LECTURE 12

STATIC HEDGING OF BARRIER OPTIONS
EXTENDING THE BINOMIAL MODEL

Today: The binomial model as a basis for extending Black-Scholes

Exam: Mon 9 March and Wed 11 March during class in Room 303. No cheat sheet necessary or allowed. Bring a calculator.

Recap: Insight into Static Hedging from Valuation Formula

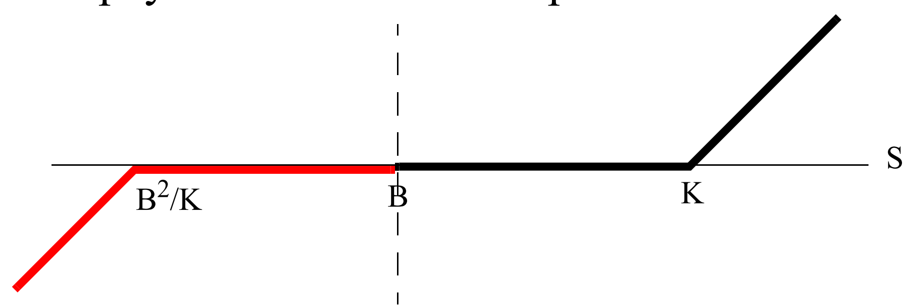
We showed above that, in a Black-Scholes world with zero drift, the fair value for a down-and-out call with strike K and barrier B is given by

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.12.1}$$

Payoff of first term: $\theta(S-K)(S-K)$

Payoff of $\frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right)$ $\quad \frac{S}{B}\left(\frac{B^2}{S} - K\right)\theta\left(\frac{B^2}{S} - K\right) = \left(B - \frac{KS}{B}\right)\theta\left(\frac{B^2}{K} - S\right) = \frac{K}{B}\theta\left(\frac{B^2}{K} - S\right)\left(\frac{B^2}{K} - S\right)$

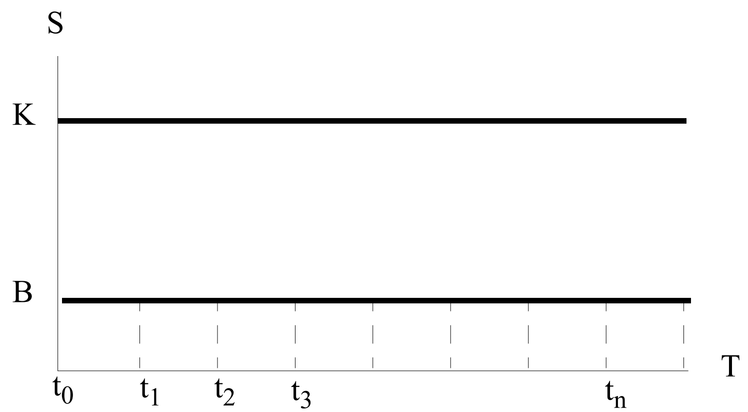
This second term represents the payoff of *K/B* standard puts with strike B^2/K. Roughly speaking



the payoff of a down-and-out-call is that of an ordinary call and its strike image reflection (in log space) in the barrier. What volatilities are you sensitive to?
What two options will replicate your barrier under your assumed volatility smile?

Read This: Weak Replication of Exotics with Standard Options

Consider a discrete down-and-out call with strike K, a barrier B, expiration time T; the option knocks out only at *n* times $\{t_1, t_2, ..., t_n\}$ between inception of the trade and expiration.



Create a portfolio of standard options that have the payoff of a call with strike K if the barrier B hasn't been penetrated, and vanishes on the boundary B at time $\{t_1, t_2, ..., t_n\}$.

We can replicate the payoff of the call at expiration with a standard call $C(S, t, K, T)$
Create a portfolio V whose *value* is the $C(S, t, K, T)$ at expiration but vanishes at each intermediate time t_i when $S = B$. The value of securities added to the portfolio must cancel the value of entire portfolio on B, but must also add have no payoff above B in order to mimic a call:
We can use puts $P(S, t, B, t_i)$ with strike B and expiration time t_i

Here we replicate with a payoff of n standard puts $P(S, t, B, t_i)$ and the call $C(S, t, K, T)$ such that

$$V(S,t) = C(S,t,K,T) + \sum_{j=1}^{n} \alpha_j P(S,t,K,t_j) \qquad \text{Eq.12.2}$$

Since both the call and the put satisfy the Black-Scholes equation, so does V, which it should. Only its boundary conditions differ from those of a standard call or put.

Solve for α_j such that V vanishes at all t_i for $i = 0$ to $n-1$ at $S = B$:

$$V(B,t_i) = C(B,t_i,K,T) + \sum_{j=1}^{n} \alpha_j P(B,t_i,K,t_j) = 0 \qquad \text{Eq.12.3}$$

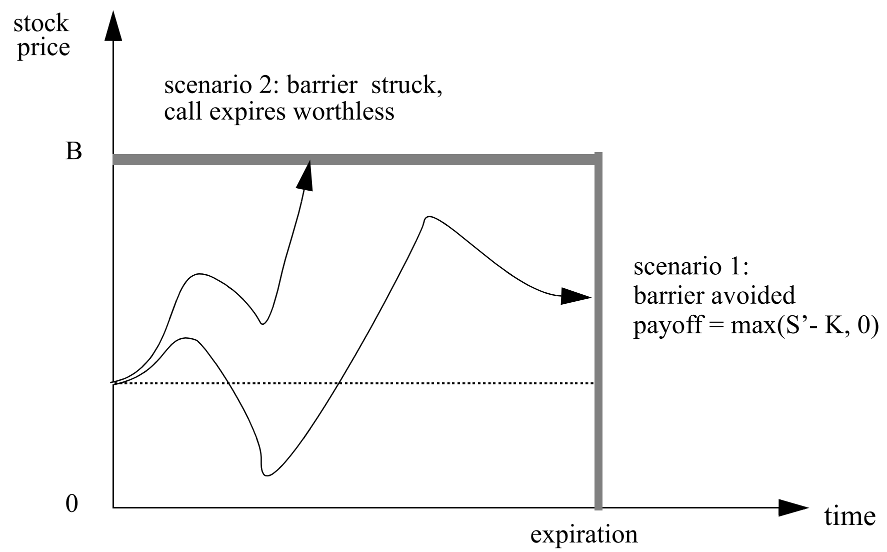
n equations for the n unknowns α_j, which can be solved in sequential order.

As n increases we get closer to a continuous barrier. I**t is still weak replication: coefficients depend on the model. It is static and uses only other options, which is good; it is weak, which is not so good.**
When the stock price hits the barrier, the replicating portfolio must be immediately unwound. This assumes that the stock price moves continuously and that there are no jumps across the barrier.

A Numerical Example: Up-and-Out Call with High Gamma

All options values are Black-Scholes.
An up-and-out call option.

Stock price:	100
Strike:	100
Barrier:	120
Time to expiration:	1 year
Up-and-Out Call Value:	0.656
Ordinary Call Value:	11.434



Portfolio 1 replicates the target up-and-out call for all scenarios which never hit the barrier.

Quantity	Type	Strike	Expiration	Value 1 year before expiration	
				Stock at 100	Stock at 120
1	call	100	**1 year**	11.434	**25.610: too big**

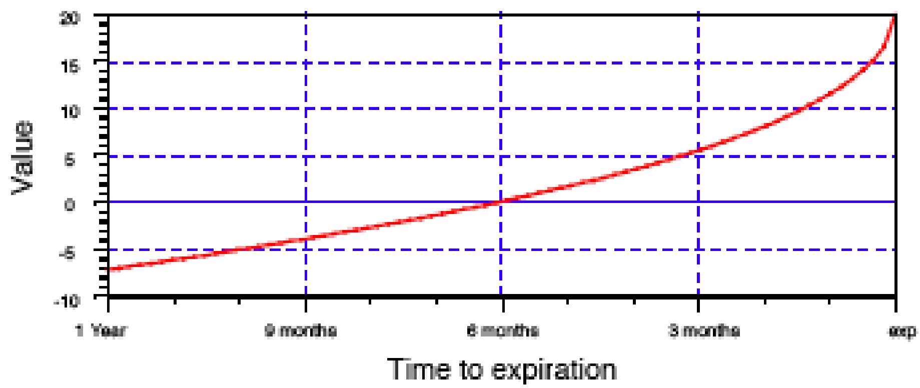
Portfolio 2 improvement.

Add a short position in *one* extra option so as to attain the correct zero value for the replicating portfolio at a stock price of 120 with 6 months to expiration, as well as for all stock prices below the barrier at expiration.

Portfolio 2. Its payoff matches that of an up-and-out call if the barrier is never crossed, or if it is crossed exactly at 6 months before expiration.

Quantity	Type	Strike	Expiration	Value 6 months before expiration	
				Stock at 100	**Stock at 120**
1.000	call	100	1 year	7.915	22.767
-2.387	call	120	1 year	-4.446	-22.767
Net				3.469	0.000

Value of Portfolio 2 on the barrier at 120.



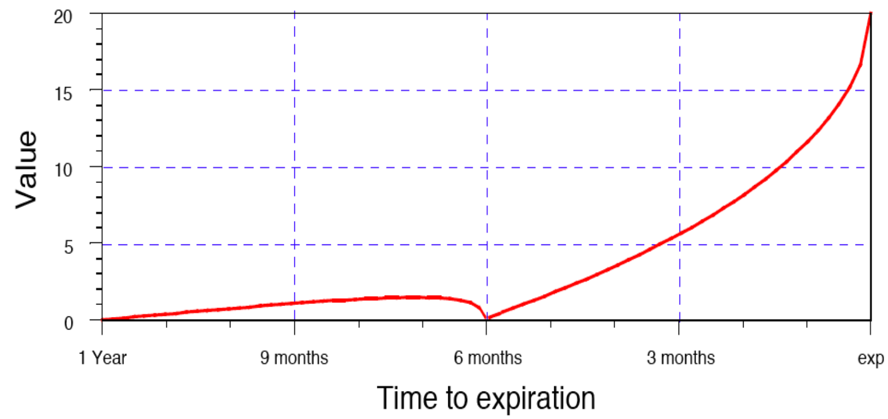
Portfolio 2 matches the zero payoff of the up-and-out call at a stock price of 120 at both six months *and* one year prior to expiration.

Portfolio 3. Its payoff matches that of an up-and-out call if barrier is never crossed, or if it is crossed exactly at 6 months or 1 year before expiration.

Quantity	Type	Strike	Expiration	Value for stock price = 120	
				6 months	1 year
1.000	call	100	1 year	22.767	25.610
-2.387	call	120	1 year	-22.767	-32.753
0.752	call	120	6 months	0.000	7.142
Net				0.000	0.000

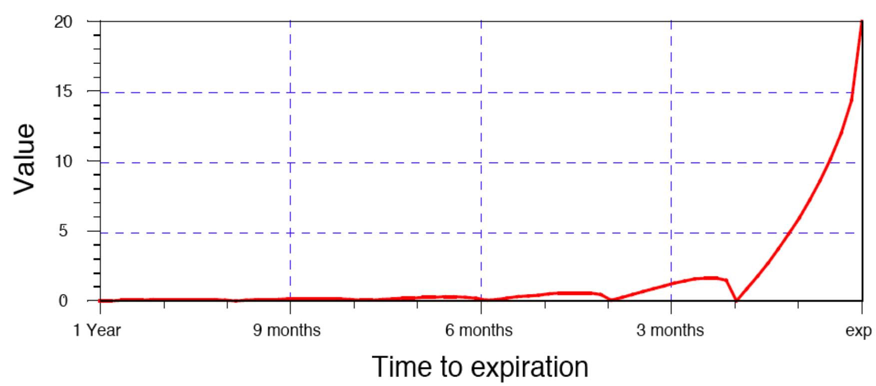
Value of Portfolio 3 on the barrier at 120

For the first six months in the life of the option, the boundary value at a stock price of 120 remains fairly close to zero.



A portfolio of seven standard options at a stock level of 120 that matches the zero value of the target up-and-out call on the barrier every two months.

Value on the barrier at 120 of a portfolio of standard options that is constrained to have zero value every two months.



Replication Accuracy

An up-and-out call option.

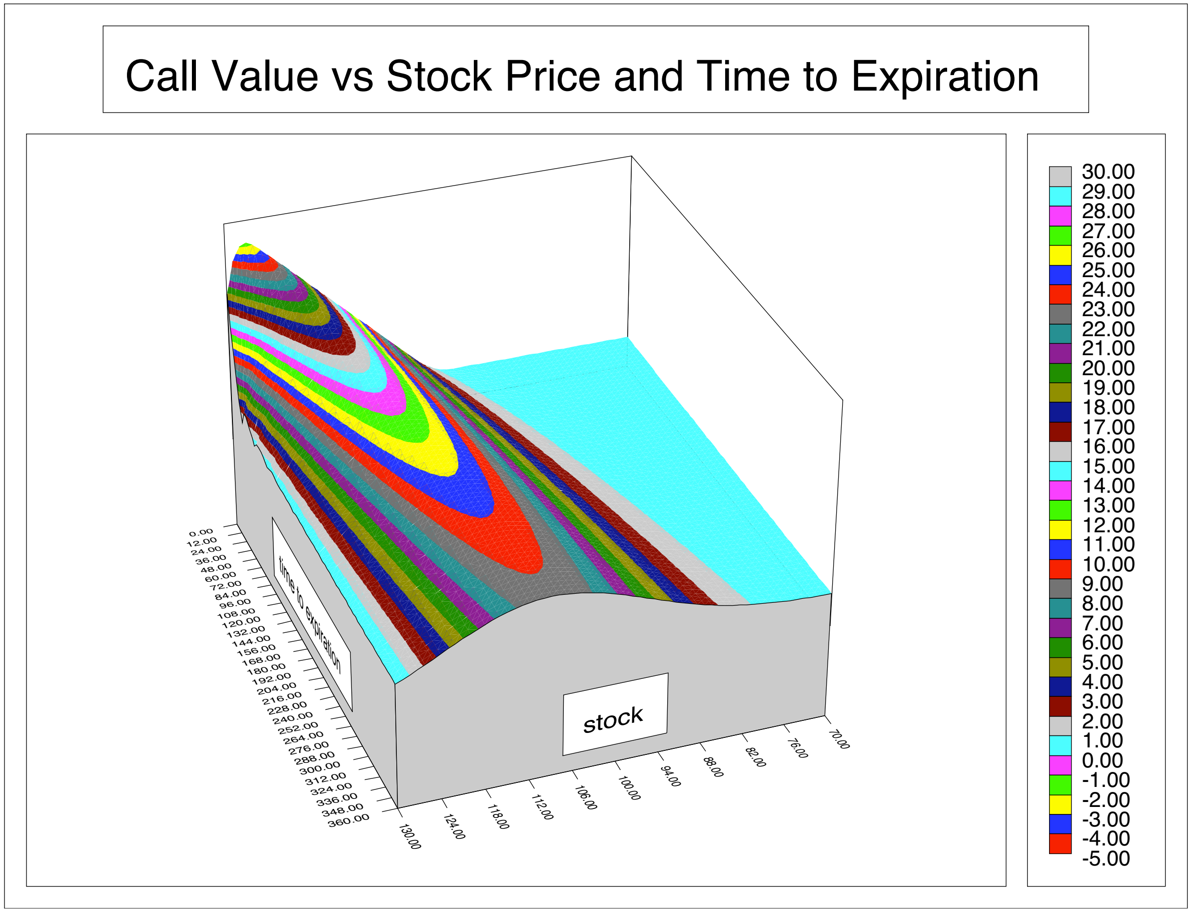
Stock price:	100
Strike:	100
Barrier:	120
Time to expiration:	1 year
Up-and-Out Call Value:	1.913

The replicating portfolio of 7 options.

Quantity	Option Type	Strike	Expiration (months)	Value (Stock = 100)
0.16253	Call	120	2	0.000
0.25477	Call	120	4	0.018
0.44057	Call	120	6	0.106
0.93082	Call	120	8	0.455
2.79028	Call	120	10	2.175
-6.51351	Call	120	12	-7.140
1.00000	Call	100	12	6.670
Total				2.284

The theoretical value of the replicating portfolio in Table at a stock price of 100, one year from expiration, is 2.284, about 0.37 or 19% off from the theoretical value of the target option.
Cost of hedge know. Then unwind if you get too close and take your losses.

Here's the behavior over all stock prices and time prior to expiration of a 24-option replicating portfolio.



You can see it looks a lot like the value of the payoff of an up and out call option. But it's been created using Black-Scholes to match the coefficients, assuming future volatilities are the same as today, and so it's imperfect.

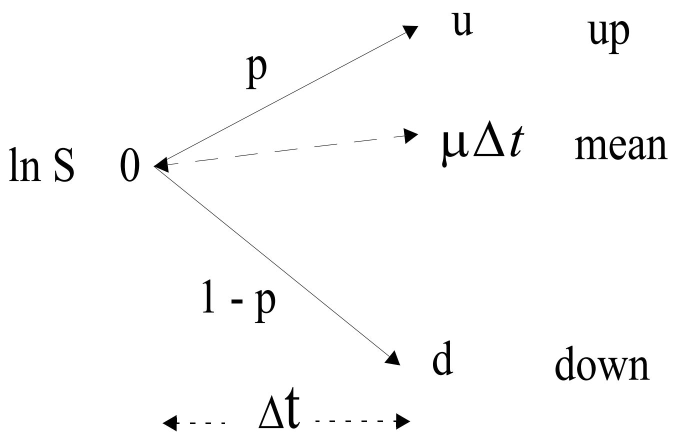
The Binomial Model for Stock Evolution

Search for models of stock price evolution that can account for the smile. It's easiest to begin in the binomial framework where intuition is clearer.

In Black-Scholes framework $\quad d(\ln S) = \mu dt + \sigma dZ.$

Expected return on the stock price is $\mu + \sigma^2/2$. The total variance in time Δt is $\sigma^2 \Delta t$.

We model the actual evolution of the stock price over an instantaneous time Δt by means of a one-period binomial tree.



How do we choose p, u and d to match the continuous-time $d(\ln S) = \mu dt + \sigma dZ$?
Match the mean and variance of the return:

$$pu + (1-p)d = \mu \Delta t$$

$$p[u - \mu \Delta t]^2 + (1-p)[d - \mu \Delta t]^2 = \sigma^2 \Delta t$$

$$pu + (1-p)d = \mu \Delta t$$

$$p(1-p)(u-d)^2 = \sigma^2 \Delta t$$

Two constraints on the three variables *p, u,* and *d*. Pick convenient ones.

First Solution: The Cox-Ross-Rubinstein Convention

Choose $u + d = 0$: stock price always returns to the same level; center of the tree fixed.

$$(2p-1)u = \mu\Delta t$$

$$4p(1-p)u^2 = \sigma^2\Delta t$$

Squaring the first equation and dividing by the second leads to $\frac{(2p-1)^2}{4p(1-p)} = \frac{\mu^2\Delta t}{\sigma^2}$

If Δt is zero then $p = 1/2$. Write $p = 1/2 + \varepsilon$, then

$$\varepsilon \approx \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad u = \sigma\sqrt{\Delta t}$$

$$p \approx \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad d = -\sigma\sqrt{\Delta t}$$

Check: mean return of the process is $\left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) - \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) = \mu\Delta t$ perfect!

The variance is $p(1-p)(u-d)^2 \approx \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right)4\sigma^2\Delta t \approx \sigma^2\Delta t - \mu^2(\Delta t)^2$ a little small.

The convergence to the continuum limit is a little slower than if it matched the variance exactly.
For small enough Δt there is no riskless arbitrage with this convention

Another Solution: The Jarrow-Rudd Convention

We must satisfy the constraints

$$\begin{aligned} pu + (1-p)d &= \mu\Delta t \\ p(1-p)(u-d)^2 &= \sigma^2\Delta t \end{aligned}$$

Choose $p = 1/2$, so that the up and down moves have equal probability:

$$\begin{aligned} u + d &= 2\mu\Delta t \\ u - d &= 2\sigma\sqrt{\Delta t} \end{aligned}$$

$$\begin{aligned} u &= \mu\Delta t + \sigma\sqrt{\Delta t} \\ d &= \mu\Delta t - \sigma\sqrt{\Delta t} \end{aligned}$$

The mean return is exactly μ; the volatility of returns is exactly σ, convergence is faster than.

$$E[S] = \frac{(e^u + e^d)}{2}S = e^{\mu\Delta t}\frac{\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{2} \approx e^{\mu\Delta t}\left(1 + \frac{\sigma^2\Delta t}{2}\right) \approx e^{\left(\mu + \frac{\sigma^2}{2}\right)\Delta t}$$

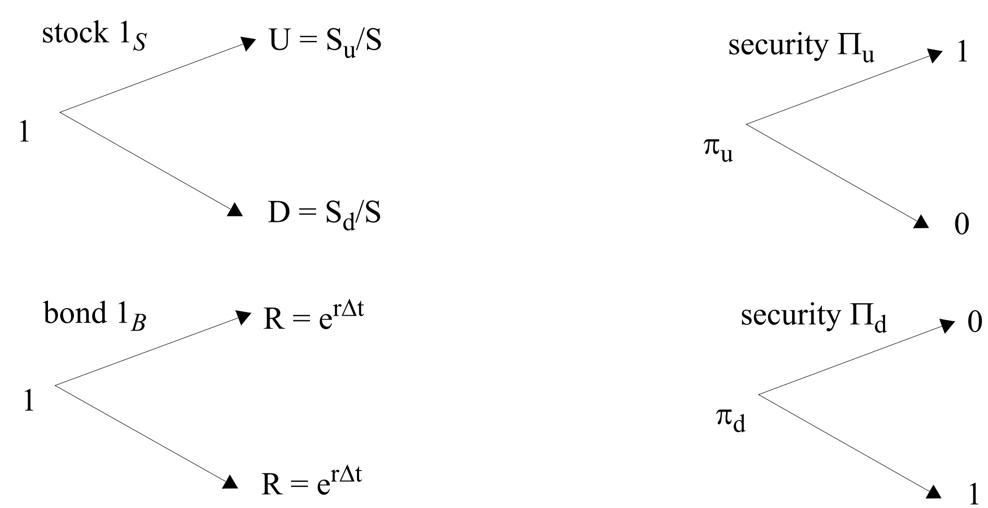
In the limit $\Delta t \to 0$, both the CRR and the JR convention describe the same process, and there are many other choices of u, d, and q that do so too.

We will use these binomial processes as a basis for modeling more general stochastic processes that can perhaps explain the smile.

The Binomial Model for Options (Precursor to Local Vol)

Options Valuation in the q-measure

One can decompose the stock S and the bond B into two securities Π_u and Π_d that pay out only in the up or down state.



Define $\Pi_u = \alpha \times \overrightarrow{1}_S + \beta \times \overrightarrow{1}_B$. Note that because it is riskless, the sum $\Pi_u + \Pi_d = 1/R$

Then $\begin{aligned} \alpha U + \beta R &= 1 \\ \alpha D + \beta R &= 0 \end{aligned}$ so that $\begin{aligned} \alpha &= \frac{1}{(U-D)} \\ \beta &= \frac{-D}{R(U-D)} \end{aligned}$ $\quad \overrightarrow{\Pi_u} = \dfrac{R \times \overrightarrow{1}_S - D \times \overrightarrow{1}_B}{R(U-D)} \quad \overrightarrow{\Pi_d} = \dfrac{U \times \overrightarrow{1}_B - R \times \overrightarrow{1}_S}{R(U-D)}$

The values are $\pi_u = \frac{R-D}{R(U-D)} \equiv \frac{q}{R}$ $\qquad \pi_d = \frac{U-R}{R(U-D)} \equiv \frac{1-q}{R}$

$q = \frac{R-D}{U-D}$ $\qquad 1-q = \frac{U-R}{U-D}$ are the no-arbitrage probabilities that don't depend on p.

The first equation can be rewritten as $qU + (1-q)D = R$, or

$$S = \frac{qS_u + (1-q)S_d}{R}$$

so that in this measure the expected future stock price is the forward price.

Any option C which pays C_u (C_d) in the up (down)-state is replicated by $C = C_u\Pi_u + C_d\Pi_d$ with

$$C = \frac{qC_u + (1-q)C_d}{R}$$

Regard the stock equation as *defining* the measure q given the values of S, S_u and S_d; the second equation specifies the value C in terms of the option payoffs and the value of p. This is why probability theory seems to be important in options pricing, because of complete markets.

The Black-Scholes Partial Differential Equation and the Binomial Model

The Black-Scholes PDE comes from taking the limit of the binomial pricing equation as $\Delta t \to 0$. Cox-Ross-Rubinstein choice of q, u & d (or in another binomial convention too).

$$u = \sigma\sqrt{\Delta t} \qquad d = -\sigma\sqrt{\Delta t}$$

$$RC = qC_u + (1-q)C_d$$

$$q = \frac{RS - S_d}{S_u - S_d} \qquad 1 - q = \frac{S_u - RS}{S_u - S_d}$$

Now substitute $S_u = e^u S$, $S_d = e^d S$ and $R = e^{r\Delta t}$ in the two equation directly above, so that all terms are re-expressed in terms of the variables r, σ and S.

$$e^{r\Delta t}C = qC\left(e^{\sigma\sqrt{\Delta t}}S, t + \Delta t\right) + (1-q)C\left(e^{-\sigma\sqrt{\Delta t}}S, t + \Delta t\right)$$

Substituting for p and performing a Taylor expansion to leading order in Δt, one can show that

$$Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$

Extending Black-Scholes to Time-dependent Deterministic Volatility

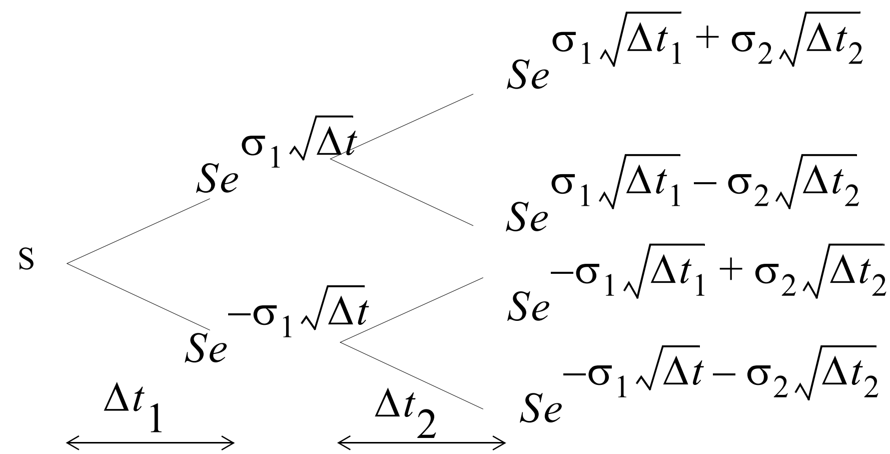
Black-Scholes and the binomial model assume that σ is constant.
Suppose now that the stock volatility σ is a function of (future) time t.

$$\frac{dS}{S} = \mu dt + \sigma(t) dZ$$

How do we modify Black-Scholes or the binomial tree method for term structure $\sigma(t)$?

Suppose we try to build a CRR tree with σ_1 in period 1 and σ_2 in period 2.

The tree doesn't "close" in the second period unless σ_i is constant. It needn't but it's computationally convenient.



To make the tree close

$$\sigma_1\sqrt{\Delta t_1} = \sigma_2\sqrt{\Delta t_2} = \dots = \sigma_N\sqrt{\Delta t_N}$$

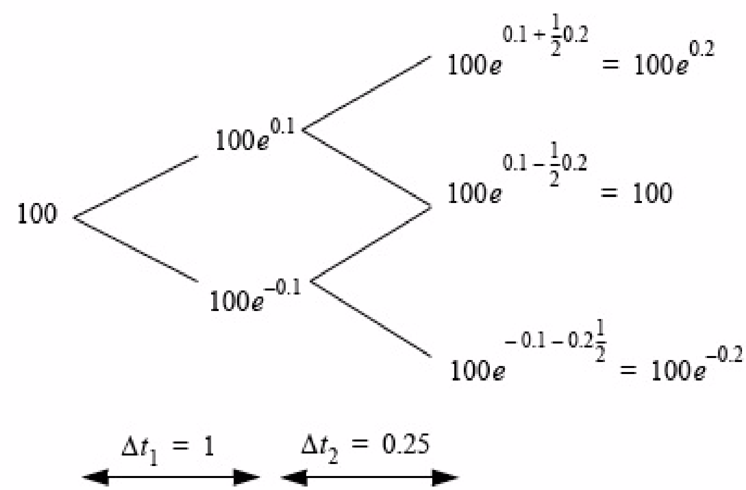
Thus, though the tree looks the same from a topological point of view, each step between levels involves a step in time that is smaller when volatility in the period is larger, and vice versa.

How many time steps needed? Given the term structure of volatilities, solve for the number of time steps needed.

Example: suppose we believe volatility will be 10% in year 1 and 20% in year 2. We choose the first period to be one year long and then solve for the second period.

	period 1	period 2
σ	0.1	0.2
σ^2	0.01	0.04
Δt	1	**1/4**

CRR convention up & down moves given by $\sigma_i\sqrt{\Delta t_i}$:



In essence, we build a standard binomial tree with price moves generated by $e^{\pm\sigma\sqrt{\Delta t}}$, where $\sigma\sqrt{\Delta t}$ is constant, and then choose σ to match the term structure of volatility, and then adjust Δt. The tree and node prices are topologically identical to a constant volatility tree. However, we reinterpret the times at which the levels occur, and the volatilities that took them there. One tree with same prices at each node can represent different stochastic processes with different volatilities moving through different amounts of time.

The tree in the illustration above extended to 1.25 years. We would need a total of 4 periods to span the entire second year at a volatility of 0.2, but only one period for the first year, so that 5 steps are necessary to span two years.

More generally, if you have a definite time T to expiration, then $T = \sum_{i=1}^{N} \Delta t_i = \Delta t_1 \sum_{i=1}^{N} \frac{\sigma_1^2}{\sigma_i^2}$

Solve for N.

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Note 1: Even though the nodes in the tree above have *prices* corresponding to a CRR tree with $\sigma_i \sqrt{\Delta t_i} = 0.1$, irrespective of volatility term structure, *the binomial no-arbitrage probabilities vary with* Δt_i, because for each fork in the tree

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

Even though $e^{\sigma\sqrt{\Delta t}}$ is the same over all time steps Δt, the factor $e^{r\Delta t}$ varies from step to step with the value of Δt, so that p varies from level to level.

Note 2: The total variance at the terminal level of the tree is the same as before

$$\equiv \sum \sigma_i^2 \Delta t_i \rightarrow \int \sigma^2(s) ds$$

Valuing an option on this tree leads to the Black-Scholes formula with the relevant time to expiration, the relevant interest rates and dividends at each period, and a total variance

$$\Sigma^2 = \frac{1}{T-t} \int_t^T \sigma^2(s) ds$$

Constant volatility of 20%

sigma		0.2	0.2	0.2	0.2	0.2
delta t		0.25	0.25	0.25	0.25	0.25
Time	0	0.25	0.5	0.75	1	1.25
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

First 3-months volatility is 10%

sigma		0.1	0.2	0.2	0.2	0.2
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with constant sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

same tree in terms of prices, but different total variance and time

risk neutral stock tree CRR-style with variable sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

Time	0	0.25	0.5	0.75	1	1.25
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.536		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.200		0.200			
		0.200		0.200		
			0.200			
				0.200		

from p, u, d

Time	0	1	1.25	1.5	1.75	2
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.725		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.089		0.200			
		0.200		0.200		
			0.200			
				0.200		

from p, u, d

$$\sigma^2 = \frac{0.275(0.725)(1.1052-0.9048)^2}{1} = (0.089)^2 = \frac{p(1-p)(u-d)^2}{\Delta t}$$

Example of a CRR tree with variable volatility, 20% year one, 40% year 2

CRR Tree with variable volatility

Vol is 20% for one year, 40% for second, sqrt of annual variance is 31.6%

sigma		0.2	0.4	0.4	0.4	0.4
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.2	0.2	0.2	0.2	0.2	0.2
u		1.221	1.221	1.221	1.221	1.221
r_annual	0.1					

risk neutral stock tree CRR-style with variable sigma(t)

					271.83
				222.55	
			182.21		**182.21**
		149.18		**149.18**	
	122.14		**122.14**		**122.14**
100.00		**100.00**		**100.00**	
	81.87		**81.87**		**81.87**
		67.03		**67.03**	
			54.88		**54.88**
				44.93	
					36.79

Time	0	1	1.25	1.5	1.75	2
p-tree	0.699	0.510	0.510	0.510	0.510	

Two year put struck at 100

					0.000
				0.000	
			0.000		0.000
		1.985		0.000	
	6.318		4.149		0.000
9.309		11.139		8.672	
	19.328		18.965		18.127
		28.804		30.613	
			40.465		45.119
				52.713	
					63.212

p = (R-D)/(U-D)
= (1.1 - 1/1.22)/1.22 - 1/1.22)
= 0.7 for year one

= (1.1^0.25 - 1/1.22)/1.22 - 1/1.22)
= 0.51

Calibrating a binomial tree to term structures

How do we build a binomial tree to price options that's consistent yield and vol term structures? This is important for American-style options and early exercise. We have to make sure to use the right forward rate and the right forward volatility at each node.

Example:

	Year 1	Year 2	Year 3
Term structure of zero coupons:	5%	7.47%	9.92%
Forward rates:	5%	$10\% = \frac{(1.0747^2)}{1.05} - 1$	15%
Term structure of Implied vols:	Σ_1	Σ_2	Σ_3
	20%	25.5%	31.1%
Forward vols:	Σ_1	$\Sigma_{12} = \sqrt{2\Sigma_2^2 - \Sigma_1^2}$	$\Sigma_{23} = \sqrt{3\Sigma_3^2 - 2\Sigma_2^2}$
	20%	30%	40%

Now build a (toy) tree with different forward rates/vols:

r:	5%	10%	15%
σ	20%	30%	40%
Δt	Δt_1	$\Delta t_2 = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \Delta t_1$	$\Delta t_3 = \left(\frac{\sigma_1}{\sigma_3}\right)^2 \Delta t_1$
		$= 0.44\Delta t_1$	$0.25\Delta t_1$

A possible scheme:

For the first year use $\Delta t_1 = 0.1$ and take 10 periods of 0.1 years per step.

Then $\Delta t_2 = 0.044$ and we need about 23 periods for the second year.

Finally, $\Delta t_3 = 0.025$ and we need 40 periods for the third year.

In each period the up and down moves in the tree are generated by

$$e^{\sigma\sqrt{\Delta t}} = e^{(0.2)0.316} = 1.065.$$

Using forward rates and forward volatilities over three years produces a very different tree from using just the three-year rates and volatilities over the whole period, especially for American-style exercise

Local volatility models

In the last lecture we extended the constant-volatility geometric Brownian motion picture underlying the Black-Scholes model to account for a volatility that can vary with future time. Now we head off in a new direction for several classes.

How to make $\sigma = \sigma(S, t)$ a function of future stock price S and future time t? Why?
Realized volatility does go up when the market goes down;
We want to see if this simple extension of Black-Scholes can then lead to an explanation of the smile.
These models are very widely used.

Some references on Local Volatility Models (there are many more).

- *The Volatility Smile and Its Implied Tree*, Derman and Kani, RISK, 7-2 Feb.1994, pp. 139-145, pp. 32-39 (see www.ederman.com for a PDF copy of this.

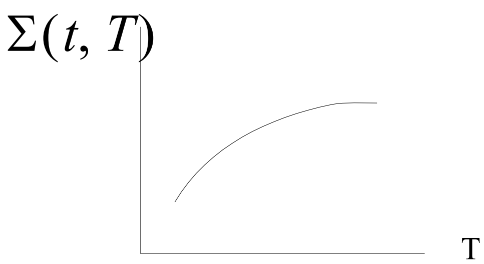
- *The Local Volatility Surface by Derman, Kani and Zou, Financial Analysts Journal, (July-Aug 1996), pp. 25-36* (see www.ederman.com for a PDF copy of this). Read this to get a general idea of where we're going.

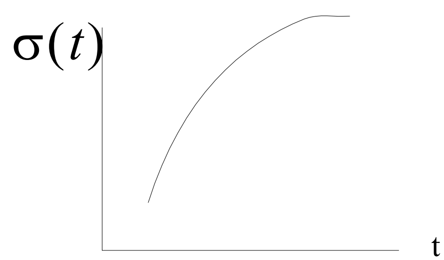
- Gatheral's book *The Volatility Surface.*

Modeling a stock with a variable volatility $\sigma(S, t)$

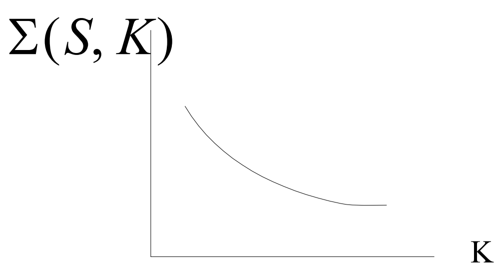
Model a stock with a variable volatility $\sigma(S, t)$, value options, examine $\Sigma(S, t, K, T)$.

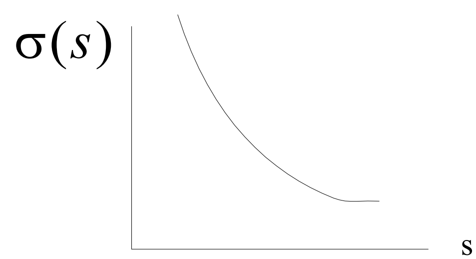
Pure term structure $\Sigma(t, T)$, calibrate the forward volatilities $\sigma(t)$, $\Sigma^2 = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.





"Sideways" volatilities $\Sigma(S, t, K, T)$ to $\sigma(S, t)$?



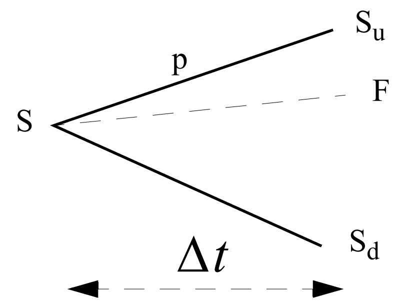


More generally, how does the local volatility $\sigma(S, t)$, influence the current implied $\Sigma(S, t, K, T)$?

- *Can we find a unique local vol surface* $\sigma(S, t)$ *to match the implied surface* $\Sigma(S, t, K, T)$*?*
- *Even if we can find the local volatilities that match the implied volatility surface, do they represent what actually goes on in the world?*
- *What do local volatility models tell us about hedge ratios, exotic values, etc.?*

Binomial Local Volatility Modeling

How do we build a binomial tree that closes (in order to avoid computational complexity)? We are going to keep Δt constant now:



where $$\frac{dS}{S} = (r-d)dt + \sigma(S,t)dZ$$

Expected value of S is the forward price $F = Se^{(r-d)\Delta t}$ or $F = Se^{r\Delta t} - D$

Binomially $$F = pS_u + (1-p)S_d$$

Furthermore, the SDE implies that $(dS)^2 = \sigma^2(S,t)S^2 dt$, so *approximately*

$$S^2\sigma^2\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2.$$

Solve:

$$p = \frac{F - S_d}{S_u - S_d}$$

$$(F - S_d)(S_u - F) = S^2\sigma^2\Delta t$$

$$S_u = F + \frac{S^2\sigma^2\Delta t}{F - S_d} \qquad \text{or} \qquad S_d = F - \frac{S^2\sigma^2\Delta t}{S_u - F}$$

Reference: *The Volatility Smile and Its Implied Tree*, by Derman and Kani.

- Build out the tree at any time level by starting from the middle node and then moving up or down to successive nodes at that level.
- If we know the local volatilities $\sigma(S, t)$ and the forward interest rates at each future period, we can determine the stock prices all the up nodes and down nodes from equations.
- Given all the nodes in the tree, we can then use equation for p to compute the risk-neutral probabilities at each node.

There are many ways to choose the central spine of a binomial tree:

- For every level with an odd number of nodes (1,3,5, etc.) choose the central node to be S.
- For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price S exactly as in the CRR tree, given by

$$U = e^{\sigma(S, t)\sqrt{\Delta t}} \qquad D = e^{-\sigma(S, t)\sqrt{\Delta t}}$$

You could equally well choose a tree whose spine corresponds to the forward price F of the stock, growing from level to level. Or anything else.

Example with the local volatility a function only of the stock price S:

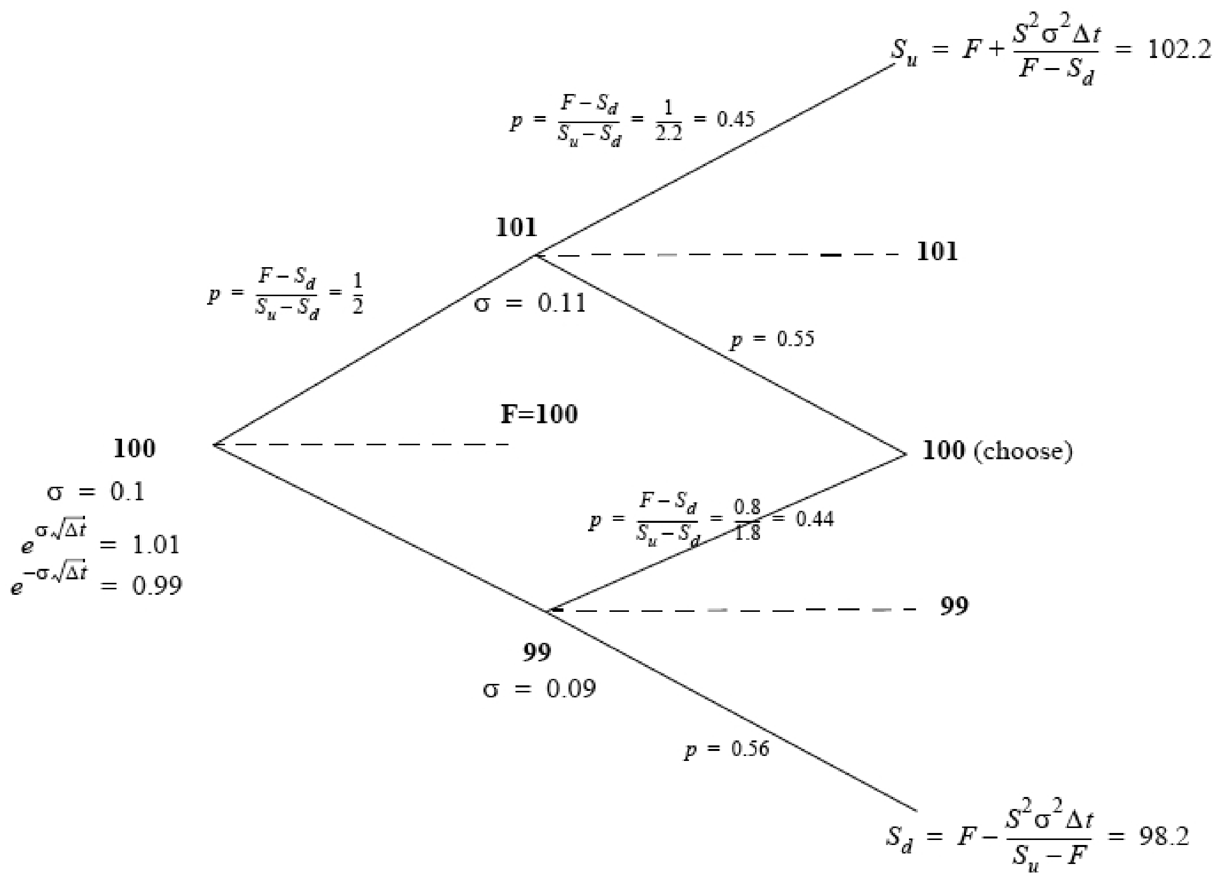
$S = 100$

$\Delta t = 0.01; d = 0, r = 0; F/S = 1; \sqrt{\Delta t} = 0.1; e^{\sigma(S)\sqrt{\Delta t}} = e^{\sigma(S)0.1}$ and

$$\sigma(S) = max\left[0.1 + \left(\frac{S}{100} - 1\right), 0\right]$$

local stock volatility starts out at 10% and increases/falls by 1 percentage point for every 1 point rise/drop in the stock price, but never goes below zero.

$\sigma(100) = 0.1$ and $\sigma(101) = 0.11$

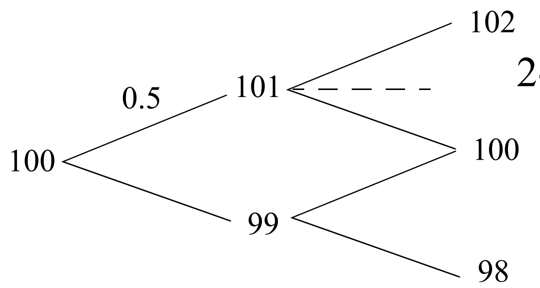


Nodes and probabilities that produce the correct discrete version of the desired diffusion.

A two-period call struck at 101:

the payoff at the top node is 1.2 with a risk-neutral probability of (0.5)(0.45) for a value of 0.27.

Compare to value of a similar call on a CRR tree with a flat 10% volatility everywhere.



2-period 101 call $= 0.5 \times 0.5 \times 1 = 0.25$

In the local volatility tree there are larger moves up and smaller moves down in the stock price.

Building a binomial tree with variable volatility is in principle possible.

In practice, one may get better (i.e. easier to calibrate, more efficient to price with, converging more rapidly as $\Delta t \rightarrow 0$,etc.) trees by using trinomial trees or other finite difference PDE approximations. Nevertheless, we will stick to binomial trees in most of our examples here because of the clarity of the intuition they provide.

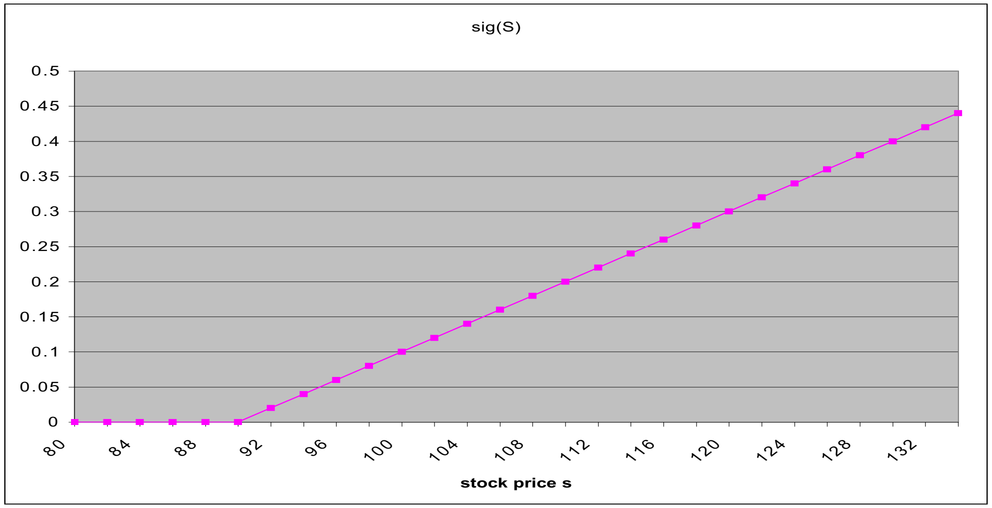
You can find more references to trinomial trees with variable volatility in Derman, Kani and Chriss, *Implied Trinomial Trees of the Volatility Smile*, The Journal of Derivatives, 3(4) (Summer 1996), pp. 7-22, and also in James' book on Option Theory which is a good general reference on much of this topic.

The Relation Between Local and Implied Volatilities.

How to build a local volatility tree that matches the smile?
What is the relation between local volatilities as a function of S and implieds as a function of K?

Intuition: Here is a graph of local volatilities that satisfy a positive skew:

$$\sigma(S) = Max[0.1 + (S/100 - 1), 0]$$



Here is the binomial local-volatility tree for the stock price, assuming $\Delta t = 0.01$, $S = 100$, $r = 0$.

This is a tree with flat volatility 0.1, usual CRR type

stock tree sig = 0.1, p = 0.5

				104.08
			103.05	
		102.02		102.02
	101.01		101.01	
100		100		100
	99		99	
		98.02		98.02
			97.04	
				96.08

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = 0.01$$

$$\sigma = 0.1$$

This is a tree with variable local volatility

stock tree sigma(s,t)

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = \frac{(1.28)(1.53)}{(103.5^2)(0.01)} = 0.018$$

$$\sigma = 0.135$$

p-tree

			0.456
		0.488	
	0.453		0.453
0.498		0.498	
	0.441		0.441
		0.509	
			0.434

vol(stock)

			0.135
		0.122	
	0.110		0.110
0.100		0.100	
	0.090		0.090
		0.082	
			0.074

vol ranges from 13.5 to 7.5 **as stock ranges from 103.5 to 97.4**

Call with strike 102 has the same value on the *local volatility tree* as it does on a *fixed-volatility CRR tree* with a volatility of 11%.

NUMERICAL ILLUSTRATION OF RELATION BETWEEN LOCAL AND IMPLIED VOL

local vol tree

				105.04
			103.51	
		102.23		**102.23**
	101.01		**101.01**	
100		**100**		**100**
	99		**99**	
		98.21		**98.21**
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 102 (sig=12%)

				3.040
			1.510	
		0.790		0.230
	0.386		0.104	
0.204		0.052		0.000
	0.023		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11% vol

				104.50
			103.36	
		102.22		102.22
	101.11		101.11	
100.00		100.00		100.00
	98.91		98.91	
		97.82		97.82
			96.75	
				95.70

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 102

				2.498
			1.355	
		0.730		0.224
	0.391		0.112	
0.208		0.055		0.000
	0.028		0.000	
		0.000		0.000
			0.000	
				0.000

11% is the average of the local volatilities between 100 and 102

The CRR implied volatility for a given strike is roughly the average of the local volatilities from spot to that strike.

Call with strike 103 on the same tree.

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 103 (sig = 13%)

				2.040
			0.929	
		0.453		0.000
	0.205		0.000	
0.102		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11.5% vol

				104.71
			103.51	
		102.33		102.33
	101.16		101.16	
100.00		100.00		100.00
	98.86		98.86	
		97.73		97.73
			96.61	
				95.50

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 103

				1.707
			0.849	
		0.422		0.000
	0.210		0.000	
0.104		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

Implied volatility is about 11.5%, the average of the local volatilities between 100 and 103.

The Rule of 2: Understanding The Relation Between Local and Implied Vols

Implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot and strike.

Cf: yields to maturity for zero-coupon bonds as an average over future short-term rates over the life of the bond.

Forward short-term rates grow twice as fast with future time as yields to maturity grow with time to maturity.

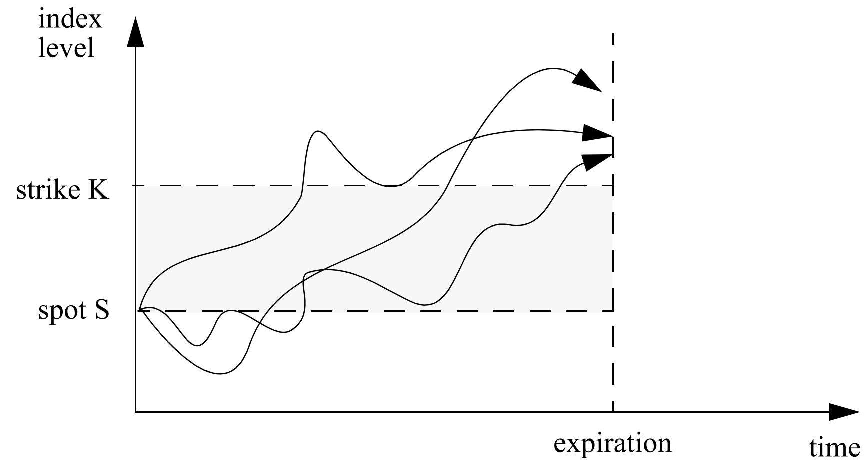
Local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

Approximate proof from *The Local Volatility Surface*. Later we'll prove it more rigorously.

Simple "sideways" vol case: $\sigma(S) = \sigma_0 + \beta S$ for all time t

$\Sigma(S, K)$: Any paths that contribute to the option value must pass between S and K

FIGURE 1.1. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



Implied volatility for the option of strike K when the index is at S ~ average of the local volatilities

$$\Sigma(S, K) \approx \frac{1}{K-S}\int_S^K \sigma(S')dS'$$

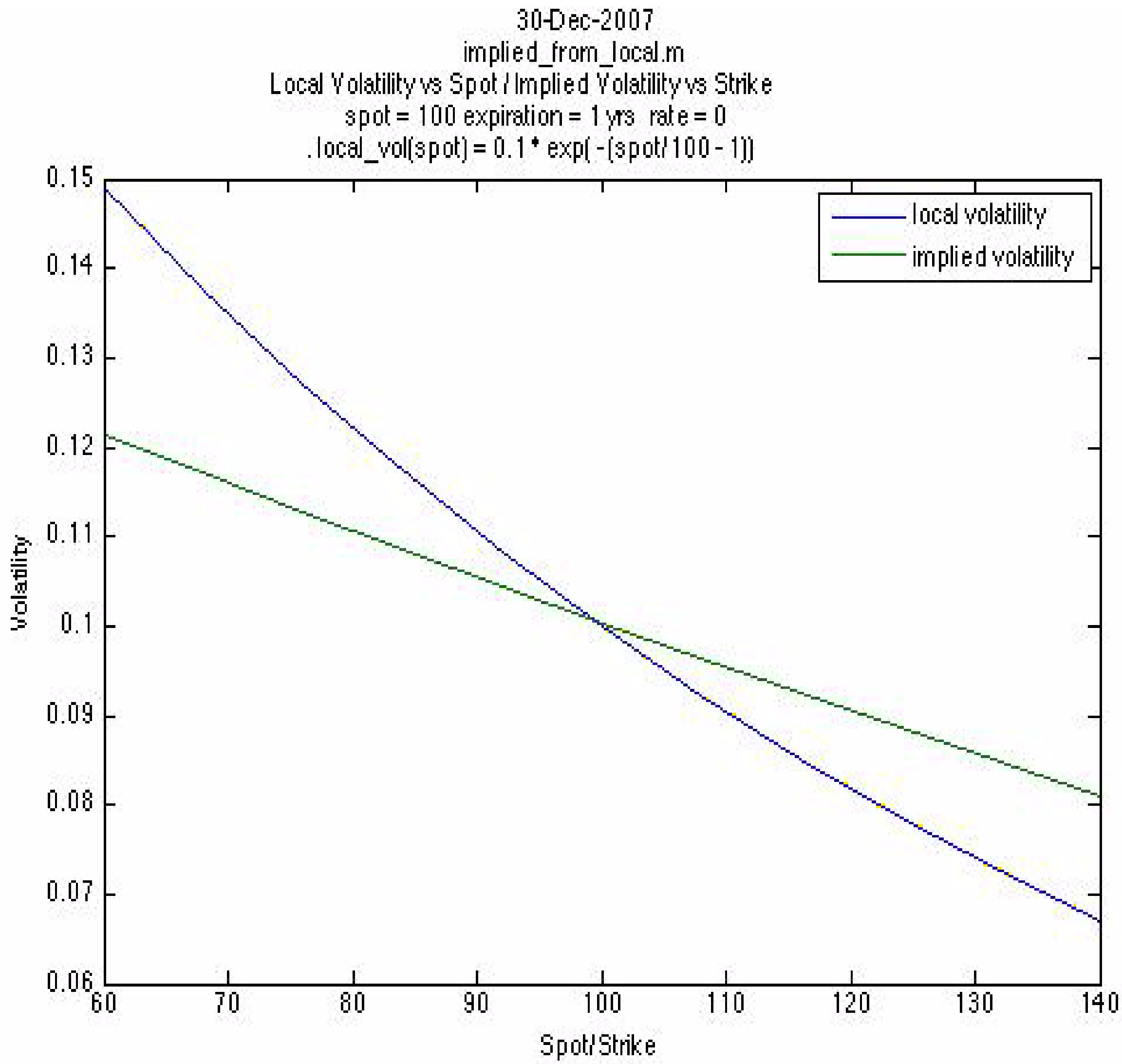
$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S+K)$$

If (implied volatility varies linearly with strike K at a fixed market level S)
then (it also varies linearly at the same rate with the index level S itself).
Local volatility varies with S at twice that rate.

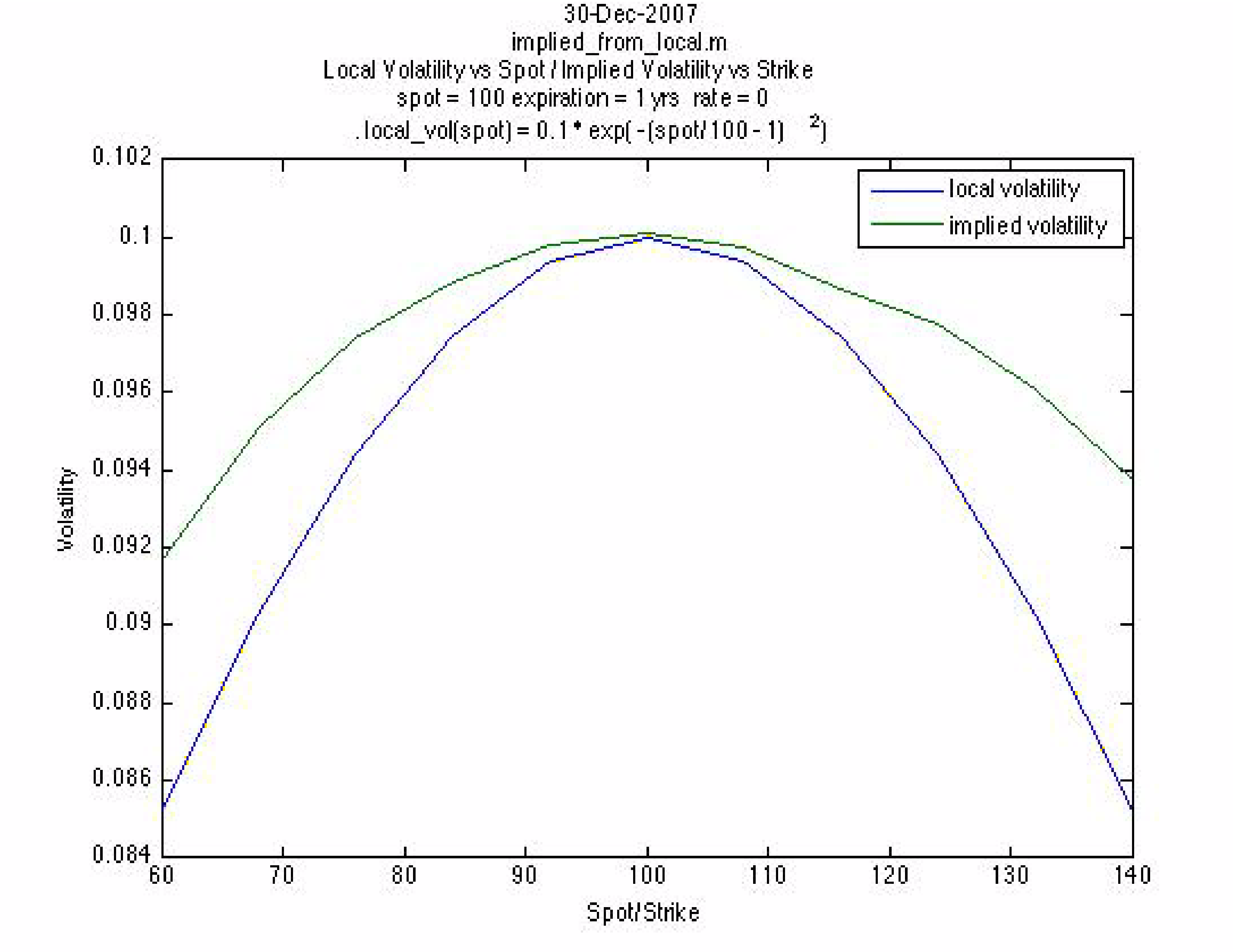
Can also write $$\Sigma(S, K) \approx \sigma(S) + \frac{\beta}{2}(K-S)$$

Some Examples of Local and Implied Volatilities.

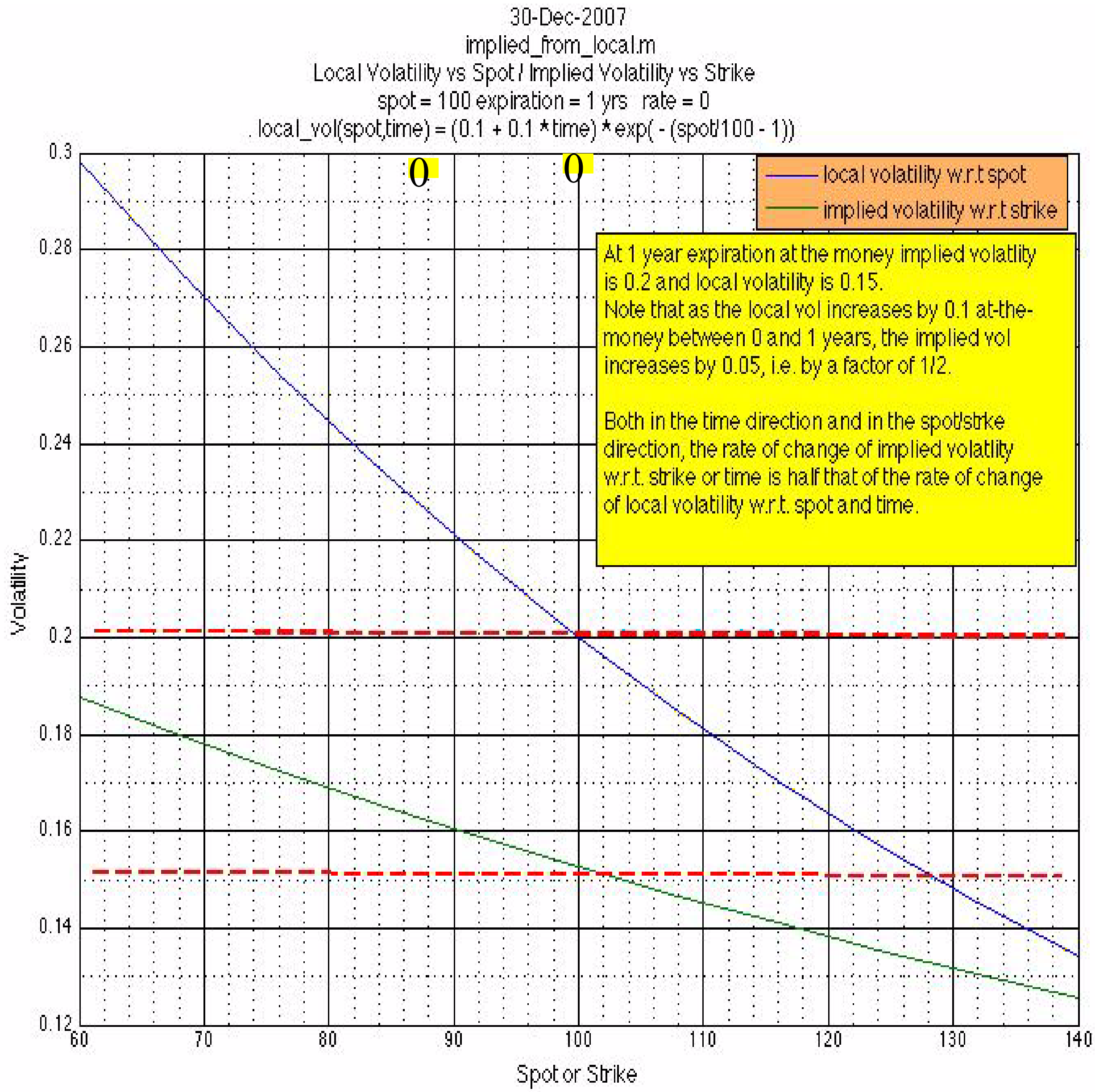
$\sigma(S, t) = 0.1\exp(-[S/100 - 1])$. Note the slopes



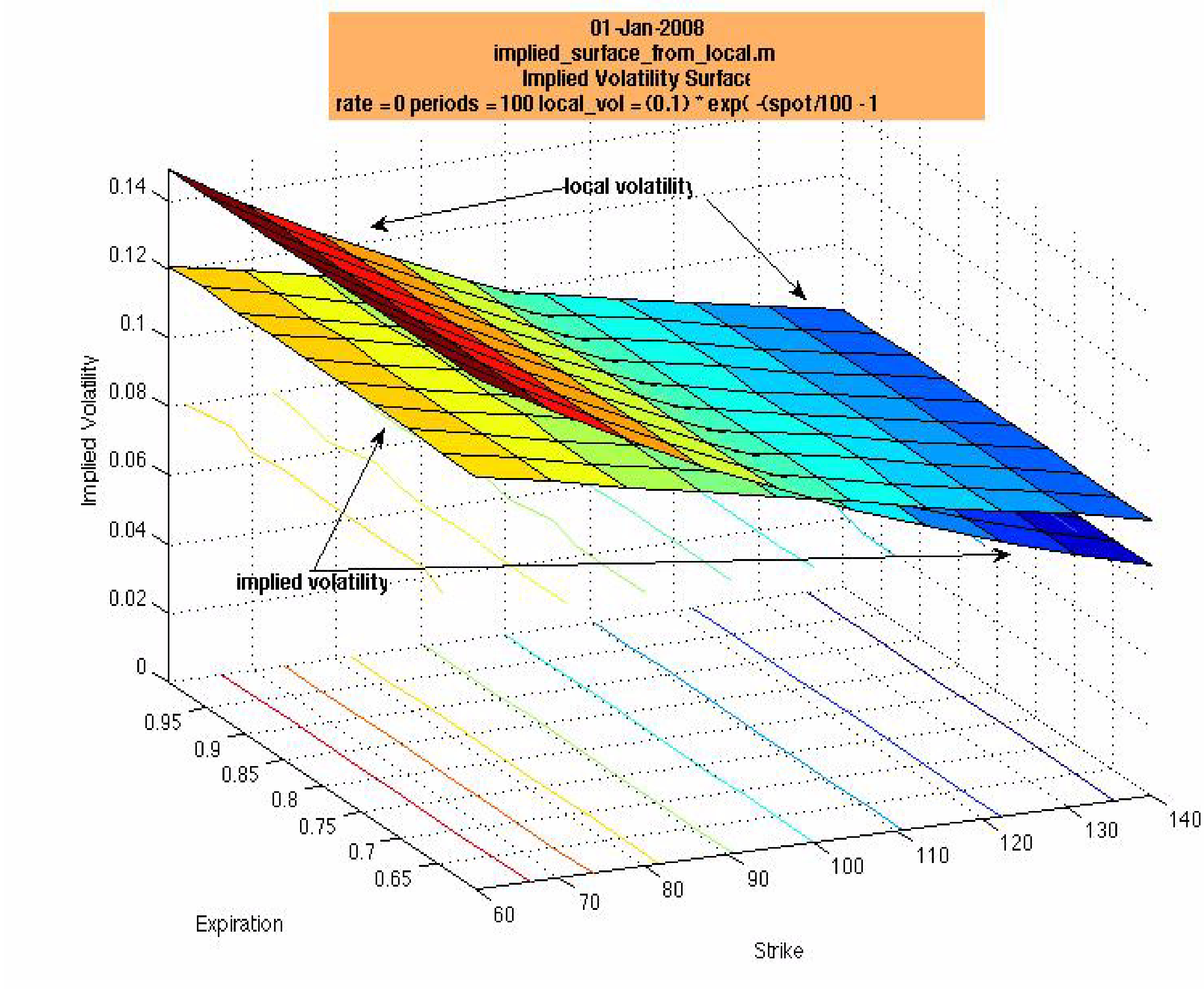
$$\sigma(S, t) = 0.1 \exp(-[S/100 - 1]^2)$$



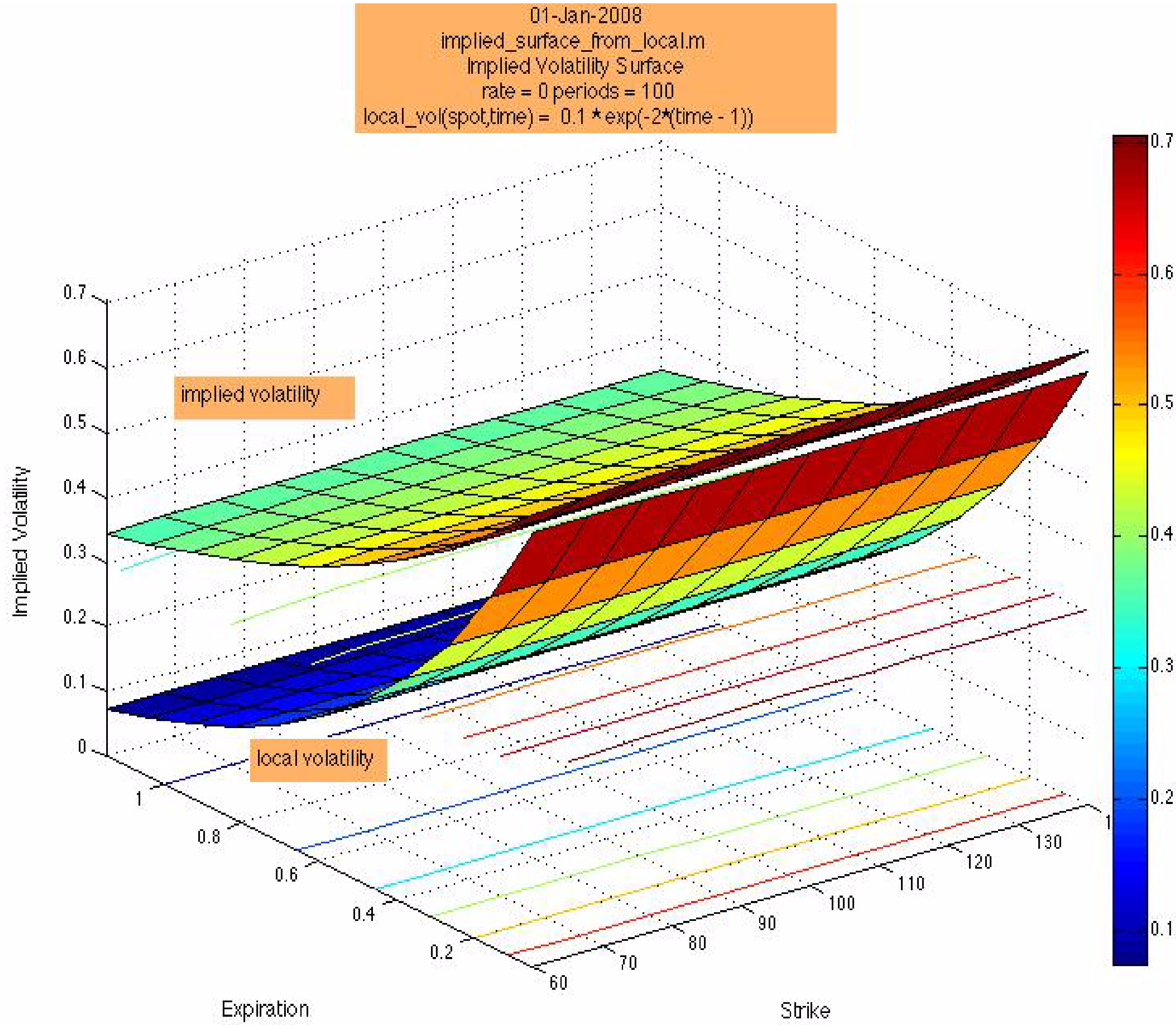
$$1\sigma(S,t) = (0.1 + 0.1t)\exp(-[S/100 - 1])$$



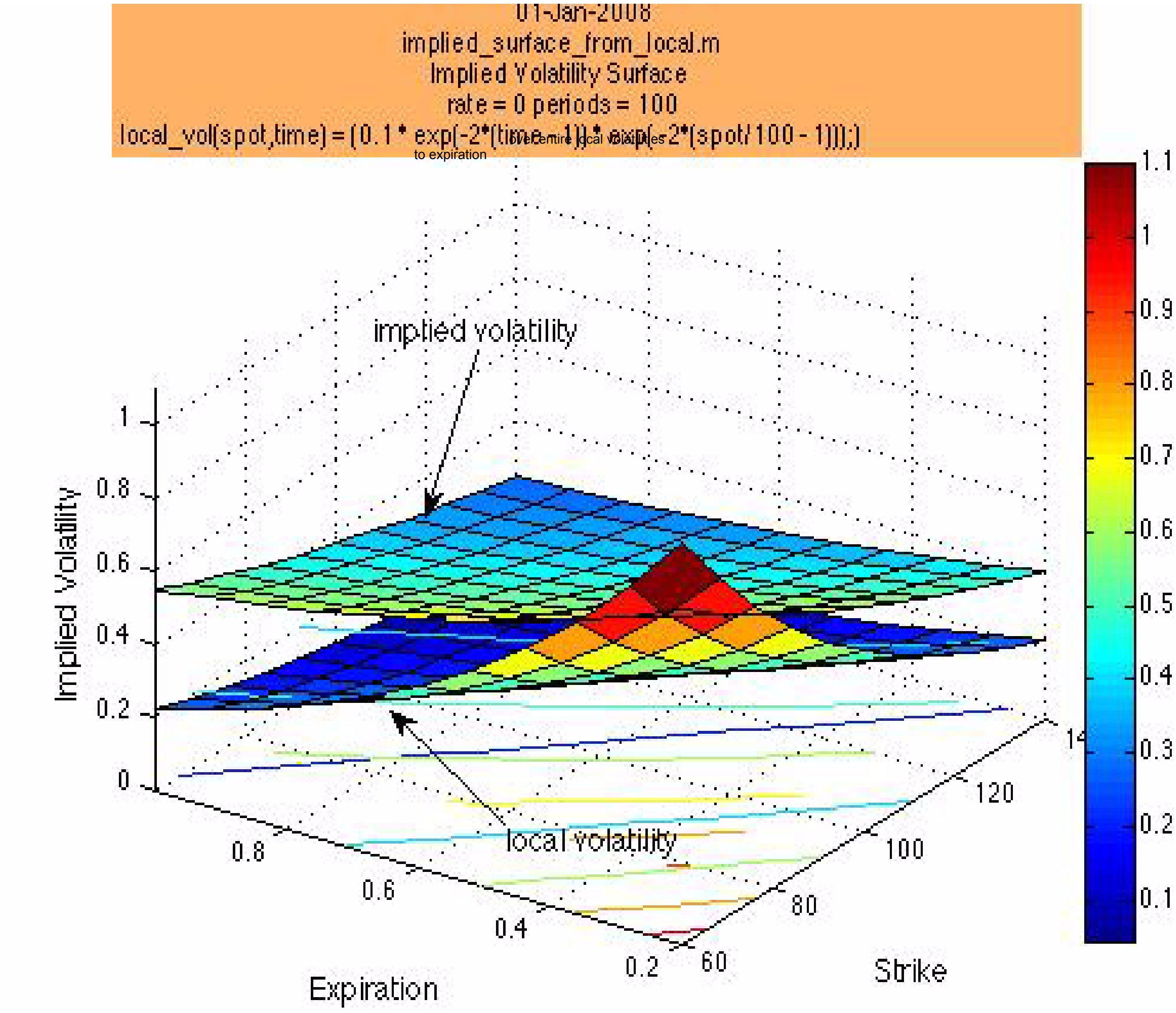
Dependent only on *S*: $\sigma(S, t) = 0.1\exp(-[S/100 - 1])$: Plot surface



Dependent only on *t*: $\sigma(S, t) = 0.1 \exp(-2[t-1])$: Plot Surface



Dependent on *S* and *t*: $\sigma(S,t) = 0.1\exp(-2[t-1])\exp(-2[S/100-1])$**. Local vol is 10% at t = 1 and S = 100.**



Difficulties with binomial trees

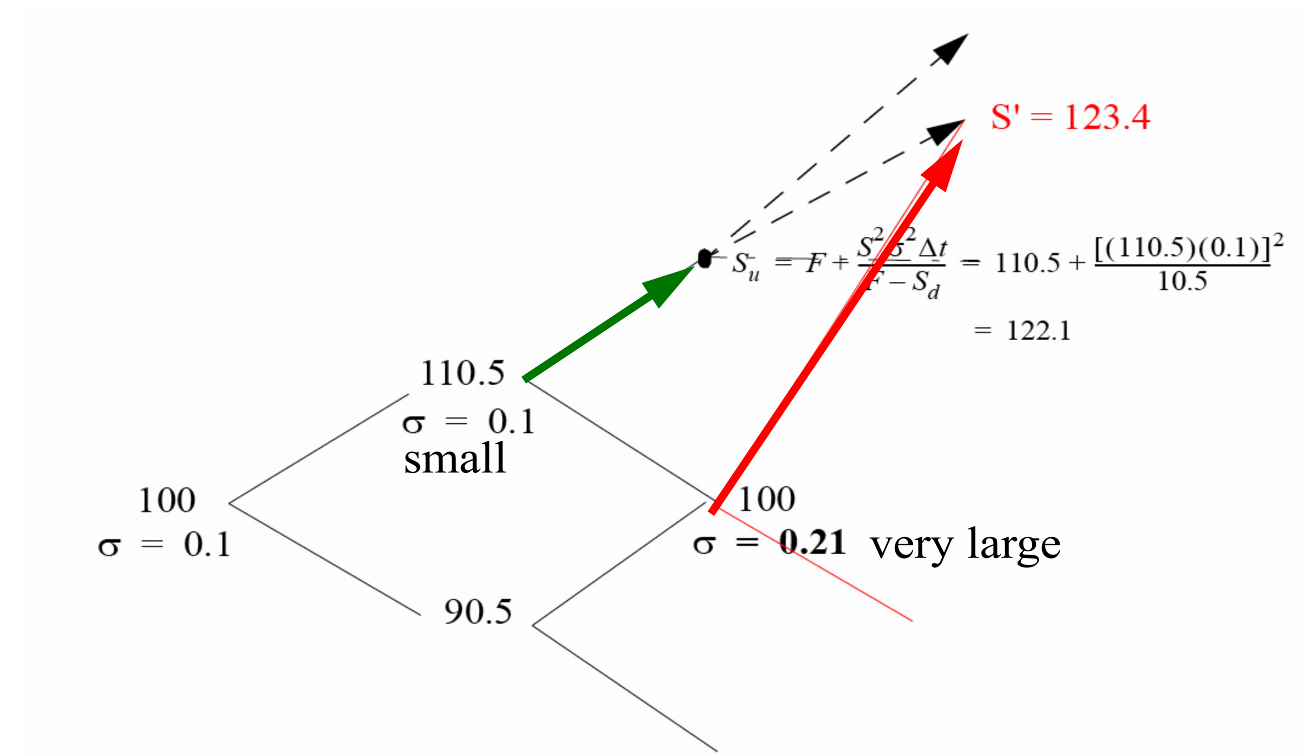
The nodes and the transition probabilities we discussed are uniquely determined by forward rates and the local volatility function we specify.

If $\sigma(S, t)$ varies too rapidly with stock price or time, then, for finite Δt, you can get binomial transition probabilities greater than 1 or less than zero.

Here is an example with $\Delta t = 1$ and $r = 0$.

The local volatility on level 3 at S = 100 is 0.21.

The S' node, S's up node in level 4 should be the down node from S_u in level 3, but it lies below S_u, but in fact lies above it, and so violates the no-arbitrage condition



You can remedy this with smaller time steps Δt, but then you are trying to extract to extract more information that is available from implied volatilities. Therefore, it's sometimes easier to use trinomial trees. They provide greater flexibility in avoiding arbitrage situations.