Lecture 23: Jump Diffusion Models

Jumps

- Why are we interested in jump models? Stocks and indexes don't diffuse smoothly, and do seem to jump. Even currencies sometimes jump. It's one of the things that happen.

- It's hard to define what a jump is, exactly.

- Jumps provide an easy way to produce the steep short-term skew that persists in equity index markets, and that indeed appeared soon after the jump/crash of 1987. They seem to play a part, behaviorally.

- Jumps are unattractive from a theoretical point of view because you cannot continuously hedge a distribution of finite-size jumps, and so risk-neutral arbitrage-free pricing isn't possible.

  As a result, most jump-diffusion models simply assume risk-neutral pricing without a thorough justification. It may make sense to think of the implied volatility skew in jump models as simply representing what sellers of options will charge to provide protection on an actuarial basis.

- Whatever the case, there have been and will be jumps in asset prices, and even if you can't hedge them, we are still interested in seeing what sort of skew they produce.

An Intuitive, Expectations View of the Skew Arising from Jumps

Assume:

Probability p_K that **a single jump** will occur taking the market from S to K sometime before option expiration T, and will then diffuse at $\sigma(T)$

Without that jump the future diffusion volatility of the index would have been $\sigma(T)$.

Assumption: The implied variance for strike K is the expected value of the realized variance along these two scenarios.

Realized daily volatility σ_d of an index S_i over a period of N days is the square root of the variance of the daily log returns r_i.

Without jump, $\sigma_d^2 \approx \frac{1}{N} \sum_{i=1}^{N} r_i^2$

With one jump, $\sigma_d^2 \approx \frac{1}{N} \left\{ \sum_{i=1}^{N-1} r_i^2 + \ln^2 \frac{S}{K} \right\} \approx \sigma_d^2(T) + \frac{\ln^2\left(\frac{S}{K}\right)}{N}$

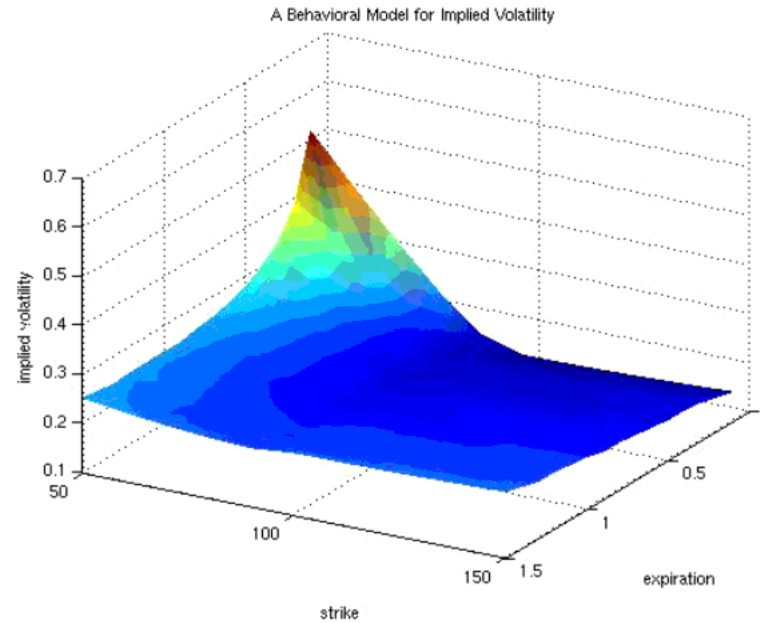
Expected daily variance is approx $p\left(\sigma_d^2(T) + \dfrac{\ln^2\left(\frac{S}{K}\right)}{N}\right) + (1-p)\sigma_d^2(T) = \sigma_d^2(T) + p_K \dfrac{\ln^2\left(\frac{S}{K}\right)}{N}$

Expected annualized variance is $365\left(\sigma_d^2 + p_K \dfrac{\ln^2\left(\frac{S}{K}\right)}{N}\right) = \sigma_a^2(T) + p_K \ln^2\left(\frac{S}{K}\right)\dfrac{365}{N}$

where $\sigma_a(T)$ is the annualized volatility and $\dfrac{N}{365} = T$ the time to expiration.

But this is supposedly the total implied variance $\Sigma^2(S, K, T) = \sigma_a^2(T) + \dfrac{p_K}{T}\ln^2\left(\frac{S}{K}\right)$

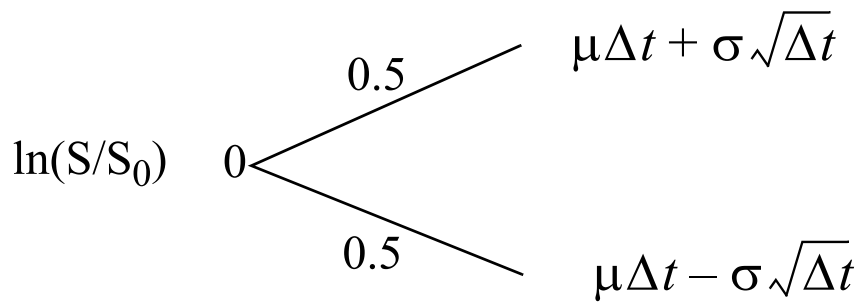
Thus $\Sigma^2(S, K, T) = \sigma_a^2(T) + \frac{p_K}{T}\ln^2\left(\frac{S}{K}\right)$



Modeling Jumps Alone

Pure Jump Processes: Calibration and Compensation Always Important

Discrete binomial approximation to a diffusion process over time Δt:

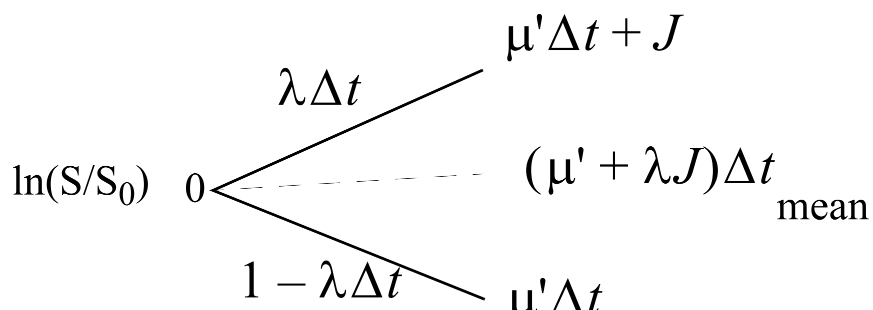


The **probabilities** of both up and down moves are **finite**, but the **moves** themselves are **small**, of order $\sqrt{\Delta t}$.

The net variance is $\sigma^2 \Delta t$ and the drift is μ. In continuous time this represents the process
$d\ln S = \mu dt + \sigma dZ$

Jumps are fundamentally different.

There the **probability of a jump *J* is small**, of order Δt, but the **jump itself is finite**.



3 parameters μ', J, λ

Mean:

$$\begin{aligned} E[\ln S] &= \lambda\Delta t[\mu'\Delta t + J] + (1 - \lambda\Delta t)\mu'\Delta t \\ &= (\mu' + \lambda J)\Delta t \end{aligned}$$

Variance

$$\begin{aligned} \text{var} &= \lambda\Delta t[J(1 - \lambda\Delta t)]^2 + (1 - \lambda\Delta t)[J\lambda\Delta t]^2 \\ &= (1 - \lambda\Delta t)J^2\lambda\Delta t[1 - \lambda\Delta t + \lambda\Delta t] \\ &= (1 - \lambda\Delta t)J^2\lambda\Delta t \\ &\to J^2\lambda\Delta t \qquad \text{as } \Delta t \to 0 \end{aligned}$$

$$\textit{Observed}\text{ drift } \mu = (\mu' + \lambda J)$$

$$\textit{Observed}\text{ volatility } \sigma = J^2 \lambda .$$

Calibration: If we *observe* a drift μ and a volatility σ, we must calibrate the jump process so that

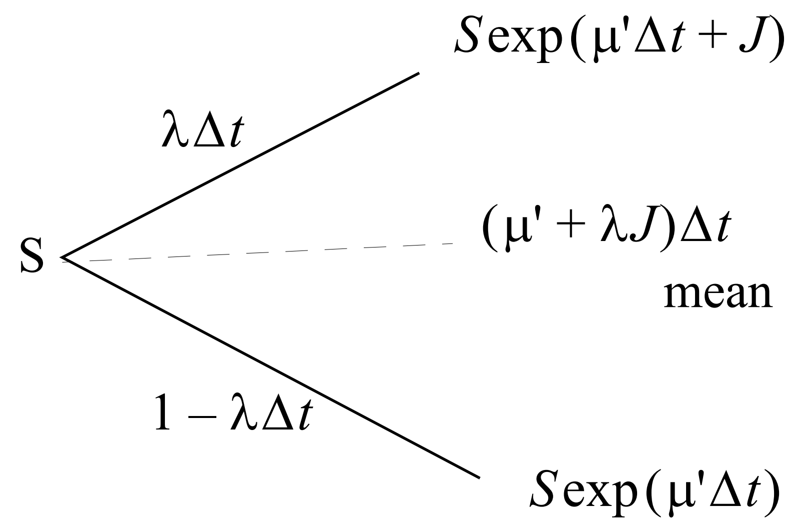
$$J = \frac{\sigma}{\sqrt{\lambda}}$$

$$\mu' = \mu - \sqrt{\lambda}\sigma$$

The one unknown is λ which is the probability of a jump in return of J in $\ln S$ per unit time.

This described how $\ln(S)$ evolves. How does S evolve?

$$\begin{aligned} E[S] &= (1-\lambda\Delta t)S\exp(\mu'\Delta t) + \lambda\Delta t S\exp(\mu'\Delta t + J) \\ &= S\exp(\mu'\Delta t)[1 + \lambda\Delta t(e^J - 1)] \\ &\approx S\exp\left[\left\{\mu' + \lambda(e^J - 1)\right\}\Delta t\right] \end{aligned}$$



Thus risk neutral growth means

$$r = \mu' + \lambda(e^{J} - 1)$$

$$\mu' = r - \lambda(e^{J} - 1)$$

We have to **compensate the drift for the jump contribution to calibrate to a total return r**.

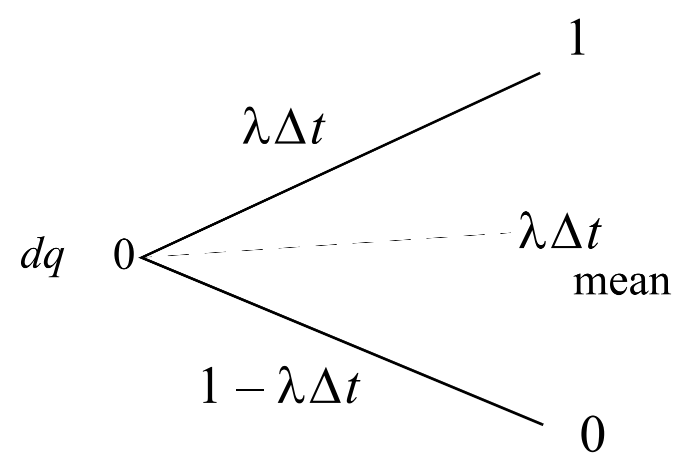
In continuous-time notation the jump can be written as a Poisson process

$$d\ln S = \mu' dt + J dq$$

Here dq is a jump or Poisson process that is modeled as follows:

The increment dq takes the valueS:

1 with probability λdt if a jump occurs
0 with probability $1 - \lambda dt$ if no jump occurs
expected value $E[dq] = \lambda dt$.



The Poisson Distribution of Jumps

λ = the constant probability of a jump J occurring per unit time.

$P(n, t)$ be the probability of n jumps occurring during time t.

$$P[0, t] = (1-\lambda dt)^{\frac{t}{dt}} = \left(1-\lambda t\frac{dt}{t}\right)^{\frac{t}{dt}} = \left(1-\frac{\lambda t}{N}\right)^{N} \to e^{-\lambda t} \quad \text{as} \quad N \to \infty$$

$$\begin{aligned} P(n, t) &= \frac{N!}{n!(N-n)!}(\lambda dt)^{n}(1-\lambda dt)^{N-n} \\ &= \frac{N!}{n!(N-n)!}\left(\frac{\lambda t}{N}\right)^{n}\left(1-\frac{\lambda t}{N}\right)^{N-n} \\ &= \frac{N!}{N^{n}(N-n)!}\frac{(\lambda t)^{n}}{n!}\left(1-\frac{\lambda t}{N}\right)^{N-n} \\ &\to \frac{(\lambda t)^{n}}{n!}e^{-\lambda t} \end{aligned}$$

as $N \to \infty$ for fixed n. Note that $\sum_{n=0}^{\infty} P(n, t) = 1$

The mean number of jumps during time t is λt so λ is the probability per unit time of one jump.

Pure jump risk-neutral option pricing

We can value a standard call option (**assuming** risk-neutral expectation) for a pure jump model:

$$C = e^{-r\tau} \sum_{n=0}^{\infty} max[Se^{\mu'\tau + nJ} - K, 0]\frac{(\lambda\tau)^n}{n!}e^{-\lambda t}$$

where $Se^{\mu'\tau + nJ}$ is the final stock price after n Poisson jumps, and the payoff of the call is multiplied by the probability of the jump occurring, and

$$\mu' = r - \lambda(e^J - 1)$$

Jumps plus Diffusion

Some comments

- You can replicate an option exactly with stock and a finite number of options for a finite number of jumps of known size.

- But with an infinite number of possible jumps, you cannot replicate; you can only minimize the variance of the P&L.

- Merton's model of jump-diffusion regards jumps as "abnormal" market events that have to be superimposed upon "normal" diffusion.
  Mandelbrot, and Eugene Stanley and his econophysics collaborators prefer a single model, rather than a "normal" and "abnormal" model.

- Variance-gamma models also provide a unified view of market moves in which all stock price movements are jumps of various sizes.

Merton's jump-diffusion model and its PDE

Poisson jumps + GBM diffusion,
$$\frac{dS}{S} = \mu dt + \sigma dZ + Jdq$$

$$E[dq] = \lambda dt$$
$$var[dq] = \lambda dt$$

J is like a random dividend, **not** paid to stockholder.. Later we'll make J a normal random variable.

You can derive a partial differential equation for options valuation:

Option $C(S, t)$ and usual hedged portfolio $\pi = C - nS$

$$ndS = nS(\mu dt + \sigma dZ + Jdq)$$

$$\Delta C = \left[C_t + \frac{1}{2}C_{SS}(\sigma S)^2\right]dt + C_S(\mu S dt + \sigma S dZ) + [C(S + JS, t) - C(S, t)]dq$$

$$\begin{aligned}\Delta\pi &= \Delta C - n[\mu S dt + \sigma S dZ + JS dq] \\ &= \left[C_t + C_S \mu S + \frac{1}{2}C_{SS}(\sigma S)^2 - n\mu S\right]dt + (C_S - n)\sigma S dZ \\ &\quad + [C(S + JS, t) - C(S, t) - nSJ]dq\end{aligned}$$

Choose n to hedge the diffusion: $n = C_S$.

$$\Delta\pi = \left[C_t + \frac{1}{2}C_{SS}(\sigma S)^2\right]dt + [C(S + JS, t) - C(S, t) - C_S SJ]dq$$

The partially hedged portfolio is still risky because of the possibility of jumps.

Imagine that we can diversity our portfolio over many different stocks and their options, where the stocks have uncorrelated jumps, so that jump risk becomes diversifiable and can be eliminated?

$$E[\Delta V] = rV\Delta t \qquad\qquad E[dq] = \lambda\Delta t$$

$$C_t + \frac{1}{2}C_{SS}(\sigma S)^2 + E[C((1 + J)S, t) - C(S, t) - C_S SJ]\lambda = (C - SC_S)r$$

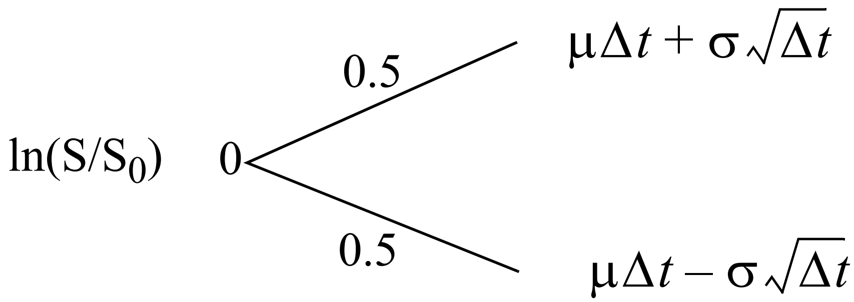
averaging over all jump sizes.

$$C_t + \frac{1}{2}C_{SS}(\sigma S)^2 + rSC_S - rC + E[C((1 + J)S, t) - C(S, t) - C_S SJ]\lambda = 0$$

This is a mixed difference/partial-differential equation for a standard call with terminal payoff $C_T = max(S_T - K, 0)$. For $\lambda = 0$ it reduces to the Black-Scholes equation. We will solve it a little later by the Feynman-Kac method as an expected discounted value of the payoffs.

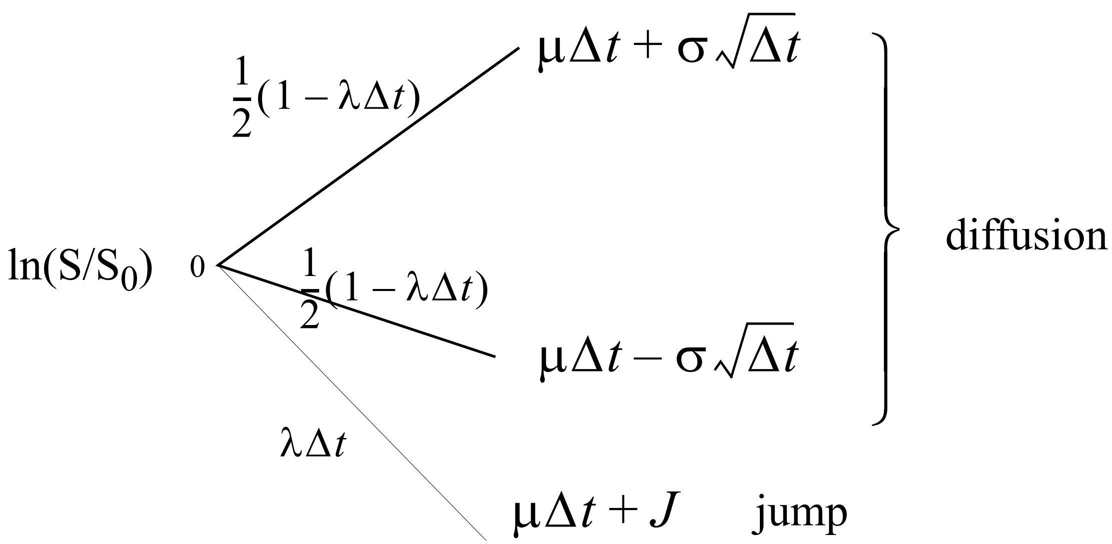
Trinomial Jump-Diffusion and Compensation

Diffusion can be modeled binomially, as in



The volatility σ of the log returns adds an Ito $\sigma^2/2$ term to the drift of the stock price S itself, so that for pure risk-neutral diffusion one must choose $\mu = r - \sigma^2/2$.

To add jumps one J needs a third, trinomial, leg in the tree:



Just as diffusion modifies the drift of the stock price, so do jumps.

The expected log return after time Δt:

$$E\left[\log\frac{S}{S_0}\right] = \left(\frac{1-\lambda\Delta t}{2}\right)2\mu\Delta t + \lambda\Delta t[\mu\Delta t + J]$$

$$= (\mu + J\lambda)\Delta t$$

Thus the effective drift of the jump-diffusion process will be $\mu_{JD} = \mu + J\lambda$.

The variance at time Δt is

$$var \equiv \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} - J\lambda\Delta t]^2 + \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} + J\lambda\Delta t]^2$$

$$+ \lambda\Delta t[J(1-\lambda\Delta t])]^2$$

$$= \left(\frac{1-\lambda\Delta t}{2}\right)[2\sigma^2\Delta t + 2J^2\lambda^2(\Delta t)^2] + \lambda\Delta t J^2(1-\lambda\Delta t)^2$$

$$= (1-\lambda\Delta t)[\sigma^2\Delta t] + (1-\lambda\Delta t)J^2\lambda\Delta t(\lambda\Delta t + 1 - \lambda\Delta t)$$

$$= (1-\lambda\Delta t)[\sigma^2 + J^2\lambda]\Delta t$$

so that, as $\Delta t \to 0$, the variance of the jump diffusion process is $\sigma_{JD}^2 = [\sigma^2 + J^2\lambda]$

the sum of the diffusion variance plus the jump variance. The drift and variance are both affected by the fractional jump J and its probability λ of occurring per unit time.

The Compensated Process

How must we choose/calibrate the diffusion and jumps so that $E[dS] = Srdt$?

First let's compute the stock growth rate under jump diffusion.

$$E\left[\frac{S}{S_0}\right] = \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t + \sigma\sqrt{\Delta t}} + \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t - \sigma\sqrt{\Delta t}} + \lambda\Delta t e^{\mu\Delta t + J}$$

$$= e^{\mu\Delta t}\left[\frac{(1-\lambda\Delta t)}{2}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right) + \lambda\Delta t e^{J}\right]$$

One can show by expanding this to keep terms of order Δt that

$$E\left[\frac{S}{S_0}\right] = \exp\left(\left\{\mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)\right\}\Delta t\right) + \text{higher order terms}$$

so that, if we want the stock to grow risk-neutrally, we must set $r = \mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)$

$$\mu_{JD} = r - \frac{\sigma^2}{2} - \lambda(e^J - 1)$$

diffusion compensation

jump compensation

Valuing a Call in the Jump-Diffusion Model

The process we are considering is $\frac{dS}{S} = \mu dt + \sigma dZ + J dq$

where
$$E[dq] = \lambda dt$$
$$var[dq] = \lambda dt$$

J is assumed to be a fixed jump size now, but will later be generalized to a normal variable.

Risk neutrality: $$\mu = r - \frac{\sigma^2}{2} - \lambda(e^J - 1)$$

The value of a standard call in this model is given by

$$C_{JD} = e^{-r\tau} E[(S_T - K, 0)]$$

The risk-neutral terminal value of the stock price is $S_T = Se^{\mu\tau + Jq + \sigma\sqrt{\tau}Z}$

Sum over 0, 1, ... n ... jumps plus the diffusion, where the probability of n jumps $\frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$

Thus,
$$C_{JD} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{\lambda\tau^{n}}{n!} e^{-\lambda\tau} E\left[max\left(S_T^n - K, 0\right)\right]$$

where S_T^n is the terminal lognormal distribution of the stock price that started with initial price S and underwent n jumps as well as the diffusion.

This is an expectation over a lognormal stock price that, after time τ, has undergone n jumps, and therefore is simply related to a Black-Scholes expectation with a jump-shifted distribution or different forward price.

In the compensated world, the expected return on a stock that started at an initial price S and suffered n jumps is

$$\mu_n = r - \frac{\sigma^2}{2} - \lambda(e^J - 1) + \frac{nJ}{\tau}$$

where the last term in the above equation adds the drift corresponding to n jumps to the standard compensated risk-neutral drift $r - \frac{\sigma^2}{2}$, which appears in the Black-Scholes formula via the terms $d_{1,2}$.

Thus, since S_T is lognormal with a shifted center moved by *n* jumps,

$$E\left[max\left(S_T^n - K, 0\right)\right] \equiv e^{r_n \tau} C_{BS}(S, K, \tau, \sigma, r_n)$$

where $C_{BS}(S, K, \tau, \sigma, r_n)$ is the standard Black-Scholes formula for a call with strike K and volatility σ with the drift rate r_n given by

$$r_n \equiv \mu_n + \frac{\sigma^2}{2} = r - \lambda(e^J - 1) + \frac{nJ}{\tau}$$

The $\sigma^2/2$ term is omitted because the Black-Scholes formula for a stock with volatility σ already includes the term $\sigma^2/2$ in the $N(d_{1,\,2})$ terms as part of the definition of C_{BS}.

$$C_{JD} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{r_n \tau} C_{BS}(S, K, \tau, \sigma, r_n)$$

$$= e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{\left(r - \lambda(e^J - 1) + \frac{nJ}{\tau}\right)\tau} C_{BS}(S, K, \tau, \sigma, r_n)$$

$$= e^{-(\lambda e^J \tau)} \sum_{n=0}^{\infty} \frac{(\lambda e^J \tau)^n}{n!} C_{BS}\left(S, K, \tau, \sigma, r + \frac{nJ}{\tau} - \lambda(e^J - 1)\right)$$

Writing $\bar{\lambda} = \lambda e^J$ as the “effective” probability of jumps, we obtain

$$C_{JD} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sigma, r + \frac{nJ}{\tau} - \lambda(e^J - 1)\right)$$

This is a **mixing formula**. The jump-diffusion price is a mixture of Black-Scholes options prices with compensated drifts. This is similar to the result we got for stochastic volatility models with zero correlation -- a mixing theorem -- but here we had to appeal to the diversification of jumps or actuarial pricing rather than perfect riskless hedging.

Generalize, as Merton did, to a distribution of normal jumps with

$$E[J] = \bar{J} \qquad var[J] = \sigma_J^2$$

Then $$E[e^J] = e^{\bar{J} + \frac{1}{2}\sigma_J^2}$$

Incorporating the expectation over this distribution of jumps has two effects:

- J gets replaced by $\bar{J} + \frac{1}{2}\sigma_J^2$

- second, the variance of the jump process adds to the variance of the entire distribution in the Black-Scholes formula by blurring the mean of each subdistribution, so that we must replace σ^2 by $\sigma^2 + \frac{n\sigma_J^2}{\tau}$ because n jumps adds $\frac{n\sigma_J^2}{\tau}$ amount of variance. (The division by τ is necessary because variance is defined in terms of geometric Brownian motion and grows with time, but the variance of normally distributed J is independent of time.)

The general formula is therefore:

$$C_{JD} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r + \frac{n\left(\bar{J} + \frac{1}{2}\sigma_J^2\right)}{\tau} - \lambda\left(e^{\bar{J} + \frac{1}{2}\sigma_J^2} - 1 \right) \right)$$

where $\bar{\lambda} = \lambda e^{\bar{J} + \frac{1}{2}\sigma_J^2}$

If $\bar{J} = -\frac{1}{2}\sigma_J^2$ so that $E[e^J] = 1$ and the jumps add no drift to the process, then we get the simple intuitive formula

$$C_{JD} = e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r \right)$$

in which we simply sum over an infinite number of Black-Scholes distributions, each with identical riskless drift but differing volatility dependent on the number of jumps and their distribution.