LECTURE 11

IMPLIED DISTRIBUTIONS AND STATIC HEDGING

Breeden Litzenberger From Put Prices Too

$$\exp(r\tau) \times P = \int_0^{\infty} dS'(K - S')\theta(K - S')p(S, t, S', T)$$

Now differentiate under the integral

$$\exp(r\tau) \times \frac{\partial P}{\partial K} = \int_0^K p(S, t, S', T)dS'$$

Differentiate again to obtain the Breeden-Litzenberger result:

$$\exp(r\tau) \times \frac{\partial^2 P}{\partial K^2} = p(S, t, K, T) \qquad \text{Eq.11.1}$$

Static Replication: Valuing arbitrary payoffs at a fixed expiration

For any W: $$W(S, t) = \int_0^\infty \frac{\partial^2 C}{\partial K^2}(S, t, K, T) W(K, T) dK$$

strike terminal stock price

If we know call prices (or put prices) and their derivatives for all strikes at a fixed expiration, we can find the value of any other European-style derivative security at that expiration.

This **involves no use of option theory at all, and no use of the Black-Scholes equation**. It works even if there is a smile or skew or jumps. As long as the options are honored.

Replicating by standard options

Integration by parts to get V as the sum of portfolios of zero coupon bonds, forwards, puts & calls.

European payoff $W(K, T)$. K represents **the terminal stock price**.
Use puts below strike A and for calls above strike A.

$$
\begin{aligned}
W(S,t) &= e^{-r\tau}\int_0^{\infty}\rho(S,t,K,T)W(K,T)dK \\
&= e^{-r\tau}\left[\int_0^{A}\rho(S,t,K,T)W(K,T)dK + \int_A^{\infty}\rho(S,t,K,T)W(K,T)dK\right] \\
&= \int_0^{A}\frac{\partial^2 P}{\partial K^2}W(K,T)dK + \int_A^{\infty}\frac{\partial^2 C}{\partial K^2}W(K,T)dK
\end{aligned}
$$

Integrate by parts twice to get

$$
W(S,t) = \begin{array}{l}
\displaystyle\int_0^{A}\frac{\partial^2}{\partial K^2}W(K,T)P(S,K)dK + \int_A^{\infty}\frac{\partial^2 W}{\partial K^2}C(S,K)dK \\
\displaystyle+\left(\left(W\frac{\partial P}{\partial K} - P\frac{\partial W}{\partial K}\right)\Bigg|_{K=0}^{K=\mathrm{A}} + \left(W\frac{\partial C}{\partial K} - C\frac{\partial W}{\partial K}\right)\Bigg|_{K=A}^{K=\infty}\right)
\end{array}
$$

where $P(S, K)$ and $C(S, K)$ are the current values at time *t* and stock price *S* of a put and call with strike *K* and expiration *T*.

Use the following conditions for the current call and put prices.

$$P[S, 0] = 0$$

$$\frac{\partial}{\partial K} P[S, 0] = 0$$

$$C[S, \infty] = 0$$

$$\frac{\partial}{\partial K} C[S, \infty] = 0$$

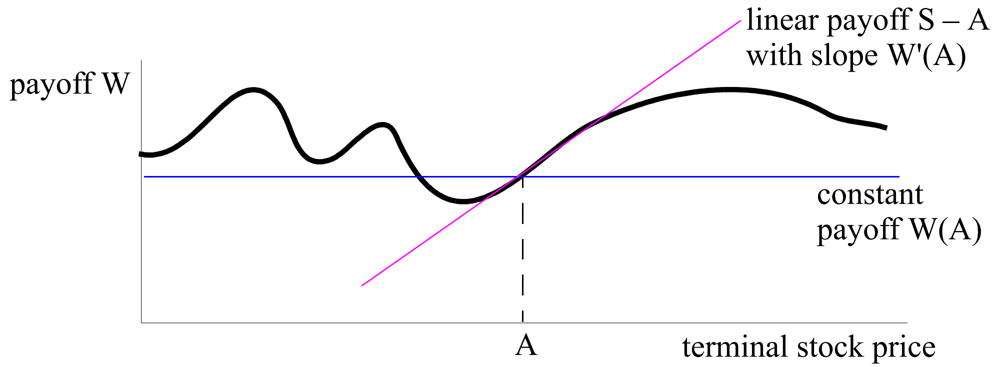
$$P[S, K] - C[S, K] = Ke^{-r\tau} - S$$

$$\frac{\partial}{\partial K} P[S, K] - \frac{\partial}{\partial K} C[S, K] = e^{-r\tau}$$

Then

$$W(S, t) = W(A)e^{-r\tau} + W'(A)[S - Ae^{-r\tau}] + \int_0^A P(K)W''(K)dK + \int_A^\infty C(K)W''(K)dK. \qquad \text{Eq.11.2}$$

zero coupon   forward   calls and puts



Two views of static replication.

• If you know the risk-neutral density ρ then you W(S,t) is an integral over the terminal payoff.

• Alternatively, W(S,t) as an integral over call and put prices with different strikes.

If you can buy every option in the continuum you need from someone who will never default on their payoff, then you have a perfect static hedge. No math involved.

If you cannot buy every single option, then you have only an approximate replicating portfolio whose value will deviate from the value of the target option's payoff. Picking a reasonable or tolerable replicating portfolio is up to you.

This works even if there is volatility skew.

Note: The Black-Scholes risk-neutral probability density

In the BS evolution, returns $\ln S_T / S_t$ are normal with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$.

Therefore,

$$x = \frac{\ln S_T / S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}} \qquad \text{Eq.11.3}$$

is normally distributed with mean 0 and standard deviation 1, with a probability density $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The returns $\ln S_T / S_t$ can range from $-\infty$ to ∞. From Eq.10.6,

$$\frac{dS_T}{S_T} = \sigma\sqrt{\tau}\,dx$$

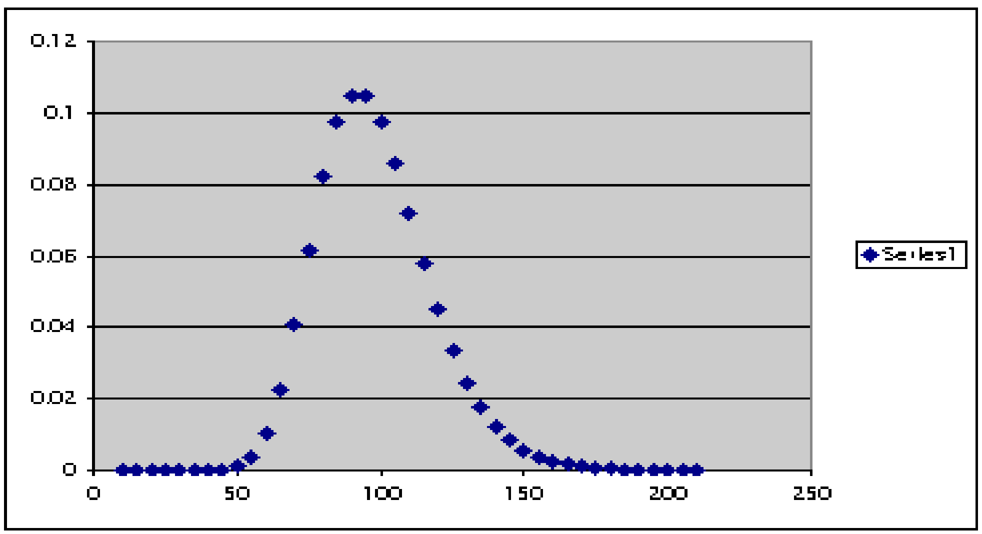
The risk-neutral value of the option is given by

$$e^{r\tau}C = \frac{1}{\sqrt{2\pi}}\int_{-\mathrm{d}_2}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{K}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)\frac{dS_T}{S_T}$$

where

$$\frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi\tau}\sigma S_T}$$

is the risk-neutral density function to be used in integrating payoffs over S_T, plotted below



A Static Replication Example in the Presence of a Skew

Consider an exotic option, strike B, quadratic payoff:

$$V(s) = s \times max[s-B, 0] = s \times (s-B)\theta(s-B)$$

Replicate by adding together a collection of vanilla calls with strikes starting at B, and then adding successively more of them to create a quadratic payoff, as illustrated below.

$$\mathrm{V(S)} = \int_0^{\infty} q(K)\theta(K-B)\mathrm{C(S,K)dK}$$

where $q(K)$ is the unknown density of calls with strike *K*. A in the formula is chosen to be B.

$$\begin{aligned}\frac{\partial V}{\partial s}(s) &= \frac{\partial}{\partial s}[s \times (s-B)\theta(s-B)] \\ &= (s-B)\theta(s-B) + s\theta(s-B) + s(s-B)\delta(s-B) \\ &= (s-B)\theta(s-B) + s\theta(s-B)\end{aligned}$$

Second derivative:

$$\begin{aligned}\frac{\partial^2 V}{\partial s^2} &= (s-B)\delta(s-B) + 2\theta(s-B) + s\delta(s-B) \\ &= 2\theta(s-B) + s\delta(s-B)\end{aligned}$$

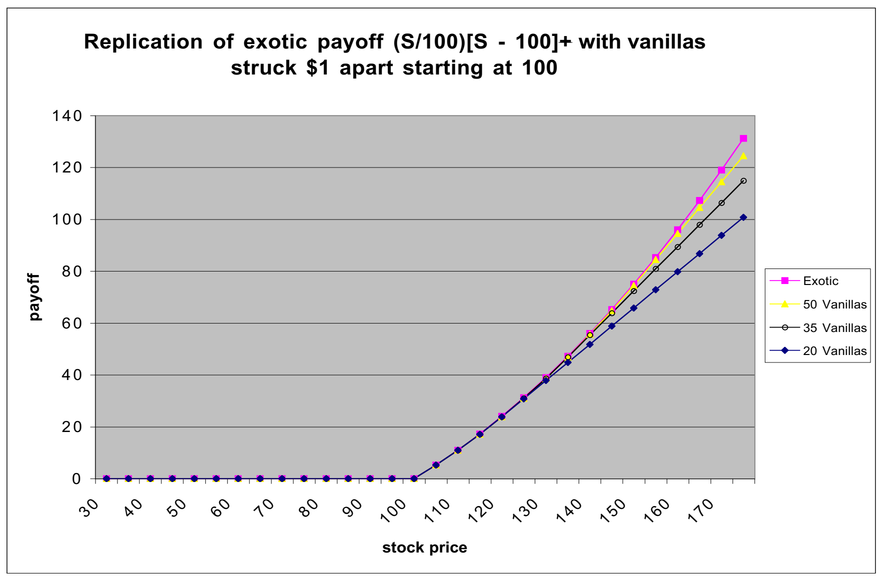
Integrate over calls with this density: $V(S, t) = \int_A^{\infty} \frac{\partial^2 V}{\partial K^2} C(S, K) dK$

S***ecurity*** **V** in terms of call options $C(K)$ of various strikes K:

Security: $\boldsymbol{V} = B\mathbf{C}(\mathbf{B}) + \int_B^{\infty} 2\mathbf{C}(\mathrm{K})\mathrm{dK}$

Value: $V(S, t) = BC(S, t, B, T) + 2\int_B^{\infty} C(S, t, K, T) dK$

Payoff of 50 calls with strikes equally spaced and \$1 apart between 100 and 150.



Convergence of the value of the replicating formula to the correct no-arbitrage value for two different smiles.

$$\Sigma(K) = 0.2\left(\frac{K}{100}\right)^{\beta}$$

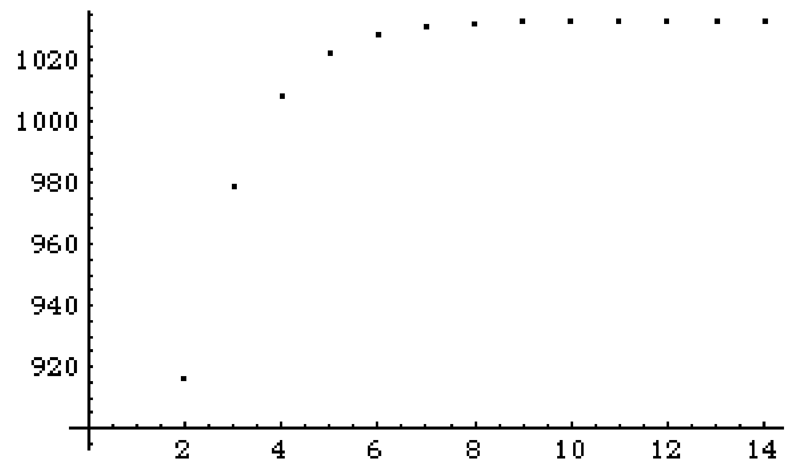
$\beta = -0.5$ “negative” skew. Implied volatility increases with decreasing strike.

$\beta = 0$ corresponds to no skew at all.

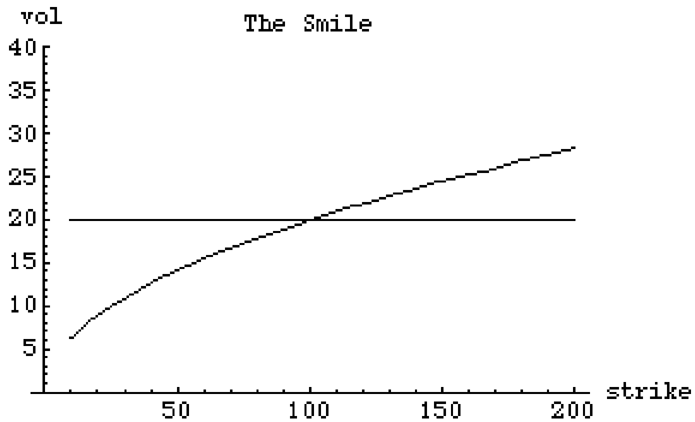
$\beta = 0.5$ corresponds to a positive skew.

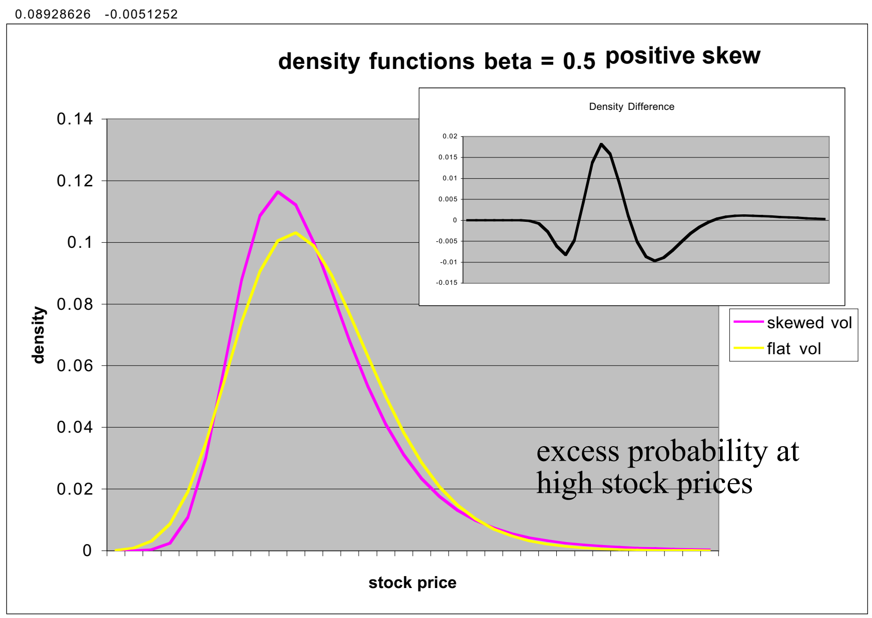
For $\beta = 0$ the fair value of V when replicated by an infinite number of calls is 1033. With 10 strikes the value has virtually converged.

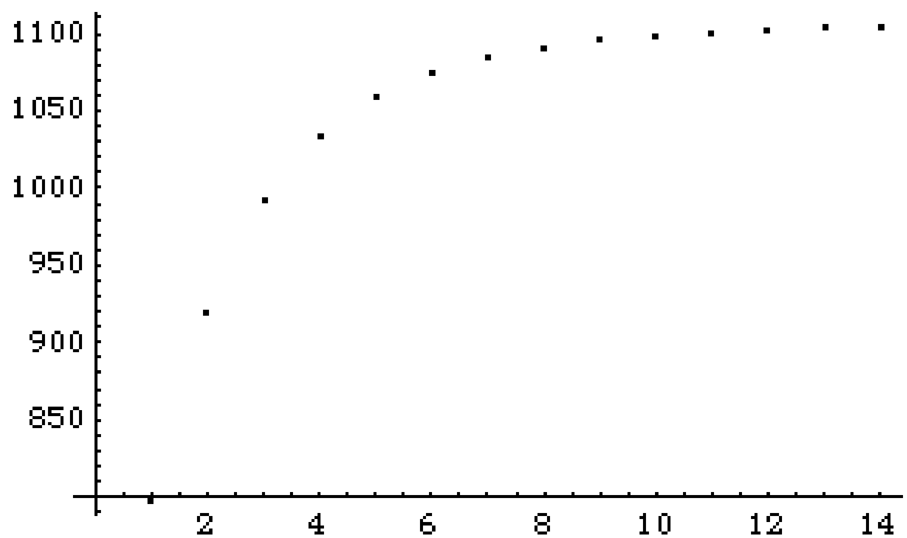
<u>Convergence as we increase number of strikes for flat 20% volatility</u>



Positive skew $\beta = 0.5$

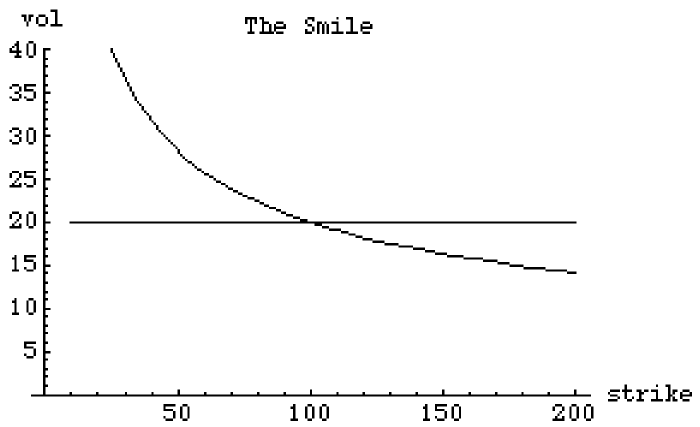


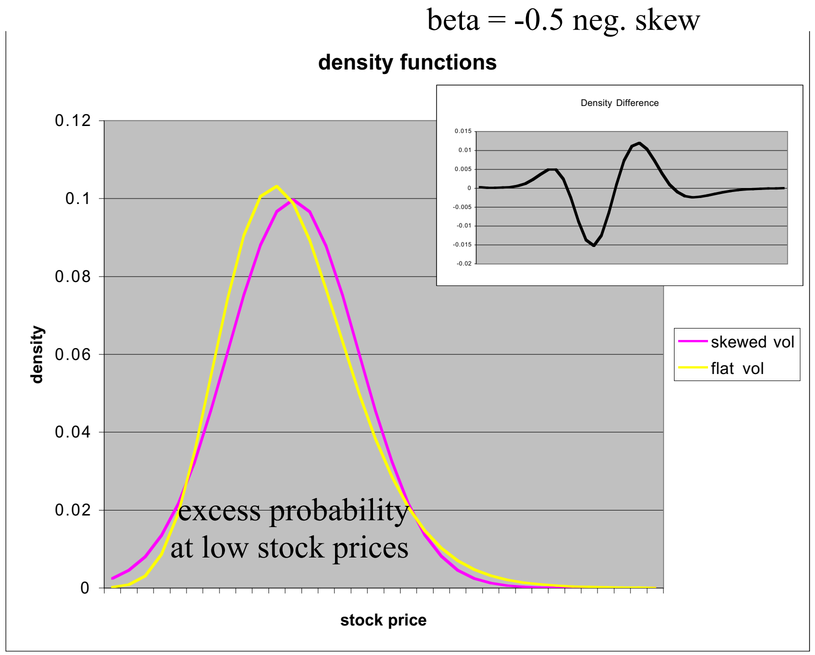


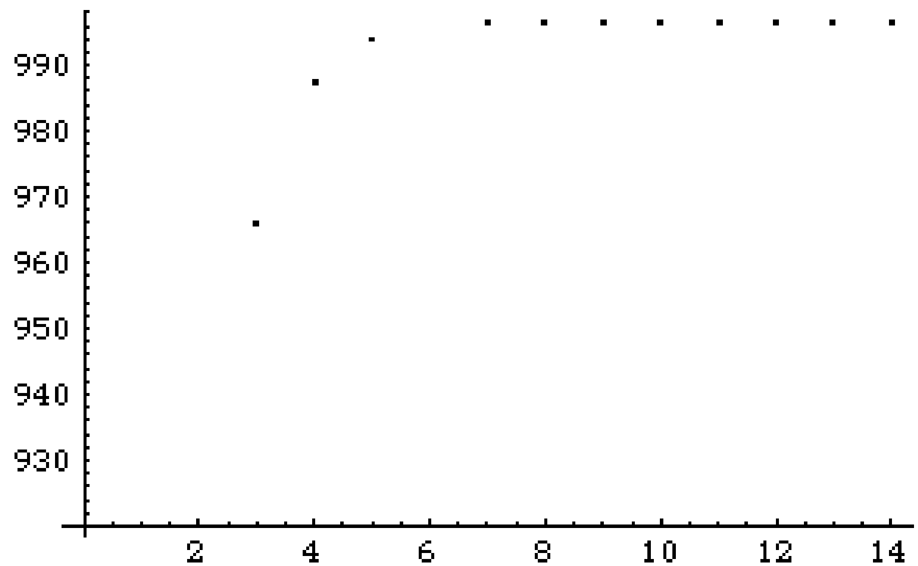


Convergence for a positive skew to a fair value of 1100 is slower and requires more strikes

Negative skew $\beta = -0.5$







Convergence for a negative skew to a fair value 996 is faster and requires fewer strikes. GS.

Static Replication of Non-European Options

- **Strong static replication:** Replication is independent of model.

  What we just did.

- **Weak static replication:** Weak replication needs a model and an assumption about the future smile. The method relies on the assumptions behind the Black-Scholes theory, or any other theory you used to replace it. The more and more liquid options you use to replicate the target portfolio, the better you can do.The costs of replication and transaction are embedded in the market prices of the standard options employed in the replication.

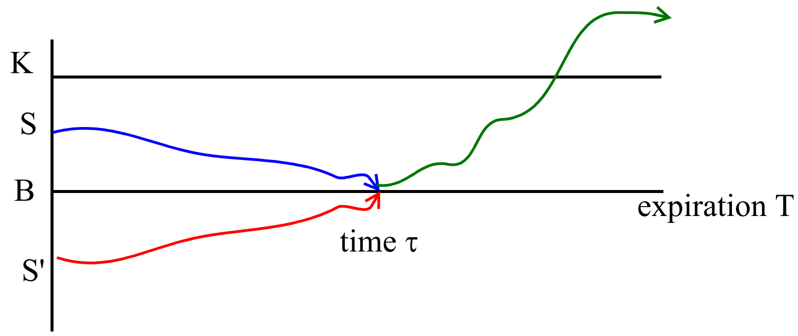
  What we are about to do.

Valuing Barrier Options

This is important because the valuation suggests a method of replication.

Valuing a Barrier Option for GBM with Zero Risk-Neutral Stock Drift

A **down-and-out** option with strike K and barrier B.



- Choose a “reflected” imaginary stock S': The blue trajectory from S and the red trajectory from S' have equal probability to get to any point on B at time τ,.
- From there, they have equal probability of taking the future green trajectory that finishes in the money.
- For any green trajectory finishing in the money, the paths beginning at S and S' have the same probability of producing the green trajectory.
- Subtract the two probability densities and then above the barrier B, the contribution from every path emanating from S that touched the barrier at any time τ will be cancelled by a similar path emanating from S'.

Where is S'?

The probability to get from S to S' in a GBM world depends only on $\ln S/S'$; the reflection S' of S in the barrier B must be a log reflection, that is

$$\ln\frac{S}{B} = \ln\frac{B}{S'} \text{ or } S' = \frac{B^2}{S}$$

The density for getting from S to a stock price S_τ a time τ later is therefore

$$n' = n\left(\frac{\ln S_\tau/S + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \qquad \text{Eq.11.4}$$

for some coefficient α, where $n(x)$ is a normal distribution with mean 0 and standard deviation 1, and we want this density to vanish when $S_\tau = B$, which requires $\alpha = \left(\frac{S}{B}\right)$ independent of τ

Thus the option price is $C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K\right)$

C_{DO} vanishes on boundary S = B at any time. If S > K > B at expiration, $B^2/S < K^2/S < K$ and second option finishes out of the money. Thus C_{DO} has the correct boundary conditions.

Valuation for non-zero risk-neutral drift $\mu = r - 0.5\sigma^2$

When the drift is non-zero then probabilities for reaching B from both S and S' differ, since the drift distorts the symmetry. Pick a superposition of densities and S and the same reflection $S' = B^2/S$.

Trial down-and-out density for reaching a stock price S_τ a time τ later is

$$n' = n\left(\frac{\ln S_\tau/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 - \mu\tau}{\sigma\sqrt{\tau}}\right)$$

$$n\left(\frac{\ln B/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln S/B - \mu\tau}{\sigma\sqrt{\tau}}\right) = 0 \quad \text{implies} \quad \alpha = \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} \quad \text{independent of } \tau$$

$$C_{DO} = C_{BS}(S, t, \sigma, K) - \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} C_{BS}\left(\frac{B^2}{S}, t, \sigma, K\right)$$

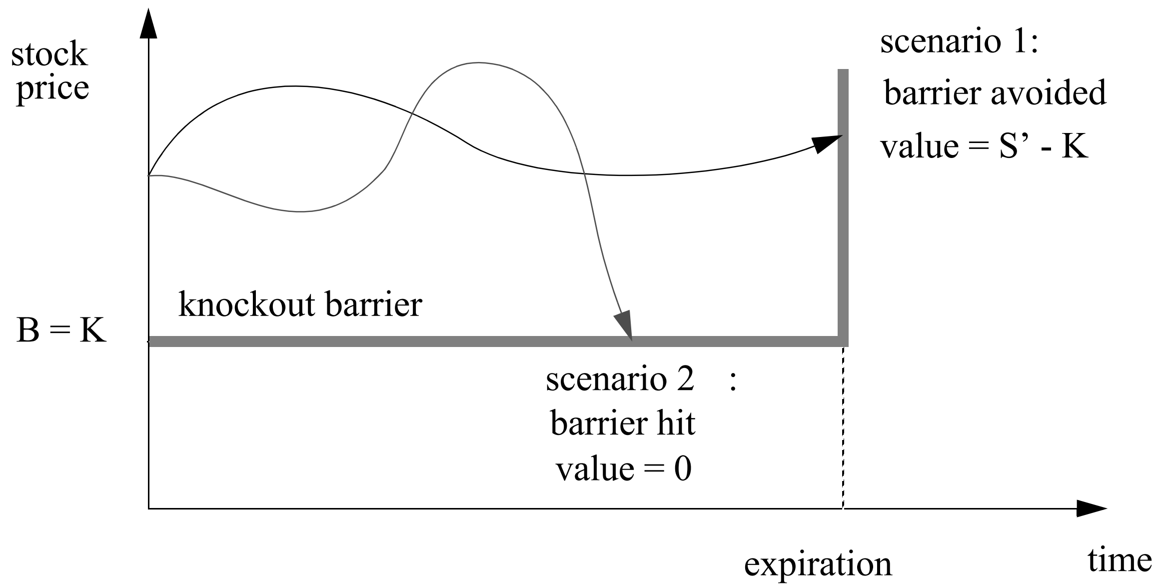
A superposition of solutions. Method of images in electrostatics.

First Steps: Some Exact Static Hedges in Simple Cases

Sometimes you can statically replicate a barrier option with a position in stocks and bonds alone.

European Down-and-Out Call with Barrier at Strike

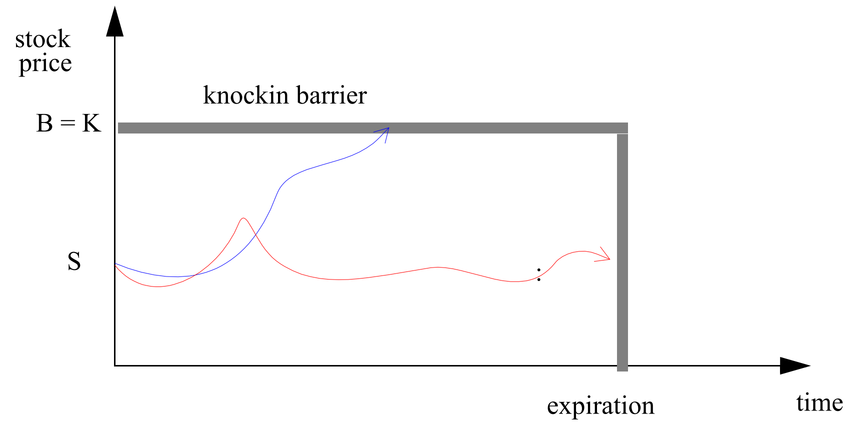
Scenario 1 in which the barrier is avoided and the option finishes in-the-money;
Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



Replicate with a forward $F = Se^{-dt} - Ke^{-rt}$ *if the stock moves continuously.*

European Up-and-In Put with Barrier at Strike

Now consider an up-and-in put with strike K equal to the barrier B, as illustrated below.



Blue trajectories that hit the barrier generate a standard put $P(\mathrm{S{=}K}, K, \sigma, \tau)$
Red trajectories that avoid the barrier expire worthless.

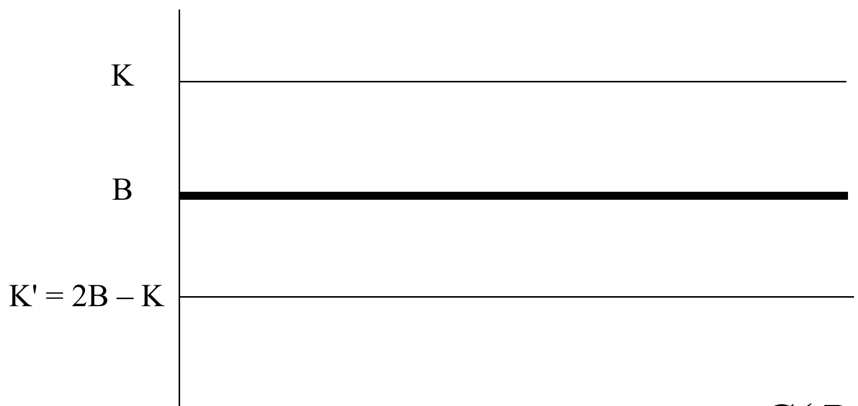
To replicate we need a security that expires worthless if the barrier is avoided and has the value of the put $P(K, K, \sigma, \tau)$ on the barrier.

A standard call $C(S, K, \sigma, \tau)$ bought at the beginning will expire worthless for all values of the stock price below K at expiration. And, on the boundary $S = K$, the value
$C(\mathrm{S{=}K}, K, \sigma, \tau) = P(\mathrm{S{=}K}, K, \sigma, \tau)$ *if interest rates and dividend yields are zero*.
At the barrier, you must *sell* the standard call and immediately buy a standard put.

Hedging a General Down-and-Out Call via Black-Scholes Put-Call Symmetry

In a Black-Scholes world, in the special circumstances where $r = d = 0$, it's possible to create more static hedges for barrier options.

Assume **arithmetic Brownian motion**, $dS = \sigma dW$.
The probability of moving from B up towards K through $K - B$ = the probability of moving from B down to $K' = B - (K - B) = 2B - K$



$$C(B, K) = P(B, K')$$

So, the portfolio $W = C(S, K) - P(S, K')$ for $S \geq B$ will have the same payoff as an ordinary call struck at K (since the put will expire out of the money when the call is in the money), and, will have value zero when S = B. In other words, W has the same boundary conditions as a down-and-out call with barrier B.

Now geometric Brownian motion.

Then the diffusion is in the log of S, so reflection means $\ln K/B = \ln B/K'$ or $K' = B^2/K$.

However $C(B, K) \neq P(B, K')$ because of geometric rather than arithmetic evolution. Instead, because of the homogeneity of the solution to the Black-Scholes equation,

$$\frac{C(B, K)}{B} = F\left(\ln\frac{K}{B}\right) = F\left(\ln\frac{B}{K'}\right) = \frac{P(B, K')}{K'} \qquad \text{Eq.11.5}$$

Therefore, $$P(B, K') = \frac{K'}{B}C(B, K) \equiv \frac{B}{K}C(B, K)$$

On the barrier B, $$C(B, K) = \frac{K}{B}P\left(B, \frac{B^2}{K}\right)$$

So, $$W = C(S, K) - \frac{K}{B}P\left(S, \frac{B^2}{K}\right) :$$ Note reflection!

has the payoff of only the call at expiration when $S > K$ and vanishes everywhere on the barrier S = B, and so is a perfect static hedge.

Again, you must unwind as soon as the stock price hits the barrier, and it must hit the barrier while moving continuously.

Insight into Static Hedging from Valuation Formula

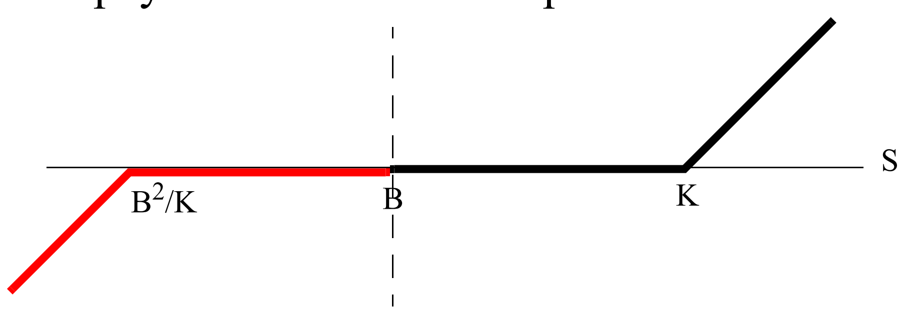
We showed above that, in a Black-Scholes world with zero drift, the fair value for a down-and-out call with strike K and barrier B is given by

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.11.6}$$

Payoff of first term: $\theta(S-K)(S-K)$

Payoff of $\frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right)$ $\qquad \frac{S}{B}\left(\frac{B^2}{S} - K\right)\theta\left(\frac{B^2}{S} - K\right) = \left(B - \frac{KS}{B}\right)\theta\left(\frac{B^2}{K} - S\right) = \frac{K}{B}\theta\left(\frac{B^2}{K} - S\right)\left(\frac{B^2}{K} - S\right)$

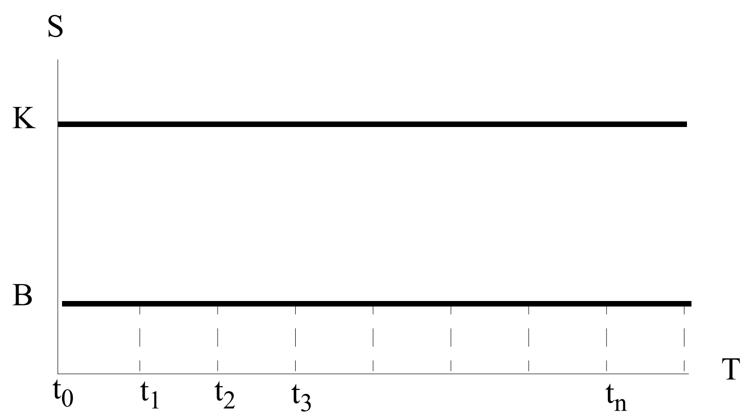
This second term represents the payoff of *K/B* standard puts with strike B^2/K.



1.1 Roughly speaking the payoff of a down-and-out-call is that of an ordinary call and its strike image reflection (in log space) in the barrier. What volatilities are you sensitive to?

Weak Replication of Exotics with Standard Options

Consider a discrete down-and-out call with strike K, a barrier B, expiration time T; the option knocks out only at n times $\{t_1, t_2, ..., t_n\}$ between inception of the trade and expiration.



Create a portfolio of standard options that have the payoff of a call with strike K if the barrier B hasn't been penetrated, and vanishes on the boundary B at time $\{t_1, t_2, ..., t_n\}$.

We can replicate the payoff of the call at expiration with a standard call $C(S, t, K, T)$
Create a portfolio V whose *value* is the $C(S, t, K, T)$ at expiration but vanishes at each intermediate time t_i when $S = B$. The value of securities added to the portfolio must cancel the value of

entire portfolio on B, but must also add have no payoff above B in order to mimic a call:

We can use puts $P(S, t, B, t_i)$ with strike B and expiration time t_i

Here we replicate with a payoff of n standard puts $P(S, t, B, t_i)$ and the call $C(S, t, K, T)$ such that

$$V(S, t) = C(S, t, K, T) + \sum_{j=1}^{n} \alpha_j P(S, t, K, t_j) \qquad \text{Eq.11.7}$$

Since both the call and the put satisfy the Black-Scholes equation, so does V, which it should. Only its boundary conditions differ from those of a standard call or put.

Solve for α_j such that V vanishes at all t_i for $i = 0$ to $n-1$ at $S = B$:

$$V(B, t_i) = C(B, t_i, K, T) + \sum_{j=1}^{n} \alpha_j P(B, t_i, K, t_j) = 0 \qquad \text{Eq.11.8}$$

n equations for the n unknowns α_j, which can be solved in sequential order.

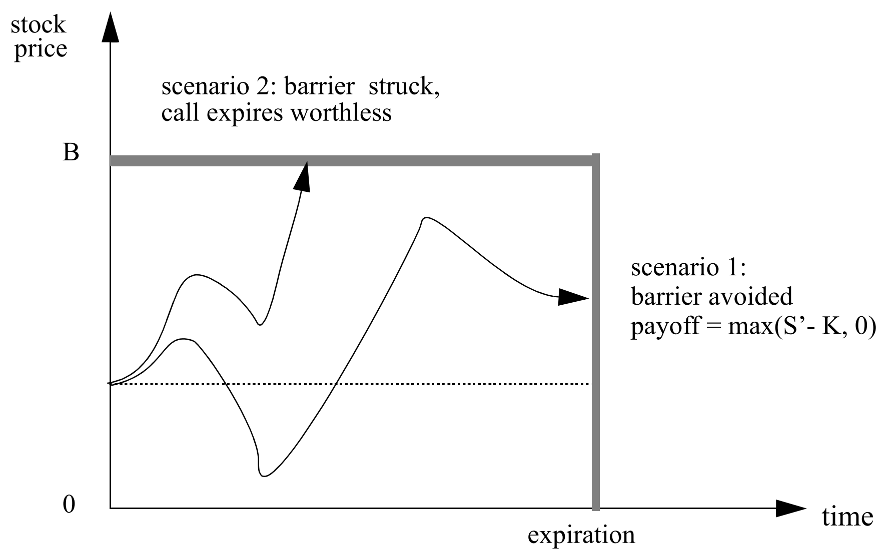
As n increases we get closer to a continuous barrier. I**t is still weak replication: coefficients depend on the model. It is static and uses only other options, which is good; it is weak, which is not so good.**

When the stock price hits the barrier, the replicating portfolio must be immediately unwound. This assumes that the stock price moves continuously and that there are no jumps across the barrier.

1.2 A Numerical Example: Up-and-Out Call with High Gamma

All options values are Black-Scholes.
An up-and-out call option.

Stock price:	100
Strike:	100
Barrier:	120
Time to expiration:	1 year
Up-and-Out Call Value:	0.656
Ordinary Call Value:	11.434



Portfolio 1 replicates the target up-and-out call for all scenarios which never hit the barrier.

Quantity	Type	Strike	Expiration	Value 1 year before expiration	
				Stock at 100	Stock at 120
1	call	100	**1 year**	11.434	**25.610: too big**

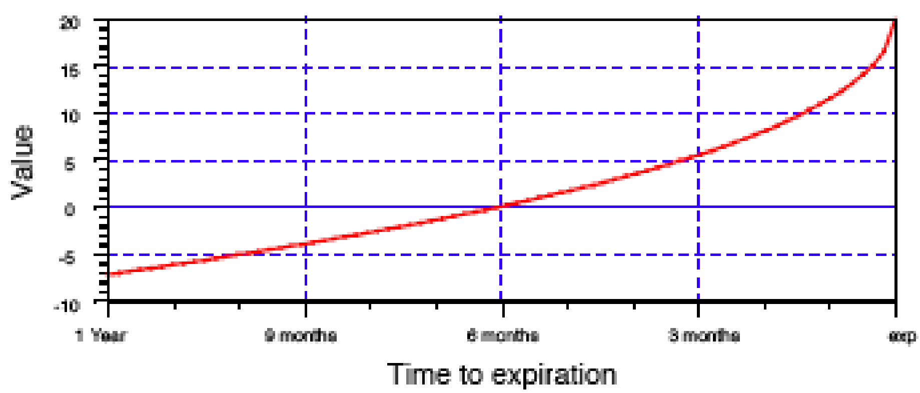
Portfolio 2 improvement.

Add a short position in *one* extra option so as to attain the correct zero value for the replicating portfolio at a stock price of 120 with 6 months to expiration, as well as for all stock prices below the barrier at expiration.

Portfolio 2. Its payoff matches that of an up-and-out call if the barrier is never crossed, or if it is crossed exactly at 6 months before expiration.

Quantity	Type	Strike	Expiration	Value 6 months before expiration	
				Stock at 100	**Stock at 120**
1.000	call	100	1 year	7.915	22.767
-2.387	call	120	1 year	-4.446	-22.767
Net				3.469	0.000

Value of Portfolio 2 on the barrier at 120.

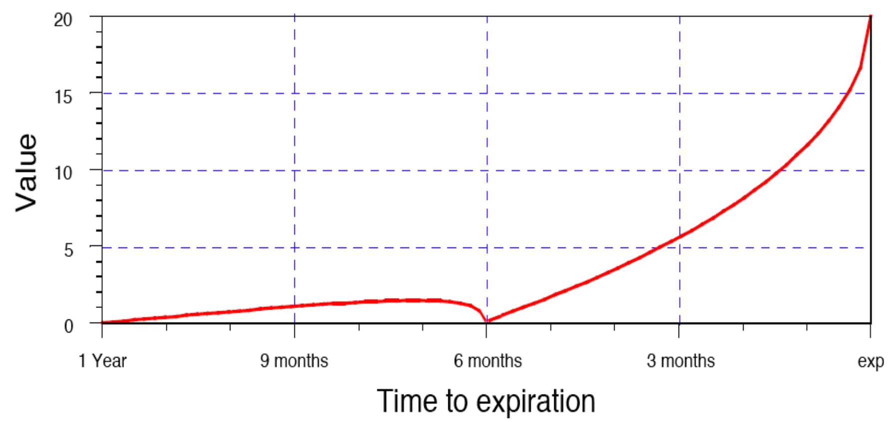


Portfolio 2 matches the zero payoff of the up-and-out call at a stock price of 120 at both six months *and* one year prior to expiration.

Portfolio 3. Its payoff matches that of an up-and-out call if barrier is never crossed, or if it is crossed exactly at 6 months or 1 year before expiration.

Quantity	Type	Strike	Expiration	Value for stock price = 120	
				6 months	1 year
1.000	call	100	1 year	22.767	25.610
-2.387	call	120	1 year	-22.767	-32.753
0.752	call	120	6 months	0.000	7.142
Net				0.000	0.000

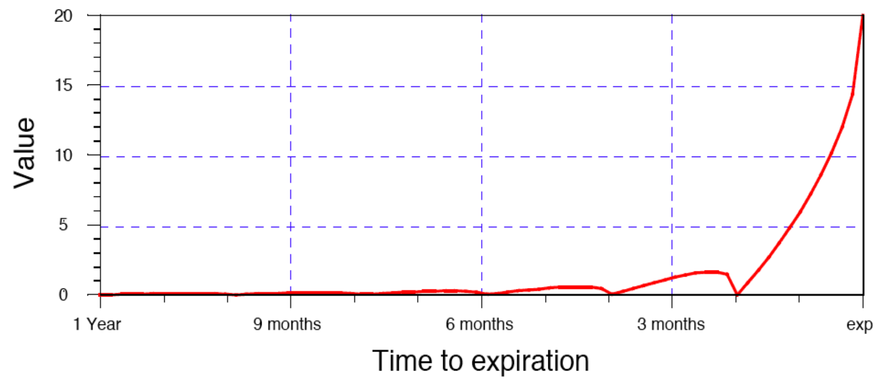
Value of Portfolio 3 on the barrier at 120



For the first six months in the life of the option, the boundary value at a stock price of 120 remains fairly close to zero.

A portfolio of seven standard options at a stock level of 120 that matches the zero value of the target up-and-out call on the barrier every two months.

Value on the barrier at 120 of a portfolio of standard options that is constrained to have zero value every two months.



1.3 Replication Accuracy

An up-and-out call option.

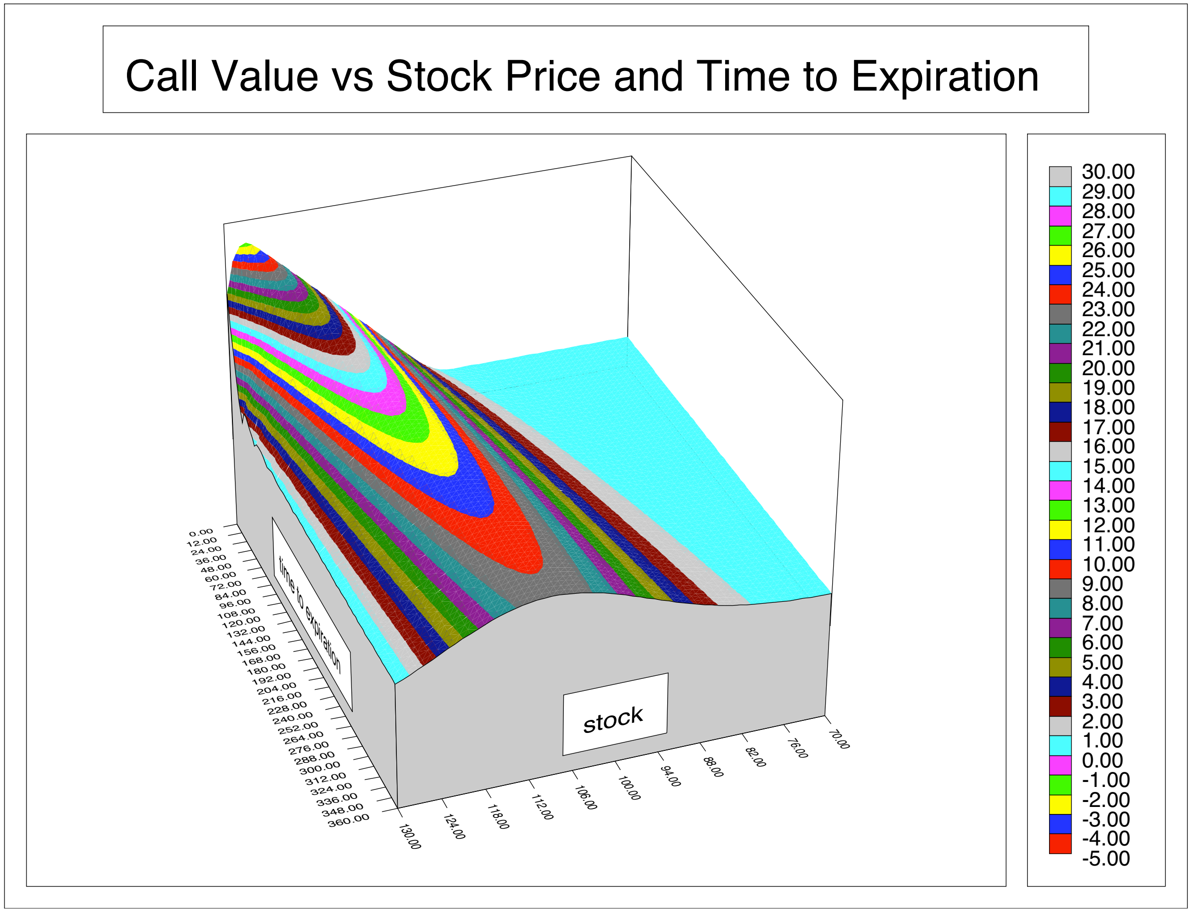
Stock price:	100
Strike:	100
Barrier:	120
Time to expiration:	1 year
Up-and-Out Call Value:	1.913

The replicating portfolio of 7 options.

Quantity	Option Type	Strike	Expiration (months)	Value (Stock = 100)
0.16253	Call	120	2	0.000
0.25477	Call	120	4	0.018
0.44057	Call	120	6	0.106
0.93082	Call	120	8	0.455
2.79028	Call	120	10	2.175
-6.51351	Call	120	12	-7.140
1.00000	Call	100	12	6.670
Total				2.284

The theoretical value of the replicating portfolio in Table at a stock price of 100, one year from expiration, is 2.284, about 0.37 or 19% off from the theoretical value of the target option.
Cost of hedge know. Then unwind if you get too close and take your losses.

Here's the behavior over all stock prices and time prior to expiration of a 24-option replicating portfolio.



You can see it looks a lot like the value of the payoff of an up and out call option. But it's been created using Black-Scholes to match the coefficients, assuming future volatilities are the same as today, and so it's imperfect.

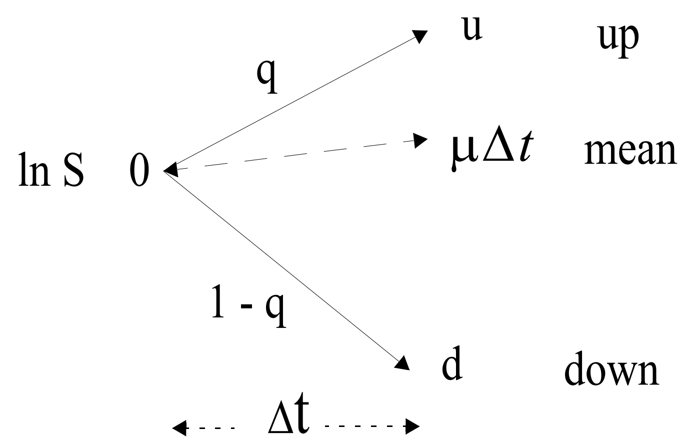
1.4 The Binomial Model for Stock Evolution

Search for models of stock price evolution that can account for the smile. It's easiest to begin in the binomial framework where intuition is clearer.

In Black-Scholes framework $\quad d(\ln S) = \mu dt + \sigma dZ.$

Expected return on the stock price is $\mu + \sigma^2/2$. The total variance in time Δt is $\sigma^2 \Delta t$.

We model the actual evolution of the stock price over an instantaneous time Δt by means of a one-period binomial tree.



How do we choose q, u and d to match the continuous-time $d(\ln S) = \mu dt + \sigma dZ$?
Match the mean and variance of the return:

$$qu + (1-q)d = \mu \Delta t$$

$$q[u - \mu\Delta t]^2 + (1-q)[d - \mu\Delta t]^2 = \sigma^2 \Delta t \qquad \text{Eq.11.9}$$

$$qu + (1-q)d = \mu \Delta t$$

$$q(1-q)(u-d)^2 = \sigma^2 \Delta t$$

Two constraints on the three variables *q, u,* and *d*. Pick convenient ones.

First Solution: The Cox-Ross-Rubinstein Convention

Choose $u + d = 0$: stock price always returns to the same level; center of the tree fixed.

$$(2q-1)u = \mu\Delta t$$
$$4q(1-q)u^2 = \sigma^2\Delta t$$

Squaring the first equation and dividing by the second leads to $\frac{(2q-1)^2}{4q(1-q)} = \frac{\mu^2\Delta t}{\sigma^2}$

If Δt is zero then $q = 1/2$. Write $q = 1/2 + \varepsilon$, then

$$\varepsilon \approx \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad u = \sigma\sqrt{\Delta t}$$
$$q \approx \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad d = -\sigma\sqrt{\Delta t} \qquad \text{Eq.11.10}$$

Check: mean return of the process is $\left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) - \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) = \mu\Delta t$ perfect!

The variance is $q(1-q)(u-d)^2 \approx \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right)4\sigma^2\Delta t \approx \sigma^2\Delta t - \mu^2(\Delta t)^2$ a little small.

The convergence to the continuum limit is a little slower than if it matched the variance exactly.For small enough Δt there is no riskless arbitrage with this convention

Another Solution: The Jarrow-Rudd Convention

We must satisfy the constraints

$$\begin{aligned} qu + (1-q)d &= \mu\Delta t \\ q(1-q)(u-d)^2 &= \sigma^2\Delta t \end{aligned}$$

Choose $q = 1/2$, so that the up and down moves have equal probability:

$$\begin{aligned} u + d &= 2\mu\Delta t \\ u - d &= 2\sigma\sqrt{\Delta t} \end{aligned}$$

$$\begin{aligned} u &= \mu\Delta t + \sigma\sqrt{\Delta t} \\ d &= \mu\Delta t - \sigma\sqrt{\Delta t} \end{aligned} \qquad \text{Eq.11.11}$$

The mean return is exactly μ; the volatility of returns is exactly σ, convergence is faster than.

$$E[S] = \frac{(e^u + e^d)}{2}S = e^{\mu\Delta t}\frac{\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{2} \approx e^{\mu\Delta t}\left(1 + \frac{\sigma^2\Delta t}{2}\right) \approx e^{\left(\mu + \frac{\sigma^2}{2}\right)\Delta t}$$

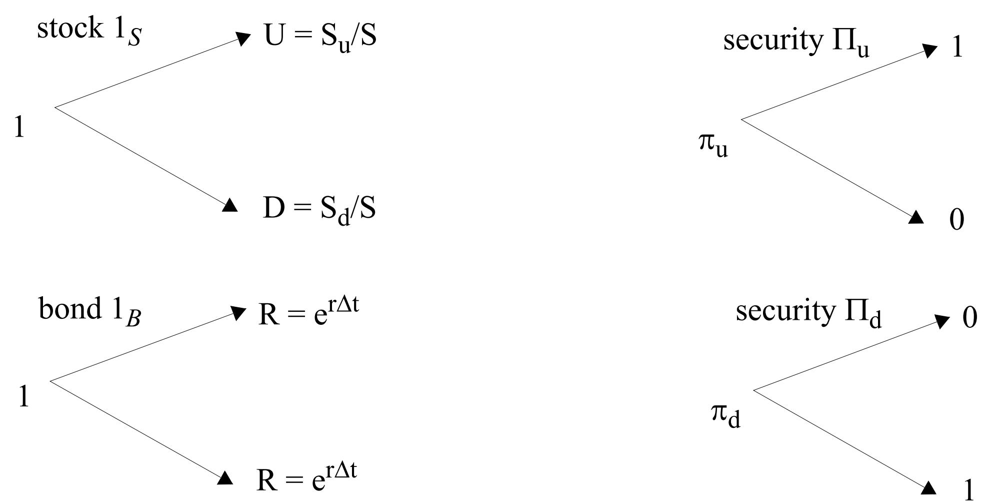
In the limit $\Delta t \rightarrow$, both the CRR and the JR convention describe the same process, and there are many other choices of u, d, and q that do so too.

We will use these binomial processes as a basis for modeling more general stochastic processes that can perhaps explain the smile.

1.5 The Binomial Model for Options Valuation

Options Valuation

One can decompose the stock S and the bond B into two securities Π_u and Π_d that pay out only in the up or down state.



Define $\Pi_u = \alpha \times 1_S + \beta \times 1_B$. Note that because it is riskless, the sum $\Pi_u + \Pi_d = 1/R$

Then $\begin{aligned} \alpha U + \beta R &= 1 \\ \alpha D + \beta R &= 0 \end{aligned}$ so that $\begin{aligned} \alpha &= \frac{1}{(U-D)} \\ \beta &= \frac{-D}{R(U-D)} \end{aligned}$ $\quad \Pi_u = \frac{R \times 1_S - D \times 1_B}{R(U-D)} \quad \Pi_d = \frac{U \times 1_B - R \times 1_S}{R(U-D)}$

The values are $\pi_u = \frac{R-D}{R(U-D)} \equiv \frac{p}{R}$ $\qquad \pi_d = \frac{U-R}{R(U-D)} \equiv \frac{1-p}{R}$

$p = \frac{R-D}{U-D}$ $\qquad 1-p = \frac{U-R}{U-D}$ are the no-arbitrage probabilities that don't depend on q.

The first equation can be rewritten as $pU + (1-p)D = R$, or

$$S = \frac{pS_u + (1-p)S_d}{R} \tag{Eq.11.12}$$

so that in this measure the expected future stock price is the forward price.

Any option C which pays C_u (C_d) in the up (down)-state is replicated by $C = C_u\Pi_u + C_d\Pi_d$ with

$$C = \frac{pC_u + (1-p)C_d}{R} \tag{Eq.11.13}$$

Regard the stock equation as *defining* the measure p given the values of S, S_u and S_d; the second equation specifies the value C in terms of the option payoffs and the value of p. This is why probability theory seems to be important in options pricing, because of complete markets.

The Black-Scholes Partial Differential Equation and the Binomial Model

The Black-Scholes PDE comes from taking the limit of the binomial pricing equation as $\Delta t \to 0$. Cox-Ross-Rubinstein choice of q, u & d:

$$u = \sigma\sqrt{\Delta t} \qquad d = -\sigma\sqrt{\Delta t}$$

$$RC = pC_u + (1-p)C_d \tag{Eq.11.14}$$

$$p = \frac{RS - S_d}{S_u - S_d} \qquad 1 - p = \frac{S_u - RS}{S_u - S_d} \tag{Eq.11.15}$$

Now substitute $S_u = e^u S$, $S_d = e^d S$ and $R = e^{r\Delta t}$ in the two equation directly above, so that all terms are re-expressed in terms of the variables r, σ and S.

$$e^{r\Delta t}C = pC\left(e^{\sigma\sqrt{\Delta t}}S, t + \Delta t\right) + (1-p)C\left(e^{-\sigma\sqrt{\Delta t}}S, t + \Delta t\right)$$

Substituting for p and performing a Taylor expansion to leading order in Δt, one can show that

$$Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$