# LECTURE 5

# VARIANCE SWAPS CONTINUED

# DYNAMIC REPLICATION

# Recap: What Should You Pay for Convexity?

Suppose we think we know the future volatility of the stock, $\Sigma$.

Binomially, this corresponds to $\Delta S = \pm\Sigma S\sqrt{\Delta t}$ with $(\Delta S)^2 = \Sigma^2 S^2 \Delta t$.

Change in value from the movement in stock price $= \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t)$

Change in value from passage of time $= \Theta(\Delta t)$ where $\Theta = \frac{\partial C}{\partial t}$

Total change in value of the hedged position is $d\text{P\&L} = d(C - \Delta S) = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t) + \Theta(\Delta t)$

If we know $\Sigma$, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it behaves like a riskless bond and must earn interest:

$$\left(\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2\right)dt = rdt\left(C - \frac{\partial C}{\partial S}S\right)$$

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

# Recap: Hedging an Option Means Betting On Volatility

$$\text{Net P\&L} = \frac{1}{2}\int \Gamma S^2(\sigma^2 - \Sigma^2)\Delta t$$

In a BS world, you can capture pure volatility if you own a derivative O whose curvature satisfies

$$\Gamma_O = 1/S^2 \quad \text{P\&L(O)} = \int \frac{1}{2}(\sigma^2 - \Sigma^2)\Delta t$$

The security with this gamma is the "log contract" with value $O = -\ln S$ and a hedge ratio $\Delta = -1/S$, **independent** of volatility! You hedge it by owning \$1 worth of stock always.

# Volatility and Variance Swap Contracts

A variance swap is a forward contract on realized variance. It pays

$$\left(\sigma_R^2 - K_{var}\right) \times N$$

**Exhibit 1.1.1 — Variance Swap on S&P 500 : sample terms and conditions**

# VARIANCE SWAP ON S&P500

**SPX INDICATIVE TERMS AND CONDITIONS**

| | |
|---|---|
| Instrument: | Swap |
| Trade Date: | TBD |
| Observation Start Date: | TBD |
| Observation End Date: | TBD |
| Variance Buyer: | TBD (e.g. JPMorganChase) |
| Variance Seller: | TBD (e.g. Investor) |
| Denominated Currency: | USD ("USD") |
| Vega Amount: | 100,000 |
| Variance Amount: | 3,125 ( determined as Vega Amount/(Strike*2) ) |
| Underlying: | S&P500 (Bloomberg Ticker: SPX Index) |
| **Strike Price:** | **16** |
| Currency: | USD |
| Equity Amount: | **T+3** after the Observation End Date, the Equity Amount will be calculated and paid in accordance with the following formula: |

*Final Equity payment = Variance Amount * (Final Realized Volatility$^2$ – Strike Price$^2$)*

If the Equity Amount is positive the Variance Seller will pay the Variance Buyer the Equity Amount.
If the Equity Amount is negative the Variance Buyer will pay the Variance Seller an amount equal to the absolute value of the Equity Amount.

where

$$\textit{Final Realised Volatility} = \sqrt{\frac{252 \times \sum_{t=1}^{t=N}\left(\ln\frac{P_t}{P_{t-1}}\right)^2}{\text{Expected\_N}}} \times 100$$

***Expected_N*** = [number of days], being the number of days which, as of the Trade Date, are expected to be Scheduled Trading Days in the Observation Period

$P_0$ = The Official Closing of the underlying at the Observation Start Date

$P_t$ = Either the Official Closing of the underlying in any observation date t or, at Observation End Date, the Official Settlement Price of the Exchange-Traded Contract

| | |
|---|---|
| Calculation Agent: | JP Morgan Securities Ltd. |
| Documentation: | ISDA |

# Volatility Derivatives in Practice: Activity and Impact

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**Abstract:**

We use unique regulatory data to examine open positions and activity in both listed and OTC volatility derivatives. Gross vega notional outstanding for index variance swaps is over USD 2 billion, with dealers short vega in order to supply the long vega demand of asset managers. For maturities less than one year, VIX futures are far more actively traded and have a higher notional amount outstanding than S&P 500 variance swaps. To the extent that dealers take on risk when facilitating trades, we estimate that the long volatility bias of asset managers puts upward pressure on VIX futures prices. Hedge funds have offset this potential impact by actively taking a net short position in nearby contracts. In our 2011-2014 sample, the net impact added less than half a volatility point, on average, to nearby VIX futures contracts but added between one and two volatility points for contracts in less liquid, longer-dated parts of the curve. We find no evidence that this price impact forces VIX futures outside no-arbitrage bounds.

# Intuitive Approach to Variance Replication in a BS World

Zero interest rates for simplicity, so $C = C(S, K, \nu)$ where $v = \sigma\sqrt{\tau}$.

$$C_{BS} = SN(d_1) - KN(d_2) \qquad d_{1,2} = \frac{\ln S/K \pm v^2/2}{v}$$

Then the exposure to volatility is given by

$$\kappa \equiv \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

You can see that the option has sensitivity to S and σ, and is therefore not a good way to make a clean bet on volatility. What we want is a portfolio whose exposure κ to volatility is independent of the stock price $S$, so that we can bet on volatility no matter what the stock price does.

Construct a portfolio $\pi(S) = \int_0^\infty \rho(K) C(S, K, v) dK$ such that $\kappa = \frac{\partial \pi}{\partial \sigma^2}$ is independent of S.

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^\infty \rho(K) \frac{S\sqrt{\tau}}{2\sigma} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} dK \sim \int_0^\infty \rho(K) Sf\left(\frac{K}{S}, v\right) dK$$

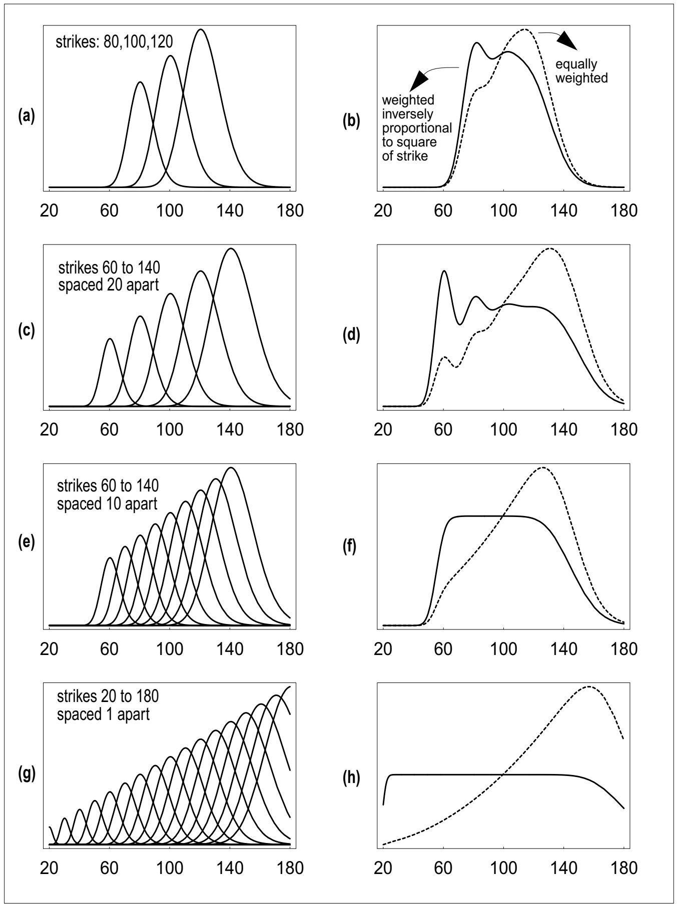
We can make the S-dependence of this explicit by changing variable to $x = K/S$ so that

$$\frac{\partial\pi}{\partial\sigma^2} = \int_0^{\infty} \rho(xS)S^2 f(x, v)dx$$

In order for this to be independent of S, we require that $\rho(K) \sim 1/K^2$

A density of options whose weights decrease as $K^{-2}$ will give the correct volatility dependence.

**FIGURE 1. The variance exposure, $V$, of portfolios of call options of different strikes as a function of stock price S. Each figure on the left shows the individual $V_i$ contributions for each option of strike $K_i$. The corresponding figure on the right shows the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally-weighted sum of options; the solid line corresponds to weights inversely proportional to $K^2$, and becomes totally independent of stock price S inside the strike range**



# What Payoff Are We Replicating?

Use liquid puts below some strike $S^*$ and use calls with strikes above $S^*$. The payoff at expiration is

$$\pi(S, S^*, v) = \int_{(K > S^*)} C(S, K, v)\frac{dK}{K^2} + \int_{(K < S^*)} P(S, K, v)\frac{dK}{K^2}$$

What does this payoff look like at expiration when $\tau = 0$: Call has S > K; Put has S < K

$$\pi(S, S^*, v) = \left(\int_{S^*}^{S} (S-K)\frac{dK}{K^2}\right) \text{ for S>S* and } \int_{S}^{S^*} (K-S)\frac{dK}{K^2} \text{ for S<S*}$$

$$= -\ln\frac{S}{S^*} + \left(\frac{S-S^*}{S^*}\right)$$

**In order to be exposed purely to volatility, we need to short a log contract L and own a forward contract with delivery price $S^*$, which has no volatility dependence and can be replicated statically**

# Value of Log Contract in a Black-Scholes World

Solve the Black-Scholes equation $\frac{\sigma^2 S^2}{2}\frac{\partial L}{\partial S^2} + \frac{\partial L}{\partial t} = 0$ for r =0, with the boundary condition for the log payoff $L(S, S^*, 0) = -\ln\frac{S}{S^*}$.

Solution: $L(S, S^*, \tau) = -\ln S/S^* + (\sigma^2\tau)/2$. with volatility exposure $\kappa = \tau/2$.

The delta of the contract is $-1/S$.

Going long 1/S shares at any instant – i.e. by owning exactly \$1 worth of shares at any instant – you have exactly a $\Gamma = 1/S^2$ and the right vol. exposure.

At the start of the trade, when $t = 0$ and $\tau = T$, you need to buy 2/T contracts to have $\kappa = 1$, a variance exposure of \$1 for the whole trade.

$\Pi(S, S_*, t, T) = \frac{2}{T}\left[\frac{S - S_*}{S_*} - \ln\frac{S}{S_*}\right] + \frac{T-t}{T}\sigma^2$, after hedging, captures $\sigma_R^2 - \sigma_I^2$

If $S^* = S_0$ then $\Pi(S, S_0, t, T) = \frac{2}{T}\left[\frac{S - S_0}{S_0} - \ln\frac{S}{S_0}\right] + \frac{T-t}{T}\sigma^2$ and $\Pi(S_0, S_0, 0, T) = \sigma^2$

## Proof that the fair value of a log contract with $S^*=S_0$ is actually the variance. (Assume r = 0)

Consider a log contract that pays out $\log(S_T/S_0)$ at expiration time T. Let its value today be denoted by $L_0$. Look at the trading strategy below that starts with a short position in one log contract and long \$1 worth of shares, and then maintains this dollar value of shares by rehedging as below.

| time | stock | #shares | value | Dollars | log contracts | Total value |
|---|---|---|---|---|---|---|
| $t_0$ | $S_0$ | $1/S_0$ | 1 | 0 | -1 worth $L_0$ | $-L_0 + 1$ |
| $t_1$ | $S_1$ | $1/S_0$ | $S_1/S_0$ | 0 | -1 worth $L_1$ | $-L_1 + S_1/S_0$ |

Now rebalance to own \$1 worth of shares:. buy $(1/S_1 - 1/S_0)$ shares by borrowing $(1/S_1 - 1/S_0)S_1 =$ $(S_0 - S_1)/S_o$ dollars. You then own $1/S_1$ shares worth \$1, and you have borrowed (that is, you are short) $(S_0 - S_1)/S_o$ dollars. Then, after rebalancing,

| time | stock | #shares | value | Dollars | log contracts | Total value |
|---|---|---|---|---|---|---|
| $t_1$ | $S_1$ | $1/S_1$ | 1 | $-(S_0 - S_1) \div S_o$ | -1 worth $L_1$ | $-L_1 + 1 + (S_1 - S_0) \div S_0$ |

Now move to time $t_2$ and rebalance again, to get

| time | stock | #shares | value | Dollars | log contracts | Total value |
|---|---|---|---|---|---|---|
| $t_2$ | $S_2$ | $1/S_2$ | 1 | $-(S_0 - S_1)/S_o$ <br> $-(S_1 - S_2)/S_1$ | -1 worth $L_2$ | $-L_2 + 1 + {-(S_0 - S_1) \div S_o \atop -(S_1 - S_2) \div S_1}$ |

Repeat rehedging N times to expiration:

$$1 - L_T + \frac{S_1 - S_0}{S_0} + \frac{S_2 - S_1}{S_1} + \ldots + \frac{S_N - S_{N-1}}{S_{N-1}} = 1 - L_T + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_{N-1}}\frac{S_{N-1}}{S_{N-2}}\ldots\frac{S_1}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \sum_{i=0}^{N-1}\left(\log\frac{S_{i+1}}{S_i}\right) + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$\approx 1 - \sum_{i=0}^{N-1}\left[\frac{\Delta S_i}{S_i} - \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2\right] + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$ in a Taylor expansion to second order

$$= 1 + \sum_{i=0}^{N-1}\frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 = 1 + \sum\frac{\sigma_i^2 \Delta t_i}{2}$$

Thus, if you assume zero interest rates, we've shown that an initial investment at time t = 0 of value

$-L_0 + 1$, by dynamic rehedging, leads to a final value at time t = T of $1 + \sum \frac{\sigma_i^2 \Delta t_i}{2}$.

Therefore, the fair value of $L_0$ at the beginning must be $L_0 = -\sum_i \frac{\sigma_i^2 \Delta t_i}{2}$.

**Being short a log contract with strike $S_0$ and being long \$1 worth of stock, dynamically rehedging as the stock moves, will guarantee you a final payoff equal to the realized volatility (assuming GBM with a variable volatility).**

## Problems with Replication

If you could buy a log contract you'd have exactly what you want. Instead you have to buy a continuum of calls and puts, which doesn't exist. You can only buy a discrete number in a discrete range, so you have no sensitivity to volatility outside the strike range.

## 1.1 More Rigorous Results

Everything so far has been in a Black-Scholes world, unlike the real one, which has skew, stochastic volatility and jumps. Does this still work?

As long as there is continuous diffusion (no jumps), the log contract still captures realized volatility.

$$\frac{dS}{S} = \mu dt + \sigma(t, \ldots)dZ_t$$

$$d\ln S = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t$$

$$\frac{dS}{S} - d\ln S = \frac{1}{2}\sigma^2 dt$$

$$\text{total variance} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\underbrace{\int_0^T \frac{dS_t}{S_t}}_{\text{rebalanced hedge}} - \underbrace{\ln\frac{S_T}{S_0}}_{\text{short log contract}}\right]$$

**No expectations have been taken here; replication is enough to capture variance!** Thus

$$-\ln\frac{S_T}{S_0} = \frac{1}{2}\int_0^T \sigma^2 dt - \int_0^T \frac{dS_t}{S_t} \text{ and } \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right]$$

# Replicate $\ln(S_T/S_0)$ by Forward Plus Weighted Calls And Puts

$$-\ln \boldsymbol{S_T/S_0} = -\ln S_*/S_0 - \ln \boldsymbol{S_T/S_*}$$ **and by payoff replication**

$$-\ln \boldsymbol{S_T/S_*} = -\frac{(S_T - S_*)}{S_*} + \int_{(K>S^*)} C(S_T, K, v)\frac{dK}{K^2} + \int_{(K<S^*)} P(S_T, K, v)\frac{dK}{K^2}$$

Therefore

$$-\ln \boldsymbol{S_T/S_0} = -\ln S_*/S_0 - \frac{(S_T - S_*)}{S_*} + \int_{(K>S^*)} C(S_T, K, v)\frac{dK}{K^2} + \int_{(K<S^*)} P(S_T, K, v)\frac{dK}{K^2}$$

Therefore, adding in the integral, we have **a prescription for generating realized volatility**:

$$\frac{1}{T}\int_0^T \sigma^2 dt =$$

"rehedging" \$1

$\ln(S_T/S_0)$

options value at expiration

$$= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_*}{S_0} - \frac{(S_T - S_*)}{S_*} + \int_{(K>S^*)} C(S_T, K, 0)\frac{dK}{K^2} + \int_{(K<S^*)} P(S_T, K, 0)\frac{dK}{K^2}\right]$$

The RHS consists of a strategy trade and some payoffs of forwards and options that tell us how to replicate the variance.

**Its fair value is given by the risk-neutral expectation.** If in the risk-neutral world

$$\frac{dS_t}{S_t} = r\,dt + \sigma_t dZ_t$$

- $\frac{dS_t}{S_t} = \frac{S_{n+1} - S_n}{S_n}$ is the adjustment in the position that always holds \$1 worth of stock.
-
- $E\left[\int_0^T \frac{dS_t}{S_t}\right] = rT$ and $E[S_T] = S_0 e^{rT}$
- The current call value $C(S, K) = e^{-rT} E[C(S_T, K, 0)]$ so $E[C(S_T, K, 0)] = e^{rT} C(S, K)$

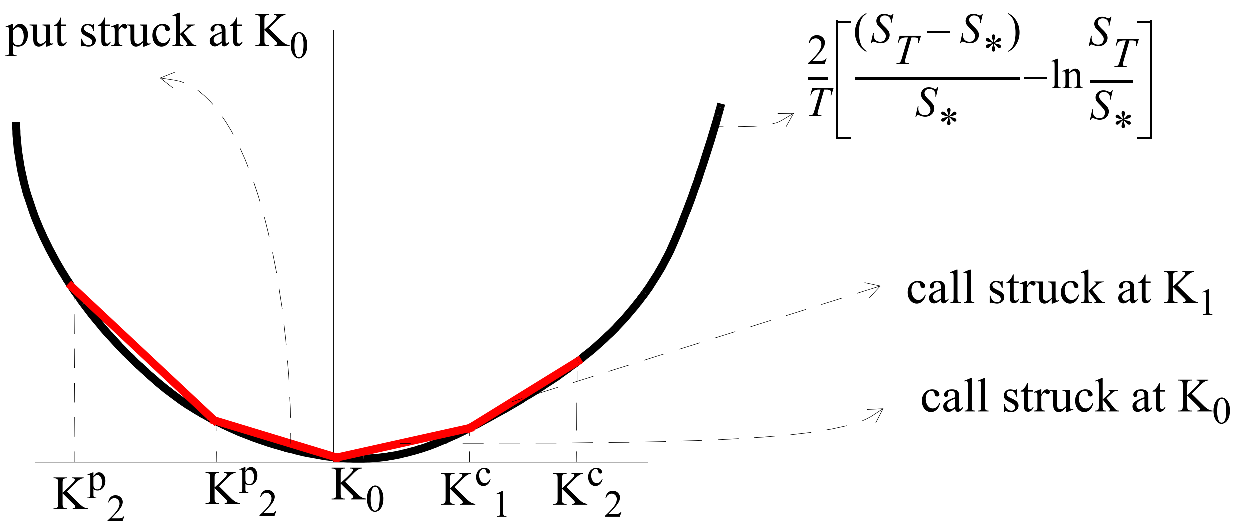
Thus the fair value of the total variance is

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{rT}}{S_*} - 1\right) + e^{rT} \int_{(K > S^*)} C(S, K)\frac{dK}{K^2} + e^{rT} \int_{(K < S^*)} P(S, K)\frac{dK}{K^2}\right]$$

Every option's price can be taken from the marketplace, even with a skew, and we can value the variance almost independent of theory.

# Fair variance in a skew

Replicate $\frac{2}{T}\left[\frac{(S_T - S_*)}{S_*} - \ln\frac{S_T}{S_*}\right]$ by linear-payoff calls and puts that dominate it, with strikes $K_i^{c,p}$ for the calls and puts.




See Demeterfi et al paper: *More than you ever wanted to know* ... posted on Courseworks.

# Imperfections in Valuation by Replication

- Discrete strikes with a limited range capture less variance than the true variance. You gamble by omitting some strikes because when/if the stock price gets to those strikes, you have no options to capture the variance.
- Effect of jumps
  The log contract doesn't capture the true variance if jumps occur, for two reasons.

  1. Jumps can move the stock price out of the range of replication.
  2. Jumps contribute to the realized variance proportional to $J^2$, but jumps contribute to the log contract with a $J^3$ term too.

  The log contract captures

$$\sum \frac{\Delta S_i}{Si} - \log\frac{S_T}{S_0} = \sum\left[\frac{\Delta S_i}{Si} - \log\frac{S_{i+1}}{S_i}\right] = \sum\left[\frac{\Delta S_i}{Si} - \log\left(1 + \frac{\Delta S_i}{S_i}\right)\right]$$

$$\approx \sum \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 - \frac{1}{3}\left(\frac{\Delta S_i}{S_i}\right)^3 + \ldots$$

The first term is the true variance contribution; the second is normally negligible, but for a large jump $(\Delta S_i)/S_i = J$ will add an asymmetric term to the P&L that is absent from the true variance

## 1.2 Valuing Volatility Swaps

**Volatility is the square root of variance, a derivative.** You can replicate it with the continuous dynamic trading of portfolios of variance swaps, just as you can replicate $\sqrt{S}$ by trading $S$.

Expand about $V_E$, the expected variance.

$$\sigma = \sqrt{\sigma^2} = \sqrt{V} \equiv \sqrt{V_E + \{V - V_E\}}$$

$$= \sqrt{V_E}\left(1 + \frac{V - V_E}{V_E}\right)^{1/2}$$

$$\approx \sqrt{V_E}\left[1 + \frac{V - V_E}{2V_e} - \frac{1}{8}\left(\frac{V - V_E}{V_E}\right)^2 + \dots\right]$$ The square root has negative convexity therefore worth less.

$$\approx \sqrt{V_E} + \frac{V - V_E}{2\sqrt{V_E}} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$$

Taking risk-neutral expectations: $E(\sigma) \approx \sqrt{V_E} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$

Thus the fair volatility is smaller than the square root of the variance, and depends on the volatility of variance, like an option on variance.

## 1.3 The VIX Volatility Index

The VIX, from 1993 - 2003, used to defined as the weighted average of various atm and otm implied volatilities. This was rather arbitrary. In 2003 the CBOE changed the definition of the VIX to be the square root of the fair delivery price of variance as captured by a variance swap, using the formula from this paper with stock
dividends.

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[(r-d)T - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{(r-d)T}}{S_*} - 1\right) + e^{rT}\int_{(K > S^*)} C(S,K,0)\frac{dK}{K^2} + e^{rT}\int_{(K < S^*)} P(S,K,0)\frac{dK}{K^2}\right]$$

The RHS is

$$\frac{2}{T}\left\{\ln\frac{F}{S_0} - \ln\frac{S_*}{S_0} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls above S* plus puts below S*}]\right\}$$

$$= \frac{2}{T}\left\{\ln\frac{F}{S^*} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$= \frac{2}{T}\left\{\ln\left(1 + \frac{F}{S^*} - 1\right) - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$\approx \frac{2}{T}\left\{e^{rT}[\text{sum of calls and puts}] - \frac{1}{2}\left(\frac{F}{S_*} - 1\right)^2\right\}$$

The CBOE uses a finite sum over traded options at two expirations near 30 days, and then interpolates/extrapolates to thirty day volatility.

Some advantages of the new VIX

- The VIX is an estimate of one-month future realized volatility based on listed options prices. The value of the VIX depends on implied volatility.
- The estimate is independent of market level because it involves the sum of different options prices.
- It is relatively insensitive to model issues, because it assumes only continuous underlier movement, but doesn't assume Black-Scholes.
- It is hedgeable because it involves a portfolio of listed options.

You can therefore in principle price futures, forwards and options on the VIX.

### Future Extensions

Many variance swaps are capped and implicitly contain embedded volatility options.

Valuing options on volatility is the big challenge.

Modeling the VIX and VIX futures because it's the most liquid measure of volatility.

# Aside: The Black-Scholes Equation and Sharpe Ratios

Valuation by perfect replication. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

$$
\begin{aligned}
dS_t &= \mu_S S_t dt + \sigma_S S_t dZ_t \\
dB_t &= B_t r_t dt
\end{aligned}
\qquad \text{Eq.5.1}
$$

The option price $C(S_t, t)$ whose evolution is given by

$$
\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2 dt \\
&= \left\{ \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2 \right\} dt + \frac{\partial C_t}{\partial S}\sigma_S S_t d \\
&\equiv \mu_C C_t dt + \sigma_C C_t dZ_t
\end{aligned}
$$

where by definition

$$\mu_C = \frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2\right\} \qquad \text{Eq.5.2}$$

$$\sigma_C = \frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_S S_t\right)$$

Riskless portfolio $\pi = \alpha S + C$

Then

$$\begin{aligned} d\pi &= \alpha\{\mu_S S_t dt + \sigma_S S_t dZ_t\} + \{\mu_C C_t dt + \sigma_C C_t dZ_t\} \\ &= (\alpha\mu_S S_t + \mu_C C_t)dt + (\alpha\sigma_t S_t + \sigma_C C_t)dZ_t \end{aligned} \qquad \text{Eq.5.3}$$

Riskless necessitates

$$\alpha = -\frac{\sigma_C C}{\sigma_S S} \qquad \text{Eq.5.4}$$

That no riskless arbitrage: $d\pi = \pi r dt$.

Requires $\alpha\mu_S S + \mu_C C = (\alpha S + C)r$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for $\alpha$ from Equation 5.4 leads to the relation

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S} \qquad \text{Eq.5.5}$$

This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 4.2 into Equation 4.5 for $\mu_C$ and $\sigma_C$ we obtain

$$\frac{\frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_S S_t)^2\right\} - r}{\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_S S_t\right)} = \frac{\mu_S - r}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \qquad \text{Black-Scholes equation, no drift} \qquad \text{Eq.5.6}$$

It's good to get very familiar with manipulating this solution and its derivatives.

The solution, the Black-Scholes formula and its implied volatility, is the quoting currency for trading prices of vanilla options.

You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$C(S, K, t, T, r, \sigma) = e^{-r(T-t)} \times [S_F N(d_1) - K N(d_2)]$$

$$S_F = e^{r(T-t)} S$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm 0.5\sigma_S^2 (T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.5.7}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy$$

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma_S^2 (T-t)$. If you rewrite the formula in terms of the prices of traded securities – the present value of the bond $K_{PV}$ and the stock price $S$ – then indeed time and volatility always appear together:

$$C(S, K, t, T, \sigma) = [SN(d_1) - K_{PV}N(d_2)]$$

$$K_{PV} = e^{-r(T-t)}K$$

$$d_{1,2} = \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq.5.8}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

Note that the time to expiration appears in the formulas in two different combinations, $r(T-t)$ the discount factor and $\sigma^2(T-t)$ the total variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.

**Next: The P&L of Hedged Trading Strategies**

# NEW TOPIC:
# P&L HEDGING

## 1.4 The P&L of Hedged Trading Strategies

Consider an initial position at time $t_0$ in an option C that is $\Delta$-hedged with borrowed money which earns interest *r*, and then rehedged using in discrete steps at times $t_i$ and stock prices $S_i$.

**Notation:** $C_n = C(S_n, t_n)$   $\Delta_n = \Delta(S_n, t_n)$.

| $t_n,S_n$ | Hedging action | No. Shares | Share Value | Dollars Received From Shares Traded | Net Value of Position: Option + Stock + Cash |
|---|---|---|---|---|---|
| $t_0,S_0$ | Buy $C_0$, short $\Delta_0$ shares | $-\Delta_0$ | $-\Delta_0 S_0$ | $\Delta_0 S_0$ | $C_0$ |
| $t_1,S_1$ | none | $-\Delta_0$ | $-\Delta_0 S_1$ | $\Delta_0 S_0 e^{r\Delta t}$ | $C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\Delta t}$ |
| | get short $\Delta_1$ shares by shorting $\Delta_1 - \Delta_0$ shares | $-\Delta_1$ | $-\Delta_1 S_1$ | $\Delta_0 S_0 e^{r\Delta t}$<br>$+ (\Delta_1 - \Delta_0) S_1$ | $C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\Delta t}$<br>$+ (\Delta_1 - \Delta_0) S_1$ |
| $t_2,S_2$ | none | $-\Delta_1$ | $-\Delta_1 S_2$ | $\Delta_0 S_0 e^{2r\Delta t}$<br>$+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ | $C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\Delta t}$<br>$+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ |

| $t_n,S_n$ | Hedging action | No. Shares | Share Value | Dollars Received From Shares Traded | Net Value of Position: Option + Stock + Cash |
|---|---|---|---|---|---|
| $t_2,S_2$ | get short $\Delta_2$ shares by shorting $\Delta_2 - \Delta_1$ shares | $-\Delta_2$ | $-\Delta_2 S_2$ | $\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$ | $C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$ |
| etc. | | | | | |
| $t_n,S_n$ | get short $\Delta_n$ shares by shorting $\Delta_n - \Delta_{n-1}$ shares | $-\Delta_n$ | $-\Delta_n S_n$ | $\Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$ | $C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1}) S_n$ |

The initial value of the positions was $C_0$ and would have generated $C_0 e^{r(T-t)}$

The final value is $C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$

where the subscript $b$ at the end of the formula denotes a backwards Ito integral.

Therefore, the fair value of $C_0$ is given by equating these two quantities:

$$e^{r(T-t)}C0 = C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$$

or

$$(C_0 - \Delta_0 S_0)e^{r(T-t)} = (C_T - \Delta_T S_T) + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b \qquad [A]$$

You can integrate by parts using the relation

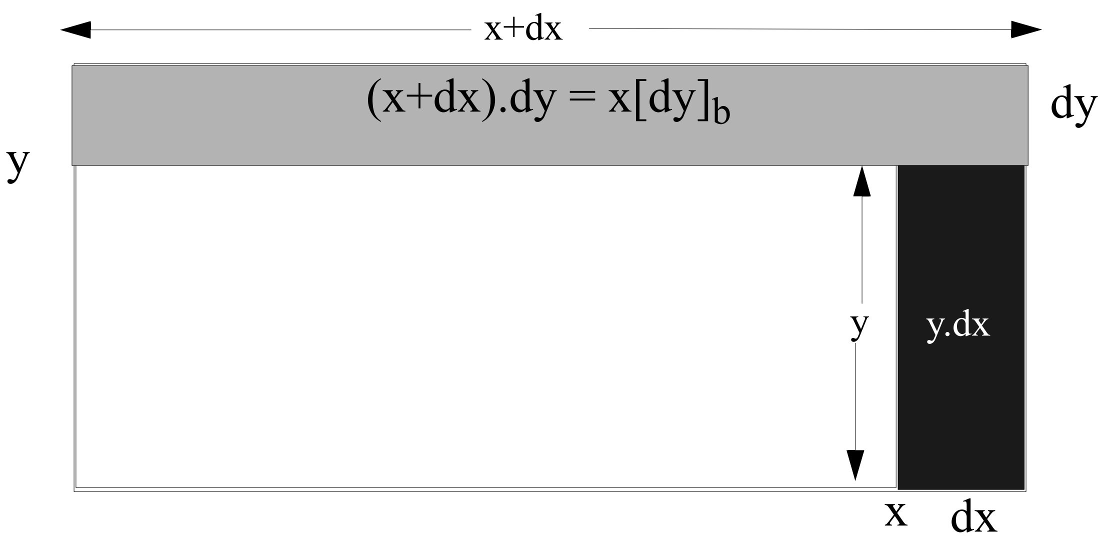
$$e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b = d\left[e^{r(T-\tau)} S_\tau \Delta_\tau\right] + re^{r(T-\tau)} \Delta_\tau S_\tau d\tau - e^{r(T-\tau)} \Delta_\tau dS_\tau$$

to obtain

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[dS_\tau - S_\tau r d\tau]e^{-r(T-\tau)} \qquad [B]$$

[A] and [B] provide a way to calculate the value of the call in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration.

# Backward Ito Integral




$$d[xy] = ydx + x[dy]_b$$