LECTURE 9

BACK TO THE SMILE

DAX Implied Volatility Surface 2008

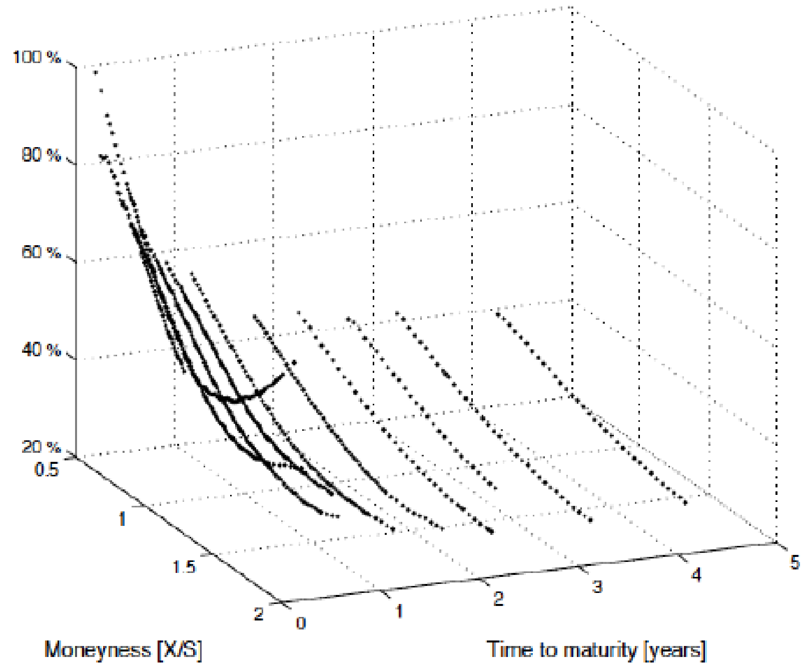


Fig. 1. IV surface of DAX index options from 28 Oct. 2008, traded at the EUREX. IV given in percent across a spot moneyness metric, time to expiry in years.

Implied Volatility as a Function of Strike/Spot for Different Expirations. (Crash-o-phobia: A Domestic Fear Or A Worldwide Concern? Foresi & Wu JOD Winter 05

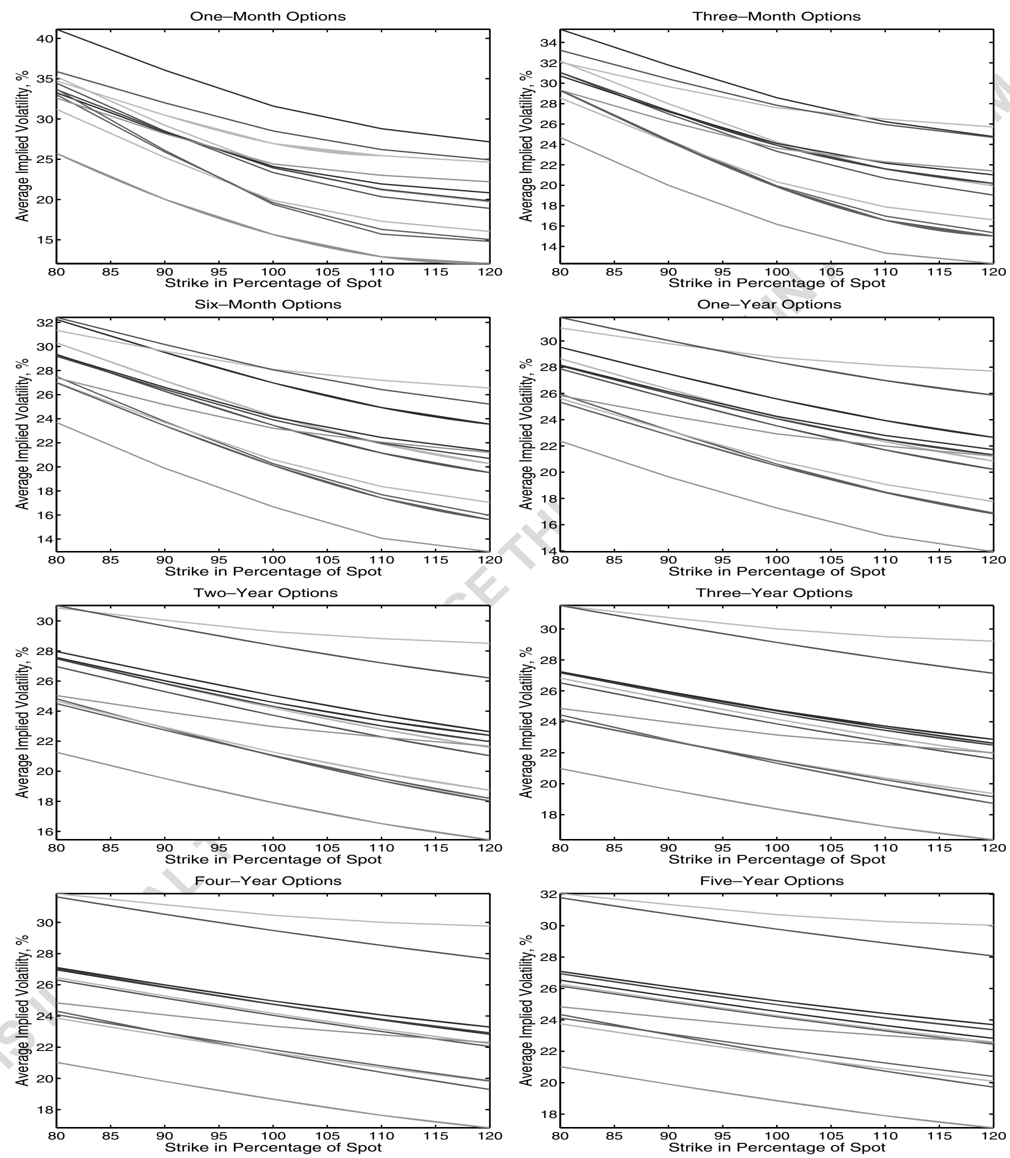
The quoting convention is the Black-Scholes implied volatility

EXHIBIT 2
Implied Volatility Smirk on Major Equity Indexes

Notice the patterns that persist across all indexes:

- out-of-the-money puts have higher implied Black-Scholes volatilities than out-of-the-money calls. (Why?)

- The slope of implied volatil-



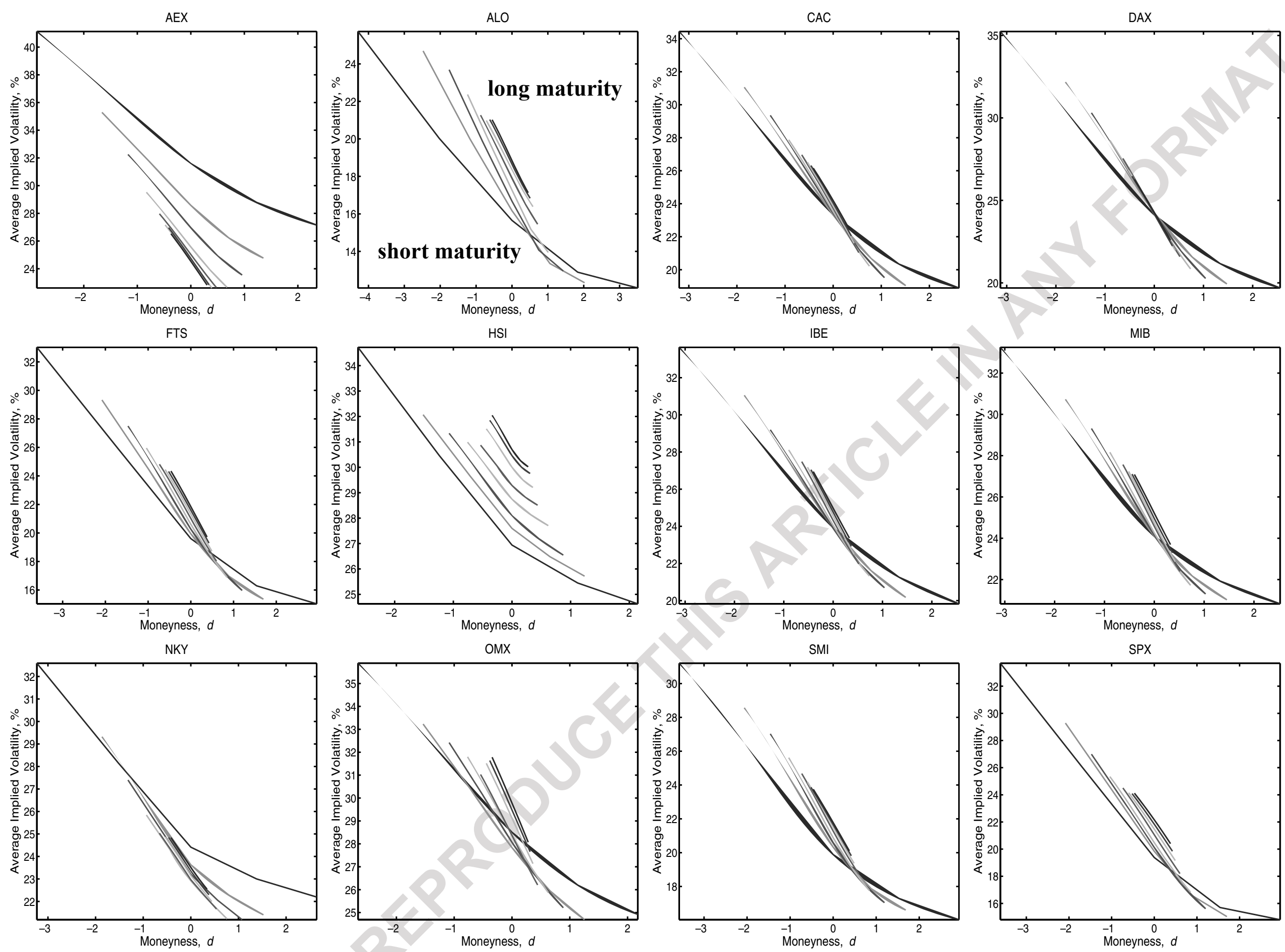
Lines represent the sample averages of the implied volatility quotes plotted against the fixed moneyness levels defined as strike prices as per

Implied Volatility as a Function of $\left(\log\dfrac{\text{Strike}}{\text{Spot}}\right)/(\sigma\sqrt{\tau})$

related to d_1

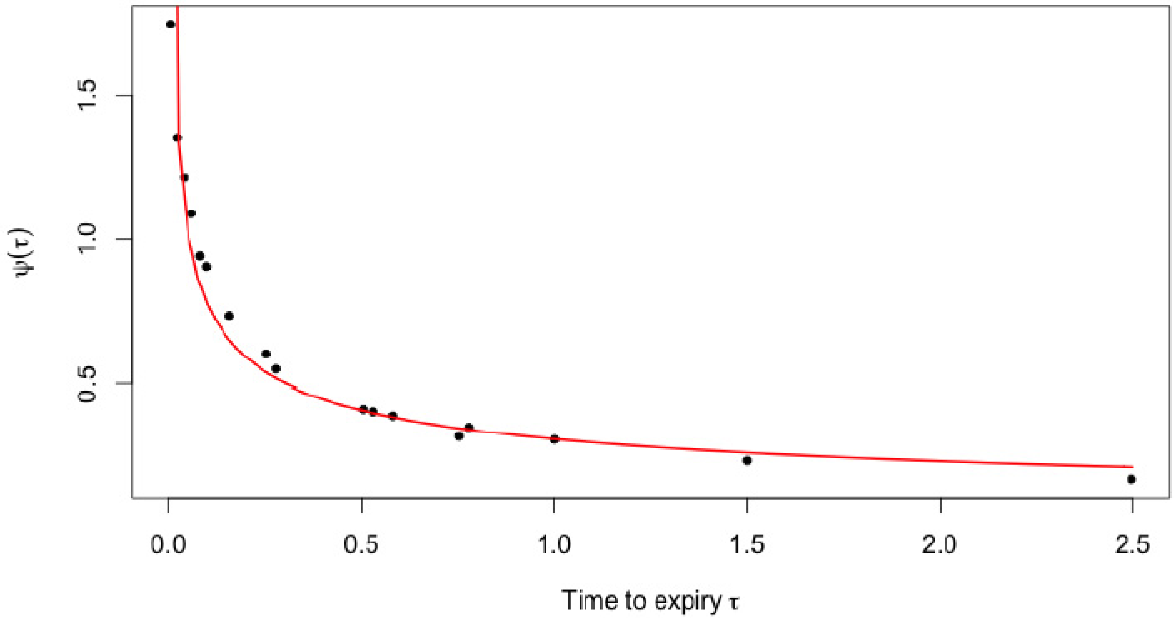
EXHIBIT 3

Maturity Pattern of Implied Volatility Smirks



When plotted against the number of standard deviations between the log of the strike and the log of the spot price for a lognormal process, the slope of the skew actually increases with expiration. Whatever is happening to cause this doesn't fade away with future time.

But plotted just against log moneyness, we see the slope decreases.

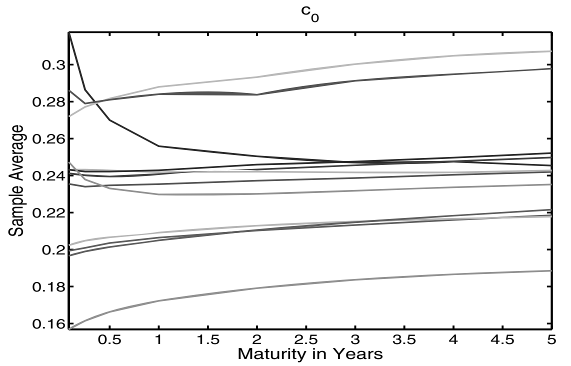


The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013 ; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.4}$.

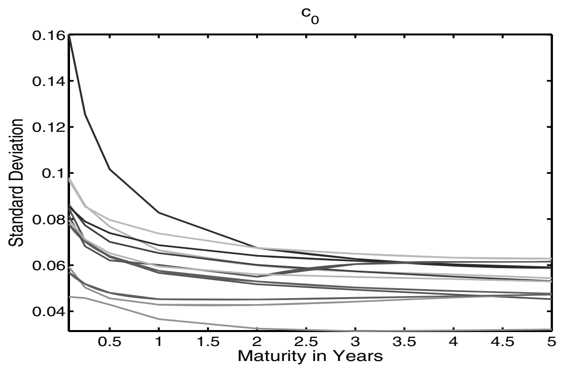
which is consistent by change of variable.

$$\frac{d\sigma}{d\left(\frac{m}{\sqrt{\tau}}\right)} = \sqrt{\tau}\frac{d\sigma}{d(m)} \sim \sqrt{\tau}\frac{1}{\tau^{0.4}} \sim \tau^{0.1}$$

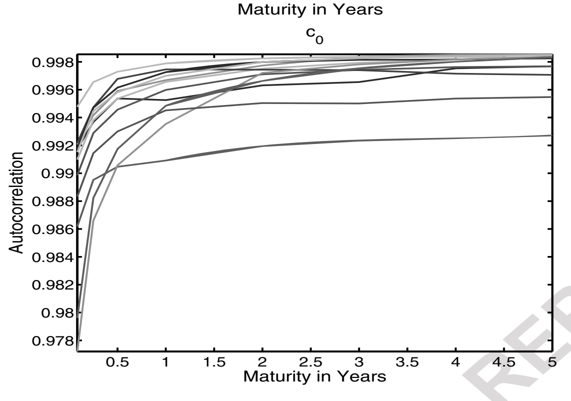
Behavior of implied volatility level c_0 as a function of option expiration.



term structure of implied volatility is roughly flat



volatility of volatility decreases with expiration, suggesting mean reversion or stationarity for the instantaneous volatility evolution

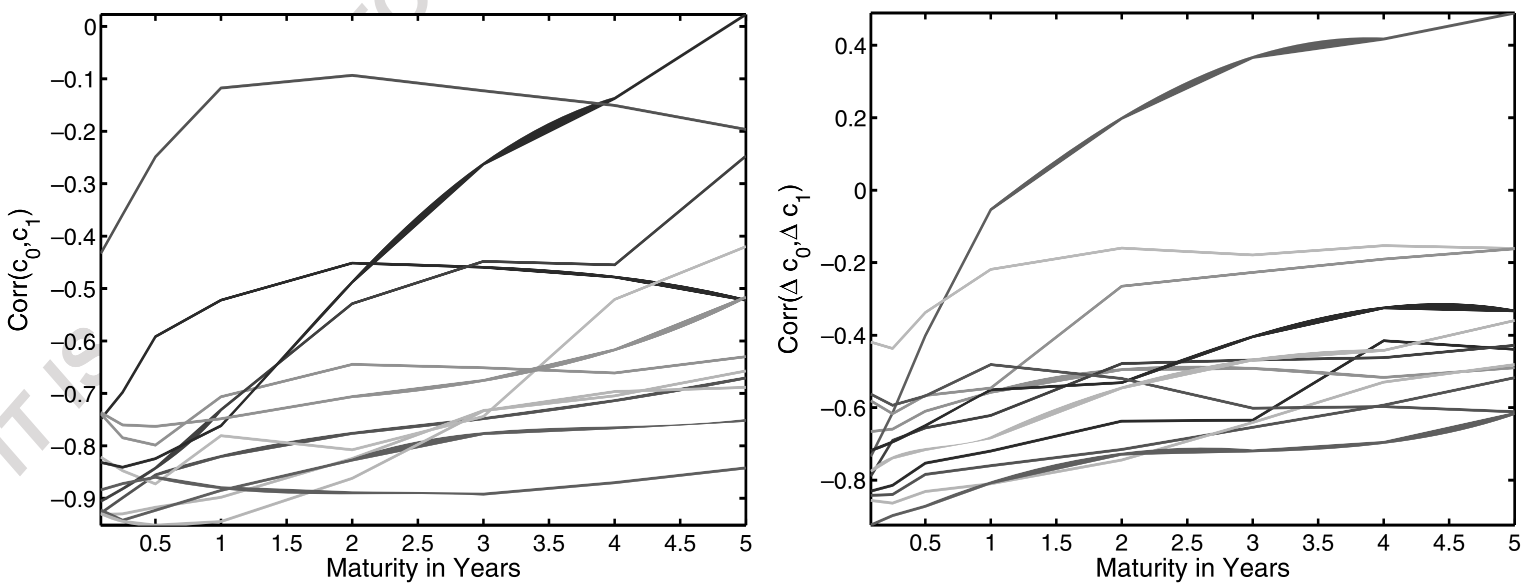


daily autocorrelation of implied volatility is large, and larger for longer maturities
[excitement or depression tends to continue]

The cross-correlation between volatility level and slope of the skew is large.

E X H I B I T 5

Cross Correlations between Volatility Level and Smirk Slope



Lines denote the cross-correlation estimates between the volatility level proxy (c_0) and the volatility smirk slope proxy (c_1). The left panel measures the correlation based on daily estimates, the right panel measures the correlation based on daily changes of the estimates.

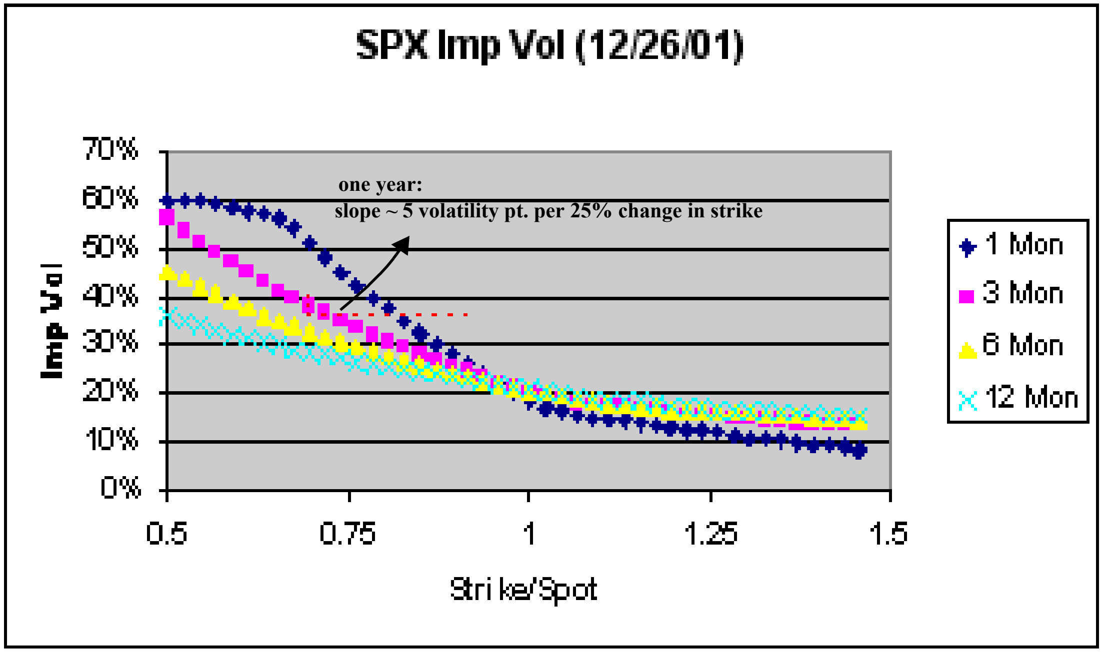
short-term slope tends to get more negative as volatility increases

Some characteristics of the equity implied volatility smile

- Volatilities are steepest for small expirations as a function of strike, shallower for longer expirations.
- The minimum volatility as a function of strike occurs near atm strikes or strikes corresponding to slightly otm call options.
- Low strike volatilities are usually higher than high-strike volatilities but high strike volatilities can rise a little too.
- The term structure can be up or down.
- The volatility of implied volatility is greatest for short maturities, as with Treasury rates.
- There is a negative correlation between changes in implied atm volatility and changes in the underlying asset itself. [Fengler: $\rho = -0.7$ for the DAX in the 2000's]
- Implied volatility appears to be mean reverting.
- Implied volatility tends to rise fast and decline slowly.
- Shocks across the surface are highly correlated. There are a small number of principal components or driving factors. We'll study these effects more closely later in the course.
- Implied volatility is usually greater than recent historical volatility.

Different Smiles in Different Markets

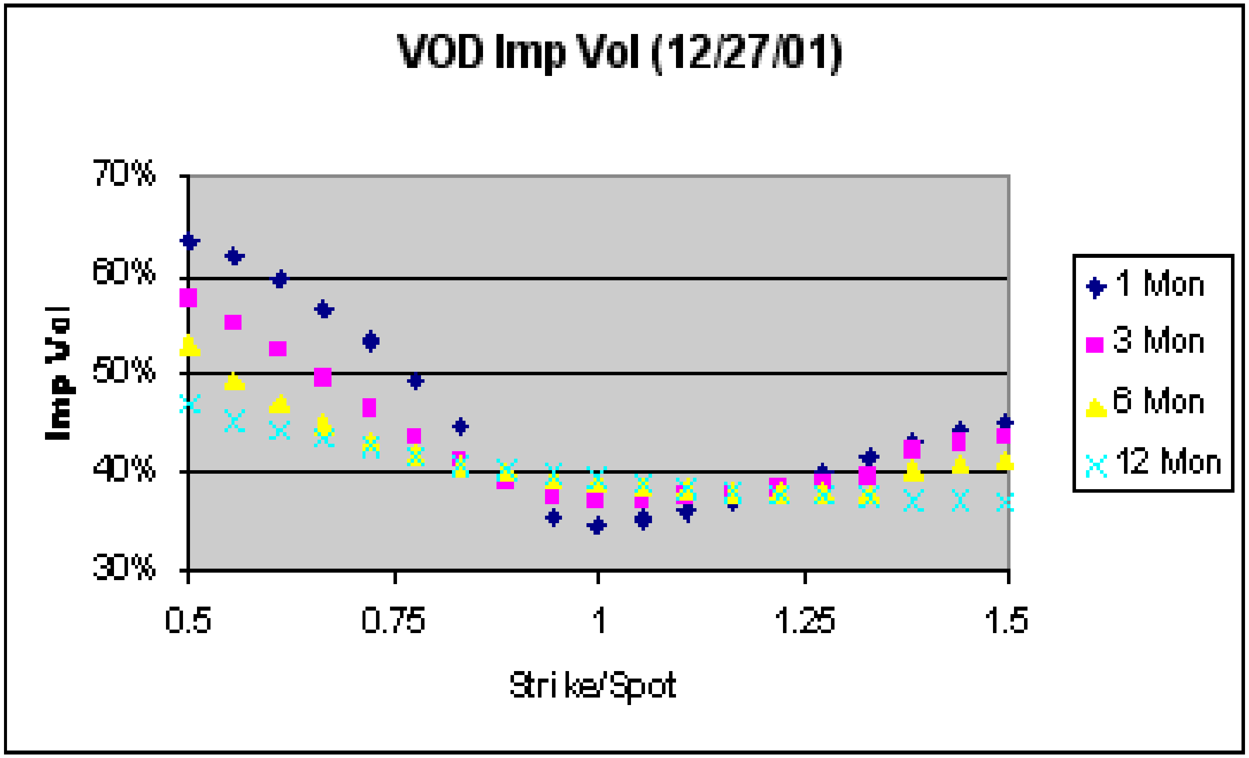
Here are some old smiles for the S&P 500, plotted a little differently:



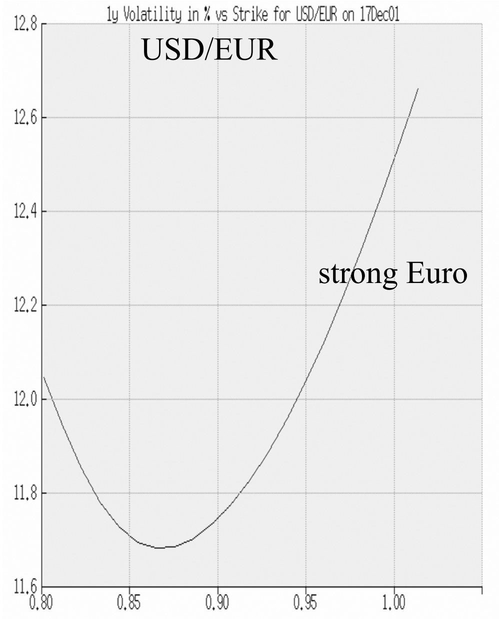
Indexes generally have a negative skew. The slope here for a one-year option is of order 5 volatility points per 250 S&P points, or about $\frac{0.05}{250} = 0.0002$. Note that the slope for a 3-month option is about twice as much, which roughly confirms the idea that the smile depends on $\frac{(\ln K/S)}{(\sigma\sqrt{\tau})}$, because a four-fold decrease in time to expiration then implied a doubling of the slope of the smile. The magnitude of the slope of the one-month option volatility is about 23 volatility points per 250 S&P points, or about 0.001.

Single stock smiles

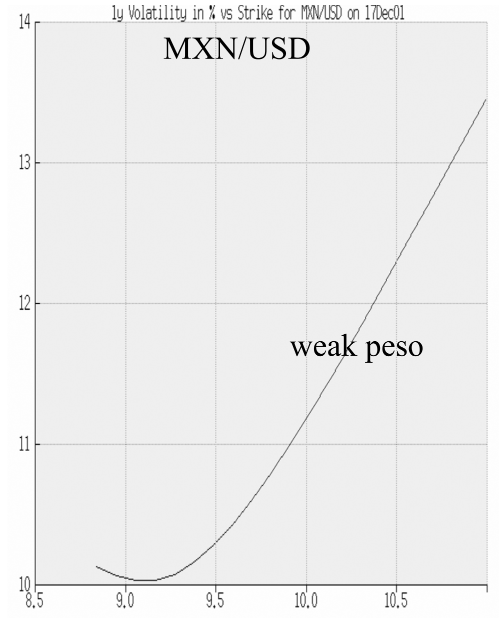
A single stock smile is more of an actual smile with both sides turning up.



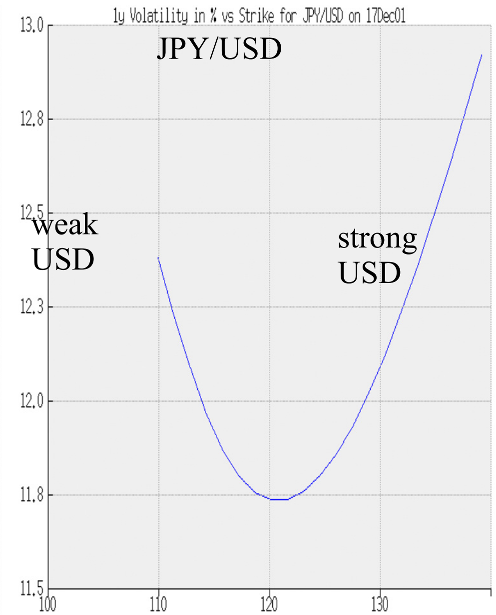
Some currency smiles....



ATM Strike = 0.90



9.85



123.67

The smiles are more symmetric for “equally powerful” currencies, less so for “unequal” ones. Equally powerful currencies are likely to move up or down.

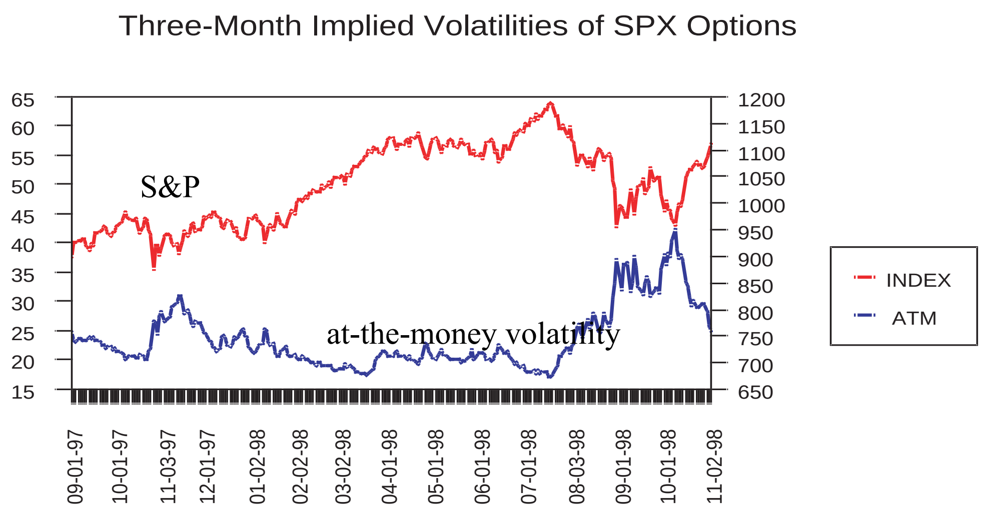
Equity index smiles tend to be skewed to the downside. The big painful move for an index is a downward move, and needs the most protection. Upward moves hurt almost no-one. An option on index vs. cash is very different and much more asymmetric than an option on JPY vs. USD.

Single-stock smiles tend to be more symmetric than index smiles. Single stock prices can move dramatically up or down. Indexes like the S&P when they move dramatically, move down.

Interest-rate or swaption volatility, which we will not consider much in this course, tend to be more skewed and less symmetric, with higher implied volatilities corresponding to lower interest rate strikes. This can be partially understood by the tendency of interest rates to move normally rather than lognormally as rates get low.

Variation of implied volatility and the smile over time

Example: here is the behavior of “volatility” itself as time passes.



Why do traders talk most about atm volatility?

ATM volatility is therefore not the volatility of a particular option you own.

Some of the apparent correlation in the figure above would occur even if $\Sigma(S, t, K, T)$ didn’t change with S at all. How much of the correlation is true co-movement and not incidental?

People in the market often talk about how “volatility changed.” One must be very careful in speaking about volatility

realized volatility, at-the-money implied volatility, and implied volatility for a *definite strike K and tenor T - t*, $\Sigma = \Sigma(S, t; K, T)$

When you talk about the change in Σ, what are you keeping fixed?

Consequences of the Smile for Trading

- Liquid standard call and put options prices are simply *quoted* via the Black-Scholes formula, so the model doesn't really matter for pricing.

- The model does matter if you wanted to generate your own idea of fair options values and then arbitrage them against market prices, but that is a very risky long-term business. This is a buy-side view.

- It does matter for hedge ratios.

- The model matters for pricing illiquid OTC exotic options.

- The question in both of these cases is of course: which model?

How to Graph the Smile

You see the yield curve at one instant and wonder what will happen to it later.

The snapshot $\Sigma(S_0, t_0, K, T)$. What is $\Sigma(S, t; K, T)$?

Plot $\Sigma(\ )$ vs. strike K, moneyness K/S, forward moneyness K/S_F, $(\ln K/S_F)/(\sigma\sqrt{\tau})$, or even more generally against $\Delta = N(d_1)$, which depends on S, K, t and *implied volatility*, itself.

Traders like to plot the smile against Δ because they believe it's more invariant. Also:

- Plotting implied volatilities against Δ immediately indicates the hedge.

- Since Δ depends on both strike and expiration, you can compare the implied volatilities of differing expirations and strikes as a function of single variable.

- Finally, d_2 is roughly the number of standard deviations the stock price must move to expire in the money and $N(d_2)$ is the risk-neutral probability of this happening. An "actuarial" measure.

Using the wrong quoting convention can distort the simplicity of the underlying dynamics. Perhaps the Black-Scholes model uses the wrong dynamics for stocks and therefore the smile looks peculiar in that quoting convention: ABM vs. GBM: constant arithmetic volatility corresponds to variable lognormal volatility. Plotting lognormal volatility against stock price would obscure the simplicity of the underlying evolution.

Δ and the Smile

The meaning of delta

Suppose that

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

$$d\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ$$

$$\ln\frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\varepsilon$$

The risk-neutral probability of $S_t > K$ is $P(S_t > K)$ given by

$$P(\ln S_t > \ln K) = P\left(\ln\frac{S_t}{S_0} > \ln\frac{K}{S_0}\right) = P\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\varepsilon > \ln\frac{K}{S_0}\right]$$

$$= P\left[\varepsilon > \frac{\ln K/S_0 - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right] = P[\varepsilon > -d_2] = P[\varepsilon < d_2] = N(d_2) \approx N(d_1)$$

$$= \Delta$$

Delta is approximately the risk-neutral probability of the option finishing in the money at expiration.

The Relationship between Δ and Strike

- The most popular and liquid option is $\Delta \sim 0.5$. Why?

- Far out-of-the-money options are also popular for buyers. Trading desks don't like to sell them. Why?

A standard measure of the skew is the *risk reversal*: difference in volatility between an out-of-the-money call option with a 25% Δ and an out-of-the-money put with a –25% Δ.

Moneyness rather than strike because relative rather than absolute.

What percentage of moneyness corresponds to a given Δ?

For simplicity set $r = 0$.

$$\Delta = N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} \exp\left(-\frac{y^2}{2}\right)dy + \int_{0}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy\right] \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}$$

$$d_{1,2} = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} \pm \frac{\Sigma\sqrt{\tau}}{2} \text{ and } \tau = T - t$$

At the money $S = K$ $\qquad \Delta \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\Sigma\sqrt{\tau}}{2} \approx 0.5 + (0.4)(0.5)\Sigma\sqrt{\tau}$

For 20% volatility 1 year expiration $\Delta \approx 0.5 + 0.04 = 0.54$.

Slightly out of the money: $K = S + \delta S$ $\ln\left(\frac{S}{S+\delta S}\right) = -\ln(1+\delta S/S) \approx -\frac{\delta S}{S}$

$$d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} \approx -\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

Then for a slightly out-of-the-money option, a fraction J away from the at-the-money level,

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{\Sigma\sqrt{\tau}}{2} - \frac{\overbrace{(\delta S)/S}^{\text{percent move}}}{\underbrace{\Sigma\sqrt{\tau}}_{\text{total variance}}}\right)$$

Suppose $(\delta S)/S = 0.01$, $T = 1$ year $\Sigma = 0.2$.

Then $\Delta \approx 0.54 - \frac{(0.4)(0.01)}{0.2} = 0.54 - 0.02 = 0.52$

Thus, Δ decreases by two basis points for every 1% that the strike moves out of the money.

The difference between a 50-delta and a 25-delta option therefore corresponds to about a 12% or 13% move in the strike price.

The move δS to decrease the delta from atm 0.54 to 0.25 is approximately given by

$$\frac{1}{\sqrt{2\pi}}\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} \approx 0.29 \text{ or } (\delta S)/S = 0.29\sqrt{2\pi}\Sigma\sqrt{\tau} \approx 0.29 \times 2.5 \times 0.2 \approx 0.15$$

Thus the strike of the 25-delta call is about 115. Actually it's about 117 if you use the exact Black-Scholes formula to compute deltas.

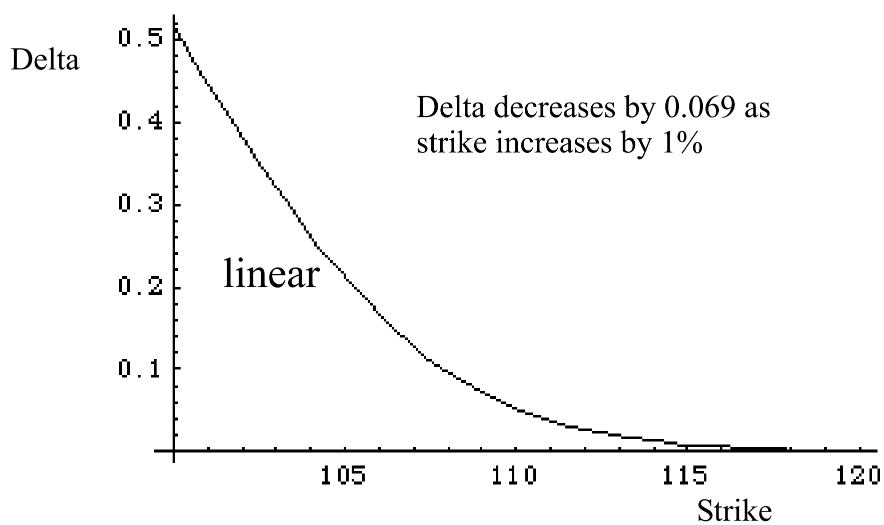
More generally

$$\text{change in Delta} \approx \frac{1}{\sqrt{2\pi}}\left(-\frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

and a one-basis point change in Δ corresponds to a change in $(\delta S)/S$ of about $0.025\Sigma\sqrt{\tau}$.

Key is the percent move in stock price divided by the square root of the annual variance. For a greater volatility or time to expiration and you need a bigger move in the strike to get to the same Δ.

A 1-month call with zero interest rates, 20% volatility



1.1 No-Arbitrage Bounds on the Smile

Yield to maturity: the parameter that determines bond prices: $B_T = 100\exp(-y_T T)$

Σ: the parameter that determines options prices in Black-Scholes.

There are no-arbitrage bounds on bond yields.

For example suppose $B_1 = 90$ and $B_2 = 91$.

$\pi = \frac{91}{90}B_1 - B_2$ has zero cost.

After one year the long position is worth more than \$100, so if you wait for B_2 to mature and pay off the face value you have a riskless profit so there is something wrong with these yields.

Similar constraints on options implied volatilities

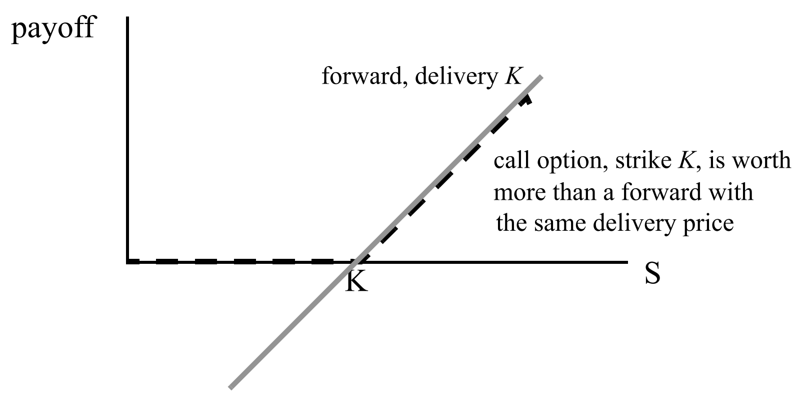
1.1.1 Some of the Merton Inequalities for Strike

Assume zero dividends, European calls.

1. A call is always worth more than a forward: $C \geq S - Ke^{-r(T-t)}$:

   Proof: An option is always worth more than a forward, because it has the same payoff when $S_T > K$, and is worth more when $S_T < K$.

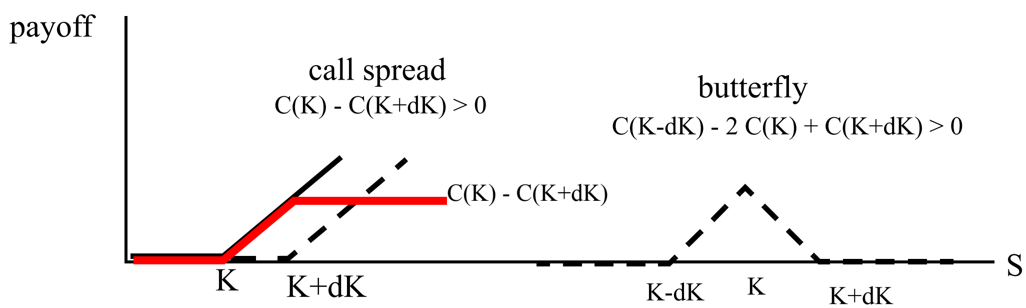
   Diagrammatically:



2. For the same expiration, options prices satisfy two constraints on their derivatives:

$$\frac{\partial C}{\partial K} < 0 \text{ and } \frac{\partial^2 C}{\partial K^2} > 0,$$

Proof: Look at payoff of a *call spread* and a *butterfly*.



There are similar constraints on European put prices:

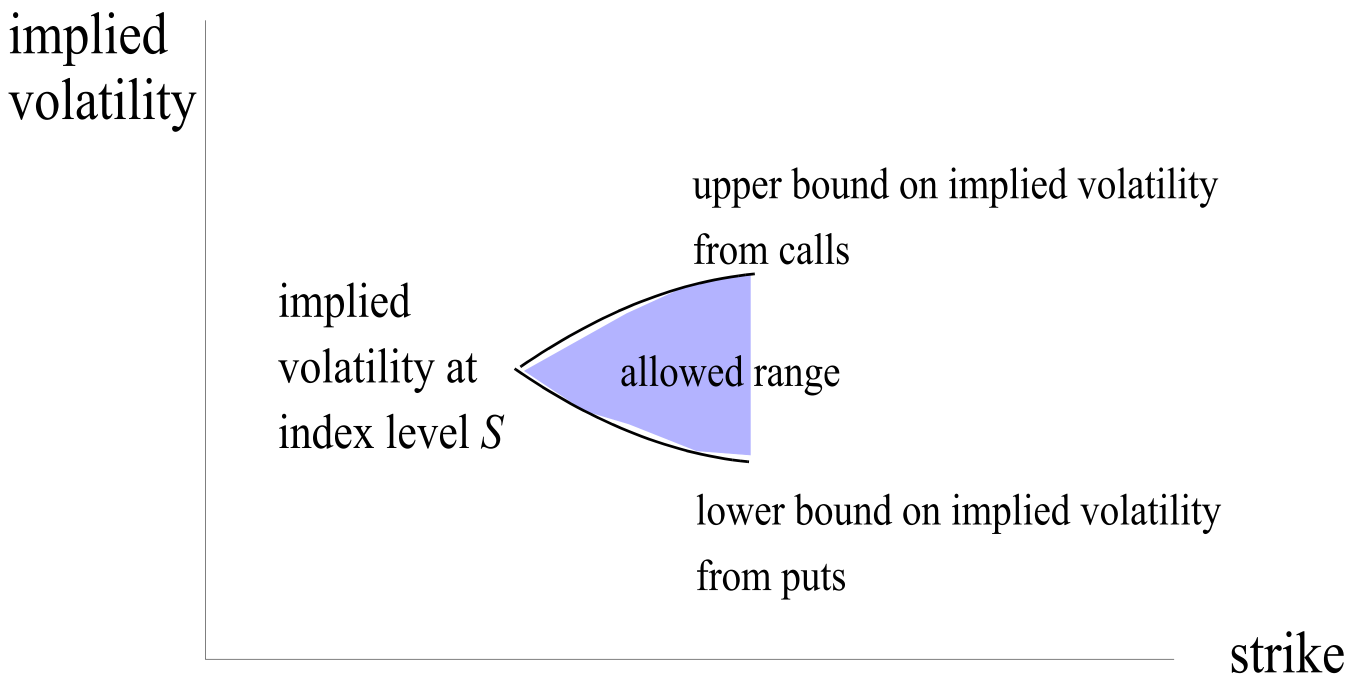
$$\frac{\partial P}{\partial K} > 0 \text{ and } \frac{\partial^2 P}{\partial K^2} > 0$$

Not just partials.

1.1.2 Inequalities for the slope of the smile

The constraints on $\frac{\partial C}{\partial K} < 0$ and $\frac{\partial P}{\partial K} > 0$ put limits on the slope of the smile independent of model.

Therefore they put constraints on the implied volatility parameters as a function of strike.



These constraint are true in the Black-Scholes formula with strike-independent volatility.

Now let’s develop this idea more quantitatively.

$$C = C_{BS}(S, t, K, T, r, \Sigma)$$

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} < 0$$

$$\frac{\partial C_{BS}}{\partial \Sigma} = S\sqrt{\tau}N'(d_1) \equiv Ke^{-r\tau}\sqrt{\tau}N'(d_2) \qquad \text{Eq.1.1}$$

$$\frac{\partial \Sigma}{\partial K} \le -\frac{\frac{\partial C_{BS}}{\partial K}}{\frac{\partial C_{BS}}{\partial \Sigma}} = \frac{e^{-r\tau}N(d_2)}{Ke^{-r\tau}\sqrt{\tau}N'(d_2)} = \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

For small volatility, at the money: $d_2 \approx 0$, $N(d_2) \approx 0.5$ and $N'(d_2) \approx \frac{1}{\sqrt{2\pi}}$:

$$\frac{\partial \Sigma}{\partial K} \le \sqrt{\frac{\pi}{2}}\frac{1}{K\sqrt{\tau}} \approx \frac{1.25}{K\sqrt{\tau}} \qquad \text{Eq.1.2}$$

For 1-month options on the S&P $\qquad \frac{\partial \Sigma}{\partial K} < 0.0043$.

For a 1% change in strike ($\Delta K \approx 10$) volatility must change less than 4.3 volatility points.

Recall: the S&P skew slope for one-month options was ~ 0.001, or 1 volatility point for a 1% change in the strike, only a factor of 4 below the arbitrage limit.

Asymptotically short expiration

$$\frac{\partial \Sigma}{\partial K} \le \frac{N(d_2)}{K\sqrt{\tau} N'(d_2)}$$

At-the-money forward, as $\tau \to 0$

$$d_2 \to -\frac{\Sigma\sqrt{\tau}}{2} \to 0$$

$$N(d_2) \to \frac{1}{2}$$

$$N'(d_2) \to \frac{1}{\sqrt{2\pi}}$$

and so

$$\frac{\partial \Sigma}{\partial K} \le O(\tau^{-1/2}) \qquad \text{as } \tau \to 0.$$

As the time to expiration $\tau \to 0$, the slope steepness can increase no faster than $O(\tau^{-1/2})$.

Asymptotically long expiration

At the other extreme, as $\tau \to \infty$, $d_2 \to -\infty$, and therefore

$$\frac{\partial \Sigma}{\partial K} \le \frac{1}{K\sqrt{\tau}} \frac{N(d_2)}{N'(d_2)} \sim O\left(\frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{\tau}}\right) \sim O\left(\frac{1}{\tau}\right)$$

To prove the line above we have made use of the asymptotic relation

$N(d_2)/N'(d_2) \sim O(\tau^{-0.5})$ as $\tau \to \infty$.

The area under the tail gets smaller faster than the height of the tail.

Thus, the slope of the smile can decrease with time to expiration no more slowly than $O(\tau^{-1})$.

Reference: *Arbitrage Bounds on the Implied Volatility Strike and Term Structures of European-Style Options*. Hardy M. Hodges, Journal of Derivatives, Summer 1996, pp. 23-35.

1.2 Some Behavioral Reasons for an Implied Volatility Skew

Think of options trading as the trading of volatility as an asset, and also the buying of protection.

- Knowledge of past behavior in options markets suggests a skew in options would be wise. (How much, though? What's the fair value?) Implied and realized volatilities go up after a crash.

- Expectation of future changes in volatility naturally gives rise to a term structure.

- Expectation of changes in volatility as support or resistance levels in currencies and interest rates suggests that realized volatility will decrease as those levels are approached.

- Expectation of an increase in the cross-sectional correlation between the returns of constituent stocks in the index as the market drops can cause an increase in the volatility of the entire index. (Some volatility arbitrage dispersion strategies are based on this.)

- Dealers' tend to be short options because they sell zero-cost collars (short otm call-long otm put) to investors who want protection against a decline.Think of options trading as the trading of volatility as an asset, and also the buying of protection.

1.3 An Aside: Why Black-Scholes is Robust

Black-Scholes works for options much better than the EMM works for stocks.

The EMM is at best a simplistic model for risk that neglects the subtleties of stock price behavior.

Black's great idea, that an option and a stock should share the same risk premium when markets are in equilibrium, is close to a more general truth.

Irrespective of subtleties, the risk of the stock and the risk of the option are sufficiently related so that equating their risk premiums gives a sensible constraint on their relative prices. That makes the Black-Scholes model robust.

The better you can describe the risk, the better the extension of BS.

But the EMM works much less well for stock valuation, because stock prices suffer risks more diverse and wild than those associated with diffusion

1.4 An Overview of Smile-Consistent Models

Two choices:

(i) Model the stochastic evolution of the underlying asset S and its realized volatility, and then deduce $\Sigma(S, t, K, T)$: fundamental, avoid arbitrage violations; but hard to get right equation;

(ii) directly model the dynamics of the parametric surface $\Sigma(S, t, K, T)$.
more intuitive but we are modeling a parameter in a bad model, not a price, and hard to avoid arbitrage violations.

(iii) pragmatic "model-less" models.

In the end, different markets have different smiles and it is unlikely that one grand replacement for Black-Scholes will cover all smiles in all markets.

1.4.1 Local Volatility Models -- the first smile models.

Black-Scholes: $\Sigma(S, t, K, T) = \sigma$ is independent of strike and expiration.

Local volatility models: $$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ$$

$\sigma(S, t)$ is a *deterministic* function of a *stochastic variable* S.

One-factor model so replication and risk-neutral valuation still works. But is it true?
Calibration: how to choose $\sigma(S, t)$ to match market values of $\Sigma(S, t, K, T)$?
Provide great intuition. People make use of them for trading and as a proxy for other models.

What might account for local volatility being a function $\sigma(S, t)$?
The leverage effect: leverage makes volatility increase as the stock price moves lower:

$$S = A - B \qquad \text{assets - liabilities}$$

$$\frac{dA}{A} = \sigma dZ$$

$$\frac{dS}{S} = \frac{dA}{S} = \frac{A\sigma dZ}{S} = \frac{(S+B)}{S}\sigma dZ$$

$$\sigma_S = \sigma(1 + B/S)$$

Constant Elasticity of Variance (CEV) models:

$$dS = \mu(S, t)dt + \sigma S^{\beta} dZ$$

$\beta = 1$ lognormal; $\beta = 0$ normal evolution. β needs to be large and negative, but then model has problems.

CEV is a parametric models and cannot fit an arbitrary smile; local volatility models are non-parametric and $\sigma(S, t)$ can be calibrated numerically.

1.4.2 Stochastic Volatility Models

Volatility is random too.

$$
\begin{aligned}
dS &= \mu_S(S, V, t)dt + \sigma_S(S, V, t)dZ_t \\
dV &= \mu_V(S, V, t)dt + \sigma_V(S, V, t)dW_t \\
V &= \sigma^2 \\
E[dWdZ] &= \rho dt
\end{aligned}
$$

Perfect replication is impossible if you can hedge only with the stock. Unpleasant but may be true.

If you can hedge with options, and ***assuming (!?)*** you know the stochastic process for volatility, then you can hedge one option's exposure to volatility with another option and derive an arbitrage-free formula for options values.

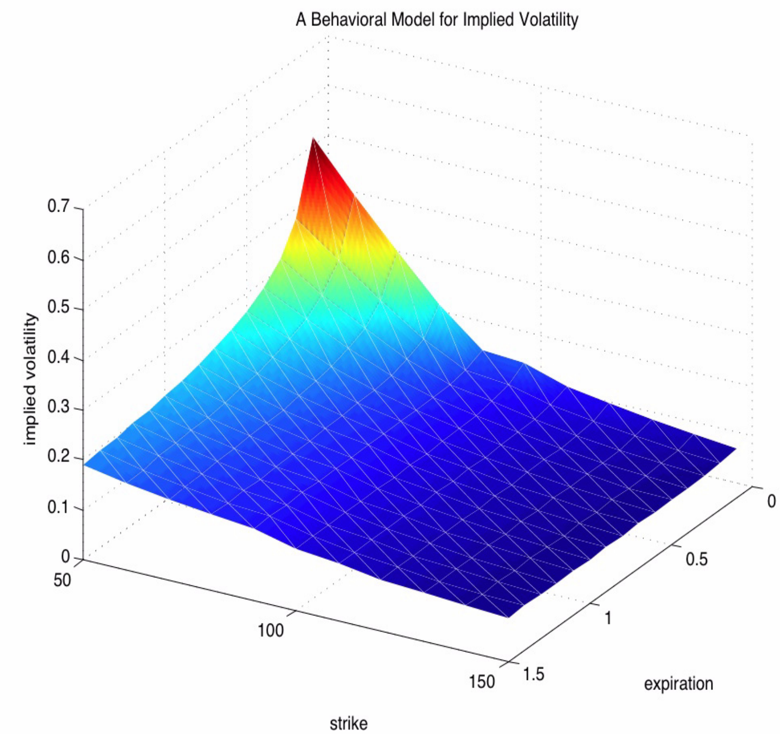
Stochastic volatility models assume that the correlation ρ is constant but that is stochastic too.

1.4.3 Jump-Diffusion Models

Black-Scholes ignores discontinuous jumps. Merton model allows an arbitrary number of jumps plus diffusion.

With a finite number of jumps of known size in the model, one can replicate any payoff perfectly by dynamic trading in a finite number of options, the stock and the bond, and so achieve risk-neutral pricing.

With arbitrary number of jumps, one cannot, but people use risk-neutral pricing anyway.



1.4.4 A Plenitude of Other Models

There are many other smile models too, which we may discuss later: mixing models, variance gamma models, stochastic volatility models of other types, stochastic implied volatility models …

In practise, one has to see which model best describes the market one is working in.

In the real world there is indeed diffusion, jumps and stochastic volatility!

There are too many different ways of fitting the observed smile that the model is non-parsimonious and offers too many choices.

In the end, you want to model the market with reasonable (but not perfect) accuracy via a fairly simple model that captures most of the important behavior of the asset.

A model is only a model, not the real thing.

1.5 Problems Caused By The Smile

There cannot be many GBM volatilities for the same stock.

Black-Scholes is often simply being used as a quoting mechanism, rather than a pure valuation model. Right price, wrong model, wrong hedge ratio.

Note: yield to maturity is used in quoting rather than calculating bond or mortgage prices.

What problems does this cause?

1.5.1 Fluctuations in the P&L from incorrect hedging of standard options

If we have the wrong model, then, even if liquid vanilla options prices are forced to be correct, the hedge ratio is wrong. This causes a variance in the P&L.

Estimate using the chain rule,

$$\Delta = \frac{dC_{BS}(S, t, K, T, \Sigma)}{dS} = \frac{\partial C_{BS}}{\partial S} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \qquad \text{Eq.1.3}$$

At the money, the vega for the S&P 500 index assuming $S \sim 1000$ and $T = 1$ year is given by

$$\frac{\partial C}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 400 \text{ dimensional}$$

Estimate $\frac{\partial \Sigma}{\partial S}$ of order $\frac{\partial \Sigma}{\partial K}$ on dimensional grounds: $\frac{\partial \Sigma}{\partial S} \sim \frac{\partial \Sigma}{\partial K} \sim \frac{0.02}{100} \sim 0.0002$

$$\Delta - \frac{\partial C_{BS}}{\partial S} = \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \sim 400 \times 0.0002 = 0.08$$

Daily index move of 1% or 10 S&P points:

P&L fluctuation from 0.08 of a share when the index moves 10 points is about 0.8 index points.

The incremental P&L from hedging with a volatility of 0.2 when the index moves 10 points is order

$$\Gamma \times \frac{\delta S^2}{2} \sim \frac{1}{S\Sigma\sqrt{T}}\frac{\delta S^2}{2} \sim \frac{1}{200}(50) = 0.25 \quad \text{points}$$

The mismatch in Δ can cause a large distortion in the incremental P&L from hedging at each step.

1.5.2 Errors in the Valuation of Exotic Options

Values of exotics need models.

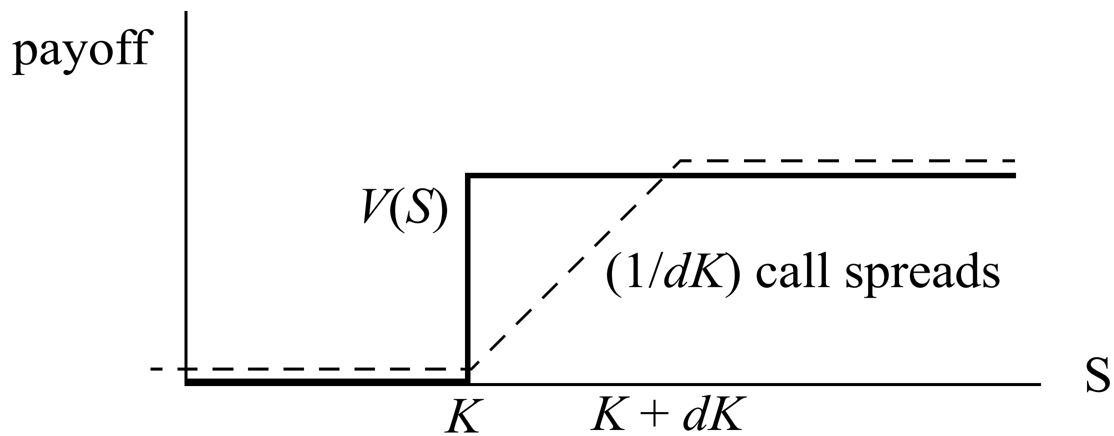
European-style pseudo-exotic option V which pays \$1 if $S \geq K$ at time T, and zero otherwise.

This serves as insurance against a fixed loss above the strike K, but not against a proportional loss as in the case of a vanilla call.

It is very hard to hedge this because the payoff oscillates between 0 and 1.

Approximately replicate V with a call spread with strikes separated by dK.

In the limit as $dK \to 0$ the call spread's payoff converges to that of the exotic option. (In practice, this is often how exotic traders hedge themselves, choosing a small value of dK.)



The value of the call spread at stock price S and time to time t is

$$\frac{-C_{BS}(S, K+dK, t, T, \Sigma(K)) + C_{BS}(K, S, t, T, \Sigma(K))}{dK} \approx -\frac{d}{dK}C_{BS}(S, K, t, T, \Sigma(K))$$

The total derivative with respect to K includes the change of all variables with K, including that of the implied volatility.

We can estimate the current value $V(S, K, t, T, \Sigma(K))$ if we know how call prices vary with strike K:

$$V(S, K, t, T) = -\frac{d}{dK}C_{BS}(S, K, t, T, \Sigma(K)) = -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \Sigma} \times \frac{\partial \Sigma}{\partial K}$$

For $r = 0$, $\Sigma = 20\%$, $T - t = 1$ year, $K = S = 1000$, and a skew slope 0.0002,

$$-\frac{\partial C_{BS}}{\partial K} = N(d_2) = N\left(-\frac{\Sigma}{2}\right) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\Sigma}{2} = 0.46$$

$$\frac{\partial C_{BS}}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 400$$

$$\begin{aligned} V(100, 100, 0, 1, 0.2) &\approx (0.46) + (400 \times 0.0002) \\ &= 0.46 + 0.08 = 0.54 \end{aligned}$$

The non-zero slope of the skew adds about 16% to the value of the option. This is a significant difference.

Why does the skew *add* to the value of the derivative V?

How can we “fix” it or extend Black-Scholes to match the skew and allow us to calculate all these quantities correctly? What changes can we make? Or, how, as we did in the above example, can we tread carefully and so avoid our lack of knowledge about the right model and still get reasonable estimates of value? Those are the questions we will tackle later.