# Lecture 5:
# Static Hedging and Implied Distributions

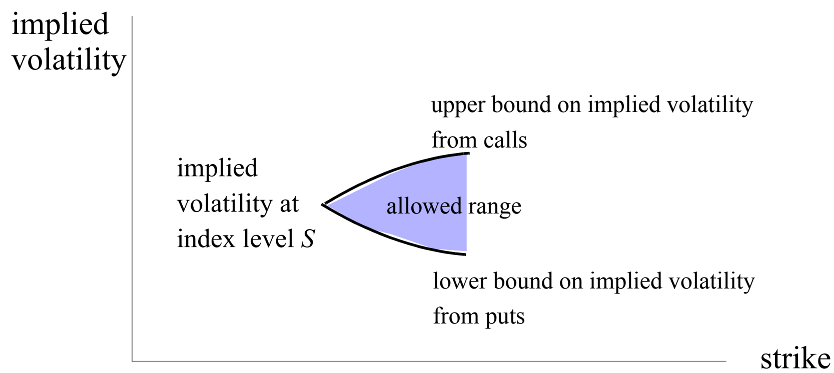
**Recapitulation of Lecture 4:**

Plotting the smile against $\Delta$ is enlightening and useful.

For a slightly out-of-the-money option a fraction $J$ away from at-the-money,

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{\Sigma\sqrt{\tau}}{2} - \frac{J}{\Sigma\sqrt{\tau}}\right)$$

Arbitrage constraints on the smile:




**Problems caused by the smile**:

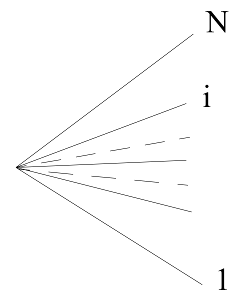
The smile manifest in the market values of standard options is inconsistent with the Black-Scholes model. Without the right model, who knows how to hedge vanilla options or value and hedge exotic options? The errors can be sizeable. Here are some classes of models:

- Local volatility: $\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ$
- Stochastic volatility:
$$\begin{aligned} dS &= \mu_S(S, V, t)dt + \sigma_S(S, V, t)dZ_t \\ dV &= \mu_V(S, V, t)dt + \sigma_V(S, V, t)dW_t \\ V &= \sigma^2 \\ E[dWdZ] &= \rho dt \end{aligned}$$
- Jump diffusion

## 5.1 Static Hedging and Implied Distributions

The Black-Scholes formula calculates options prices as the expected discounted value of the payoff over a lognormal stock distribution in a risk-neutral world, and – trivially, because a lognormal stock distribution has a single volatility – produces an implied volatility skew that is flat, independent of strike level.

We can ask the inverse question: for a fixed expiration, what risk-neutral stock distribution (the so-called *implied distribution*) matches the observed smile when options prices are computed as expected risk-neutrally discounted payoffs? Let's look at this when the world has only a discrete and finite number of possible future states.




At time $t$, consider a security $\pi_i$ that pays \$1 when the stock is in state $i$ with price $S_i$ at a future time $T$, and pays zero if the stock price takes any other value. Suppose you know the market price $\pi_i$ for each of these securities.

The portfolio that consists of all of these $\pi_i$ is effectively a riskless bond because it pays off \$1 in every future state, and its value is therefore given by

$$\sum_1^N \pi_i = \exp[-r(T-t)] \equiv \frac{1}{R}$$

where $r$ is the continuously compounded riskless rate.

Then the pseudo-probabilities $p_i \equiv R\pi_i$ have the characteristics of probabilities because $\sum p_i = 1$ and we can write $\pi_i = \frac{p_i}{R}$

If there is one state-contingent security $\pi_i$ for each state $i$ in the market at time $T$, then these securities provide a complete basis that span the space of future payoffs, and the market is said to be complete. In terms of this basis we can replicate the payoff of any security $V$ if we know its payoff $V_i$ in all states $i$.

The replicating portfolio is $V = \sum V_i \pi_i$ and its current value is $V = \sum_i \frac{p_i}{R} V_i$.

In more elegant continuous-state notation, we can write the current value of a derivative $V$ in terms of its terminal payoffs $V(S', T)$ at time $T$

$$V(S,t) = e^{-r(T-t)}\int_0^\infty p(S,t,S',T)V(S',T)dS' \qquad \text{Eq.5.1}$$

Here $p(S,t,S',T)$ is the risk-neutral (pseudo-) probability density.

We define

$$\pi(S,t,S',T) = e^{-r(T-t)}p(S,t,S',T)$$

Then $\pi(S,t,S',T)dS'$ is the price at time $t$ of a state-contingent security that pays $1 if the stock price at time $T$ lies between $S'$ and $S'+dS'$. Since the integral over all final stock prices of a security that pays $1 at expiration is equivalent to a zero-coupon bond with a face value of $1,

$$\int_0^\infty \pi(S,t,S',T)dS' = e^{-r(T-t)}$$

and

$$\int_0^\infty p(S,t,S',T)dS' = 1$$

an appropriate constraint on a probability density.

If we know the probability density $p(S,t,S',T)$, we can determine the value of all European-style payoffs at time $T$ by weighting the probability by the payoff.

In particular, we can write the value of any European option at time $T$ as an integral over the risk-neutral probability density. For a standard call option C with strike $K$,

$$C(S',T) = [S'-K]_+ = max(S'-K,0) = [S'-K]\theta(S'-K)$$

where $\theta(x)$ is the Heaviside or indicator function, equal to 1 when $x$ is greater than 0 and 0 otherwise.

Therefore

$$
\begin{aligned}
C_K(S,t) &= e^{-r(T-t)}\int_K^{\infty} p(S,t,S',T)(S'-K)dS' \\
&= e^{-r(T-t)}\int_0^{\infty} dS'(S'-K)\theta(S'-K)p(S,t,S',T)
\end{aligned}
\qquad \text{Eq.5.2}
$$

It turns out that a knowledge of call prices (or put prices) for all strikes $K$ at expiration time $T$ are enough to determine the density $p(S, t, S', T)$ for all $S'$

Therefore one can statically replicate any known payoff at time $T$ through a combination of zero-coupon bonds, forwards, calls and puts.

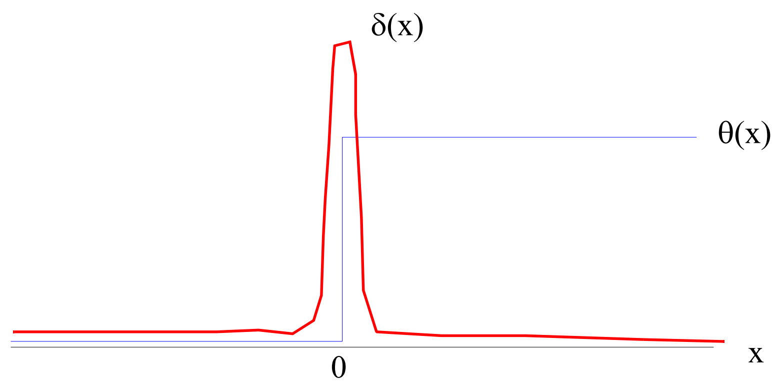
**One big caveat.** Remember though, that the risk-neutral distribution at expiration is insufficient for valuing all options on the underlyer. To value an option on a stock, one must hedge it; to hedge it, one must hedge against the changes caused by the stochastic process driving the stock price; the risk-neutral distribution at expiration tells you nothing about the evolution of the stock price on its way to expiration. Hence, implied distributions are not useful in determining dynamic hedges. Nevertheless, implied distributions are useful for statically replicating European-style payoffs at a fixed expiration.

### 5.1.1 The Heaviside and Dirac Delta functions

The derivative of the Heaviside function is the Dirac *delta function*:

$$\frac{\partial}{\partial x}\theta(x) = \delta(x)$$

$\delta(x)$ is a distribution, the generic name for a very singular function that only makes sense when used within an integral. $\delta(x)$ is zero everywhere except at $x = 0$, where its value is infinite. Its integral over all $x$ is 1.



There are three important properties of the delta function:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

$$x\delta(x) = 0$$

The latter equality holds formally because $\delta(x)$ is zero everywhere except at the origin, and $x$ itself is zero there.

### 5.1.2 Finding the risk-neutral probability density from call prices: the Breeden-Litzenberger formula

From Equation 5.2

$$\exp(r\tau) \times C(S, t, K, T) = \int_K^{\infty} dS'(S' - K)p(S, t, S', T)$$
$$\equiv \int_0^{\infty} dS'(S' - K)\theta(S' - K)p(S, t, S', T)$$

where $\tau = T - t$ .

Now differentiate the equation above with respect to *K,* taking the derivative on the right hand side under the integral sign, so that

$$\exp(r\tau) \times \frac{\partial C}{\partial K} = -\int_K^{\infty} p(S, t, S', T)dS' = -(1 - F(K))$$

Here we have made use of the identity $(S' - K)\delta(S' - K) = 0$, and $F(K)$

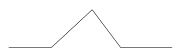
is the cumulative distribution function

$$F(K) = \int_0^K p(S, t, S', T)dS'$$

Differentiate w.r.t $K$ again to obtain the Breeden-Litzenberger formula:

$$\exp(r\tau) \times \frac{\partial^2 C}{\partial K^2} = p(S, t, K, T) \qquad \text{Eq.5.3}$$

The second derivative with respect to $K$ of call prices is the risk-neutral probability distribution, and hence must be positive. In fact, we know that the second derivative must be positive from our earlier discussion of the no-arbitrage bounds on the skew.

$\frac{\partial^2 C}{\partial K^2}$ is a butterfly spread, proportional to $C_{K+dK} - 2C_K + C_{K-dK}$ with terminal payoff ~  whose height is $dK$ and whose payoff area is $(dK)^2$.In the limit that $dK \to 0$, $\frac{\partial^2 C}{\partial K^2}$ has a payoff with area 1 if $S = K$ and zero otherwise; it behaves like a state-contingent security.

Note that at any time *t:*

$$\int_0^\infty p(S, t, K, T)dK = e^{r\tau}\int_0^\infty \frac{\partial^2 C}{\partial K^2}dK = e^{r\tau}\left[\left.\frac{\partial C}{\partial K}\right|_\infty - \left.\frac{\partial C}{\partial K}\right|_0\right] = 1$$

because

- $\left.\frac{\partial C}{\partial K}\right|_\infty = 0$ as the strike gets very large and calls become worthless; and
- for $K \to 0$ the call becomes a forward with value $S - Ke^{-r\tau}$, so that $\left.\frac{\partial C}{\partial K}\right|_0 = -e^{-r\tau}$.

## 5.2 Static Replication: valuing arbitrary payoffs at a fixed expiration using implied distributions.

From Equation 5.1 and Equation 5.3 we can write

$$V(S, t) = \int_0^\infty \frac{\partial^2 C}{\partial K^2}(S, t, K, T) V(K, T) dK \qquad \text{Eq.5.4}$$

If we know call prices and their derivatives for all strikes at a fixed expiration, we can find the value of any other European-style derivative security at that expiration in terms of its payoff and the derivatives of the call prices. Alternatively, one can use the derivatives of put prices.

**Note: this involves no use of option theory at all, and no use of the Black-Scholes equation**. It just assumes you can get all the option prices you need to get the market's state-contingent prices irrespective of any modeling issues. It works even if there is a smile or skew or jumps.

### 5.2.1 Replicating by standard options

Equation 5.4 involves calculating the expected value of the European-style payoff over the risk-neutral density function corresponding to the implied distribution.

You can use integration by parts to show that the integral of any European payoff $V$ over the risk-neutral density function can be converted into a sum of portfolios of zero coupon bonds, forwards, puts and calls that together replicate the payoff of $V$.

Consider an exotic European payoff $W(K, T)$. Then using the density for puts below strike $A$ and for calls above strike $A$, we can write

$$\begin{aligned}
W(S, t) &= e^{-r\tau} \int_0 \rho(S, t, K, T) W(K, T) dK \\
&= e^{-r\tau} \left[ \int_0^A \rho(S, t, K, T) W(K, T) dK + \int_A^\infty \rho(S, t, K, T) W(K, T) dK \right] \\
&= \int_0^A \frac{\partial^2 P}{\partial K^2} W(K, T) dK + \int_A^\infty \frac{\partial^2 C}{\partial K^2} W(K, T) dK
\end{aligned}$$

Now integrate by parts twice to get

$$W(S,t) = \int_0^A \frac{\partial^2}{\partial K^2} W(K,T) P(S,K) dK + \int_A^\infty \frac{\partial^2 W}{\partial K^2} C(S,K) dK$$
$$= \left(W\frac{\partial P}{\partial K} - P\frac{\partial W}{\partial K}\right)\Bigg|_{K=0}^{K=\mathrm{A}} + \left(W\frac{\partial C}{\partial K} - C\frac{\partial W}{\partial K}\right)\Bigg|_{K=A}^{K=\infty} \qquad \text{Eq.5.5}$$

where $P(S,K)$ is the current value at time $t$ and stock price $S$ of a put with strike $K$ and expiration $T$, and $C(S,K)$ is the corresponding call value.

We can evaluate all these boundary terms as a function of strike $K$, using the following conditions for the current call and put prices.

$$P[S,0] = 0$$
$$\frac{\partial}{\partial K} P[S,0] = 0$$
$$C[S,\infty] = 0$$
$$\frac{\partial}{\partial K} C[S,\infty] = 0$$
$$P[S,K] - C[S,K] = Ke^{-r\tau} - S$$
$$\frac{\partial}{\partial K} P[S,K] - \frac{\partial}{\partial K} C[S,K] = e^{-r\tau}$$

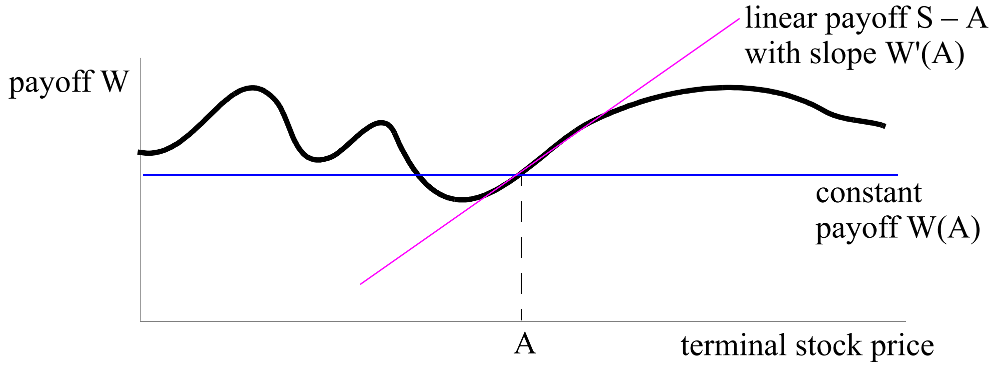
We then obtain

$$W = W(A)e^{-r\tau} + W'(A)[S - Ae^{-r\tau}] + \int_0^A P(K)W''(K)dK + \int_A^\infty C(K)W''(K)dK \qquad \text{Eq.5.6}$$

This formula[1] demonstrates that you can decompose an arbitrary payoff at time $T$ into a constant riskless payoff discounted like a zero-coupon bond, a linear part which has the same value as a forward contract with delivery price $A$, and a combination of puts with strikes below $A$ and calls with strikes above $A$, with densities given by $\frac{\partial^2}{\partial K^2} W(K,T)$.

1. Derived in this form by Carr and Madan.

The following figure illustrates the replication of the payoff, where the constant and linear parts of the payoff are replicated without any options, and the curved parts make use of options.




Thus there are two sides to static replication.

1. If you know the risk-neutral density $\rho$ then you can write down the value of W(S,t) as an integral over the terminal payoff, as in Equation 5.4.
2. Alternatively, if you know the second derivative of the payoff W, then you can write down the value of W(S,t) as an integral over call and put prices with different strikes, as in Equation 5.6.

The one equation is the complement of the other.

If you can buy every option in the continuum you need from someone who will never default on their payoff, then you have a perfect static hedge. You can go home and come back to work only when W expires, confident that the options C and P that you bought will exactly match its payoff. This hedge does not depend on any theory at all – it's pure mathematics (plus faith in your counterparties) that matches one payoff by the sum of a series of different ones.

If, as in life, you cannot buy every single option in the continuum because only a finite number of strikes are available for purchase, then you have only an approximate replicating portfolio whose value will deviate from the value of the target option's payoff. Picking a reasonable or tolerable replicating portfolio is up to you. There is always some residual unhedged risk.

**5.2.2 This works even if there is volatility skew.** If you can write the payoff of an exotic option at time $T$ as a sum over vanilla options, and if you know the skew Black-Scholes implied volatilities $\Sigma(K,T)$ at that instant for all $K$ – i.e the prices at which the market instantaneously values options of all strikes at that expiration – then you can value the exotic.**A Static Replication Example in the Presence of a Skew**

Consider an option of strike *B* and expiration *T* on a stock with price *S* whose payoff gives you one share of stock for every dollar the option is in the money. Its payoff in terms of the terminal stock price *s* is

$$V(s) = s \times max[s - B, 0] = s \times (s - B)\theta(s - B) \qquad \text{Eq.5.7}$$

When it is in the money, this payoff is quadratic in the stock price, but vanilla calls are linear. We can replicate the payoff of this option by adding together a collection of vanilla calls with strikes starting at B, and then adding successively more of them to create a quadratic payoff, as illustrated below.

We attempt to replicate the *security* V by means of a portfolio of call options C(K) with all strikes X greater than B, so that

$$\mathrm{V} = \int_0^{\infty} q(K)\theta(K - B)\mathrm{C(K)dK} \qquad \text{Eq.5.8}$$

where $q(K)$ is the unknown density of calls with strike *K* required to replicate the payoff of V, and we've chosen A in Equation 5.6 to be 0.

Differentiating Equation 5.7 with respect to *s* leads to

$$\begin{aligned}
\frac{\partial V}{\partial s}(s) &= \frac{\partial}{\partial s}[s \times (s - B)\theta(s - B)] \\
&= (s - B)\theta(s - B) + s\theta(s - B) + s(s - B)\delta(s - B) \\
&= (s - B)\theta(s - B) + s\theta(s - B) \\
\frac{\partial^2 V}{\partial s^2} &= (s - B)\delta(s - B) + 2\theta(s - B) + s\delta(s - B) \\
&= 2\theta(s - B) + s\delta(s - B)
\end{aligned}$$

Therefore for A = 0

$$\begin{aligned}
V(0) &= 0 \\
\frac{\partial V}{\partial s}(0) &= 0 \\
\frac{\partial^2 V}{\partial s^2}(K) &= 2\theta(K - B) + B\delta(K - B)
\end{aligned}$$

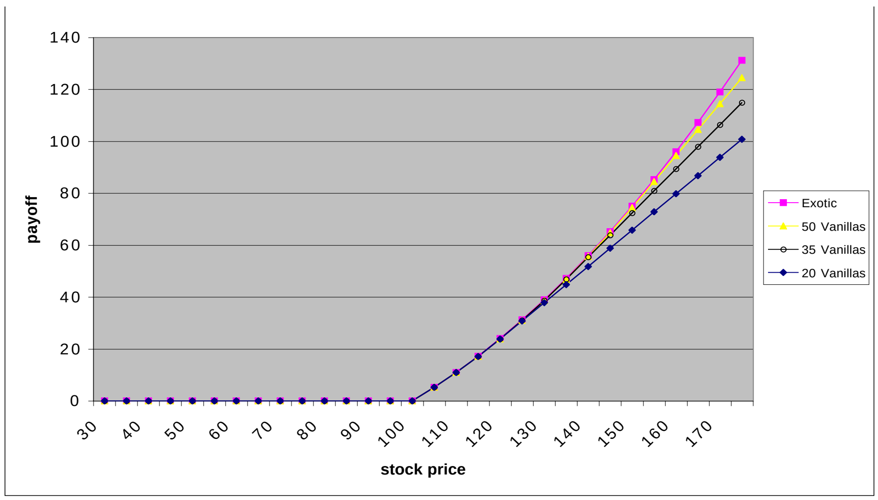
Substituting this into Eq.5.8 we obtain the decomposition of the target security V in terms of call options:

$$V = BC(B) + \int_B^\infty 2C(K)dK$$

Therefore, the current fair value of V is

$$V(S, t) = BC(S, t, B, T) + 2\int_B^\infty C(S, t, K, T)dK$$

What is this worth in real life? The quadratic payoff is a linear combination of call payoffs.The figure below shows how well the quadratic payoff as function of terminal stock price *s* is approximated by a portfolio of 50 calls with strikes equally spaced and \$1 apart between 100 and 150. The replication becomes progressively more inaccurate for stock prices greater than 150.




Now we examine the convergence of the value of the replicating formula to the correct no-arbitrage value for two different smiles.

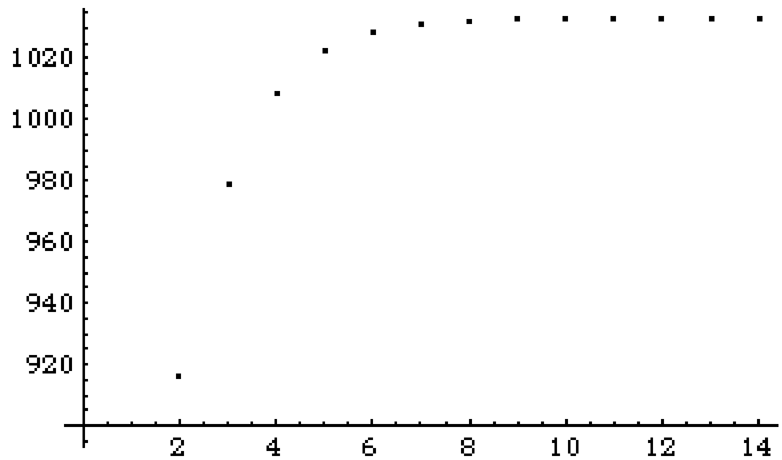
The first smile we consider is described by

$$\Sigma(K) = 0.2\left(\frac{K}{100}\right)^{\beta}$$

Here $\beta = -0.5$ corresponds to a "negative" skew in which implied volatility increases with decreasing strike; $\beta = 0$ corresponds to no skew at all; and $\beta = 0.5$ corresponds to a positive skew.

For $\beta = 0$ the fair value of V when replicated by an infinite number of calls is 1033. The graph below illustrates the convergence to fair value of the replicating portfolio as the number of strikes included in the portfolio increases. Wiith 10 strikes the value has virtually converged.

**Convergence as we increase number of strikes for flat 20% volatility**

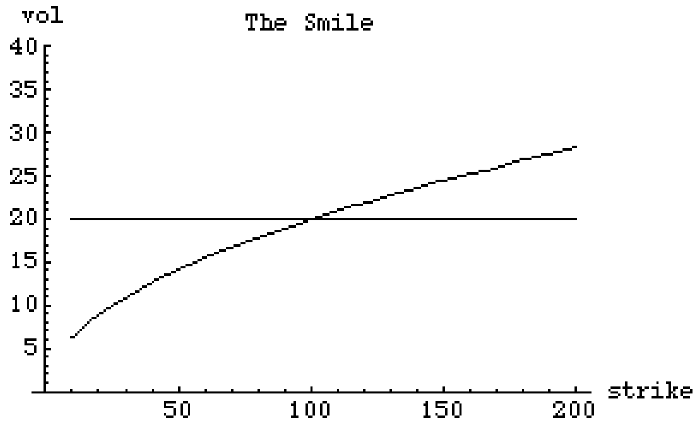


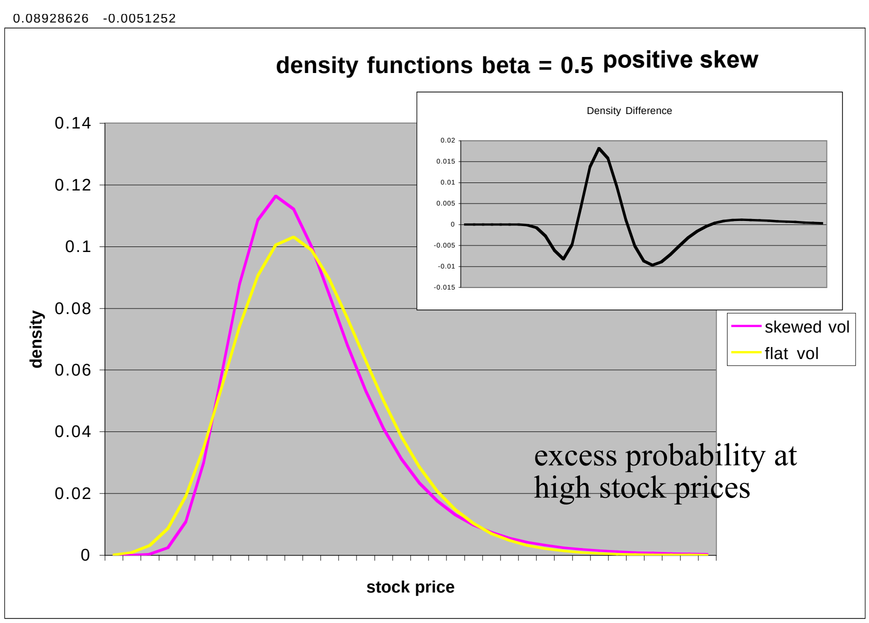

Now we examine the effect of the skew on the value and convergence of V.

For both positive and negative skews, we plot below

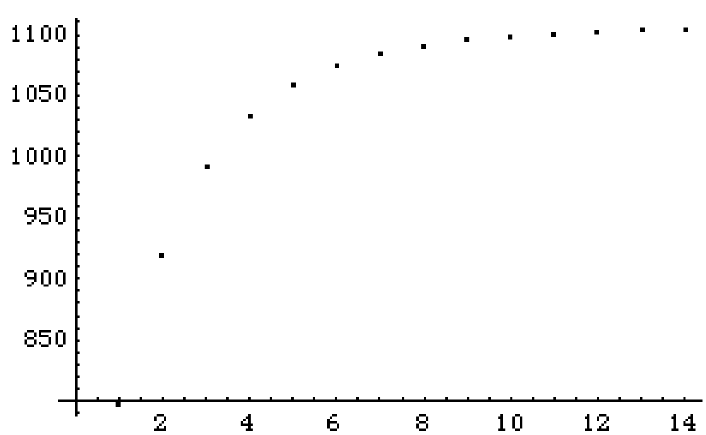
1. the implied volatility as a function of strike;
2. the implied distribution corresponding to the skew; and
3. the convergence of the value of the replicating portfolio for option V to its fair value as a function of the number of calls included in the portfolio.

**Positive skew** $\beta = 0.5$

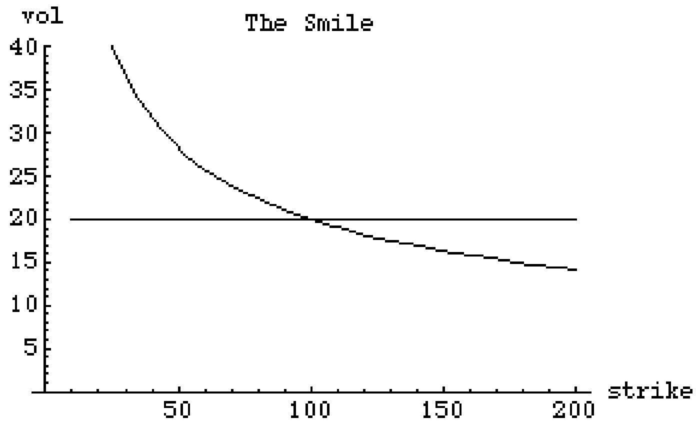


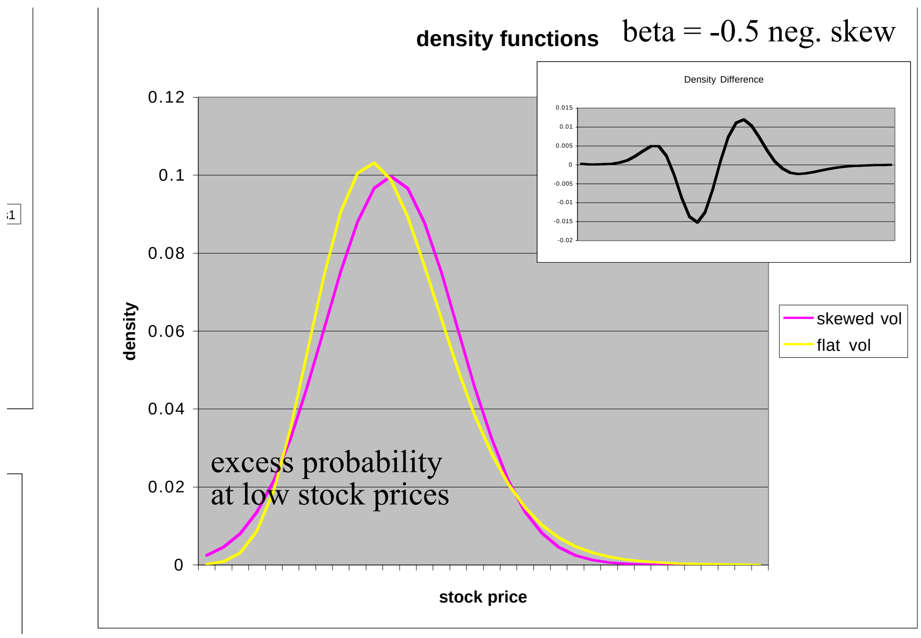




Convergence for a positive skew to a fair value of 1100 is slower and requires more strikes.

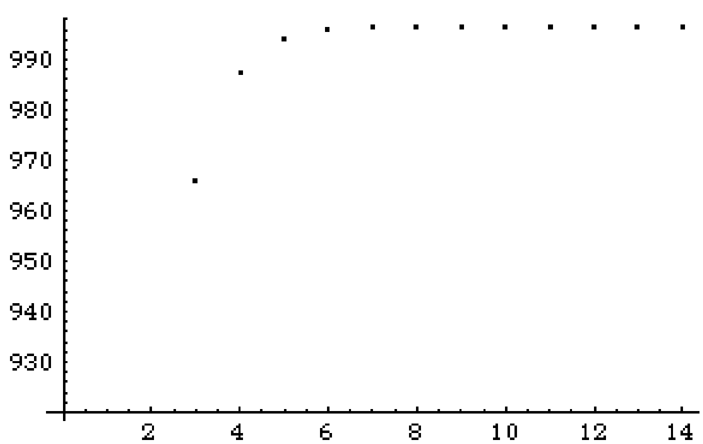


**Negative skew** $\beta = 0.5$




Convergence for a negative skew to a fair value 996 is faster and requires less strikes..



# Appendix 5.2: The Black-Scholes risk-neutral probability density

In the BS evolution, returns $\ln S_T/S_t$ are normal with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$.

Therefore,

$$x = \frac{\ln S_T/S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}} \qquad \text{Eq.5.9}$$

is normally distributed with mean 0 and standard deviation 1, with a probability density $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The returns $\ln S_T/S_t$ can range from $-\infty$ to $\infty$.

**The BS density function**

From Eq.5.9,

$$\frac{dS_T}{S_T} = \sigma\sqrt{\tau}dx$$

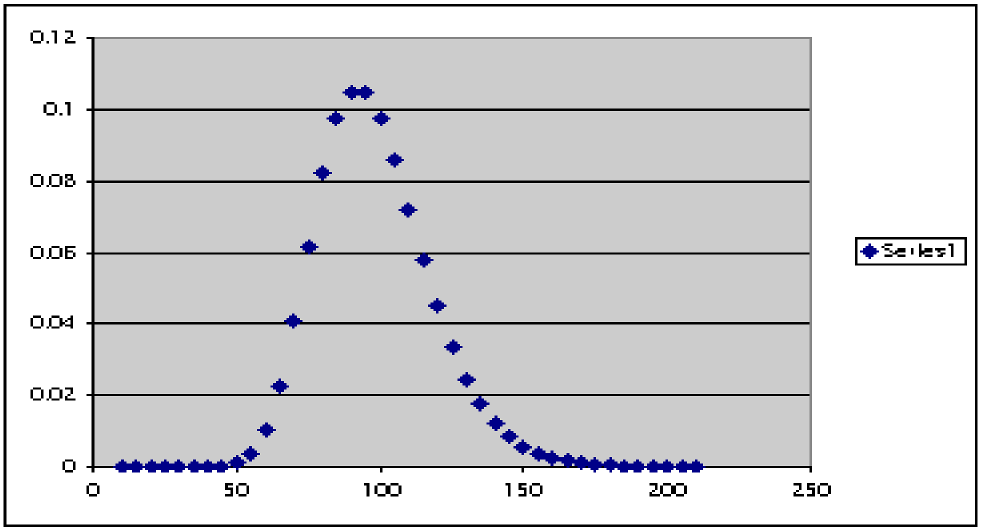
The risk-neutral value of the option is given by

$$e^{r\tau}C = \frac{1}{\sqrt{2\pi}}\int_{-\mathrm{d}_2}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_K^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)\frac{dS_T}{S_T}$$

where

$$\frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi\tau}\sigma S_T}$$

is the risk-neutral density function to be used in integrating payoffs over $S_T$, plotted below.




Let's work out the value of a call with this BS density and show that it gives the BS formula. It's tedious but worth doing once.

When $S_T = K$ then

$$x_{min} = \frac{\ln K/S_t - \left(r\tau - \frac{1}{2}\sigma^2\tau\right)}{\sigma\sqrt{\tau}} = \frac{-\ln S_t/K - \left(r\tau - \frac{1}{2}\sigma^2\tau\right)}{\sigma\sqrt{\tau}} = -\frac{\ln S_t/K + \left(r\tau - \frac{1}{2}\sigma^2\tau\right)}{\sigma\sqrt{\tau}} = -\frac{\ln S_F/K - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} = -d_2$$

**Box 1. The BS density and the BS formula (zero dividends)**

Thus

$$C = e^{-r\tau}\int_{-d_2}^{\infty} dx\,(S_T - K)\,h(x)$$

where $S_T = S_F e^{x\sigma\sqrt{\tau} - \sigma^2\tau/2}$ $\qquad S_F \equiv S_t e^{r\tau}$

$$C = e^{-r\tau}\int_{-d_2}^{\infty} dx\,\left[S_F e^{x\sigma\sqrt{\tau} - \sigma^2\tau/2} - K\right]\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$= S_t e^{-\sigma^2\tau/2}\int_{-d_2}^{\infty} dx\,\frac{e^{-x^2/2 + x\sigma\sqrt{\tau}}}{\sqrt{2\pi}} - Ke^{-r\tau}\int_{-d_2}^{\infty} dx\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$= S_t e^{-\sigma^2\tau/2}\int_{-d_2}^{\infty} \frac{dx}{\sqrt{2\pi}}\,e^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}}\,e^{\sigma^2\tau/2} - Ke^{-r\tau}\int_{-d_2}^{\infty} dx\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$= S_t\int_{-d_2}^{\infty} \frac{dx}{\sqrt{2\pi}}\,e^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}} - Ke^{-r\tau}\int_{-d_2}^{\infty} dx\,e^{-x^2/2}$$

define $y = x - \sigma\sqrt{\tau}$

$$= S_t\int_{-d_2-\sigma\sqrt{\tau}\equiv -d_1}^{\infty} \frac{dy}{\sqrt{2\pi}}\,e^{-y^2/2} - Ke^{-r\tau}\int_{-d_2}^{\infty} dx\,e^{-x^2/2}$$

$y \to -y \qquad\qquad x \to -x$

$$= S_t\int_{-\infty}^{d_1} \frac{dy}{\sqrt{2\pi}}\,e^{-y^2/2} - Ke^{-r\tau}\int_{-\infty}^{d_2} dx\,e^{-x^2/2}$$

$$C = S_t N(d_1) - Ke^{-r\tau}N(d_2)$$ The BS formula

## 5.3 Static Replication of Non-European Options

To replicate an option dynamically, you can in principle own a portfolio of stock and riskless bonds, and adjust them to obtain achieve exactly the same returns. To do so, you must *continuously* alter the weights in the replicating portfolio according to the formula as time passes and/or the stock price moves. This portfolio is called the *dynamic replicating portfolio.* Options traders ordinarily hedge options by shorting the dynamic replicating portfolio against a long position in the option to eliminate all the risk related to stock price movement.

There are three difficulties with this hedging method. First, continuous weight adjustment is impossible, and so traders adjust at discrete intervals. This causes small errors that compound over the life of the option, and result in replication whose accuracy increases with the frequency of hedging, as we've seen previously. Second, the transaction costs associated with adjusting the portfolio weights grow with the frequency of adjustment and can overwhelm the potential profit margin of the option. Traders have to compromise between the accuracy and cost. Third, the systems you need to carry out dynamic replication must be sophisticated and are costly.

What can you do about all of this? In this section we describe a method of options replication that bypasses (approximately) some of these difficulties. Given some particular exotic *target option*, we show how to construct a portfolio of standard liquid options, with varying strikes and maturities and *fixed time-independent weights* that will require no further adjustment and will (as closely as possible) replicate the value of the target option *for a chosen range of future times and market level*s. We call this portfolio the *static replicating portfolio*.

The method is not model-independent in the way that the static replication of European-style options was. The method relies on the assumptions behind the Black-Scholes theory, or any other theory you used to replace it. Therefore, the theoretical value and sensitivities of the static replicating portfolio are equal to the theoretical value and sensitivities of the target option. You can use this static replicating portfolio to hedge or replicate the target option as time passes and the stock price changes. Often, the more liquid options you use to replicate the target portfolio, the better you can do.The costs of replication and transaction are embedded in the market prices of the standard options employed in the replication.

The static replicating portfolio is not unique and usually not perfect. You can examine a variety of static portfolios available to find one that achieves other aims as well – minimizing the difference between the volatility exposures of the target and the replicating portfolio, for example. In general, a perfect static hedge requires an infinite number of standard options. In some cases, it is possible to find a portfolio consisting of only a small number of options that provides a perfect static hedge. Even so, a static hedge portfolio with only several options can provide adequate replication over a wide variety of future market conditions.

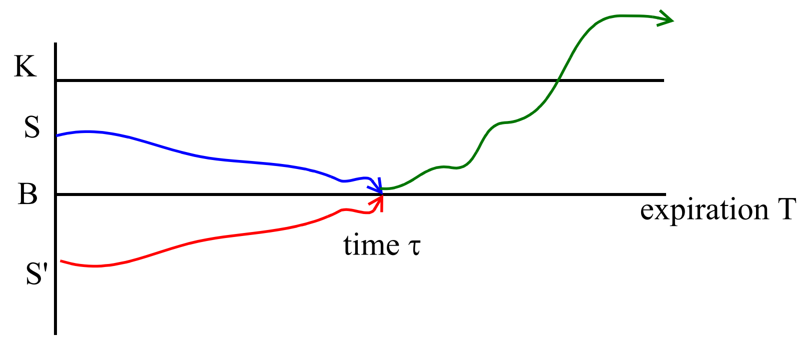
To illustrate the method we are going to consider a particular class of barrier options, namely exotic options.

## 5.4 Valuing Barrier Options

We begin by illustrating how to value a zero-rebate down-and-out barrier option under Geometrical Brownian Motion. The valuation method will suggest a replicating portfolio.

### 5.4.1 GBM with zero stock drift

Start by assuming the current stock price is S and that the Brownian motion has zero drift. Now consider a down-and-out option with strike K and barrier B.




Then, for a suitably chosen “reflected” stock price S', the blue trajectory beginning at S and the red trajectory beginning at S' have equal probability of reaching any point on the barrier B at time $\tau$, and then from that point, have equal probability of taking the future green trajectory that finishes in the money. Conversely, for any green trajectory finishing in the money, there are two trajectories starting out, one beginning at S and another beginning at S', that have the same probability of producing the green trajectory.

Thus, if we subtract the two densities corresponding to S and S', then, above the barrier B, the contribution from every path emanating from S that touched the barrier at time $\tau$ will be cancelled by a similar path emanating from S'.

For arithmetic Brownian motion we can simply subtract the two densities with initial points S and S'. But GBM is symmetric in log space, not stock space.

The probability to get from S to S' in a GBM world depends only on $\ln S/S'$, so that, intuitively, the reflection S' of S in the barrier B must be a log reflection, that is

$$\ln\frac{S}{B} = \ln\frac{B}{S'} \text{ or } S' = \frac{B^2}{S}$$

Thus the down-and-out density is the difference between a lognormal distribution from S to $S_T$ and a lognormal distribution from S' to $S_T$m, where the mean of the normal distribution of the log returns for zero rates is at $-\frac{\sigma^2\tau}{2}$

The density for reaching a stock price $S_\tau$ a time $\tau$ later is therefore

$$n' = n\left(\frac{\ln S_\tau/S + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \qquad \text{Eq.5.10}$$

for some coefficient $\alpha$, where $n(x)$ is a normal distribution with mean 0 and standard deviation 1, and we want this density to vanish when $S_\tau = B$, so that

$$n\left(\frac{\ln B/S + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln S/B + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) = 0$$

We can solve this equation for $\alpha$ to obtain

$$\alpha = \left(\frac{S}{B}\right) \qquad \text{Eq.5.11}$$

So, integrating over the payoff,

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.5.12}$$

You can see that the value of this option vanishes on the boundary S = B independent of the time at which it reaches the boundary, and, for S > K at expiration, the second option finishes out of the money. Thus $C_{DO}$ has the correct boundary conditions. The homework assigned asks that you proved that $C_{DO}$ also satisfies the Black-Scholes PDE. Given the same PDE and the correct boundary conditions, this is the correct solution.

### 5.4.2 Non-zero risk-neutral drift $\mu = r - 0.5\sigma^2$

This is a little trickier. When the drift is non-zero then we can't use the equality of the probabilities for reaching B from both S and S', since the drift distorts the symmetry. So, we try to guess our way into this.

Try to pick a superposition of densities and S and the same reflection point $S' = B^2/S$. (A more careful proof can derive the value of S' too.)

Then the trial down-and-out density for reaching a stock price $S_\tau$ a time $\tau$ later is

$$n' = n\left(\frac{\ln S_\tau/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 - \mu\tau}{\sigma\sqrt{\tau}}\right) \qquad \text{Eq.5.13}$$

for some coefficient $\alpha$, where $n(x)$ is a normal distribution with mean 0 and standard deviation 1, and we want this density to vanish when $S_\tau = B$, so that

$$n\left(\frac{\ln B/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln S/B - \mu\tau}{\sigma\sqrt{\tau}}\right) = 0$$

We can solve this equation for $\alpha$ to obtain

$$\alpha = \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} \qquad \text{Eq.5.14}$$

Notice that $\alpha$ is independent of the time $\tau$ at which the stock prices diffuse to hit the barrier, and so this trial density vanishes on the boundary for all times, for a fixed $\alpha$. Therefore, the value of a down-and-out call is given by the integration of this density over the payoff, namely

$$C_{DO} = C_{BS}(S, t, \sigma, K) - \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} C_{BS}\left(\frac{B^2}{S}, t, \sigma, K\right) \qquad \text{Eq.5.15}$$

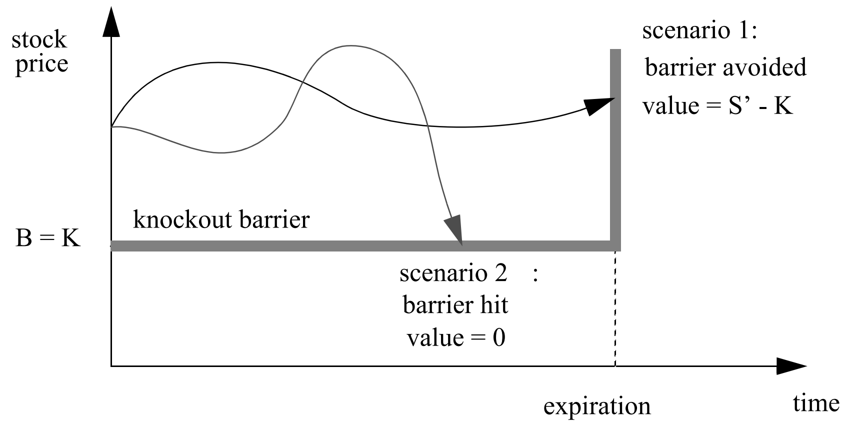
## 5.5 First Steps: Some Exact Static Hedges

Under certain limited circumstances, you can statically replicate a barrier option with a position in stocks and bonds alone, avoiding the need for options. We present and analyze several examples below.

### 5.5.1 European Down-and-Out Call

Consider a European down-and-out call option with time *t* to expiration on a stock with price *S* and dividend yield *d.* We denote the strike level by *K* and the level of the out-barrier by *B*. We assume in this particular example that *B* and *K* are equal. and that there is no cash rebate when the barrier is hit. There are two classes of scenarios for the stock price paths: scenario 1 in which the barrier is avoided and the option finishes in-the-money; and scenario 2 in which the barrier is hit before expiration and the option expires worthless. These are shown in Figure 5.1 below.

**FIGURE 5.1. A down-and-out European call option with *B = K.***



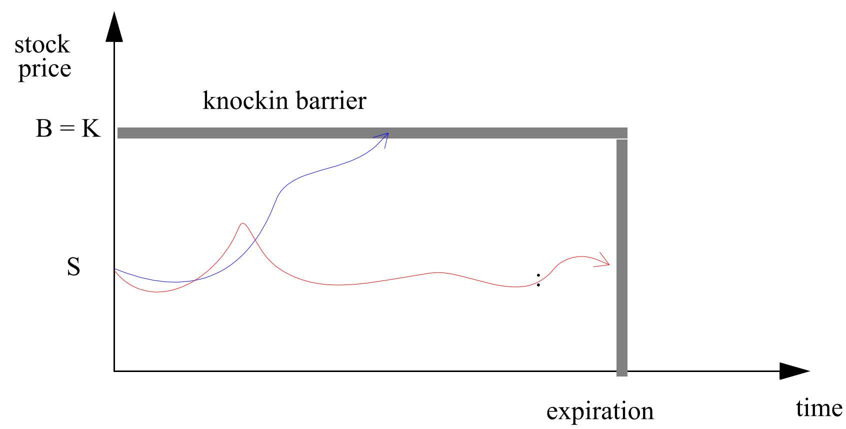

In scenario 1 the call pays out $S'-K$, where $S'$ is the unknown value of the stock price at expiration. This is the same as the payoff of a forward contract with delivery price *K*. This forward has a theoretical value $F = Se^{-dt} - Ke^{-rt}$, where *d* is the continuously paid dividend yield of the stock. You can replicate the down-and-out call under all stock price paths in scenario 1 with a long position in the forward.

For paths in scenario 2, where the stock price hits the barrier at any time $t'$ before expiration, the down-and-out call immediately expires with zero value. In that case, the above forward *F* that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$ . This matches the option value for all barrier-striking times $t'$ only if $r = d$. So, if the riskless interest rate equals the dividend yield (that is, the stock forward is close to spot[1]), a forward with delivery price *K* will exactly replicate a down-and-out call with barrier and strike at the same level *K*, no matter whether the barrier is struck or avoided.

1. In late 1993, for example, the S&P dividend yield was close in value to the short-term interest rate, and so this hedge might have been applicable to short-term down-and-out S&P options

### 5.5.2 European Up-and-Out Put[1]

Now consider an up-and-in put with strike K equal to the barrier B, as illustrated below.




Trajectories like the blue one that hit the barrier generate a standard put $P(\mathrm{S{=}K}, K, \sigma, \tau)$, whereas red trajectories that avoid the barrier expire worthless. Thus to replicate the up-and-in put we need to own a security that expires worthless if the barrier is avoided and has the value of the put $P(K, K, \sigma, \tau)$ on the barrier.

A standard call option $C(S, K, \sigma, \tau)$ bought at the beginning will expire worthless for all values of the stock price below $K$ at expiration. And, on the boundary $S = K$, the value $C(\mathrm{S{=}K}, K, \sigma, \tau) = C(\mathrm{S{=}K}, K, \sigma, \tau)$ *if interest rates and dividend yields are zero*. This put-call symmetry follows because of the symmetry of the density above and below the barrier when rates and dividend yields are zero.

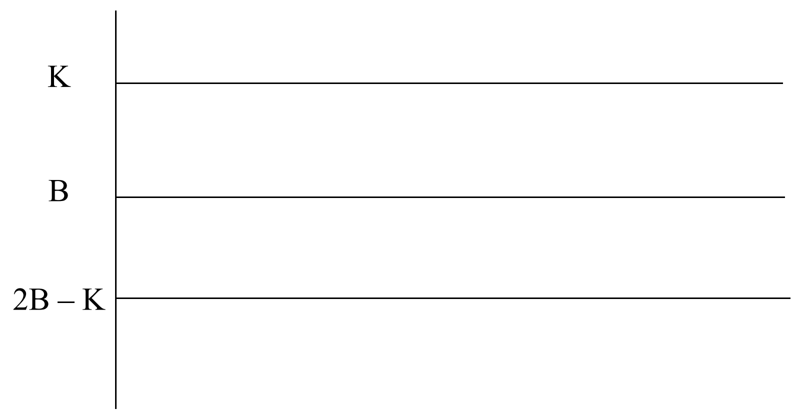
Thus, a standard call $C(S, K, \sigma, \tau)$ can replicate a down-and-in put when $B = K$. But notice, when and if the stock price hits the barrier, you must *sell* the standard call and immediately buy a standard put, which, theoretically, from the argument in the previous paragraph, should have the same value.

### 5.5.3 Hedging Using Put-Call Symmetry

In a Black-Scholes world, in the special circumstances where $r = d = 0$, it's possible to create more static hedges for barrier options.

1. Many of these examples come from papers by Peter Carr and collaborators.

Start by working with arithmetic Brownian motion, $dS = \sigma dW$. Then, as illustrated in the figure below, the probability of moving from B up towards K through a range $K - B$ is the same as the probability of moving from B down away from K to $K' = B - (K - B) = 2B - K$, i.e. through a range $K - B$ to the stock price $K'$.




Hence, by symmetry, when the stock is at B, a call struck at K has the same price as a put struck at K', i.e. $C(B, K) = P(B, K')$

So, the portfolio $W = C(S, K) - P(S, K')$ for $S \geq B$ will have the same payoff as an ordinary call struck at K (since the put will expire out of the money when t he call is in the money), and, will have value zero when S = B. In other words, W has the same boundary conditions as a down-and-out call with barrier B.

Now let's look at geometric Brownian motion. Then the diffusion symmetry is in the log of S, so that K' is determined by the condition $\ln K/B = \ln B/K'$ or $K' = B^2/K$. However $C(B, K) \neq P(B, K')$ because of the mismatch between logarithmic symmetry and linear payoff. Instead, because of the homogeneity of the solution to the Black-Scholes equation,

$$\frac{C(B, K)}{B} = F\left(\ln\frac{K}{B}\right) = F\left(\ln\frac{B}{K'}\right) = \frac{P(B, K')}{K'} \qquad \text{Eq.5.16}$$

Therefore,

$$P(B, K') = \frac{K'}{B}C(B, K) \equiv \frac{B}{K}C(B, K)$$

and so, on the barrier B,

$$C(B, K) = \frac{K}{B}P\left(B, \frac{B^2}{K}\right) \qquad \text{Eq.5.17}$$

So, the portfolio

$$W = C(S, K) - \frac{K}{B}P\left(S, \frac{B^2}{K}\right) \qquad \text{Eq.5.18}$$

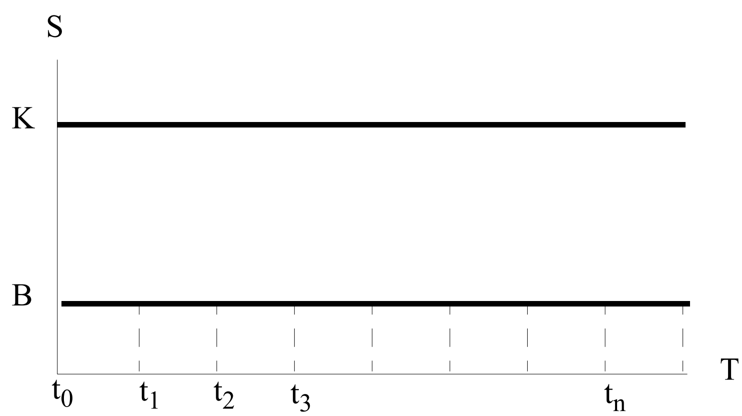
has the payoff of a call at expiration when $S > B$ and vanishes everywhere on the barrier when S = B, and so is a perfect static hedge.

This will be true even if the local volatility is not constant, but rather a function $\sigma = \sigma\left(\frac{K}{S}\right)$. because then $\Sigma\left(\frac{K}{B}\right) = \Sigma\left(\frac{B}{K'}\right)$.

## 5.6 Hedging Path-Dependent Exotics with Standard Options More Generally

References: Derman, Ergener, Kani. *Static Options Replication*, The Journal of Derivatives, 2-4 Summer 1995, pp. 78-95. Mark Joshi's book. Papers by Poulsen et al.

Consider a discrete down-and-out call with strike K, a barrier B below the strike, and an expiration time T; the options knocks out only at *n* times $\{t_1, t_2, ..., t_n\}$ between inception of the trade and expiration.




We want to create a portfolio of standard options that have the payoff of a call with strike K at expiration T if the barrier B hasn't been penetrated, and vanishes in value on the boundary B at time $\{t_1, t_2, ..., t_n\}$.

We can replicate the payoff of the call at expiration with a standard call $C(K, T)$, which denotes a security that is a call with strike K and expiration T, with value $C(S, t, K, T)$ at time t and stock price S. Now we want to put this call into a portfolio $V$ such that the portfolio *value* is the call payoff at expiration, but vanishes at each intermediate time $t_i$ when $S = B$. The value of these extra securities added to the portfolio serve to cancel the value of entire portfolio at the points on the barrier, but they must also add have no payoff above B, else they will not represent the value of the call at expiration, which has no earlier payoffs.

One solution is to use puts $P(S, t, B, t_i)$ with strike K and expiration time $t_i$, because such puts have zero value at expiration when $S > B$, since they expire out of the money. There are other possibilities too. For example we could choose all expirations to be T, and vary the strikes to lie below B.

Here we replicate with a payoff of $n$ standard puts $P(S, t, B, t_i)$ and the call $C(S, t, K, T)$ such that

$$V(S, t) = C(S, t, K, T) + \sum_{j=1}^{n} \alpha_j P(S, t, K, t_j) \qquad \text{Eq.5.19}$$

where the $\alpha_j$ are the number of puts with strike B and expiration $t_i$ in the portfolio. Note that since both the call and the put satisfy the Black-Scholes equation, so does V, which it should. Only its boundary conditions differ from those of a standard call or put.

We can now solve for the $\alpha_j$ such that the value of this portfolio vanishes at all the intermediate times $t_i$ for $i = 0$ to $n-1$ on the barrier $S = B$, namely

$$V(B, t_i) = C(B, t_i, K, T) + \sum_{j=1}^{n} \alpha_j P(B, t_i, K, t_j) = 0 \qquad \text{Eq.5.20}$$

where $P(B, t_i, K, t_j)$ is the value of a put with strike K and expiration time $t_j$ at time $t_i$. Here we have n equations for the n unknowns $\alpha_j$, which can be solved in sequential order by imposing Equation 5.20 starting with time $t_n$ and working backwards one step at a time.

Note that while the value of any put at expiration is defined by its payoff and is model-independent, the value of that put at earlier times depends on the market (in real life) and on a model (in the theory we are developing here), and so this method of replication is not truly model independent. The hope is that if we do the replication in a Black-Scholes world, or even better in a model world that matches the price of all puts to the observed volatility smile, then the perturbations to the value of the portfolio will be insensitive to the details of the model.

By letting the number $n$ of barrier points at times $t_n$ increase, we can move closer and closer to replicating a continuous barrier. The PDE for options valuation dictates that if the boundary conditions are met, the value of the options is determined. We can extend this method to more complicated boundaries too, and, importantly, **to any valuation model, not just Black-Scholes.**

When the stock price hits the barrier, the replicating portfolio must be immediately unwound. This assumes that the stock price moves continuously and that there are no jumps across the barrier.

## 5.7 A Numerical Example: Up-and-Out Call

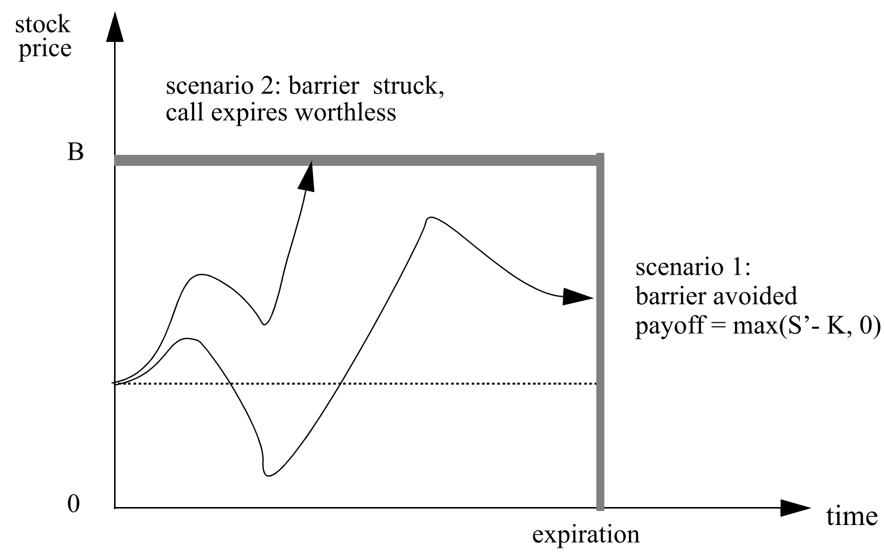
Barrier options have high gamma when the underlying stock price is in the neighborhood of the barrier. In that region, dynamic hedging is both expensive and inaccurate, and static hedging is an attractive alternative. Let's look at an up-and-out European-style call option, described in Table 1. All options values are completed with the Black-Scholes formula.

**TABLE 1. An up-and-out call option.**

| | |
|---|---|
| Stock price: | 100 |
| Strike: | 100 |
| Barrier: | 120 |
| Rebate: | 0 |
| Time to expiration: | 1 year |
| Dividend yield: | 5.0% (annually compounded) |
| Volatility: | 25% per year |
| Risk-free rate: | 10.0% (annually compounded) |
| Up-and-Out Call Value: | 0.656 |
| Ordinary Call Value: | 11.434 |

There are two different classes of stock price scenarios that determine the option's payoff, as displayed in Figure below.

**FIGURE 5.2. Stock price scenarios for an up-and-out European call option with strike K = 100 and barrier B = 120.**



From a trader's point of view, a long position in this up-and-out call is equivalent to owning an ordinary call if the stock never hits the barrier, and owning nothing otherwise. Let's try to construct a portfolio of ordinary options that behaves like this.

First we replicate the up-and-out call for scenarios in which the stock price never reaches the barrier of 120 before expiration. In this case, the up-and-out call has the same payoff as an ordinary one-year European-style call with strike equal to 100. We name this call Portfolio 1, as shown in Table 2. It replicates the target up-and-out call for all scenarios which never hit the barrier prior to expiration.

**Table 2: Portfolio 1. Its payoff matches that of an up-and-out call if the barrier is never crossed before expiration.**

| Quantity | Type | Strike | Expiration | Value 1 year before expiration | |
|---|---|---|---|---|---|
| | | | | Stock at 100 | Stock at 120 |
| 1 | call | 100 | 1 year | 11.434 | 25.610 |

The value of Portfolio 1 at a stock level of 120 is 25.610, much too large when compared with the zero value of the up-and-out call on the barrier. Consequently, its value at a stock level of 100 is 11.434, also much greater than the Black-Scholes value (0.657) of the up-and-out call with a continuous knock-out barrier.

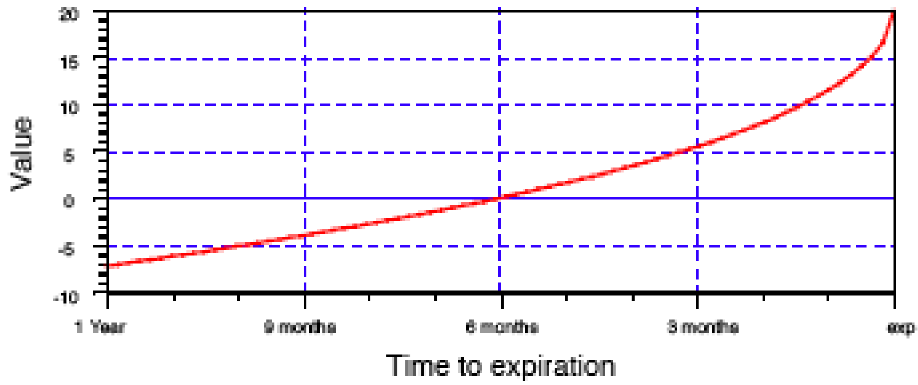
Portfolio 1 replicates the target option for scenarios of type 1.

Portfolio 2 in Table 3 illustrates an improved replicating portfolio. It adds to Portfolio 1 a short position in *one* extra option so as to attain the correct zero value for the replicating portfolio at a stock price of 120 with 6 months to expiration, as well as for all stock prices below the barrier at expiration. Figure 4 shows the value of Portfolio 2 for stock prices of 120, at all times prior to expiration. You can see that the replication on the barrier is good only at six months. At all other times, it again fails to match the up-and-out call's zero payoff.

**Table 3: Portfolio 2. Its payoff matches that of an up-and-out call if the barrier is never crossed, or if it is crossed exactly at 6 months to expiration.**

| Quantity | Type | Strike | Expiration | Value 6 months before expiration | |
|---|---|---|---|---|---|
| | | | | Stock at 100 | Stock at 120 |
| 1.000 | call | 100 | 1 year | 7.915 | 22.767 |
| -2.387 | call | 120 | 1 year | -4.446 | -22.767 |
| Net | | | | 3.469 | 0.000 |

**Table 4: Value of Portfolio 2 on the barrier at 120.**

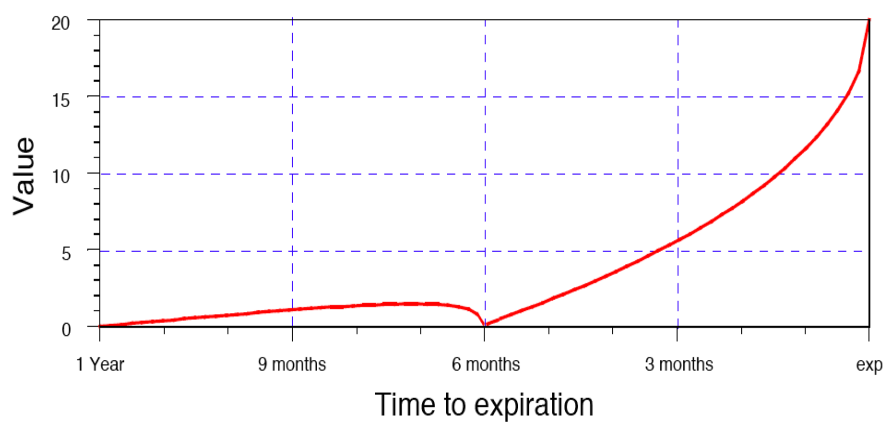



By adding one more call to Portfolio 2, we can construct a portfolio to match the zero payoff of the up-and-out call at a stock price of 120 at both six months *and* one year. This portfolio, Portfolio 3, is shown in Table 5.

**Table 5: Portfolio 3. Its payoff matches that of an up-and-out call if barrier is never crossed, or if it is crossed exactly at 6 months or 1 year to expiration.**

| Quantity | Type | Strike | Expiration | Value for stock price = 120 | |
|---|---|---|---|---|---|
| | | | | 6 months | 1 year |
| 1.000 | call | 100 | 1 year | 22.767 | 25.610 |
| -2.387 | call | 120 | 1 year | -22.767 | -32.753 |
| 0.752 | call | 120 | 6 months | 0.000 | 7.142 |
| Net | | | | 0.000 | 0.000 |

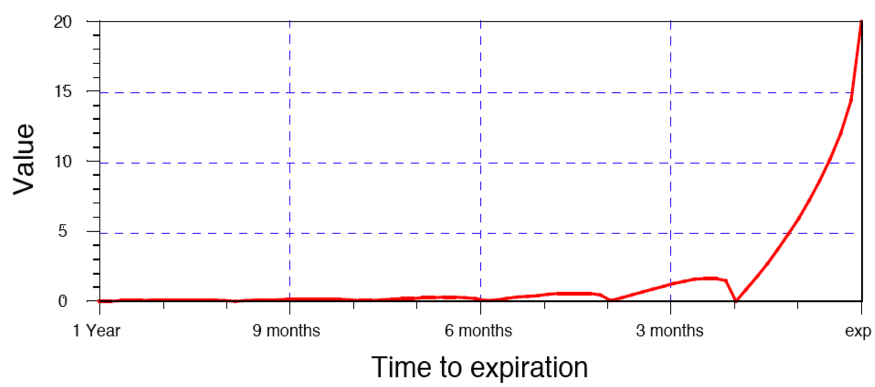
**FIGURE 5.3. .Value of Portfolio 4 on the barrier at 120**



You can see that this portfolio does a much better job of matching the zero value of an up-and-out call on the barrier. For the first six months in the life of the option, the boundary value at a stock price of 120 remains fairly close to zero.

By adding more options to the replicating portfolio, we can match the value of the target option at more points on the barrier. Figure 5.4 shows the value of a portfolio of seven standard options at a stock level of 120 that matches the zero value of the target up-and-out call on the barrier every two months. You can see that the match between the target option and the replicating portfolio on the barrier is much improved. In the next section we show that improving the match on the boundary improves the match between the target option and the portfolio for all times and stock prices.

**FIGURE 5.4. Value on the barrier at 120 of a portfolio of standard options that is constrained to have zero value every two months.**




## 5.8 Replication Accuracy

We can see how well the replicating portfolio can match the value of the option at all stock prices and times before expiration. Let's look at an option with high gamma, the up-and-out European-style call option defined in Table 6.

Out

Its theoretical value in the Black-Scholes model with one year to expiration is 1.913. We can use our method to construct a static replicating portfolio. Table 6 shows one particular example. It consists of a standard European-style call option with strike 100 that expires one year from today, plus six additional options each struck at 120. The 100-strike call replicates the payoff at expiration if the barrier is never struck. The remaining six options expire every two months between today and the expiration in one year. The position in each of them is chosen so that the total portfolio value is exactly zero at two month intervals on the barrier at 120.

**Table 6: An up-and-out call option.**

| | |
|---|---|
| Stock price: | 100 |
| Strike: | 100 |
| Barrier: | 120 |
| Rebate: | 0 |
| Time to expiration: | 1 year |
| Dividend yield: | 3.0% (annually compounded) |
| Volatility: | 15% per year |
| Risk-free rate: | 5.0% (annually compounded) |
| Up-and-Out Call Value: | 1.913 |

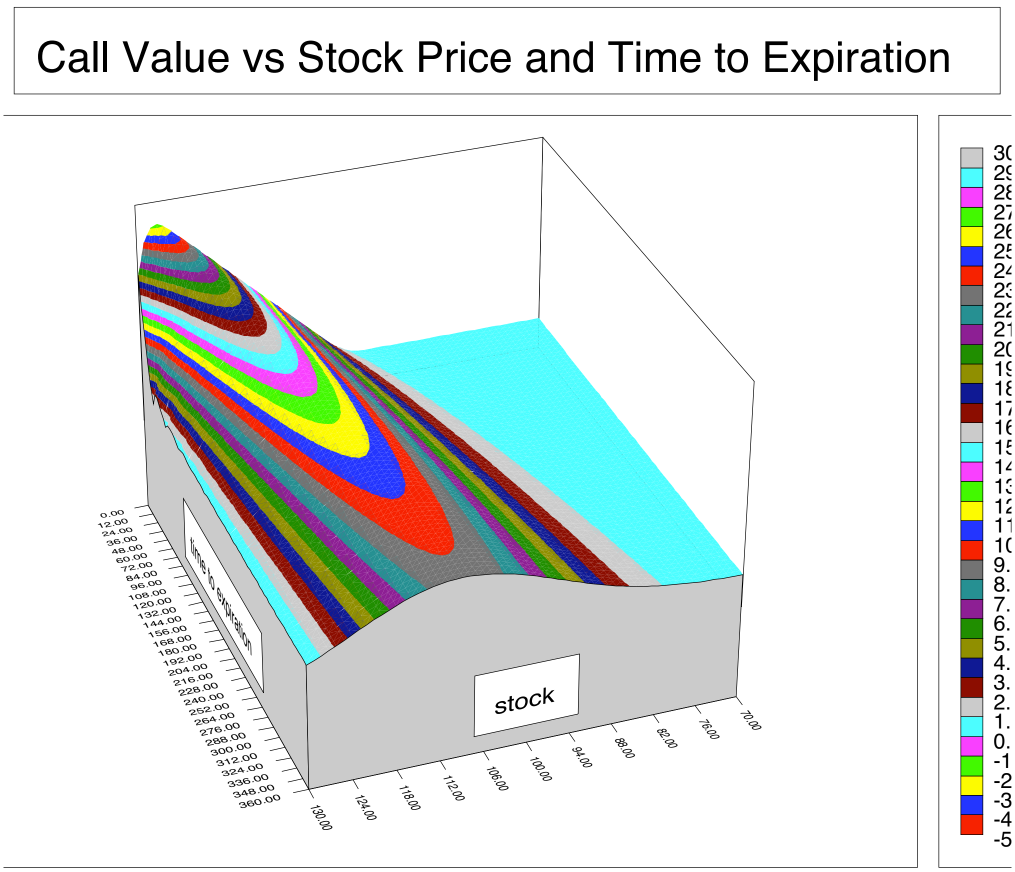
The theoretical value of the replicating portfolio in Table 7 at a stock price of 100, one year from expiration, is 2.284, about 0.37 or 19% off from the theoretical value of the target option.

**TABLE 7. The replicating portfolio.**

| Quantity | Option Type | Strike | Expiration (months) | Value (Stock = 100) |
|---|---|---|---|---|
| 0.16253 | Call | 120 | 2 | 0.000 |
| 0.25477 | Call | 120 | 4 | 0.018 |
| 0.44057 | Call | 120 | 6 | 0.106 |
| 0.93082 | Call | 120 | 8 | 0.455 |
| 2.79028 | Call | 120 | 10 | 2.175 |
| -6.51351 | Call | 120 | 12 | -7.140 |
| 1.00000 | Call | 100 | 12 | 6.670 |
| Total | | | | 2.284 |

Instead of using six options, struck at 120, to match the zero boundary value on the barrier every two months for one year, we can use 24 options to match the boundary value at half-month intervals. In that case, the theoretical value of the replicating portfolio becomes 2.01, only 0.10 away from the theoretical value of the target option. You can see that the portfolio value varies like that of an up-and-out option with barrier at 120.

Here's the behavior over all stock prices and time prior to expiration of a 24-option replicating portfolio.




You can see it looks a lot like the payoff of an up and out call option.