LECTURE 7

P&L OF TRADING STRATEGIES
THE EFFECTS OF:
HEDGING CONTINUOUSLY
HEDGING DISCRETELY
TRANSACTIONS COSTS

The P&L of Hedged Trading Strategies

Consider an initial position at time t_0 in an option C that is Δ-hedged with borrowed money which earns interest r, and then rehedged using in discrete steps at times t_i and stock prices S_i.

Notation: $C_n = C(S_n, t_n) \qquad \Delta_n = \Delta(S_n, t_n)$.

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares Traded	Net Value of Position: Option + Stock + Cash
t_0, S_0	Buy C_0, short Δ_0 shares	$-\Delta_0$	$-\Delta_0 S_0$	$\Delta_0 S_0$	C_0
t_1, S_1	none	$-\Delta_0$	$-\Delta_0 S_1$	$\Delta_0 S_0 e^{r\Delta t}$	$C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\Delta t}$
	get short Δ_1 shares by shorting $\Delta_1 - \Delta_0$ shares	$-\Delta_1$	$-\Delta_1 S_1$	$\Delta_0 S_0 e^{r\Delta t} + (\Delta_1 - \Delta_0) S_1$	$C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\Delta t} + (\Delta_1 - \Delta_0) S_1$
t_2, S_2	none	$-\Delta_1$	$-\Delta_1 S_2$	$\Delta_0 S_0 e^{2r\Delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$	$C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\Delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares Traded	Net Value of Position: Option + Stock + Cash
t_2, S_2	get short Δ_2 shares by shorting $\Delta_2 - \Delta_1$ shares	$-\Delta_2$	$-\Delta_2 S_2$	$\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$	$C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$
etc.					
t_n, S_n	get short Δ_n shares by shorting $\Delta_n - \Delta_{n-1}$ shares	$-\Delta_n$	$-\Delta_n S_n$	$\Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$	$C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$

The initial value of the positions was C_0 and would have generated $C_0 e^{r(T-t)}$

The final value is $C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$

where the subscript b at the end of the formula denotes a backwards Ito integral.

Therefore, the fair value of C_0 is given by equating these two quantities:

$$e^{r(T-t)}C0 = C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$$

or

$$(C_0 - \Delta_0 S_0)e^{r(T-t)} = (C_T - \Delta_T S_T) + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b \qquad \text{[A]}$$

You can integrate by parts using the relation

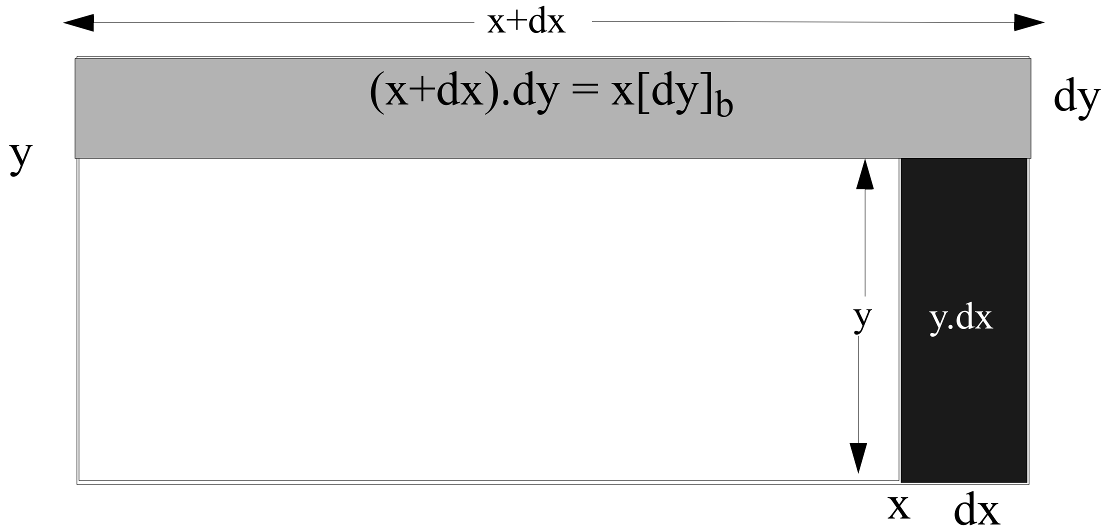
$$e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b = d\left[e^{r(T-\tau)} S_\tau \Delta_\tau\right] + re^{r(T-\tau)} \Delta_\tau S_\tau d\tau - e^{r(T-\tau)} \Delta_\tau dS_\tau$$

to obtain

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[dS_\tau - S_\tau r d\tau] e^{-r(\tau - t)} \qquad \text{[B]}$$

[A] and [B] provide a way to calculate the value of the call in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration.

Backward Ito Integral



$$d[xy] = ydx + x[dy]_b$$

The Effect of Different Hedging Strategies
How do the return profiles depend on the hedging strategy?

Realized volatility is noisy. Implied volatility is a parameter reflecting fear, hedging costs, inability to hedge perfectly, uncertainty of future volatility, the chance to make a profit, etc., usually greater than realized volatility.

Hedging with Realized (Known) Volatility

We buy the option at its implied volatility and then hedge it at the realized volatility to replicate the option perfectly. The P&L is value gained from hedging MINUS Black-Scholes implied value:

$$\text{Total P\&L} = V(S, \tau, \sigma) - V(S, \tau, \Sigma)$$

Recap: How Do We Capture $V_r - V_i$:

Table 1: Position Values when Hedging with Realized Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_r \vec{S}, -\Delta_r S$	$\Delta_r S - V_i$	0
t + dt	$\vec{V}_i(t+dt, S+dS)$, $V_i + dV_i$	$-\Delta_r \vec{S}$, $-\Delta_r(S+dS)$	$(\Delta_r S - V_i)(1+rdt)$ interest received $-\Delta_r DSdt$ dividends paid	$(V_i + dV_i - \Delta_r(S+dS))$ $(\Delta_r S - V_i)(1+rdt)$ $-\Delta_r DSdt$

$$d\text{P\&L} = dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt \qquad \text{Eq.7.1}$$

This was bought at implied, hedged at realized.

But dP&L hedged *and bought* at realized volatility is zero which is statement of Black-Scholes:

$$0 = d\boldsymbol{V_r} - \Delta_r dS - rdt(\boldsymbol{V_r} - \Delta_r S) - \Delta_r DSdt$$

So substituting in 7.1

$$d\text{P\&L} = dV_i - dV_r - rdt(V_i - V_r)$$

$$= e^{rt} d\left[e^{-rt}(V_i - V_r)\right]$$

$$d\mathrm{PV}(\mathrm{P\&L}) = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_i - V_r)\right] = e^{rt_0} d\left[e^{-rt}(V_i - V_r)\right]$$

$$\mathrm{PV}(\mathrm{P\&L}) = e^{rt_0}\int_{t_0}^{T} d\left[e^{-rt}(V_i - V_r)\right]$$

$$= 0 - (V_i - V_r) = V_r - V_i \qquad \text{if T is expiration}$$

The final P&L at the expiration of the option is known and deterministic, provided that we know the realized volatility and that we can hedge continuously.

How is this deterministic P&L realized over time? Stochastically -- like a zero coupon bond whose final principal is known but whose present value varies with interest rates.

$$d\mathrm{P\&L} = dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

Use Ito

$$d\mathrm{P\&L} = \left\{\Theta_i dt + \Delta_i dS + \frac{1}{2}\Gamma_i S^2\sigma^2 dt\right\} - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

$$= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2\sigma^2\right\}dt + (\Delta_i - \Delta_r)dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

But Black-Scholes with option bought at implied vol and with realized volatility set to implied volatility gives you $\Theta_i = -\frac{1}{2}\Gamma_i S^2\Sigma^2 + rV_i - (r-D)S\Delta_i$

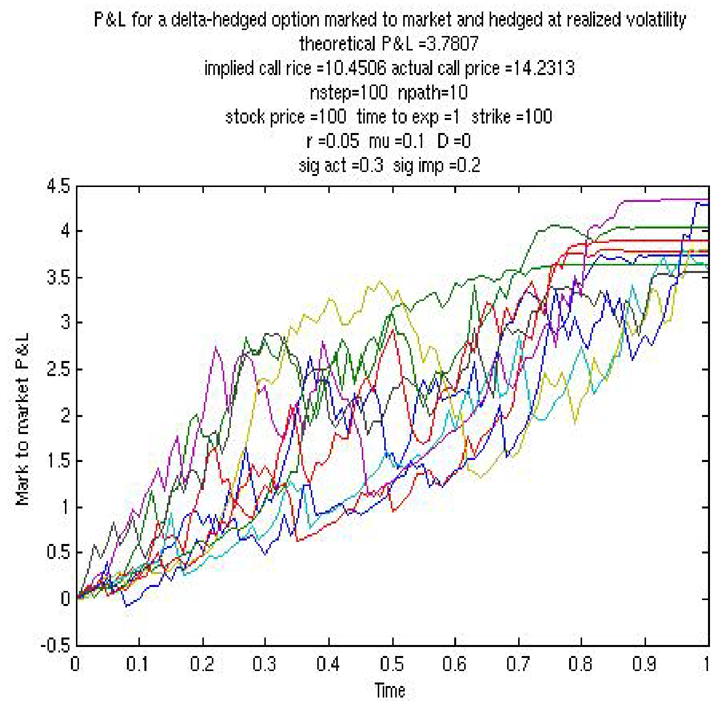
and so

$$dP\&L = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt + (\Delta_i - \Delta_r)\{(\mu - r + D)Sdt + \sigma SdZ\}$$

The total integrated P&L is deterministic but the increments have a random component dZ.

To illustrate this, plot cumulative **P&L** along ten random stock paths, 100 steps

P&L starts at zero because initial position is totally finance,

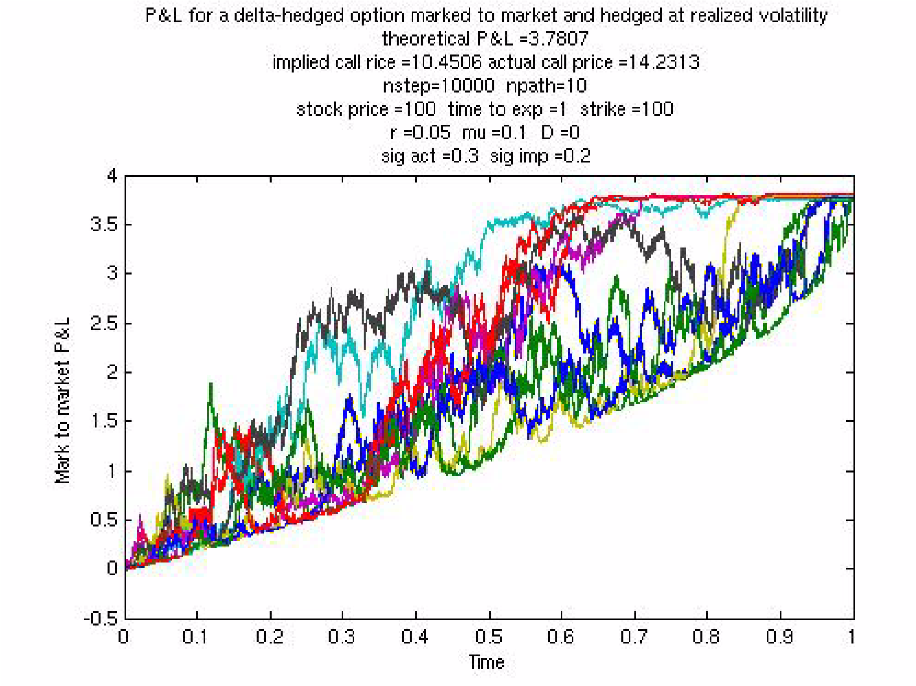


$$\sigma_r = 0.3$$

$$\sigma_i = 0.2$$

The final P&L is almost path-independent – almost, because 100 rehedgings per year is not quite the same continuous hedging.

Rehedge 10,000 times, almost independent **P&L**:



Bounds on the P&L When Hedging at the Realized Volatility

We had $\sigma > \Sigma$. Notice the upper and lower bounds. Why? We had

$$d\mathrm{PV(P\&L)} = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_i - V_r)\right] = e^{rt_0} d\left[e^{-rt}(V_i - V_r)\right]$$

Integrate from $t_0 \equiv 0$, S_0 to t, S, to obtain

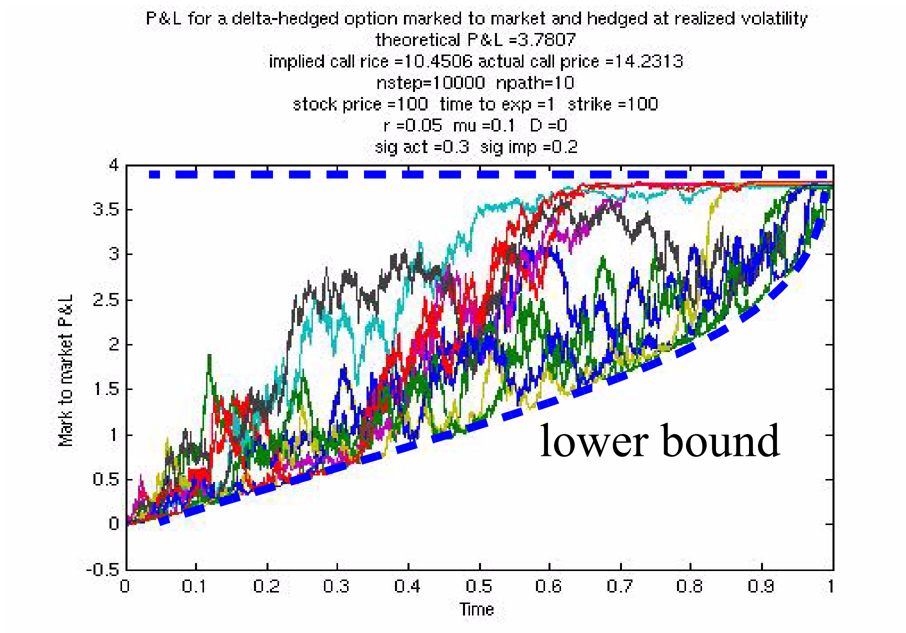
$$PV(\mathrm{P\&L(t)}) = -\underbrace{[V(\sigma, S, t) - V(\Sigma, S, t)]e^{-rt}}_{\text{value along way} > 0} + \underbrace{[V(\sigma, S_0, 0) - V(\Sigma, S, 0)]}_{\text{value at inception} > 0} \qquad \text{Eq.7.2}$$

Both terms in the square brackets in are positive.

Upper bound $[V(\sigma, S_0, 0) - V(\Sigma, S, 0)]$ occurs when the first term is zero, which occurs at $S = 0$ or $S = \infty$, and is $[V(\sigma, S_0, 0) - V(\Sigma, S, 0)]$.

The lower bound to the P&L occurs when the first term $[V(\sigma, S, t) - V(\Sigma, S, t)] \sim \frac{\partial V}{\partial \sigma}(\Sigma - \sigma)$ is a maximum, i.e. when vega is largest, close to at-the money, which turns out to be at

$$S = Ke^{-(r - 0.5\sigma\Sigma)(T-t)}$$


P&L for a delta-hedged option marked to market and hedged at realized volatility
theoretical P&L =3.7807
implied call rice =10.4506 actual call price =14.2313
nstep=10000 npath=10
stock price =100 time to exp =1 strike =100
r =0.05 mu =0.1 D =0
sig act =0.3 sig imp =0.2
Mark to market P&L
Time
lower bound

Hedging with Implied Volatility, Evolving at Realized

When you hedge with implied, the final value of the P&L depends on the path taken, and is not deterministic, but there is **no random mishedging component at each instant**.

Table 2: Position Values when Hedging with Implied Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_i \vec{S}$	$\Delta_i S - V_i$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_i \vec{S}, -\Delta_i (S + dS)$	$(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$	$(V_i + dV_i - \Delta_i (S + dS))$ $(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$

$$
\begin{aligned}
d\text{P\&L} &= [V_i + dV_i - \Delta_i (S + dS)] + (\Delta_i S - V_i)(1 + rdt) - \Delta_i DSdt \\
&= dV_i - \Delta_i dS - r(V_i - \Delta_i S)dt - \Delta_i DSdt
\end{aligned}
$$

Using Ito:

$$dP\&L = \left[\Theta_i dt + \cancel{\Delta_i dS} + \frac{1}{2}\Gamma_i S^2 \sigma^2 dt\right] \cancel{-\Delta_i dS} - r(V_i - \Delta_i S)dt - \Delta_i DSdt$$

$$= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2 \sigma^2 + (r - D)\Delta_i S - rV_i\right\}dt$$

Black-Scholes equation for all volatilities, hedging and realized, equal to σ_i, is

$$\Theta_i + \frac{1}{2}\Gamma_i S^2 \Sigma^2 + (r - D)\Delta_i S - rV_i = 0$$

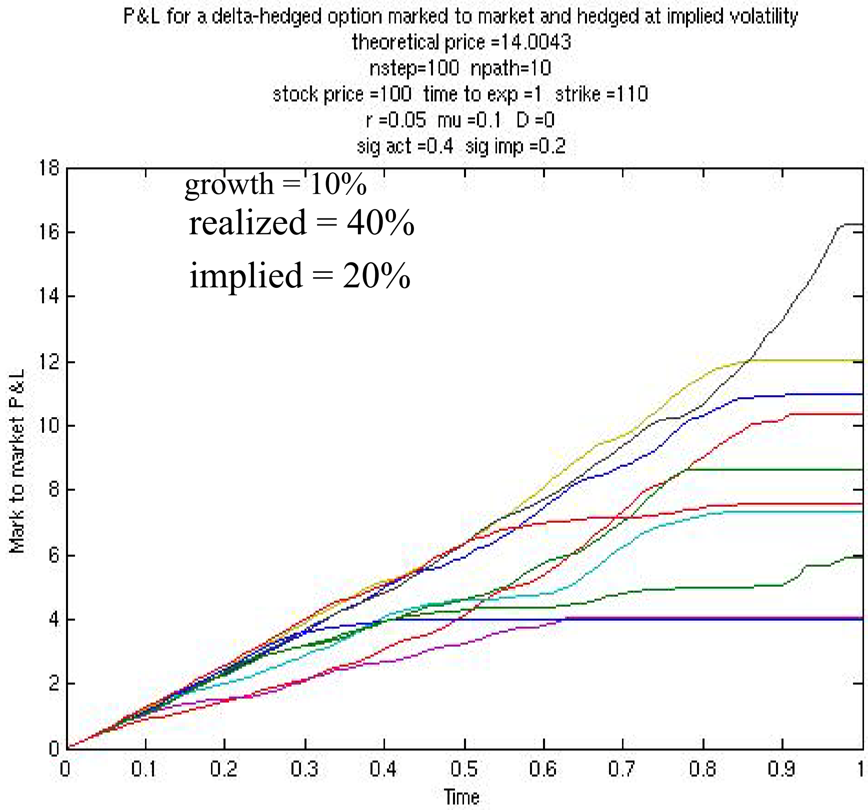
So

$$dP\&L = \frac{1}{2}\Gamma_i S^2 (\sigma^2 - \Sigma^2)dt \qquad \text{Eq.7.3}$$

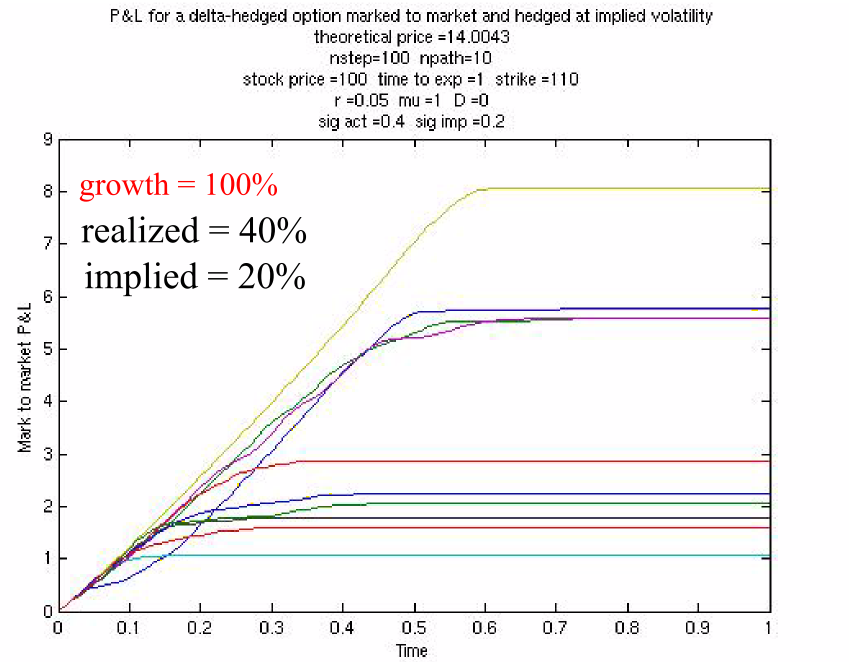
$$PV(P\&L) = \frac{1}{2}\int_{t_0}^{T} \Gamma_i S^2 (\sigma^2 - \Sigma^2) e^{-r(t - t_0)} dt \qquad \text{Eq.7.4}$$

The P&L is highly path-dependent. Although the hedging strategy captures a value proportional to $(\sigma^2 - \Sigma^2)$, it depends strongly on moneyness.

Cumulative P&L along 10 random stock paths, 100 hedging steps to expiration



worth less because more often out of the money



In practice, realized volatility isn't known in advance. A trading desk would most likely hedge at the constantly varying implied volatility which would move in synchronization with the recent realized volatility.

{Read: Hedging at an Arbitrary Constant Volatility}

Volatility is unknown or wrong:

Buy an option at implied vol Σ and hedge it to expiration at volatility σ_h, realized volatility σ_r

Table 3: Position Values when Hedging with an Arbitrary Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_h \vec{S}$	$\Delta_h S - V_i =$ $(\Delta_h S - V_h) + (V_h - V_i)$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_h \vec{S}, -\Delta_h (S + dS)$	$(\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$	$(V_i + dV_i - \Delta_h (S + dS))$ $(\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$

P&L:

$$
\begin{aligned}
d\text{P\&L} &= dV_i - \Delta_h dS - \Delta_h SDdt + \{(\Delta_h S - V_h) + (V_h - V_i)\} r dt \\
&= dV_h - \Delta_h dS - \Delta_h SDdt + (dV_i - dV_h) + \{(\Delta_h S - V_h) + (V_h - V_i)\} r dt \\
&= \left\{\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^{\ 2} + (r-D)S\Delta_h - rV_h\right\} dt + (dV_i - dV_h) + (V_h - V_i) r dt
\end{aligned}
$$

For homework you are asked to show that

$$
PV(\text{P\&L}) = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)} \Gamma_h S^2 \left(\sigma_r^{\ 2} - \sigma_h^2\right) dt \qquad \text{Eq.7.5}
$$

where $V_h = V_i$ have equal values at expiration. When σ_h is set equal to either σ_r or σ_i, Equation 7.5 reduces to our previous results.

Hedge at implied: stochastic path-dependent P&L because of Γ, deterministic (no dZ) at each S.

Hedging at realized: deterministic final P&L, but stochastic along the way because of dZ exposure

{Maximum P&L When Hedging With Arbitrary Volatility}

$$PV(\text{P\&L}) = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2\left(\sigma_r^{\ 2} - \sigma_h^2\right)dt$$

For $\sigma_r > \sigma_h$, the minimum is clearly $V_h - V_i$.

Maximum PV when path-dependent $S^2\Gamma_h$ is a maximum along the entire path. Now

$$S^2\Gamma_h = \frac{SN'(d_1)e^{-D\tau}}{\sigma_h\sqrt{\tau}} \qquad\qquad N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

and d_1 depends on σ_h through $\dfrac{\ln\dfrac{S}{Ke^{-(r-D)\tau}} - 0.5\sigma_h^2\tau}{\sigma_h\sqrt{\tau}}$

$N'(d_1)$ is a maximum when d_1 is a minimum (zero) at $S = Ke^{-(r-D+\sigma_h^2/2)\tau}$

But $S^2\Gamma_h$ is a maximum occurs when $S = Ke^{-(r - D - \sigma_h^2/2)\tau}$, ~ at the money forward.

(Each power of S adds an extra $\sigma_h^2/2$ to the place where maximum occurs.)

At this value of S (dimensionally)

$$S^2\Gamma_h = \frac{Ke^{-r\tau}}{\sigma_h\sqrt{\tau}\sqrt{2\pi}} \qquad \text{Eq.7.6}$$

Taking the path for the evolution of the stock that moves along this value of S, we maximize the P&L. By combining Equation 7.5 and Equation 7.6, we obtain

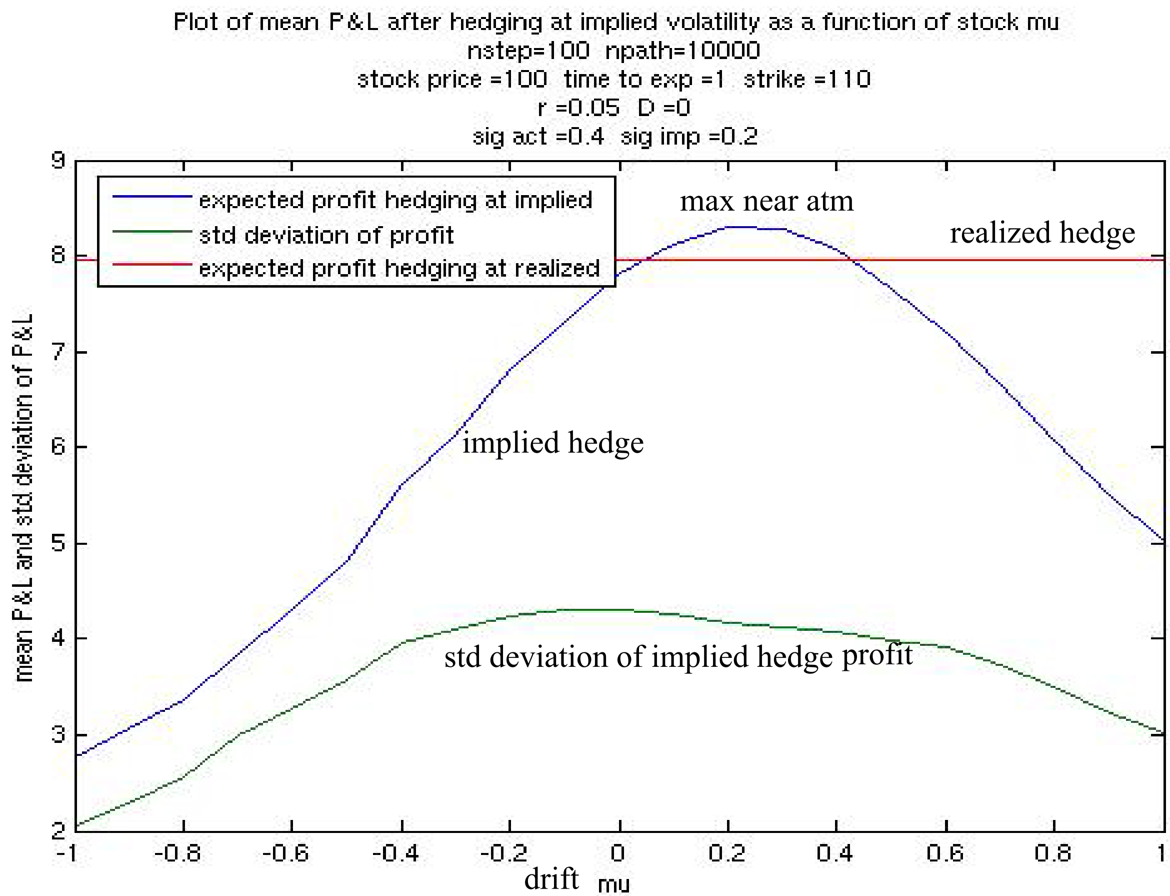
$$\begin{aligned} maxPV(\text{P\&L}) &= V_h - V_i + \frac{1}{2}\left(\sigma_r^2 - \sigma_h^2\right)\int_{t_0}^{T} e^{-r(t-t_0)}\frac{Ke^{-r(T-t)}}{\sigma_h\sqrt{T-t}\sqrt{2\pi}}dt \\ &= V_h - V_i + \frac{K\left(\sigma_r^2 - \sigma_h^2\right)e^{-r(T-t_0)}\sqrt{T-t_0}}{\sqrt{2\pi}\sigma_h} \end{aligned} \qquad \text{Eq.7.7}$$

The last term is approximately twice the difference in value of an atm option at volatility σ_h and an atm option at σ_r.

{Expected Profit after Hedging at Implied Volatility}

$$\text{P\&L} = \frac{1}{2}\int_{t_0}^{T}\Gamma_i S^2(\sigma^2 - \Sigma^2)e^{-r(t-t_0)}dt$$

This integral is path-dependent, so let's find the mean P&L over all paths via Monte Carlo



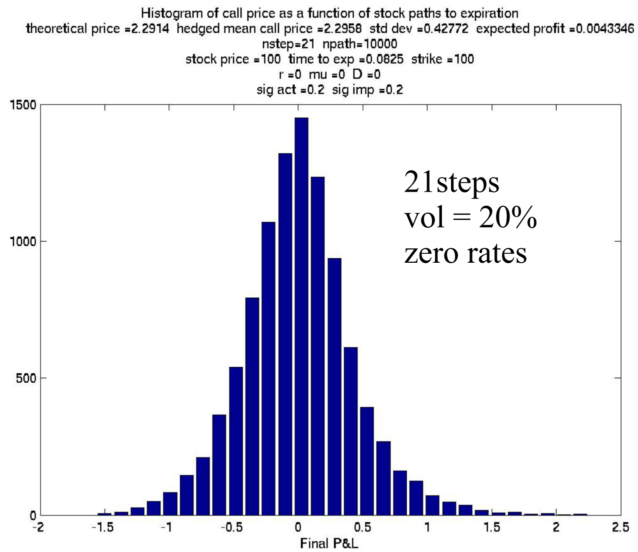
Hedging Errors from Discrete Hedging

We cannot hedge continuously:

A Simulation Approach

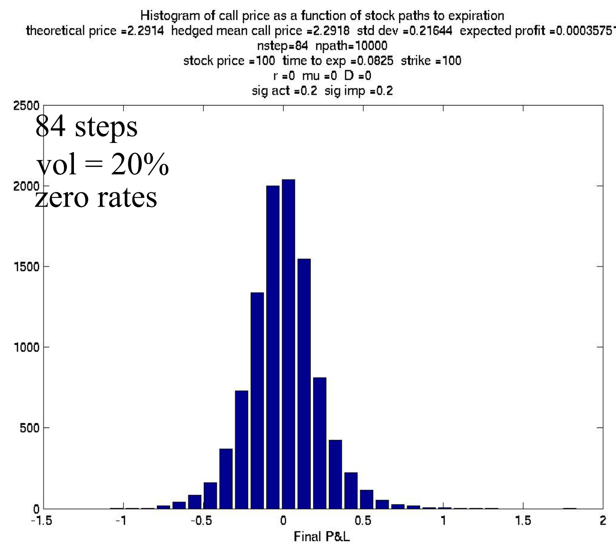
You cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss hedging at regular time intervals.

Monte Carlo: ATM option, expiration 1 month, the realized volatility is 20%, $\mu = r = 0.05$, hedged at an implied volatility of 20% equal to the realized volatility.



21 Rehedgings, Std. deviation. = 0.42

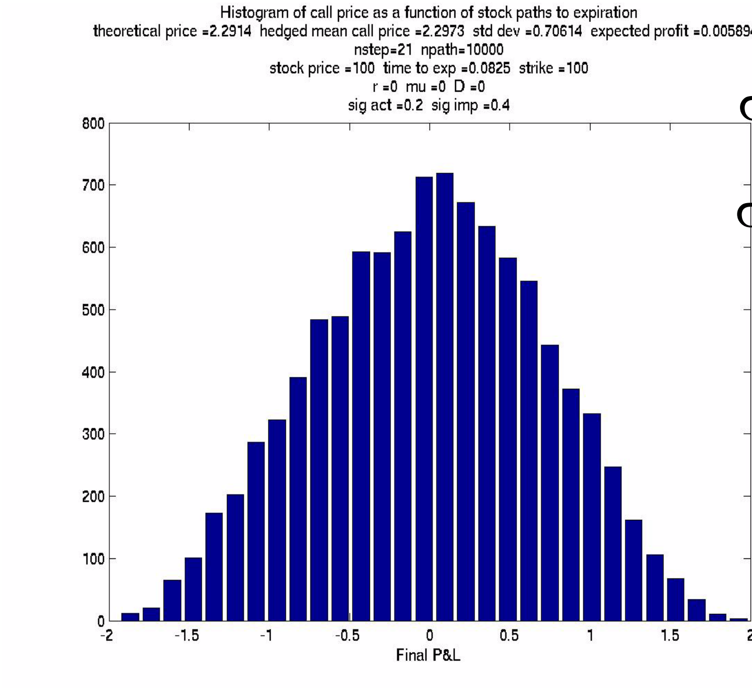
$$\sigma_i = \sigma_r$$



84 Rehedgings, Std. deviation. = 0.21

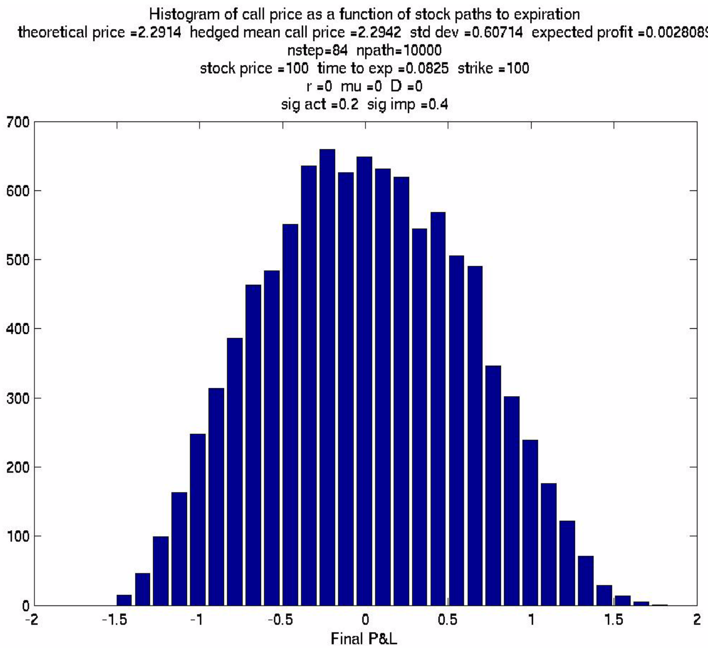
The mean P&L is zero; When we quadruple the number of hedgings, the standard deviation of the P&L halves.

Now let's see what happens $\sigma_i \neq \sigma_r$. Choose an implied volatility of 40% as the hedging volatility, that is, as the volatility used to calculate the value of Δ.



$\sigma_i = 40\%$

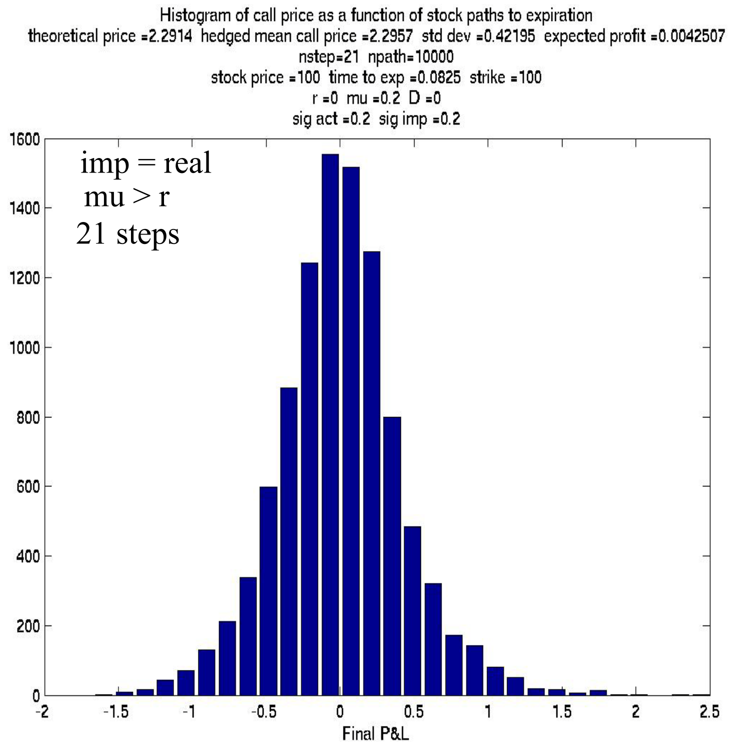
$\sigma_r = 20\%$



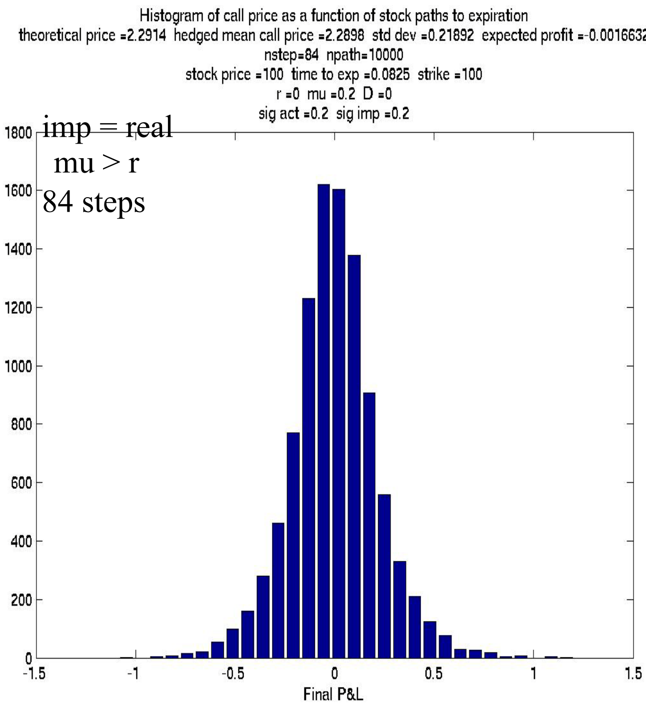
no reduction in variance with increasing rehedges unless hedge vol = realized vol

No longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

Finally let’s see what happens when the drift μ is not the same as the riskless rate, even though implied/hedging) and the realized volatility are both set equal to 0.2.

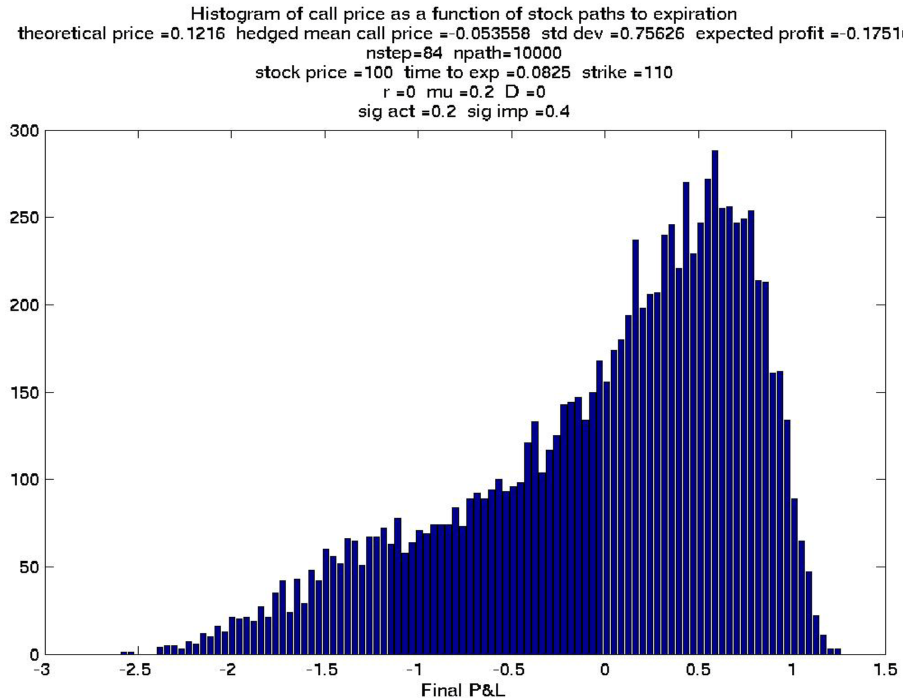


$$\sigma_i = \sigma_r$$



Std deviation again decreases by a factor of two.

Finally, for completeness, we look at the where $\sigma_i \neq \sigma_r$ *and* $\mu \neq r$. In this case the distribution is very asymmetric.



1.1 Understanding Discrete Hedging Error Analytically when $\sigma_i = \sigma_r \equiv \sigma$. (Assuming we know the future volatility)

Discrete time Δt is larger than infinitesimal $\dfrac{\Delta S}{S} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}$

$$\varepsilon \in N(0, 1)$$

Hedged portfolio $\pi = C - \left(\dfrac{\partial C}{\partial S}\right)S$; Initial long π bought with borrowed money. If we hedged continuously the P&L would be zero.

Hedging error owing to mismatch between **continuous** hedge ratio and **discrete** time step

$$HE = \pi + \Delta\pi - \pi e^{r\Delta t} \approx \Delta\pi - r\pi dt$$

$$\approx \left[C_t\Delta t + C_S\Delta S + C_{SS}\frac{\sigma^2 S^2\varepsilon^2}{2}\Delta t - C_S\Delta S\right] - r\Delta t\left[C - \left(\frac{\partial C}{\partial S}\right)S\right]$$

discrete

continuous

$$\approx \left(\boldsymbol{C_t} + C_{SS}\frac{\sigma^2 S^2\varepsilon^2}{2} - \boldsymbol{r\left[C - \left(\frac{\partial C}{\partial S}\right)S\right]}\right)\Delta t$$

Now from Black-Scholes $r\left[C - \left(\dfrac{\partial C}{\partial S}\right)S\right] = \boldsymbol{C_t} + C_{SS}\dfrac{\sigma^2 S^2}{2}$

$$HE = \frac{1}{2}C_{SS}\sigma^2 S^2(\varepsilon^2 - 1)\Delta t \text{ Gamma error} \qquad \text{Eq.7.8}$$

$\varepsilon \in Z(0, 1)$ is normal with $E(\varepsilon^2) = 1$ so $E[HE] = 0$ with a χ^2 distribution.

Over n steps to expiration, the total HE is

$$HE = \sum_{i=1}^{n} \frac{1}{2}\Gamma_i S_i^2 \sigma_i^2 \left(\varepsilon_i^2 - 1\right)\Delta t \qquad \text{Eq.7.9}$$

The variance of the hedging error can be approximately calculated and shown to be

$$\sigma_{HE}^2 = E\left[\sum_{i=1}^{n} \frac{1}{2}\left[\Gamma_i S_i^2\right]^2 \left(\sigma_i^2 \Delta t\right)^2\right] \text{ over all paths} \qquad \text{Eq.7.10}$$

Integrating over all paths starting from S_0 for an atm option

$$E\left[\Gamma_i S_i^2\right]^2 = S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Thus for constant volatility

$$\sigma_{HE}^2 = \frac{1}{2}\sum_{i=1}^{n} S_0^4\Gamma_0^2\sqrt{\frac{T^2}{T^2-t_i^2}}(\sigma^2\Delta t)^2$$

$$= \left(S_0^4\Gamma_0^2\right)(\sigma^2\Delta t)^2\frac{1}{2\Delta t}\int_0^T\sqrt{\frac{T^2}{T^2-\tau^2}}d\tau$$

$$= \left(S_0^4\Gamma_0^2\right)(\sigma^2\Delta t)^2\frac{\pi}{4}\times\frac{T}{\Delta t}\ n$$

$$= \frac{\pi}{4}n\left(S_0^2\Gamma_0\sigma^2\Delta t\right)^2$$

From BS we can interpret $S_0^2\Gamma_0 = \dfrac{1}{\sigma(T-\mathrm{t})}\dfrac{\partial C}{\partial\sigma}$

$$\sigma_{HE}^2 = \frac{\pi}{4}n\left(\frac{1}{\sigma}\frac{\partial C}{\partial\sigma}\sigma^2\frac{\Delta t}{T-t}\right)^2 = \frac{\pi}{4}n\left(\frac{1}{n}\frac{\partial C}{\partial\sigma}\sigma\right)^2 = \frac{\pi}{4n}\left(\frac{\partial C}{\partial\sigma}\sigma\right)^2$$

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial\sigma}\frac{\sigma}{\sqrt{n}} \qquad \text{Eq.7.11}$$

Thus, the hedging error is approximately $\dfrac{\partial C}{\partial\sigma}\times\dfrac{\sigma}{\sqrt{n}}$. What does this mean?

Understanding The Results Intuitively

Hedging discretely introduces uncertainty in the hedging outcome but no bias: $E[HE] = 0$

Simple analytic rule

$$\sigma_{HE} \sim \frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$$

For $S \sim K$, more simply

$$\frac{\sigma_{\text{P\&L}}}{\text{fair option value}} \sim \sqrt{\frac{\pi}{4n}} .$$

Think of this as statistical sampling error: discrete hedging samples volatility discretely and is therefore subject to error.

The standard deviation of a constant volatility σ measured discretely is $\frac{\sigma}{\sqrt{2n}}$.

This is quite a large error even assuming we know the future volatility with certainty.

In real life your hedge ratio is incorrect not just because hedging is discrete, but because you don't know the appropriate volatility to use.

The Effect of Transactions Costs

It costs money to hedge:

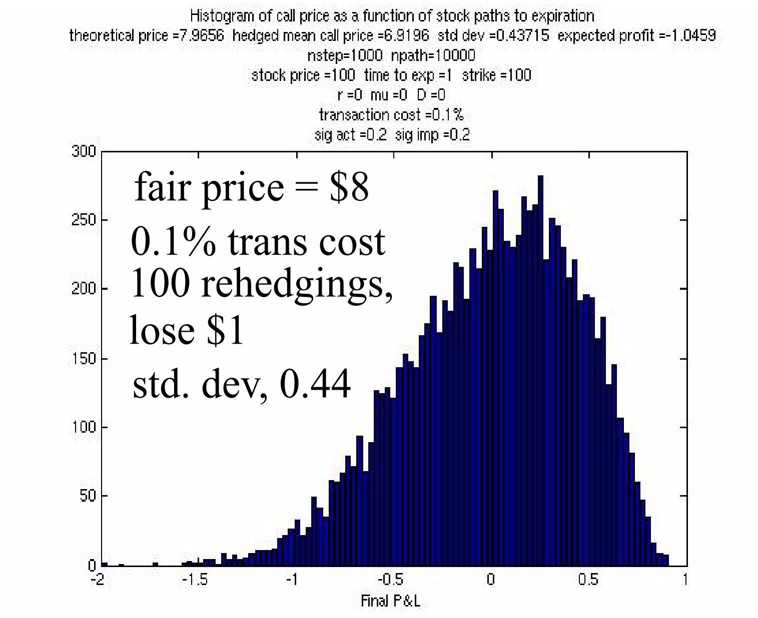
Simulation

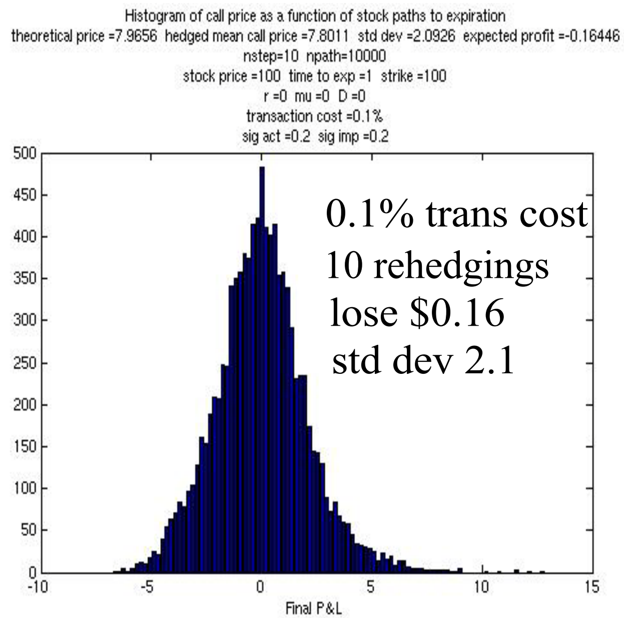
Suppose there is a fee to buy and sell the stock each time you rehedge. Then, not only is the P&L uncertain because of the discrete hedging, but the cost of hedging also lowers the fair value of the option if you buy it, and raises the cost to you if you sell it.

Assume a simple transactions cost proportional to the cost of the shares traded, and hedge at the realized volatility.

Rehedging at regular intervals

Rebalance hedge at every step, whether necessary or not.



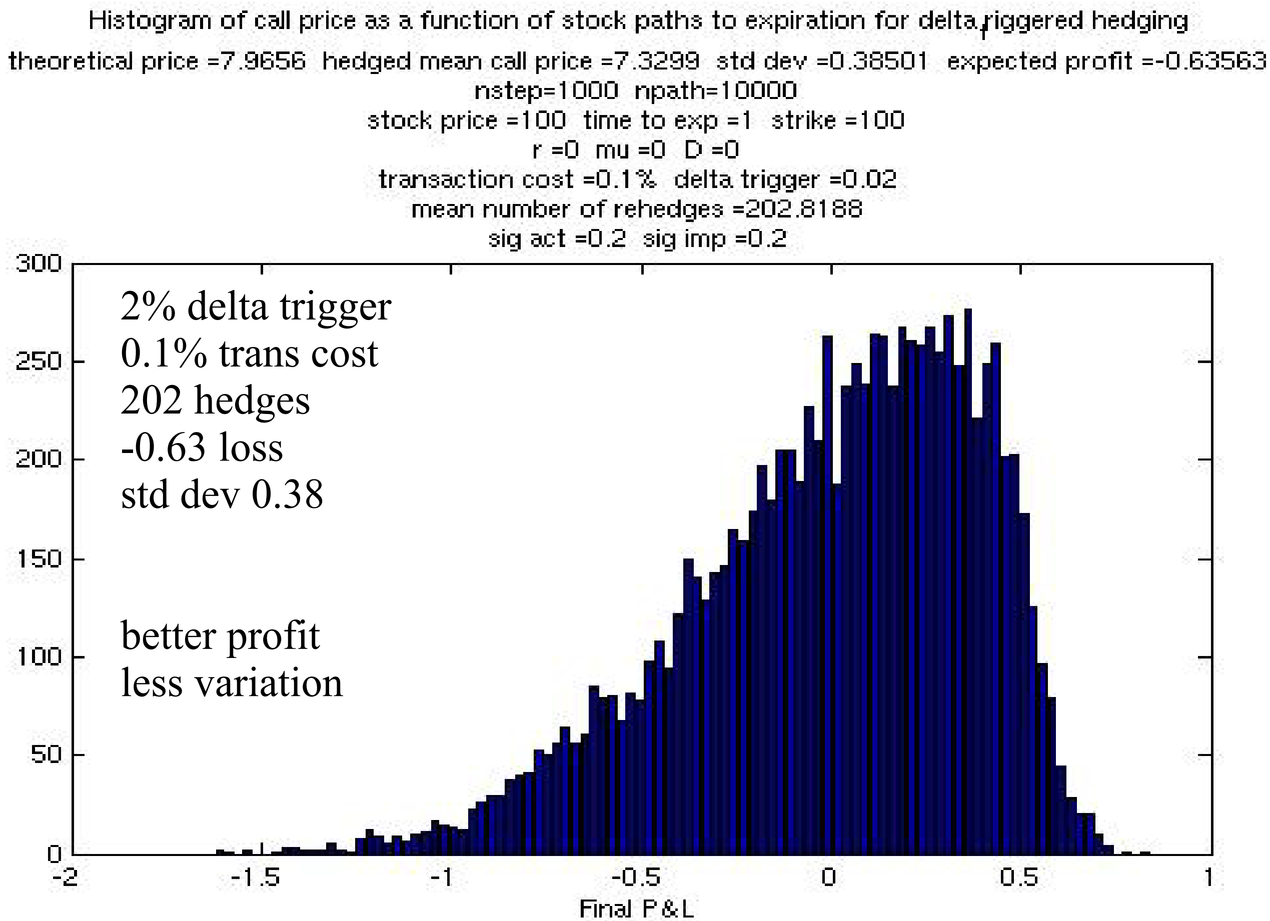


Tension between diminishing hedging error and reducing cost! What is optimal rebalancing

Rehedging triggered by changes in the hedge ratio

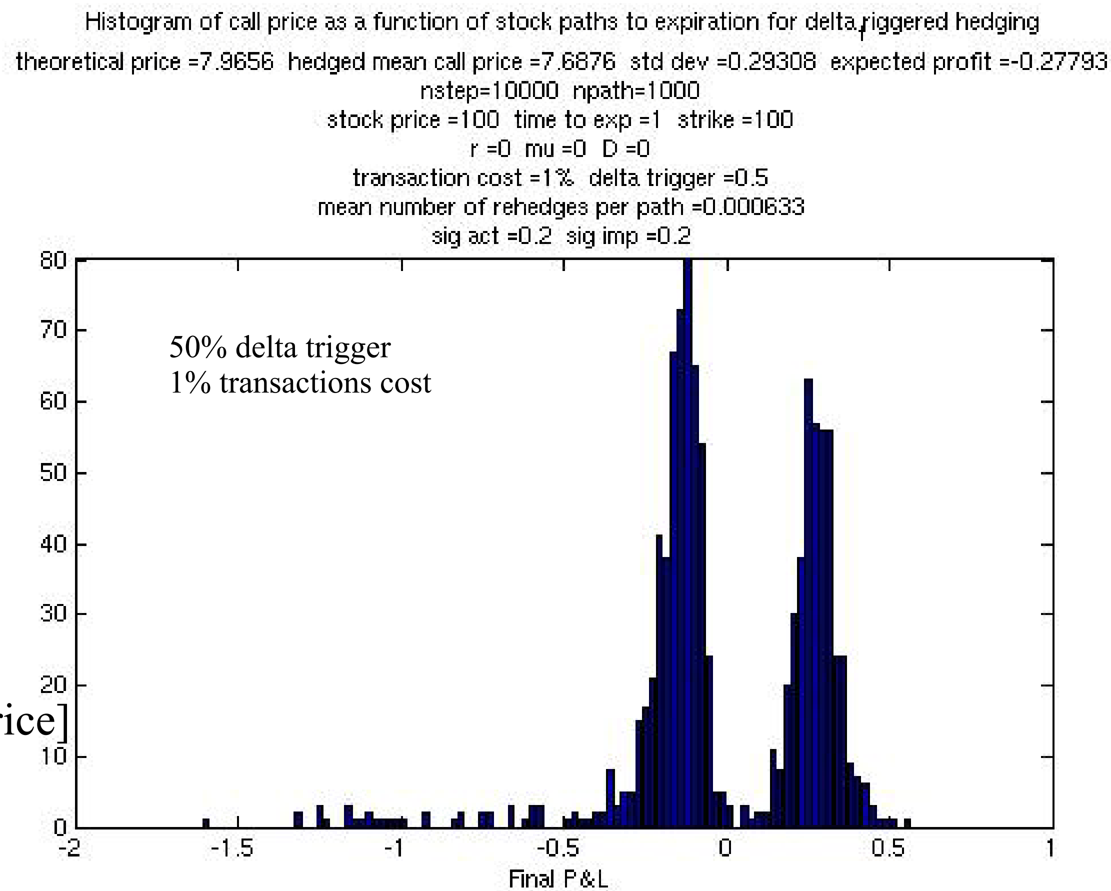
More efficiently rehedge when necessary, on a substantial change in delta.

Hedging an at-the-money call with a delta trigger of 0.02 or 2% and a transactions cost of 0.1%.



The loss owing to the transactions cost is smaller; the standard deviation of the P&L is smaller too.

Extreme case: rehedge only when the delta changes by 50 percentage points and with a transactions cost of 1%.



Some moves lead to no rehedging and high value; some moves lead to one rehedge and loss in value below the mean:
If you rehedge once, half the time, then *expected* loss in value is probability x cost =
(1/2)[k x shares traded x price]
=(0.5)(0.01)(0.5)100
=0.25

The distribution is bimodal. The reason is that if you rehedge only when the delta of the option changes by 50 points, then rehedges only occur when the stock makes a substantial move up or down in order to achieve such a large change in the delta. Hence one set of final call prices involve no transactions costs (over the paths where delta changed by less than 50 points) and hence lie above the mean; the other set of call final call prices involve one rehedging and its cost (over the paths where delta did change by 50bp or more) and hence lie below the mean.

Analytical Approximations to Transactions Cost

$$HE = \sum_{i=1}^{n} \frac{1}{2}\Gamma_i S_i^2 \sigma_i^2 \left(\varepsilon_i^2 - 1\right) \Delta t \qquad \text{Eq.7.12}$$

$$E[HE] = 0$$

$$\sigma_{HE}^2 \sim O([\Delta t]^2)$$

The total number of rehedgings is $T/(\Delta t)$

$$\sigma_{HE}^2 \sim O\left(\frac{T}{\Delta t}[\Delta t]^2\right) \sim O(T\Delta t) \to 0 \qquad \text{as } \Delta t \to 0$$

Hedging continuously captures exactly the value of the option.

Now include transactions costs. Assume that you rehedge an option C with value C every time Δt passes. Every time you trade the stock (buying *or* selling), you pay a fraction k of the cost of the shares traded.

Then, every time you rehedge, you have to trade a number of shares equal to

$$N \approx \Delta(S + \delta S, t + \Delta t) - \Delta(S, t) \approx \frac{\partial^2 C}{\partial S^2}\delta S$$

Cost is value of number of shares traded times the fraction k, that is

$$\left|\frac{\partial^2 C}{\partial S^2}\delta S \times (kS)\right|$$

where the absolute value reflects the fact that you pay a positive transaction cost irrespective of whether you buy or sell shares. (Therefore nonlinear!)

$$\delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}\varepsilon$$

To order $(\Delta t)^{1/2}$ the expected transactions cost in time Δt is

$$E\left[\left|\frac{\partial^2 C}{\partial S^2}\sigma S^2 k \varepsilon \sqrt{\Delta t}\right|\right]$$

$$E[|\varepsilon|] \neq 0$$

There are $T/(\Delta t)$ rehedgings to expiration.

Total cost of order $\frac{T}{\Delta t}\sqrt{\Delta t} \sim \frac{1}{\sqrt{\Delta t}} \to \infty$ as the time between rehedgings goes to zero.

A PDE Model of Transactions Costs

One can approach transactions costs even more analytically in the framework of Hoggard, Whaley & Wilmott. (There are many other treatments.)

Assume zero rates and dividend yield, and

$$dS = \mu S dt + \sigma S Z \sqrt{dt}$$

where ε is drawn from a standard normal distribution.

$$\begin{aligned}
d\text{P\&L} &= dV - \Delta dS - \text{cash spent on transactions costs} \\
&\approx \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \Delta dS - \kappa S|N| \\
&= \frac{\partial V}{\partial t} dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S Z \sqrt{dt}) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \kappa S|N| \\
&= \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)} \sigma S Z \sqrt{dt} + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)} + \frac{\partial V}{\partial t}\right) dt - \kappa S|N|
\end{aligned}$$

Choose the continuous hedge ratio $\Delta = \frac{\partial}{\partial S} V(S, t)$ to eliminate the first term.

After time δt we have to rehedge, so that the change in the hedge is

$$
\begin{aligned}
N(S,t) &= \frac{\partial}{\partial S}V(S+\delta S, t+\delta t) - \frac{\partial}{\partial S}V(S,t) \\
&\approx \frac{\partial^2 V}{\partial S^2}\delta S \\
&\approx \frac{\partial^2 V}{\partial S^2}\sigma S Z\sqrt{\delta t}
\end{aligned}
$$

N itself is stochastic and related to Γ of course. The P&L is not riskless.

The average number of shares traded is

$$E[N] \approx \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S E|Z| \sqrt{\delta t} = \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S \sqrt{\delta t}$$

The **average** transactions cost obtained by multiplying the above by the cost κS per share is

$$\sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa \sigma S^2 \sqrt{\delta t} = \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right| \kappa \sigma S^2 \frac{\delta t}{\sqrt{\delta t}}$$

The expected value of the change in the P&L is therefore given by

$$E[d\mathrm{P\&L}] = E\left[\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2\right)dt\right]$$

$$\approx \left[\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}Z^2 + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2\right)dt\right]$$

This isn't riskless.Nevertheless assume it expects to earn the riskless rate on the hedge, on average.

$$E[d\mathrm{P\&L}] = r\left(V - S\frac{\partial V}{\partial S}\right)dt.$$

Combining, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

Modified BS equation with nonlinear extra term proportional to the value of $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

The sum of two solutions to the equation is not necessarily a solution too; you cannot assume that the transactions costs for a portfolio of options is the sum of the transactions costs for hedging each option in isolation.

For a single long position in a call or a put, $\frac{\partial^2 V}{\partial S^2} \geq 0$, so we can drop the modulus sign.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.7.13}$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} \qquad \hat{\sigma} \approx \sigma - \kappa\sqrt{\frac{2}{\pi\delta t}}$$

This is the Black-Scholes equation with a modified reduced volatility, first derived by Leland, and the option is worth less. If you are long, you must pay less than the fair BS value since the hedging will cost you. For a short position, the effective volatility is enhanced, given by

$$\hat{\sigma} \approx \sigma + \kappa\sqrt{\frac{2}{\pi\delta t}}$$

When you sell the option you must ask for money because hedging it is going to cost you. For very small δt this expression diverges and the approximation becomes invalid.