility goes down as market goes up, so the correct delta is smaller than Black-Scholes.

Stochastic Volatility:

Implied volatility is a function of K/S,
Negative skew means that implied volatility goes up as K goes down
Then implied volatility must go up as S goes up.

Therefore, the hedge ratio will be greater than Black-Scholes, contingent on the level of the stochastic volatility remaining the same. But, remember, in a stochastic volatility model there are two hedge ratios, a delta for the stock and another hedge ratio for the volatility, so just knowing how one hedge ratio behaves doesn't tell the whole story anymore.

Best stock-only hedge in a stochastic volatility model

Although stochastic volatility models suggest a hedge ratio greater than Black-Scholes in a negative skew environment, that hedge ratio is only the hedge ratio w.r.t. the stock degree of risk, and doesn't mitigate the volatility risk.

What is the best stock-only hedge, best in the sense that you don't hedge the volatility but try to hedge away as much risk as possible with the stock alone?

Best stock-only hedge is a lot like a local volatility hedge ratio, and is indeed smaller than hedge ratio in a Black-Scholes model.

Simplistic stochastic **implied** volatility model

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \Sigma dZ \\ d\Sigma &= p dt + q dW \\ dZ dW &= \rho dt \end{aligned}$$

We have for simplicity assumed that the stock evolves with a realized volatility equal to the implied volatility of the particular option itself. Then for an option $C_{BS}(S, \Sigma)$ where both S and Σ are stochastic, we can find the hedge that minimizes the instantaneous variance of the hedged portfolio. That's as good as we can with stock alone.

This partially hedged portfolio is $\pi = C_{BS} - \Delta S$

Then in the next instant $d\pi = \left(\frac{\partial C_{BS}}{\partial S} - \Delta\right)dS + \frac{\partial C_{BS}}{\partial \Sigma}d\Sigma = (\Delta_{BS} - \Delta)dS + \kappa d\Sigma$

The instantaneous variance of this portfolio is defined by $(d\pi)^2 = var[\pi]dt$ where

$$var[\pi] = (\Delta_{BS} - \Delta)^2(\Sigma S)^2 + \kappa^2 q^2 + 2(\Delta_{BS} - \Delta)\kappa S\Sigma q\rho$$

The value of Δ that minimizes the residual variance of this portfolio is given by

$$\frac{\partial}{\partial\Delta}var[\pi] = -2(\Delta_{BS} - \Delta)(\Sigma S)^2 - 2\kappa S\Sigma q\rho = 0$$

$$\Delta = \Delta_{BS} + \rho\left(\frac{\kappa q}{\Sigma S}\right)$$

The second derivative $\frac{\partial^2}{\partial\Delta}var[\pi]$ is positive, so that this hedge produces a minimum variance. The hedge ratio Δ is less than Δ_{BS} when ρ is negative. The best stock-only hedge in a stochastic volatility model tends to resemble the local volatility hedge ratio.

Conclusion

Stochastic volatility models produce a rich structure of smiles from only a few stochastic variables. There is some element of stochastic volatility in all options markets.SV models provide a good description of currency options markets where the dominant features of the smile are consistent with fluctuations in volatility. However, the stochastic evolution of volatility is not really well understood and involves many at presently unverifiable assumptions.

Comments on Skew Models

In general, the stochastic evolution of volatility is not really well understood and involves many at presently unverifiable assumptions.

Stochastic volatility models produce a rich structure of smiles from only a few stochastic variables. There is some element of stochastic volatility in all options markets. But …is the skew caused by correlation in the market you're interested in?

Economic effects giving rise to an Equity Skew: directional effects

• Markets jump down, realized vol increases, implied vol then increases too.

• Leverage-like asymmetry. Equity is assets minus debt. These constituents have very different relative volatilities which gives rise to a leverage related skew.

• Supply and demand asymmetry. It is more natural for equity to be held long than short, which makes downwards protection more important, puts more valuable.

Economic effects giving rise to an FX Skew And Smile: less directional, level related

• Anticipated government intervention to stabilize FX rates.

• Foreign investors buy FX rate protection with options.

• Stochastic volatility may be important

Economic effects giving rise to an Interest Rate skew and smile:

• Local volatility is important. Interest rates are absolutely important and volatilities are connected to absolute levels of interest rates rather than stock prices.

• Anticipated central bank action.

Does Stochastic Volatility match the facts?

None of these economic effects are very well described by strong correlation between the asset and the uncertainty in volatility since they are more or less deterministic.

In contrast most stochastic volatility models incorporate a skew by virtue of strong correlation of volatility and stock.

But you need a lot of volatility of volatility and correlation to get a steep short-term smile, and then a lot of mean reversion to prevent wild levels of future volatility.

Not very realistic.

Perhaps better to start with a skew and then add a stochastic perturbation.

Two Extra Topics

The Corridor Variance Swap: An Extension of Variance Swaps

How to replicate realized stock variance when S is in the range [a,b]. Trying to generalize that a portfolio of options weighted with K^{-2} produces a log S whose hedging captures variance.

Intuition: Consider a portfolio π of calls expiring at time T with a continuum of strikes ranging from a to b, with weights inversely proportional to the square of the strike K and hedge them only inside the range.

Procedure: Find the value of the payoff of π at expiration as a function of the terminal stock price S for the three regions $S < a$, $a < S < b$ and $S > b$.

At expiration $\pi(S, T) = \int_a^b (S-K)\theta(S-K)\frac{dK}{K^2}$ where $\theta(x)$ is the indicator or Heaviside function.

S<a: Then since K runs from a to b but $S < a$, the Heaviside function is always zero and so $\pi = 0$

[a<S<b]: Then the Heaviside function vanishes whenever the strike is greater than S, and so only strikes from a to S contribute, so that

$$\pi(S, T) = \int_a^S (S-K)\frac{dK}{K^2} = \frac{S-a}{a} - \ln\frac{S}{a}$$

S>b: Then all strikes from a to b contribute to the integral, and so

$$\pi(S, T) = \int_a^b (S-K)\frac{dK}{K^2} = \left(\frac{b-a}{ab}\right)S - \ln\frac{b}{a}$$

Now let's find the value of $\pi(S, t)$ at an earlier time t for zero rates and zero dividends and **zero implied volatility**. Then the stock price at t is the stock price at T, since volatility is zero, and so the above values for π are the values at an earlier time t too.

Now let's find the Delta and Gamma of π in the three regions.

Differentiating twice the answers above w.r.t. S leads to:

S<a:

$$\pi = 0$$
$$\Delta = 0$$
$$\Gamma = 0$$

[a<S<b]:

$$\pi(S, T) = \frac{S-a}{a} - \ln\frac{S}{a}$$
$$\Delta = \frac{1}{a} - \frac{1}{S}$$
$$\Gamma = \frac{1}{S^2}$$

S>b:

$$\pi(S, T) = \left(\frac{b-a}{ab}\right)S - \ln\frac{b}{a}$$
$$\Delta = \frac{b-a}{ab}$$
$$\Gamma = 0$$

We see that π and Δ are continuous across the boundaries a and b.

And Γ is non-zero only in [a,b].

Now consider buying the portfolio π at an implied volatility σ_i and hedging it an arbitrary volatility σ_i when the realized volatility is σ_r. Recall that a long time ago we showed that

$$\text{P\&L} = \frac{1}{2}\int_0^T \Gamma_i S^2\left(\sigma_r^2 - \sigma_i^2\right) dt$$

Now consider at time $t = 0$ a portfolio consisting of a long position in π. We can choose any volatility to hedge it, so let's choose an implied volatility of zero: $\sigma_i = 0$.

Then according to the formula $\Delta = \frac{1}{a} - \frac{1}{S}$ inside the range [a,b], but the hedge ratio doesn't change whenever the stock goes outside the range.

Then the Γ is zero everywhere except inside the range. Inside the range, the $\Gamma_i = \frac{1}{S^2}$.

Hence, applying $\sigma_i = 0$ to $\text{P\&L} = \frac{1}{2}\int_0^T \Gamma_i S^2\left(\sigma_r^2 - \sigma_i^2\right) dt$ 1 we obtain that the P&L for this strategy is

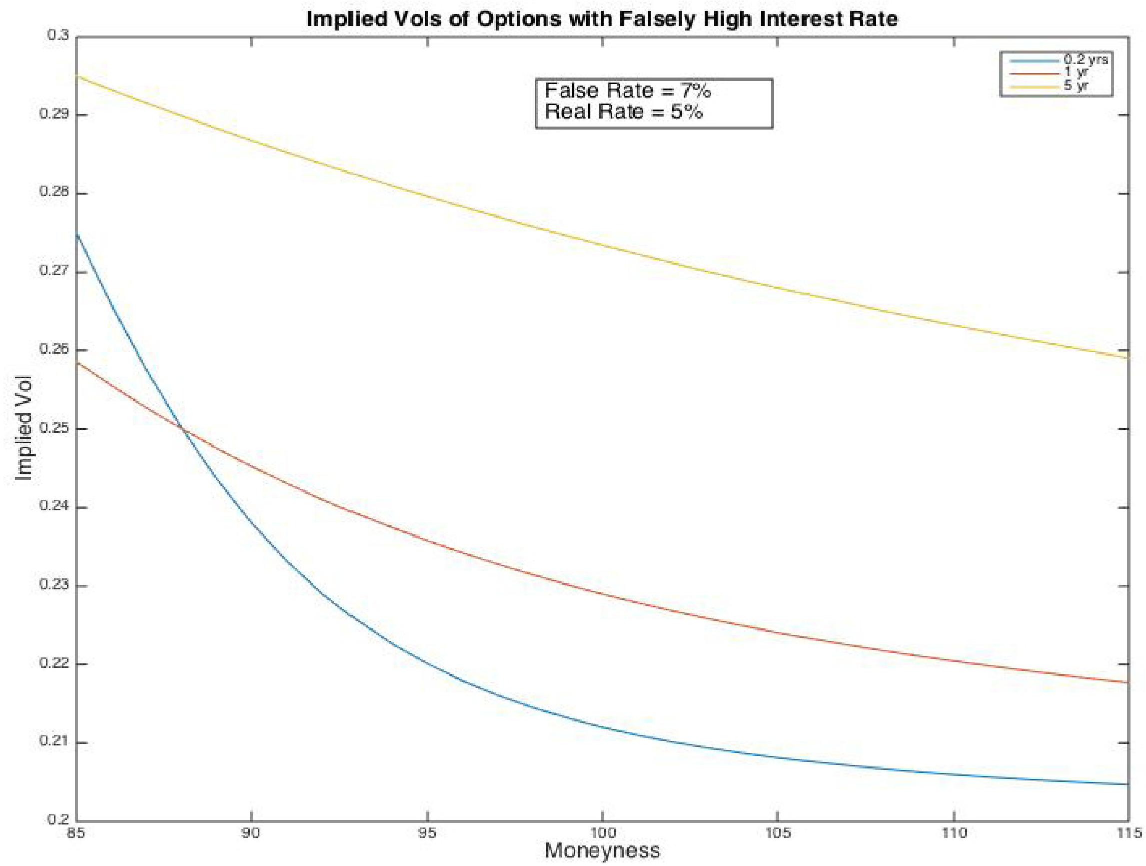
$$\text{P\&L} = \frac{1}{2}\int_0^T \left(\sigma_r^2\right) dt$$ where σ_r contributes only for paths within [a,b] because Γi is zero outside that range.

An Example of a Behavioral Skew Model

Imagine that options market participants somehow overestimate the risk of options and use a higher interest rate in Black-Scholes to calculate the option value.

Then the market calculates the implied volatility at the real interest rate.

Realistic skew that flattens at longer expirations



How do we understand this behavior?

Very roughly:

$$C(r + \varepsilon, \sigma) \approx C(r, \sigma) + \rho\varepsilon$$

But suppose we interpret this as a change in implied volatility: then

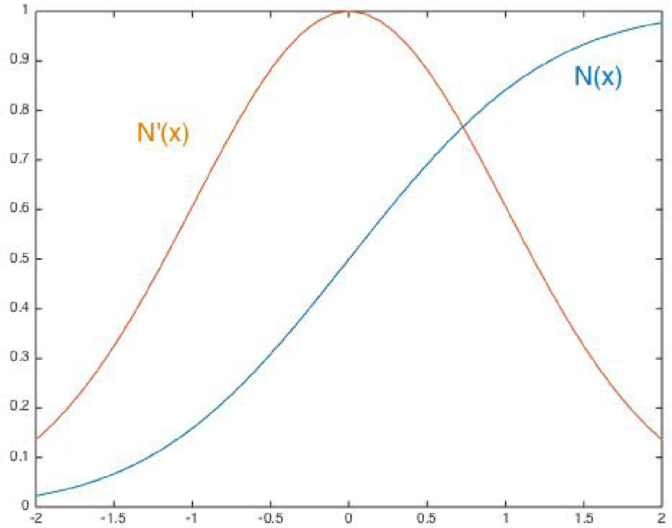
$$C(r, \sigma + \delta) \approx C(r, \sigma) + \text{vega}\,\delta$$

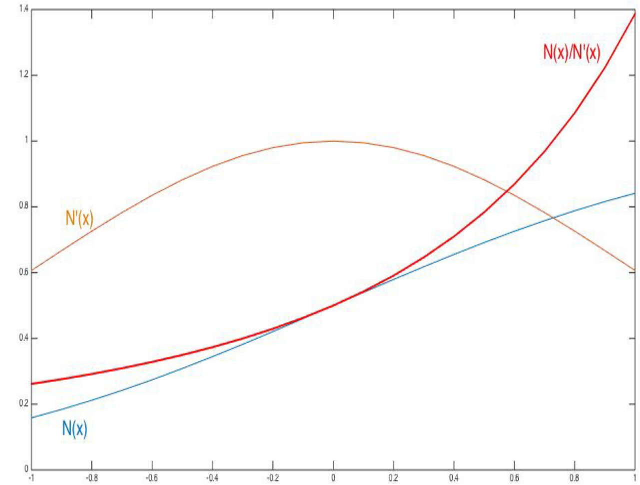
Equating the two, $\delta = \frac{\rho}{\text{vega}}\varepsilon$ and $\Sigma \approx \sigma + \frac{\rho}{\text{vega}}\varepsilon$

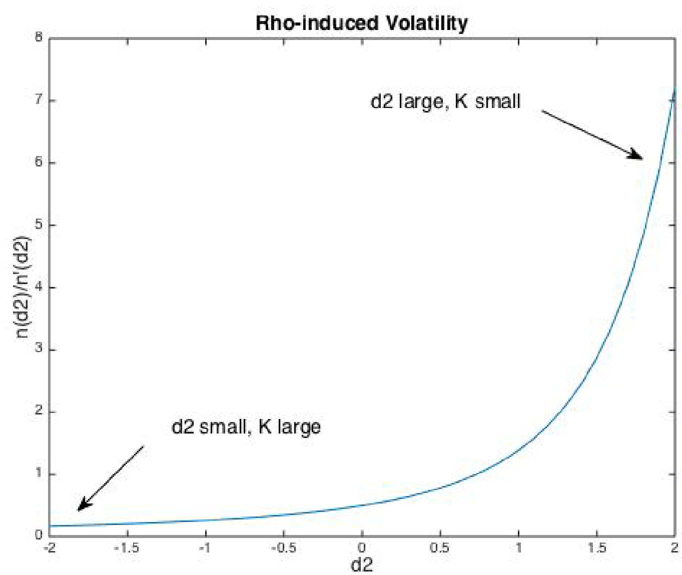
$$\rho = \tau K e^{-r\tau} N(d_2)$$

$$\text{vega} = \sqrt{\tau} K e^{-r\tau} N'(d_2)$$

$$\Sigma \approx \sigma + \sqrt{\tau}\frac{N(d_2)}{N'(d_2)}\varepsilon$$


N'(x)
N(x)


N(x)/N'(x)
N'(x)
N(x)


Rho-induced Volatility
d2 large, K small
d2 small, K large
n(d2)/n'(d2)
d2

Jump Diffusion

Jumps

- Why are we interested in jump models? Stocks and indexes don't diffuse smoothly, and do seem to jump. Even currencies sometimes jump. It's one of the things that happen.

- It's hard to define what a jump is, exactly.

- Jumps provide an easy way to produce the steep short-term skew that persists in equity index markets, and that indeed appeared soon after the jump/crash of 1987. They seem to play a part, behaviorally.

- Jumps are unattractive from a theoretical point of view because you cannot continuously hedge a distribution of finite-size jumps, and so risk-neutral arbitrage-free pricing isn't possible.

  As a result, most jump-diffusion models simply assume risk-neutral pricing without a thorough justification. It may make sense to think of the implied volatility skew in jump models as simply representing what sellers of options will charge to provide protection on an actuarial basis.

- Whatever the case, there have been and will be jumps in asset prices, and even if you can't hedge them, we are still interested in seeing what sort of skew they produce.

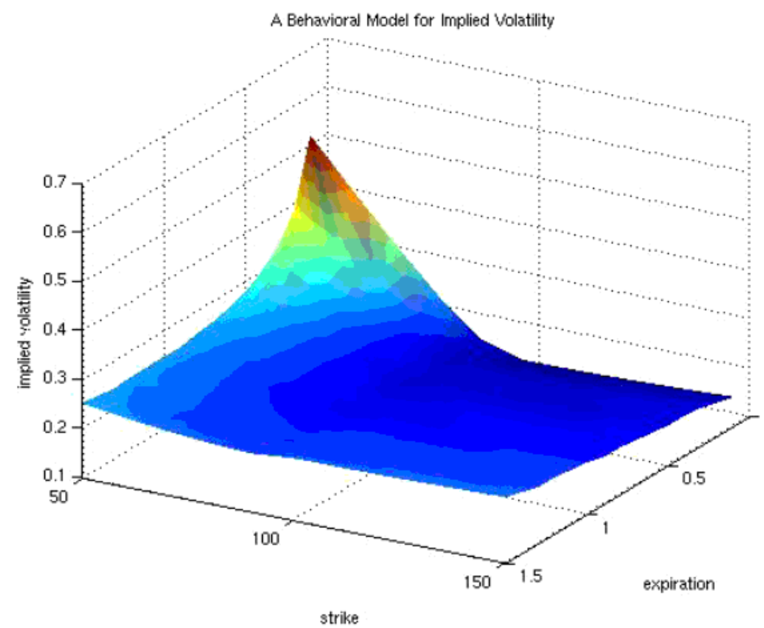
1.0.1 An Intuitive, Expectations View of the Skew Arising from Jumps

Assume:

> Probability p that **a single jump** will occur taking the market from S to K sometime before option expiration T
>
> Without that jump the future diffusion volatility of the index would have been $\sigma(T)$.

Then the expected net future realized volatility $\sigma(S, K, T)$ contingent on the market jumping to strike K via a jump and a diffusion is approximately

$$T\sigma^2(S, K, T) \approx p \times \left[\frac{(S-K)}{S}\right]^2 + (1-p) \times T\sigma^2(T)$$



If implied volatility is expected future realized volatility, then $\sigma(S, K, T)$ is also the rational value for the implied volatility of an option with strike K.

The shape is not unrealistic for index options, especially for short expirations, and can be made more realistic by allowing the diffusion volatility $\sigma(T)$ to incorporate a term structure as well, and by allowing different size jumps with different probabilities.

The key thing is that jumps affect the short-term skew more than the long-term skew.

Modeling Jumps Alone

1.0.2 Pure Jump Processes: Calibration and Compensation Always Important

Discrete binomial approximation to a diffusion process over time Δt:

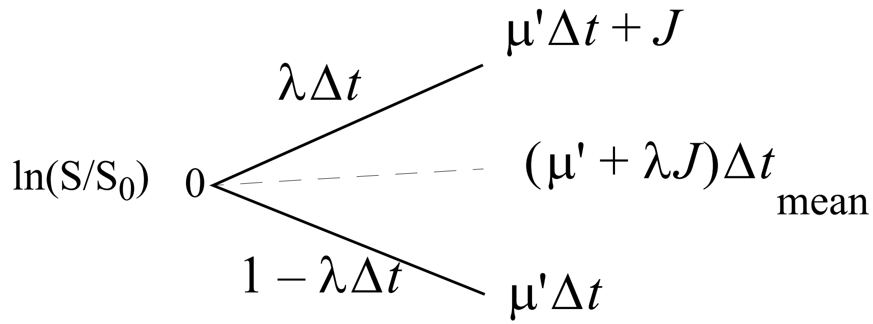
$$\begin{tikzcd}[column sep=large] & \mu\Delta t + \sigma\sqrt{\Delta t} \\ \ln(S/S_0) \quad 0 \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\ & \mu\Delta t - \sigma\sqrt{\Delta t} \end{tikzcd}$$

The **probabilities** of both up and down moves are **finite**, but the **moves** themselves are **small**, of order $\sqrt{\Delta t}$.

The net variance is $\sigma^2 \Delta t$ and the drift is μ. In continuous time this represents the process
$d\ln S = \mu dt + \sigma dZ$

Jumps are fundamentally different.

There the **probability of a jump J is small**, of order Δt, but the **jump itself is finite**.



3 parameters μ', J, λ

Mean:

$$\begin{aligned} E[\ln S] &= \lambda\Delta t[\mu'\Delta t + J] + (1-\lambda\Delta t)\mu'\Delta t \\ &= (\mu' + \lambda J)\Delta t \end{aligned}$$

Variance

$$\begin{aligned} \text{var} &= \lambda\Delta t[J(1-\lambda\Delta t)]^2 + (1-\lambda\Delta t)[J\lambda\Delta t]^2 \\ &= (1-\lambda\Delta t)J^2\lambda\Delta t[1-\lambda\Delta t+\lambda\Delta t] \\ &= (1-\lambda\Delta t)J^2\lambda\Delta t \\ &\to J^2\lambda\Delta t \qquad \text{as } \Delta t \to 0 \end{aligned}$$

$$\textit{Observed}\text{ drift } \mu = (\mu' + \lambda J)$$

$$\textit{Observed}\text{ volatility } \sigma = J^2\lambda.$$

Calibration: If we *observe* a drift μ and a volatility σ, we must calibrate the jump process so that

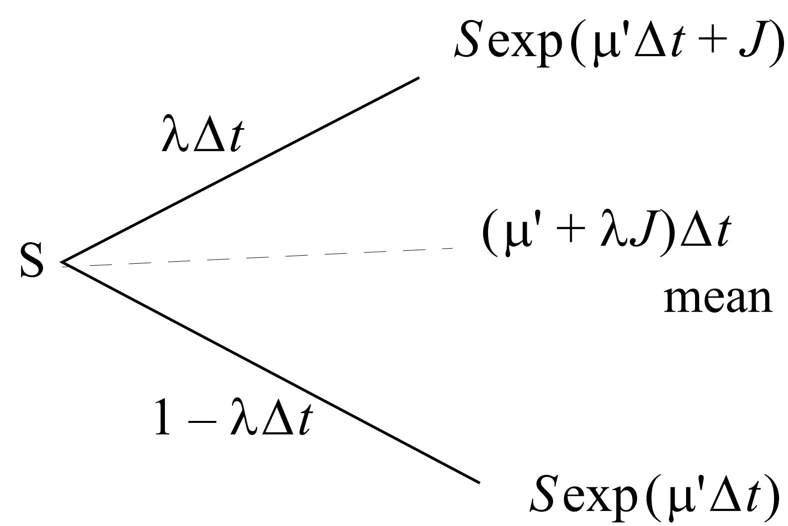
$$J = \frac{\sigma}{\sqrt{\lambda}}$$

$$\mu' = \mu - \sqrt{\lambda}\sigma$$

The one unknown is λ which is the probability of a jump in return of J in $\ln S$ per unit time.

This described how $\ln(S)$ evolves. How does S evolve?

$$\begin{aligned} E[S] &= (1-\lambda\Delta t)S\exp(\mu'\Delta t) + \lambda\Delta t S\exp(\mu'\Delta t + J) \\ &= S\exp(\mu'\Delta t)[1 + \lambda\Delta t(e^J - 1)] \\ &\approx S\exp\left[\left\{\mu' + \lambda(e^J - 1)\right\}\Delta t\right] \end{aligned}$$



Thus risk neutral growth means

$$r = \mu' + \lambda(e^J - 1)$$

$$\mu' = r - \lambda(e^J - 1)$$

We have to **compensate the drift for the jump contribution to calibrate to a total return r**.

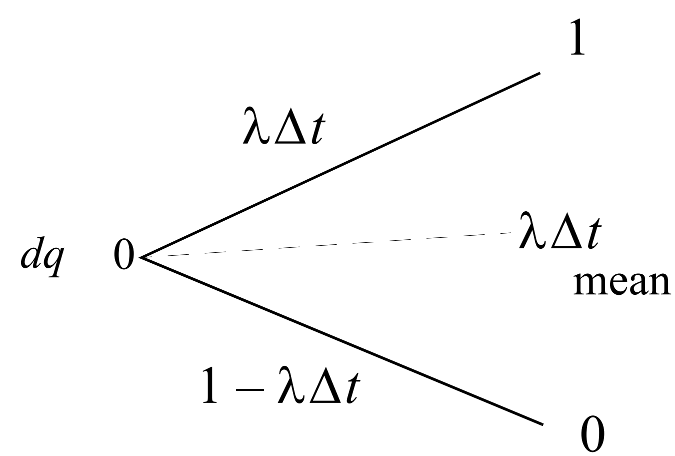
In continuous-time notation the jump can be written as a Poisson process

$$d\ln S = \mu' dt + J dq$$

Here dq is a jump or Poisson process that is modeled as follows:

The increment dq takes the valueS:

1 with probability λdt if a jump occurs
0 with probability $1 - \lambda dt$ if no jump occurs
expected value $E[dq] = \lambda dt$.



1.0.3 The Poisson Distribution of Jumps

λ = the constant probability of a jump J occurring per unit time.

$P(n, t)$ be the probability of n jumps occurring during time t.

$$P[0, t] = (1 - \lambda dt)^{\frac{t}{dt}} = \left(1 - \lambda t \frac{dt}{t}\right)^{\frac{t}{dt}} = \left(1 - \frac{\lambda t}{N}\right)^{N} \to e^{-\lambda t} \quad \text{as} \quad N \to \infty$$

$$\begin{aligned} P(n, t) &= \frac{N!}{n!(N-n)!}(\lambda dt)^{n}(1 - \lambda dt)^{N-n} \\ &= \frac{N!}{n!(N-n)!}\left(\frac{\lambda t}{N}\right)^{n}\left(1 - \frac{\lambda t}{N}\right)^{N-n} \\ &= \frac{N!}{N^{n}(N-n)!}\frac{(\lambda t)^{n}}{n!}\left(1 - \frac{\lambda t}{N}\right)^{N-n} \\ &\to \frac{(\lambda t)^{n}}{n!}e^{-\lambda t} \end{aligned}$$

as $N \to \infty$ for fixed n. Note that $\sum_{n=0}^{\infty} P(n, t) = 1$

The mean number of jumps during time t is λt so λ is the probability per unit time of one jump.