LECTURE 13

LOCAL VOLATILITY MODELS

Exam: Mon 9 March and Wed 11 March during class in Room 303. No cheat sheet necessary or allowed. Bring a calculator.

Ask me when you don’t understand something.

Local volatility models

In the last lecture we extended the constant-volatility geometric Brownian motion picture underlying the Black-Scholes model to account for a volatility that can vary with future time. Now we head off in a new direction for several classes.

How to make $\sigma = \sigma(S, t)$ a function of future stock price S and future time t? Why?
Realized volatility does go up when the market goes down;
We want to see if this simple extension of Black-Scholes can then lead to an explanation of the smile.
These models are very widely used.

Some references on Local Volatility Models (there are many more).

- *The Volatility Smile and Its Implied Tree*, Derman and Kani, RISK, 7-2 Feb.1994, pp. 139-145, pp. 32-39 (see www.ederman.com for a PDF copy of this.

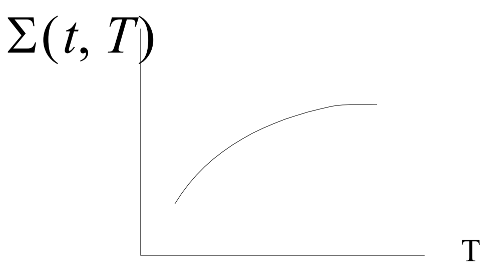
- *The Local Volatility Surface by Derman, Kani and Zou, Financial Analysts Journal, (July-Aug 1996), pp. 25-36* (see www.ederman.com for a PDF copy of this). Read this to get a general idea of where we're going.

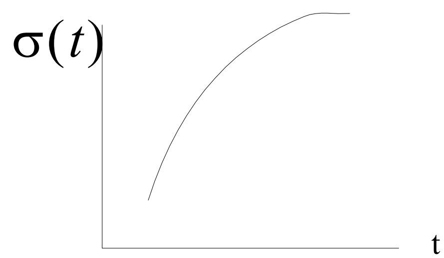
- Gatheral's book *The Volatility Surface.*

Modeling a stock with a variable volatility $\sigma(S, t)$

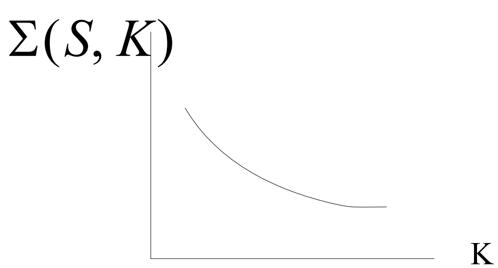
Model a stock with a variable volatility $\sigma(S, t)$, value options, examine $\Sigma(S, t, K, T)$.

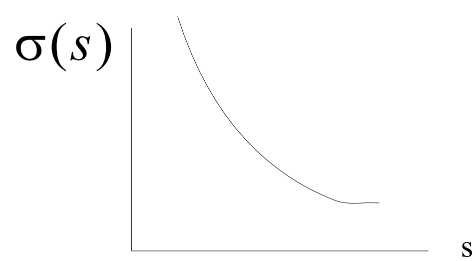
Pure term structure $\Sigma(t, T)$, calibrate the forward volatilities $\sigma(t)$, $\Sigma^2 = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.





"Sideways" volatilities $\Sigma(S, t, K, T)$ to $\sigma(S, t)$?





More generally, how does the local volatility $\sigma(S, t)$, influence the current implied $\Sigma(S, t, K, T)$?

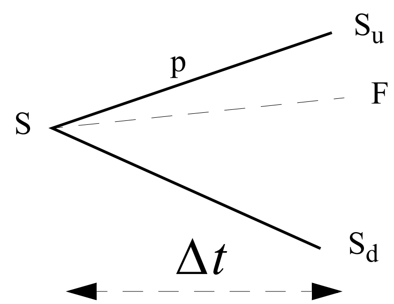
- *Can we find a unique local vol surface* $\sigma(S, t)$ *to match the implied surface* $\Sigma(S, t, K, T)$*?*

- *Even if we can find the local volatilities that match the implied volatility surface, do they represent what actually goes on in the world?*

- *What do local volatility models tell us about hedge ratios, exotic values, etc.?*

Binomial Local Volatility Modeling

How do we build a binomial tree that closes (in order to avoid computational complexity)? We are going to keep Δt constant now:



where $$\frac{dS}{S} = (r-d)dt + \sigma(S,t)dZ$$

Expected value of S is the forward price $F = Se^{(r-d)\Delta t}$ or $F = Se^{r\Delta t} - D$

Binomially $$F = pS_u + (1-p)S_d$$

Furthermore, the SDE implies that $(dS)^2 = \sigma^2(S,t)S^2 dt$, so *approximately*

$$S^2\sigma^2\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2.$$

Solve:

$$p = \frac{F - S_d}{S_u - S_d}$$

$$(F - S_d)(S_u - F) = S^2 \sigma^2 \Delta t$$

$$S_u = F + \frac{S^2 \sigma^2 \Delta t}{F - S_d} \qquad \text{or} \qquad S_d = F - \frac{S^2 \sigma^2 \Delta t}{S_u - F}$$

Reference: *The Volatility Smile and Its Implied Tree*, by Derman and Kani.

- Build out the tree at any time level by starting from the middle node and then moving up or down to successive nodes at that level.
- If we know the local volatilities $\sigma(S, t)$ and the forward interest rates at each future period, we can determine the stock prices all the up nodes and down nodes from equations.
- Given all the nodes in the tree, we can then use equation for p to compute the risk-neutral probabilities at each node.

There are many ways to choose the central spine of a binomial tree:

- For every level with an odd number of nodes (1,3,5, etc.) choose the central node to be S.
- For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price *S* exactly as in the CRR tree, given by

$$U = e^{\sigma(S,t)\sqrt{\Delta t}} \qquad D = e^{-\sigma(S,t)\sqrt{\Delta t}}$$

You could equally well choose a tree whose spine corresponds to the forward price *F* of the stock, growing from level to level. Or anything else.

Example with the local volatility a function only of the stock price *S*:

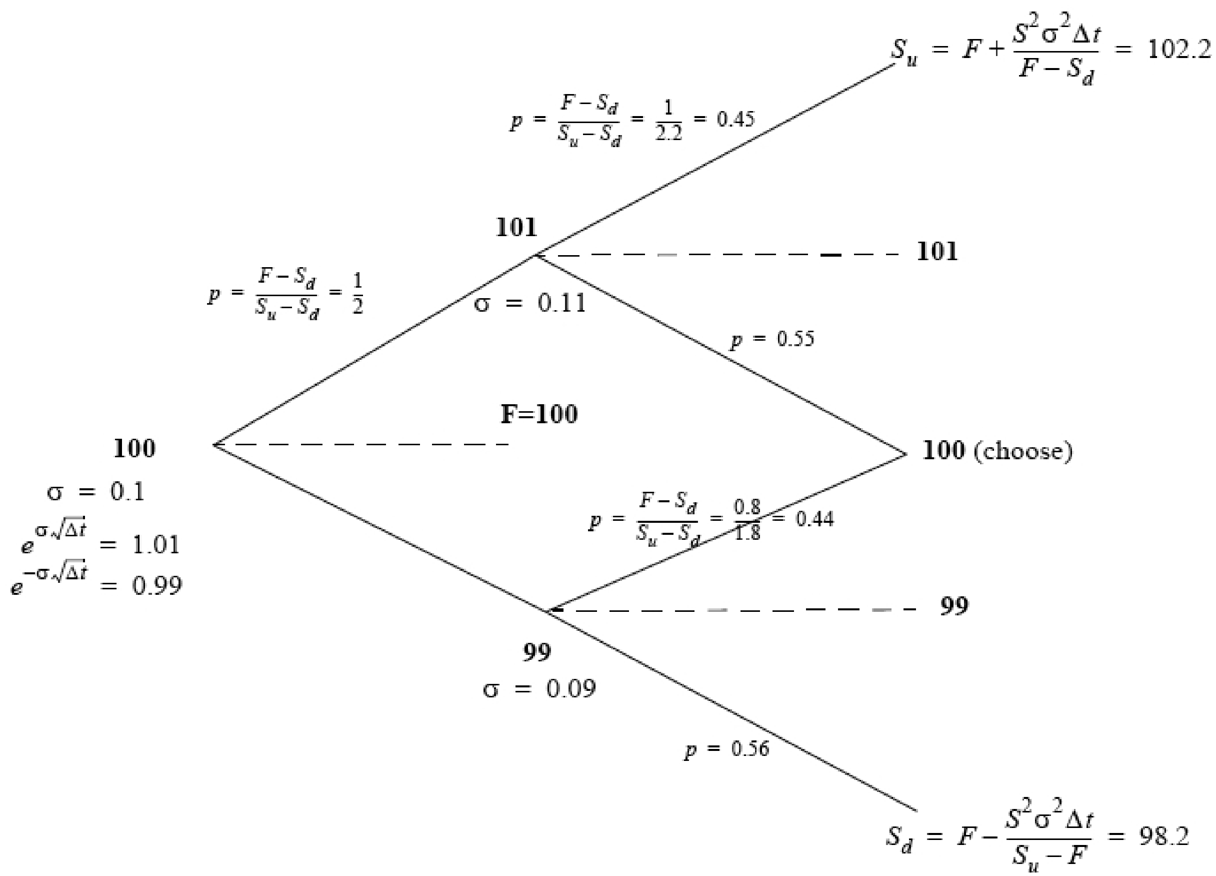
$S = 100$

$\Delta t = 0.01; d = 0, r = 0; F/S = 1; \sqrt{\Delta t} = 0.1; e^{\sigma(S)\sqrt{\Delta t}} = e^{\sigma(S)0.1}$ and

$$\sigma(S) = max\left[0.1 + \left(\frac{S}{100} - 1\right), 0\right]$$

local stock volatility starts out at 10% and increases/falls by 1 percentage point for every 1 point rise/drop in the stock price, but never goes below zero.

$\sigma(100) = 0.1$ and $\sigma(101) = 0.11$

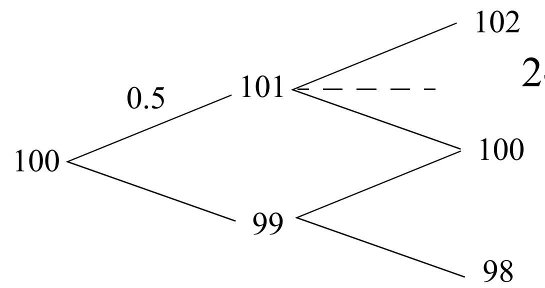


Nodes and probabilities that produce the correct discrete version of the desired diffusion.

A two-period call struck at 101:

the payoff at the top node is 1.2 with a risk-neutral probability of (0.5)(0.45) for a value of 0.27.

Compare to value of a similar call on a CRR tree with a flat 10% volatility everywhere.



2-period 101 call $= 0.5 \times 0.5 \times 1 = 0.25$

In the local volatility tree there are larger moves up and smaller moves down in the stock price.

Building a binomial tree with variable volatility is in principle possible.

In practice, one may get better (i.e. easier to calibrate, more efficient to price with, converging more rapidly as $\Delta t \to 0$,etc.) trees by using trinomial trees or other finite difference PDE approximations. Nevertheless, we will stick to binomial trees in most of our examples here because of the clarity of the intuition they provide.

I will set some HW on this next week.

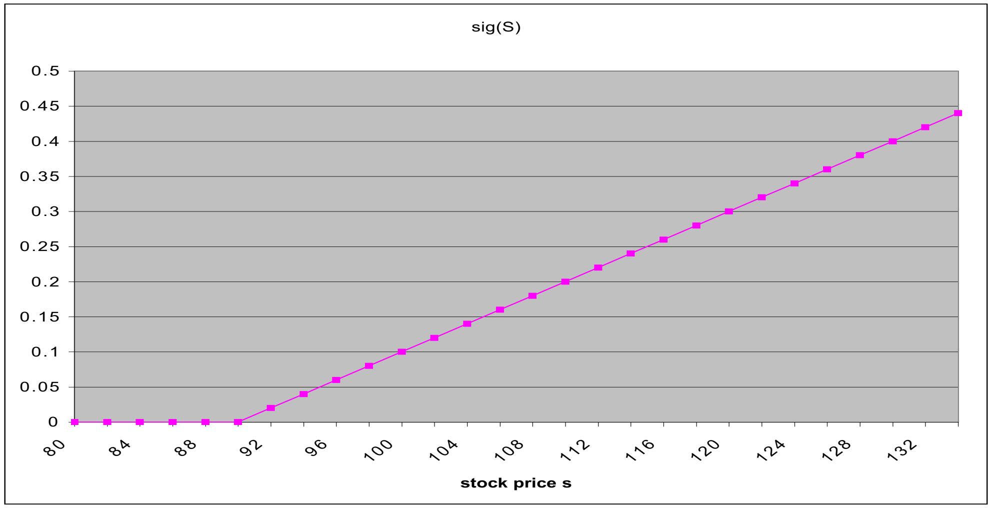
You can find more references to trinomial trees with variable volatility in Derman, Kani and Chriss, *Implied Trinomial Trees of the Volatility Smile*, The Journal of Derivatives, 3(4) (Summer 1996), pp. 7-22, and also in James' book on Option Theory which is a good general reference on much of this topic.

The Relation Between Local and Implied Volatilities.

How to build a local volatility tree that matches the smile?
What is the relation between local volatilities as a function of S and implieds as a function of K?

Intuition: Here is a graph of local volatilities that satisfy a positive skew:

$$\sigma(S) = Max[0.1 + (S/100 - 1), 0]$$



Here is the binomial local-volatility tree for the stock price, assuming $\Delta t = 0.01$, $S = 100$, $r = 0$.

This is a tree with flat volatility 0.1, usual CRR type

stock tree sig = 0.1, p = 0.5

				104.08
			103.05	
		102.02		102.02
	101.01		101.01	
100		100		100
	99		99	
		98.02		98.02
			97.04	
				96.08

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = 0.01$$

$$\sigma = 0.1$$

This is a tree with variable local volatility

stock tree sigma(s,t)

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = \frac{(1.28)(1.53)}{(103.5^2)(0.01)} = 0.018$$

$$\sigma = 0.135$$

p-tree

			0.456
		0.488	
	0.453		0.453
0.498		0.498	
	0.441		0.441
		0.509	
			0.434

vol(stock)

			0.135
		0.122	
	0.110		0.110
0.100		0.100	
	0.090		0.090
		0.082	
			0.074

vol ranges from 13.5 to 7.5
as stock ranges from 103.5 to 97.4

Call with strike 102 has the same value on the *local volatility tree* as it does on a *fixed-volatility CRR tree* with a volatility of 11%.

NUMERICAL ILLUSTRATION OF RELATION BETWEEN LOCAL AND IMPLIED VOL

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 102 (sig=12%)

				3.040
			1.510	
		0.790		0.230
	0.386		0.104	
0.204		0.052		0.000
	0.023		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11% vol

				104.50
			103.36	
		102.22		102.22
	101.11		101.11	
100.00		100.00		100.00
	98.91		98.91	
		97.82		97.82
			96.75	
				95.70

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 102

				2.498
			1.355	
		0.730		0.224
	0.391		0.112	
0.208		0.055		0.000
	0.028		0.000	
		0.000		0.000
			0.000	
				0.000

11% is the average of the local volatilities between 100 and 102

The CRR implied volatility for a given strike is roughly the average of the local volatilities from spot to that strike.

Call with strike 103 on the same tree.

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 103 (sig = 13%)

				2.040
			0.929	
		0.453		0.000
	0.205		0.000	
0.102		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11.5% vol

				104.71
			103.51	
		102.33		102.33
	101.16		101.16	
100.00		100.00		100.00
	98.86		98.86	
		97.73		97.73
			96.61	
				95.50

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 103

				1.707
			0.849	
		0.422		0.000
	0.210		0.000	
0.104		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

Implied volatility is about 11.5%, the average of the local volatilities between 100 and 103.

The Rule of 2: Understanding The Relation Between Local and Implied Vols

Implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot and strike.

Cf: yields to maturity for zero-coupon bonds as an average over future short-term rates over the life of the bond.

Forward short-term rates grow twice as fast with future time as yields to maturity grow with time to maturity.

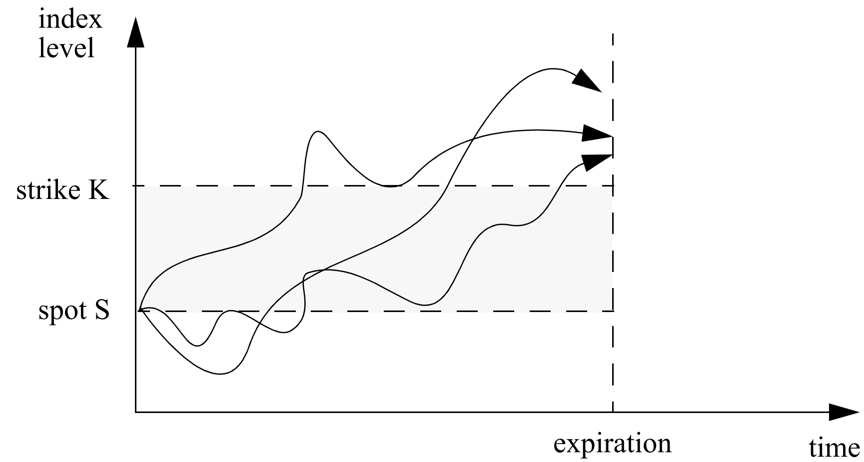
Local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

Approximate proof from *The Local Volatility Surface*. Later we'll prove it more rigorously.

Simple "sideways" vol case: $\sigma(S) = \sigma_0 + \beta S$ for all time t

$\Sigma(S, K)$: Any paths that contribute to the option value must pass between S and K

FIGURE 1.1. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



Implied volatility for the option of strike K when the index is at S ~ average of the local volatilities

$$\Sigma(S, K) \approx \frac{1}{K-S} \int_S^K \sigma(S')dS'$$

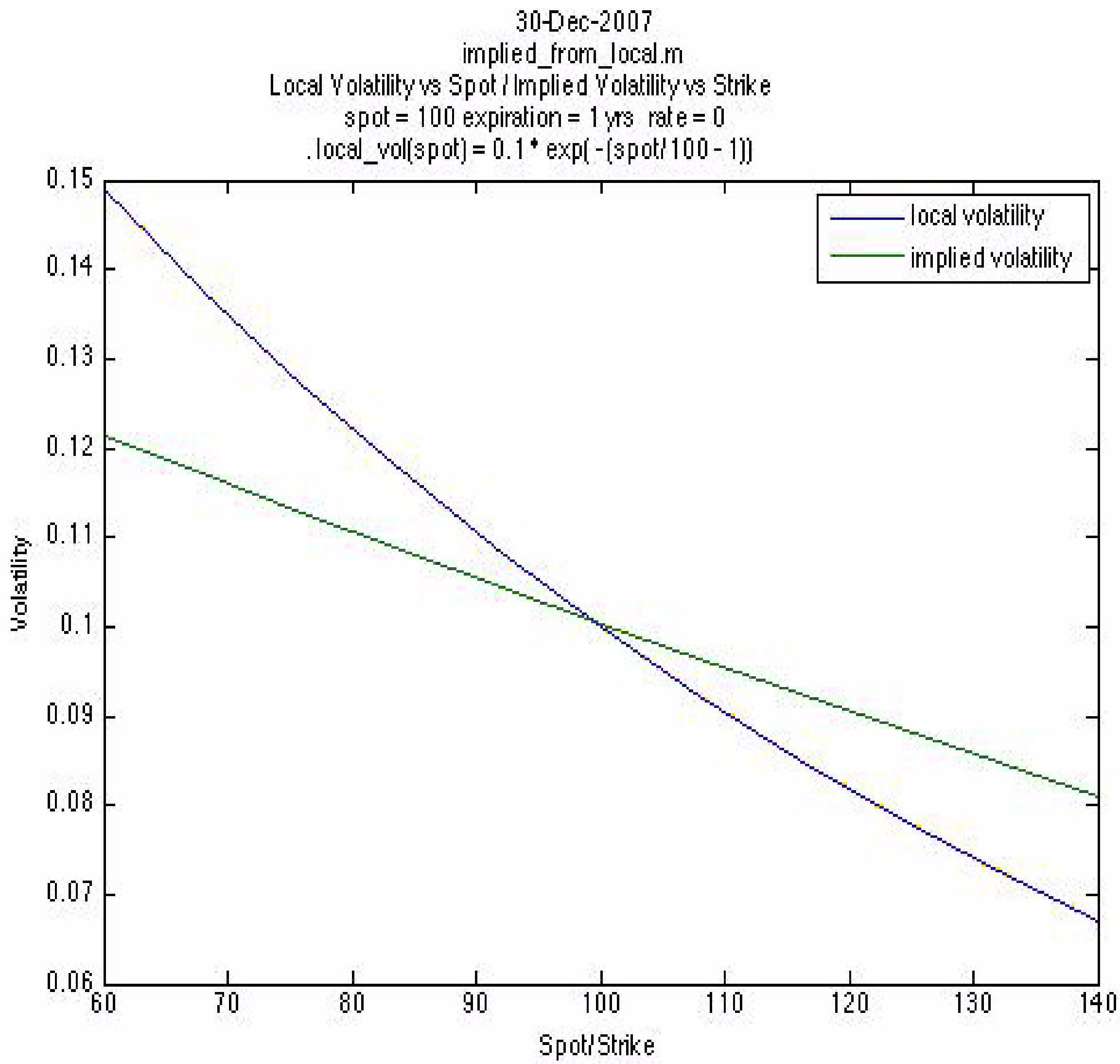
$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S+K)$$

If (implied volatility varies linearly with strike K at a fixed market level S)
then (it also varies linearly at the same rate with the index level S itself).
Local volatility varies with S at twice that rate.

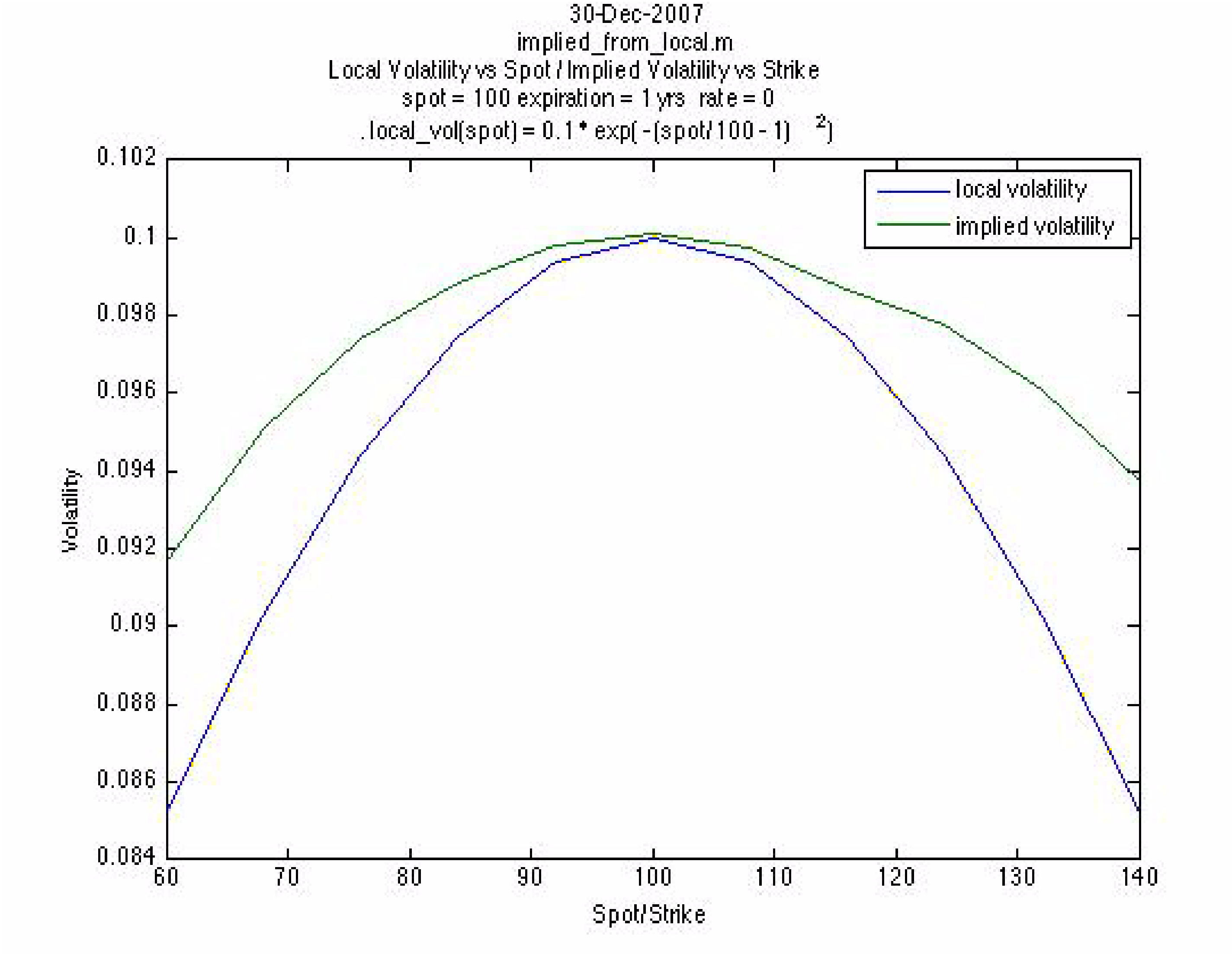
Can also write $$\Sigma(S, K) \approx \sigma(S) + \frac{\beta}{2}(K-S)$$

Some Examples of Local and Implied Volatilities.

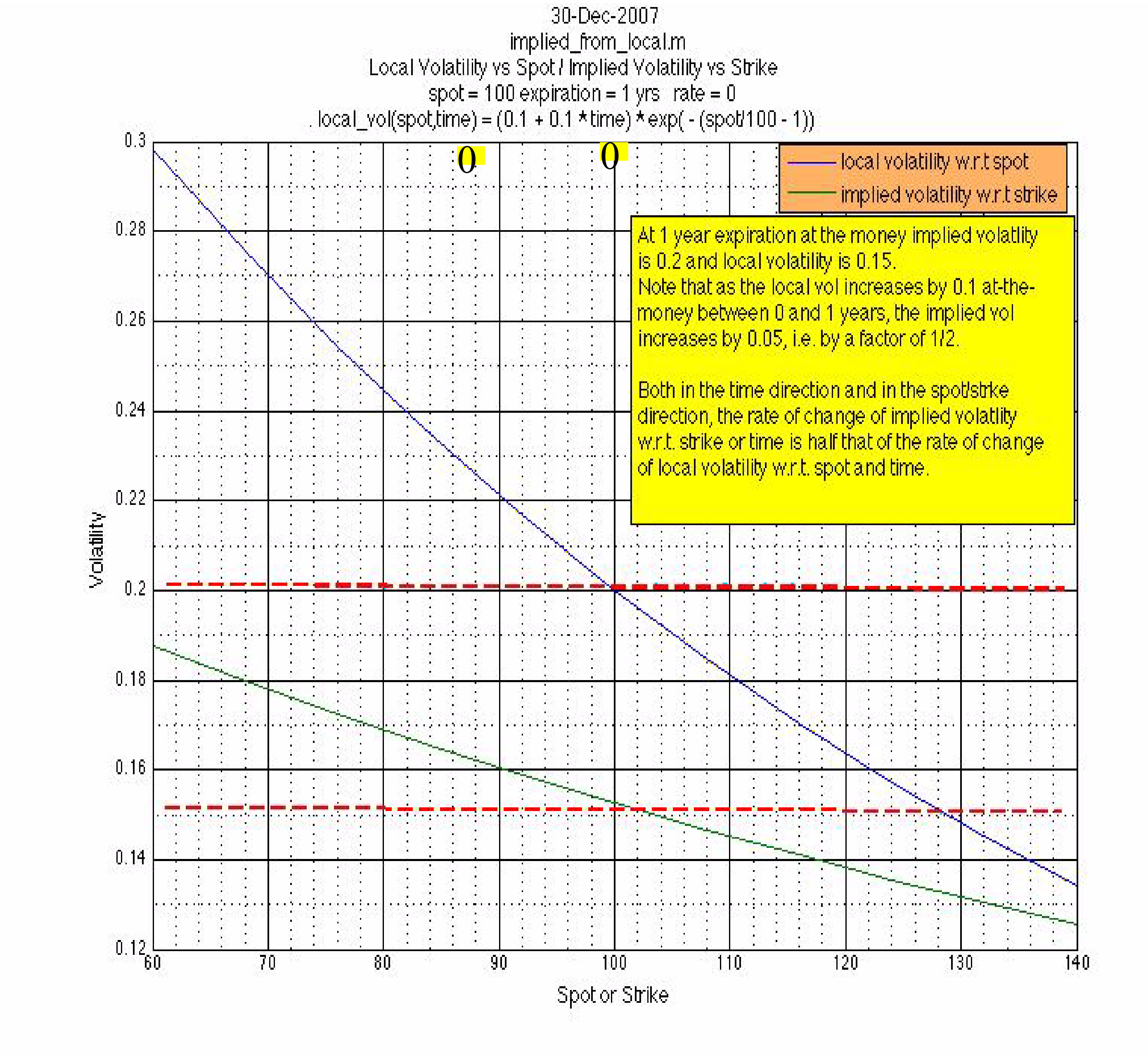
$\sigma(S, t) = 0.1 \exp(-[S/100 - 1])$. Note the slopes



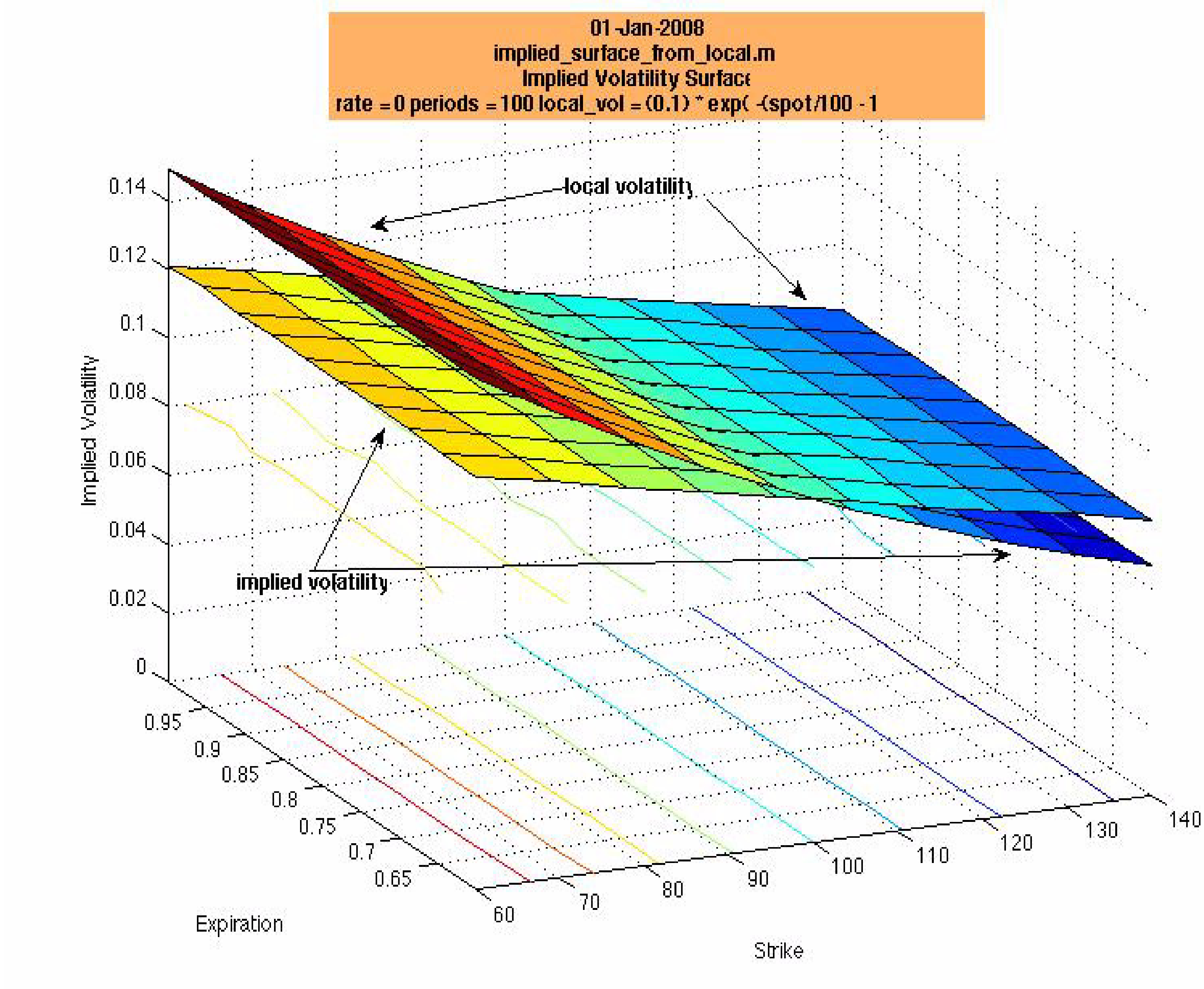
$$\sigma(S, t) = 0.1 \exp(-[S/100 - 1]^2)$$



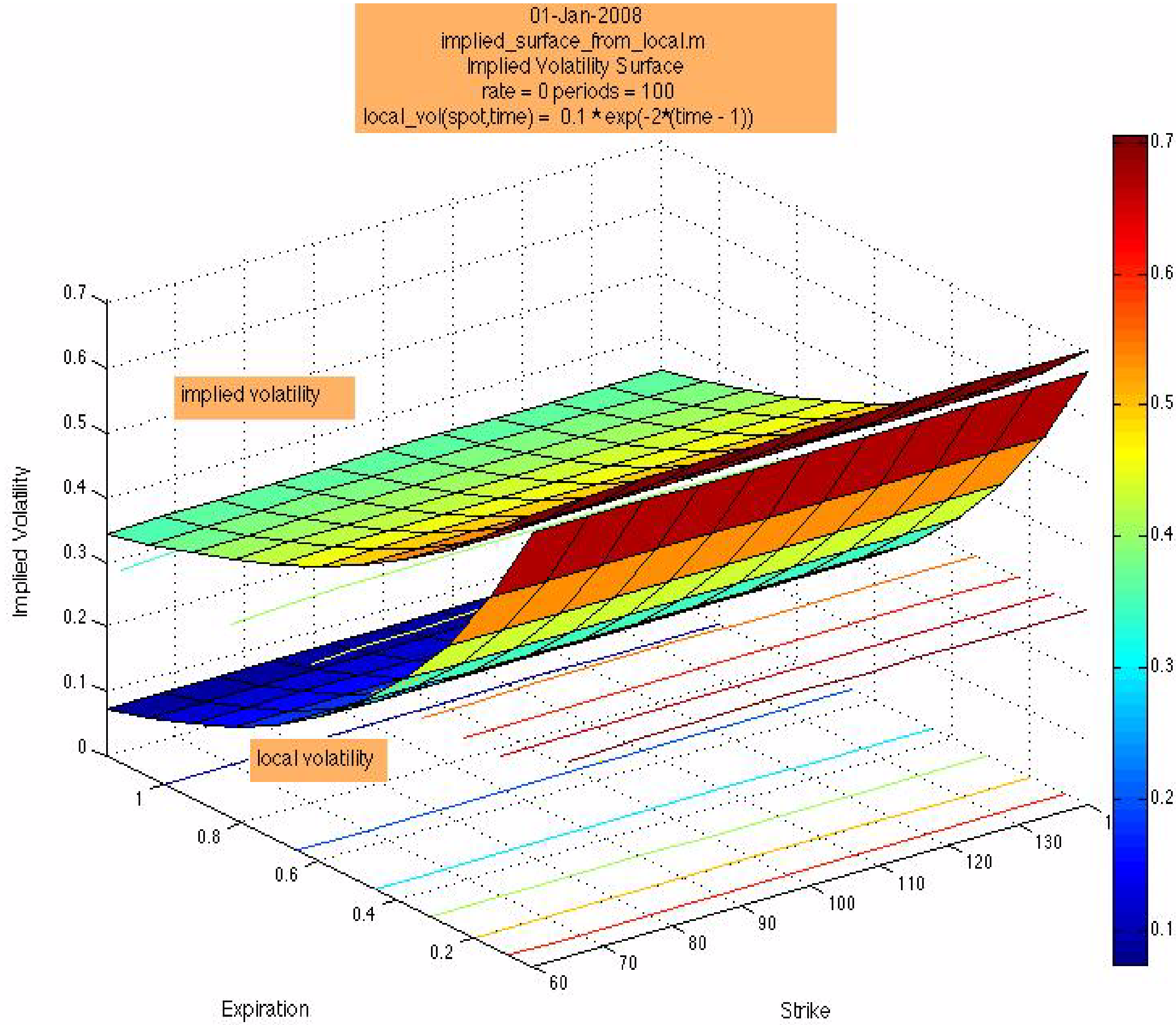
$$1\sigma(S, t) = (0.1 + 0.1t)\exp(-[S/100 - 1])$$



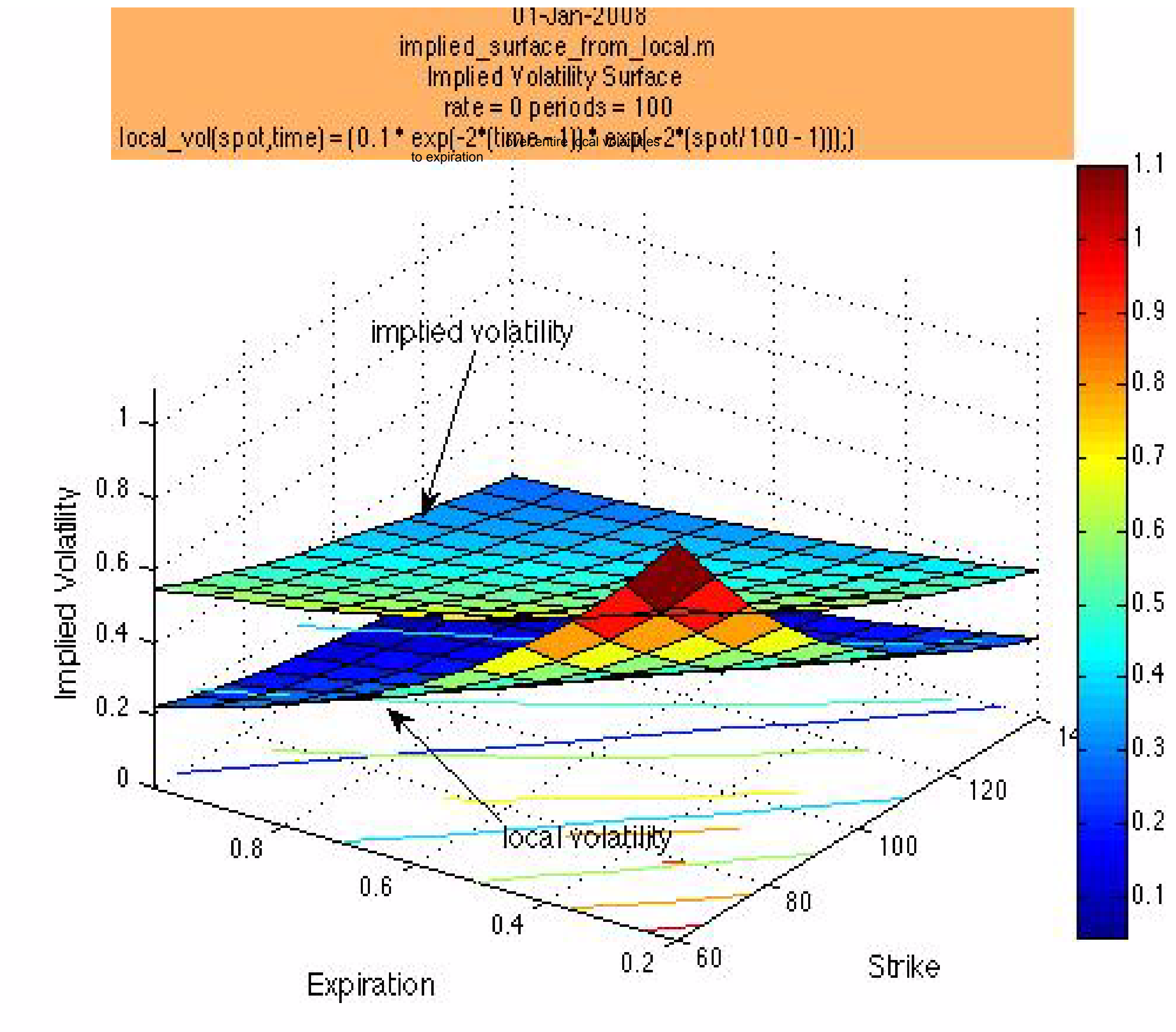
Dependent only on *S*: $\sigma(S, t) = 0.1\exp(-[S/100 - 1])$: Plot surface



Dependent only on t: $\sigma(S, t) = 0.1 \exp(-2[t-1])$: Plot Surface



Dependent on *S* and *t*: $\sigma(S, t) = 0.1\exp(-2[t-1])\exp(-2[S/100-1])$. **Local vol is 10% at t = 1 and S = 100.**



Difficulties with binomial trees

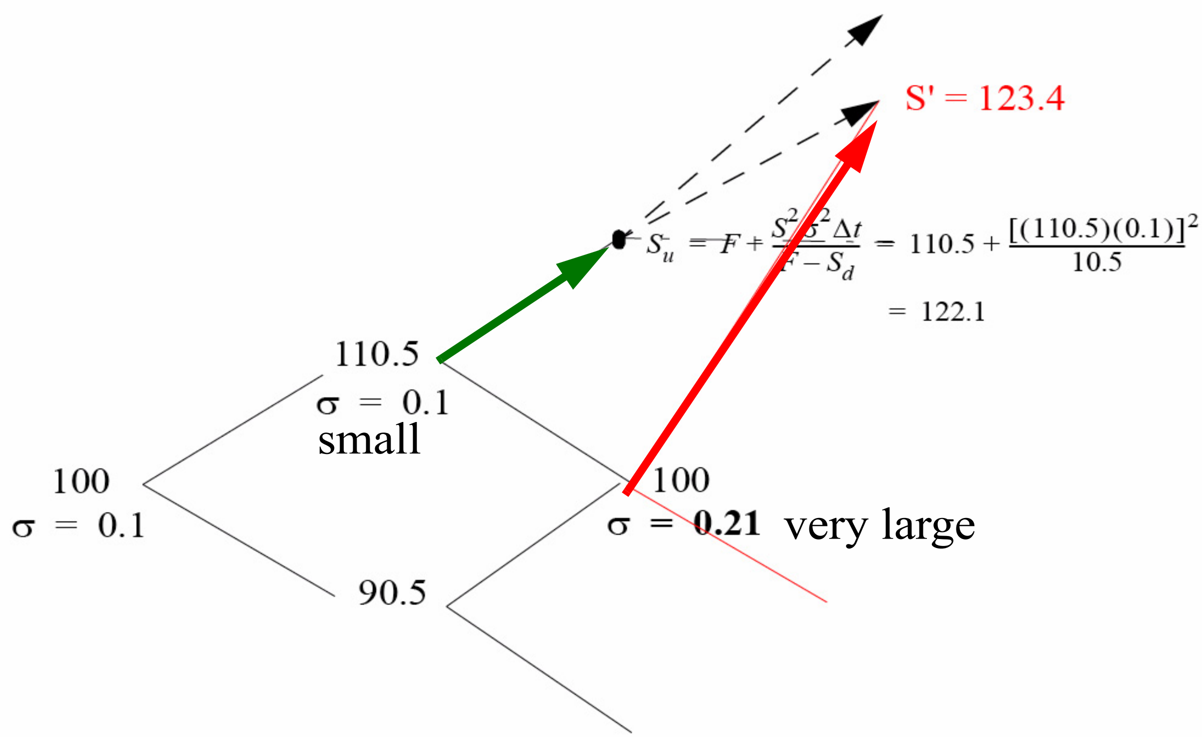
The nodes and the transition probabilities we discussed are uniquely determined by forward rates and the local volatility function we specify.

If $\sigma(S, t)$ varies too rapidly with stock price or time, then, for finite Δt, you can get binomial transition probabilities greater than 1 or less than zero.

Here is an example with $\Delta t = 1$ and $r = 0$.

The local volatility on level 3 at S = 100 is 0.21.

The S' node, S's up node in level 4 should be the down node from S_u in level 3, but it lies below S_u, but in fact lies above it, and so violates the no-arbitrage condition



You can remedy this with smaller time steps Δt, but then you are trying to extract to extract more information that is available from implied volatilities. Therefore, it's sometimes easier to use trinomial trees. They provide greater flexibility in avoiding arbitrage situations.

Implied Trees and Calibration

We went from $\sigma(S, t)$ to $\Sigma(S, t, K, T)$.

In reality, one observes discretely spaced implied volatilities $\Sigma(S, t, K_i, T_i)$ for discrete strikes K_i and expirations T_i, and one wants to calibrate a local volatility surface $\sigma(S, t)$.

"Implied trees" are a generalization of implied volatility. Implied volatility is a single variable defining a Black-Scholes tree; the implied tree is a representation of local volatility defining an evolution of a stock price consistent with options prices.

Binomial framework in the paper *The Volatility Smile and Its Implied Tree* Derman, E. & I. Kani. "Riding on a Smile." RISK, 7(2) Feb.1994, pp. 139-145. Also *The Local Volatility Surface* Continuous-time derivation of the same results: Dupire, *Pricing With A Smile*, RISK, January 1994, pages 18-20.

The key point of these papers is that there is a unique stochastic stock evolution process with a variable continuous volatility $\sigma(S, t)$ that can fit all options prices and their *continuous* implied volatilities $\Sigma(S, t, K, T)$. The tree representation of this stochastic process is called the *implied tree*.

Implied variables are widely used in finance and proprietary trading.

- *The Volatility Smile and Its Implied Tree* shows that the local volatility at each node is determined numerically and non-parametrically from options prices.
- Inverse problem: going backwards from output to input.
- Analogous to finding a potential in physics from viewing the way particles move under its influence.
- Theoretically straightforward, it turns out, but is rather difficult in practice. It's an “ill-posed” problem. Beginning with sparse implied volatilities and interpolating them into a surface, small changes in the interpolated input can cause dramatic changes in the output.
- In practice, it may be better to assume some parametric form for the local volatility function and then find the parameters that make the tree's option prices match as closely as possible the market's option prices.
- In a later class we'll discuss some methods of calibrating local volatilities to implied volatilities.

Dupire's Equation for Local Volatility

This equation describes the mathematical relationship between implied and local volatility.

Assuming continuity, you can derive the local volatility from the options prices (or their corresponding BS implied volatilities) in a simple way. Once you find the local volatilities from the implied volatilities surface, you can use them to build a tree or MC, and then use it to value options.

Breeden-Litzenberger formula: $p(S, t, K, T) = e^{r(T-t)}\frac{\partial^2}{\partial K^2}(C_{mkt}(S, t, K, T))$

p is the density or risk-neutral probability function that tells you the no-arbitrage price $p(S, t, K, T)dK$ of earning \$1 if the future stock price at time T lies between K and $K + dK$, determined by the second derivative of the market option price.

Even better: $\sigma(S, t)$ itself can be found from market prices of options and their derivatives. For zero interest rates and dividends, the local volatility at the stock price K is given by:

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T}C(S, t, K, T)}{K^2\frac{\partial^2 C}{\partial K^2}}$$

This is the continuous version of the procedure we used to construct a local volatility binomial tree.

If interest rates are non-zero,

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T} C(S, t, K, T) + rK \frac{\partial}{\partial K} C(S, t, K, T)}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

This is the mathematically correct generalization of the notion of forward stock volatilities consistent with current implied volatilities.

Understanding the Equation

We can interpret the equation in economic terms.

$$\frac{\partial C}{\partial T} = \frac{C(S, t, K, T + dT) - C(S, t, K, T)}{dT}$$

is proportional to an infinitesimal calendar spread for standard calls with strike K,

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{(dK)^2}$$

is proportional to an infinitesimal butterfly spread for standard calls with strike K.

Therefore the local variance $\sigma^2(K, T)$ at stock price K and time T is proportional to the ratio of the price of a calendar spread to a butterfly spread.

A calendar spread and a butterfly spread are combinations of tradeable options, and so the local volatility can be extracted from traded options prices (if they are available)!

Intuitively

A calendar spread

$$C(S, t, K, T + dT) - C(S, t, K, T)$$

measure the risk-neutral probability $p(S,t,K,T)$ of the stock moving from S at time t to K at time T, times the variance $\sigma^2(K, T)$ at K and T that is responsible for the adding option value.

$$\text{calendar spread} \sim p(S, t, K, T)\sigma^2(K, T).$$

But, according to the Breeden-Litzenberger, the probability

$$p(S, t, K, T) \sim \frac{\partial^2}{\partial K^2} C(S, t, K, T) \sim \text{butterfly spread}$$

So, roughly speaking, combing the two equations above, we have

$$\text{calendar spread} \approx \text{butterfly spread} \times \sigma^2(K, T)$$

or

$$\sigma^2(K, T) \approx \frac{\text{calendar spread}}{\text{butterfly spread}}$$

Using The Equation

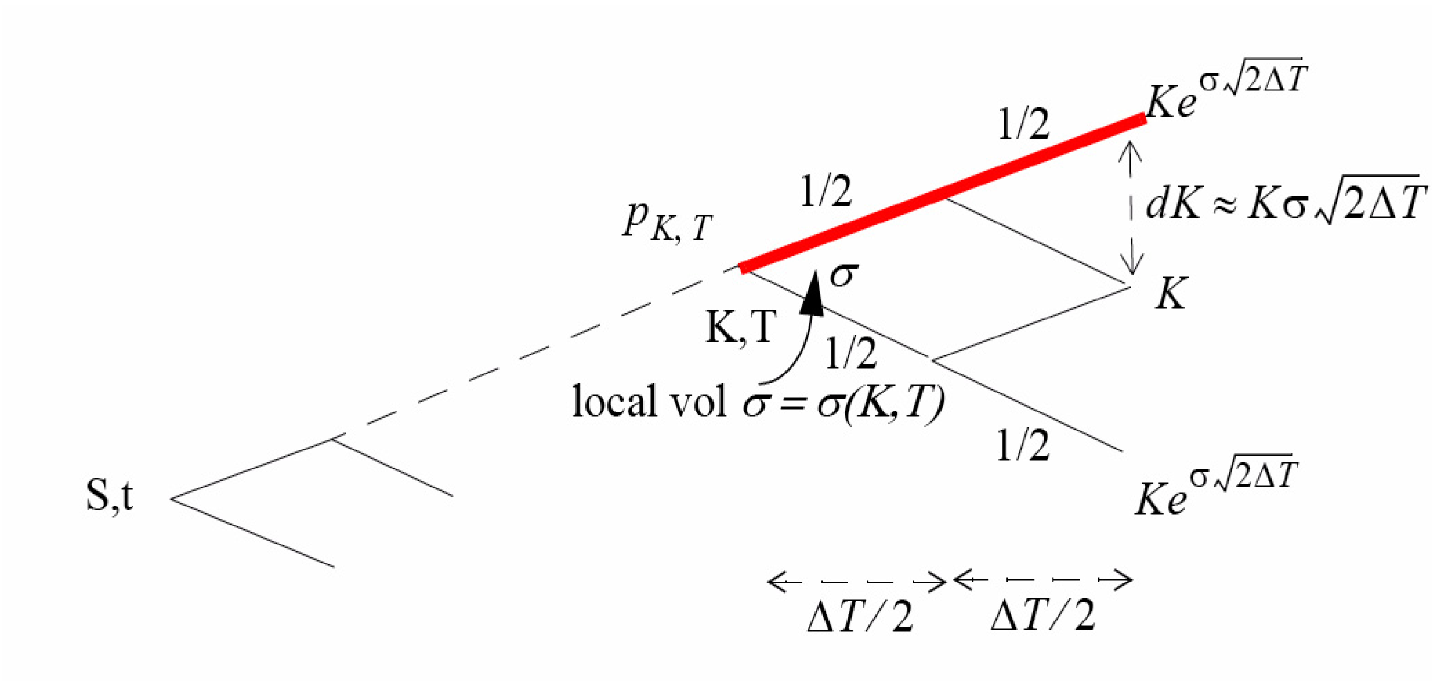
$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}} \quad \text{means} \quad \frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} - \frac{\sigma^2}{2}K^2\frac{\partial^2 C}{\partial K^2} = 0$$

resembles Black-Scholes equation with t replaced by T and S replaced by K.

- Black-Scholes equation holds for any contingent claim on S, relating the value of *any option* at S, t to the value of that option at $S + dS, T + dT$.
- This holds only for standard calls (or puts) and relates the value of a *standard* option with strike and expiration at K, T to the same option with strike and expiration $K + dK, T + dT$ when S, t is kept fixed.
- Find $\sigma(K, T)$ and hence build an implied local volatility tree from options prices and their derivatives.
- One consistent model that values all standard options correctly rather than having to use several different inconsistent Black-Scholes models with different underlying volatilities.
- Volatility arbitrage trading. You can calculate the future local volatilities implied by options prices and then see if they seem reasonable. If some of them look too low or too high in the future, you can think about buying or selling future butterfly and calendar spreads to l make a bet on future volatility.

A Poor Man's Derivation of the Dupire Equation in a Binomial Framework.

Let's use a Jarrow-Rudd tree that goes from (S,t) to (K,T) through **two half-periods of time** $\Delta T/2$,keeping interest rates zero for pedagogical simplicity.



The calendar spread obtains all its optionality from moving up the heavy red line:

$$C(S,t,K,T+dT) - C(S,t,K,T) \equiv \frac{\partial C}{\partial T}\Delta T = \overset{\text{probability}}{p_{K,T}\frac{1}{4}} \times \overset{\text{payoff}}{dK}$$

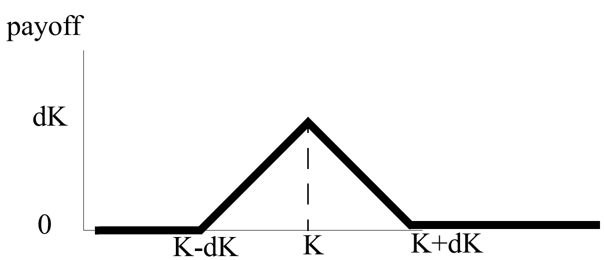
Value of the calendar spread per unit time is $\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,\,T} \times \frac{dK}{dT}$

where $p_{K,\,T}$ is the risk-neutral probability of getting to (K, T) and $dK \approx \Delta K = K\sigma\sqrt{2\Delta T}$.

We can get $p_{K,\,T}$ from a **butterfly spread portfolio**

$p_{K,\,T}$ is the value of a portfolio that pays \$1 if the stock price is at node *K*, and zero for all other nodes at time *T*.

The butterfly spread portfolio $C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K{-}dK, T)$ pays



Dividing by dK produces a payoff which is \$1 at the node K and zero at adjacent nodes.

$$p_{K,\,T} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K{-}dK, T)}{dK} = \frac{\partial^2 C}{\partial K^2} dK$$

Combining the expression for $p_{K,T}$ with $\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,T} \times \frac{dK}{dT}$

$$\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,T} \frac{\Delta K}{\Delta T} = \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{(\Delta K)^2}{\Delta T}$$

$$= \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{[K\sigma\sqrt{2\Delta T}]^2}{\Delta T}$$

$$= \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

so that the local volatility $\sigma(K, T)$ is given by

$$\sigma^2(K, T) = \frac{\partial C}{\partial T} \Big/ \left(\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}\right)$$

You can regard this as a *definition* of the effective local volatility from options prices and has meaning beyond the model.

A More Rigorous Formal Proof of Dupire's Equation

The stochastic PDE for the risk-neutral stock price with stochastic σ: $\frac{dS_t}{S_t} = r\,dt + \sigma(S_t, t, .)\,dZ_t$

Call value at time t:

$$C_t(K, T) = e^{-r(T-t)} E\left\{ [S_T - K]_+ \right\}$$

where E denotes the q-measure risk-neutral expectation over S_T and **all** other stochastic variables.

Examine the derivatives of the call value that enter the Dupire equation.

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)} E\{\theta(S_T - K)\}$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)} E\{\delta(S_T - K)\}$$

To find $\left.\frac{\partial C}{\partial T}\right|_K$ we need to take account of both the change in T and the corresponding change in S_T through Ito's Lemma.

$$C_t(K, T) = e^{-r(T-t)} E\left\{ [S_T - K]_+ \right\}$$

$$d_T C\Big|_K = E\left\{ \frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S_T} dS_T + \frac{1}{2}\frac{\partial^2 C}{\partial S_T^2}(dS_T)^2 \right\}$$ expectation in risk-neutral measure over S_T

$$= E\left\{ -rC\,dT + e^{-r\tau}\theta(S_T - K)\,dS_T + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)S_T^2\,dT \right\}$$

1 $$= E\left\{ -rC\,dT + e^{-r\tau}\theta(S_T - K)(rS_T\,dT) + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)K^2\,dT \right\}$$

$$= E\left\{ -re^{-r\tau}\theta(S_T - K)(S_T - K)\,dT + e^{-r\tau}\theta(S_T - K)rS_T\,dT + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(K, T, .)K^2\,dT \right\}$$

$$= e^{-r\tau}E\left\{ rK\theta(S_T - K)\,dT + \frac{1}{2}\delta(S_T - K)\sigma^2(K, T, .)K^2\,dT \right\}$$

$$= -rK\frac{\partial C}{\partial K}dT + \frac{1}{2}E\left\{ \sigma^2(K, T, .) \right\}\frac{\partial^2 C}{\partial K^2}K^2\,dT$$

Then the change in the value of C when S_T and T change is given by

$$\left.\frac{\partial C}{\partial T}\right|_K = -rK\frac{\partial C}{\partial K} + \frac{1}{2}E\left\{\sigma^2(K, T, .)\right\}\frac{\partial^2 C}{\partial K^2}K^2$$

$$E\left\{\sigma^2(K, T, .)\right\} = \frac{\left(\left.\frac{\partial C}{\partial T}\right|_K + rK\left.\frac{\partial C}{\partial K}\right|_T\right)}{\left.\frac{1}{2}\frac{\partial^2 C}{\partial K^2}\right|_T K^2}$$

Define the local volatility as: $\sigma^2(K, T) \equiv E\left\{\sigma^2(K, T, .)\right\}$ and this is the Dupire equation.

Denominator is positive, and one can use dominance arguments to show that the numerator is too.

For another more classic proof using the Fokker-Planck equation, see the Appendix of Derman, E. & I. Kani. “Riding on a Smile.” RISK, 7(2) Feb.1994, pp. 139-145.