LECTURE 10

SMILE MODELS
EFFECTS OF THE SMILE
STATIC HEDGING AND IMPLIED DISTRIBUTIONS

Some Behavioral Reasons for an Implied Volatility Skew

Think of options trading as the trading of volatility as an asset, and also the buying of protection.

- Knowledge of past behavior in options markets suggests a skew in options would be wise. (How much, though? What's the fair value?) Implied and realized volatilities go up after a crash.

- Expectation of future changes in volatility naturally gives rise to a term structure.

- Expectation of changes in volatility as support or resistance levels in currencies and interest rates suggests that realized volatility will decrease as those levels are approached.

- Expectation of an increase in the cross-sectional correlation between the returns of constituent stocks in the index as the market drops can cause an increase in the volatility of the entire index. (Some volatility arbitrage dispersion strategies are based on this.)

- Dealers' tend to be short options because they sell zero-cost collars (short otm call-long otm put) to investors who want protection against a decline.Think of options trading as the trading of volatility as an asset, and also the buying of protection.

An Aside: Why Black-Scholes is Robust

Black-Scholes works for options much better than the EMM works for stocks.

The EMM is at best a simplistic model for risk that neglects the subtleties of stock price behavior.

Black's great idea, that an option and a stock should share the same risk premium when markets are in equilibrium, is close to a more general truth.

Irrespective of subtleties, the risk of the stock and the risk of the option are sufficiently related so that equating their risk premiums gives a sensible constraint on their relative prices. That makes the Black-Scholes model robust.

The better you can describe the risk, the better the extension of BS.

But the EMM works much less well for stock valuation, because stock prices suffer risks more diverse and wild than those associated with diffusion

An Overview of Smile-Consistent Models

Two choices:

(i) Model the stochastic evolution of the underlying asset S and its realized volatility, and then deduce $\Sigma(S, t, K, T)$: fundamental, avoid arbitrage violations; but hard to get right equation;

(ii) directly model the dynamics of the parametric surface $\Sigma(S, t, K, T)$.
more intuitive but we are modeling a parameter in a bad model, not a price, and hard to avoid arbitrage violations.

(iii) pragmatic "model-less" models. (Vanna-volga)

In the end, different markets have different smiles and it is unlikely that one grand replacement for Black-Scholes will cover all smiles in all markets.

Local Volatility Models -- the first smile models.

Black-Scholes: $\Sigma(S, t, K, T) = \sigma$ is independent of strike and expiration.

Local volatility models: $$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ$$

$\sigma(S, t)$ is a *deterministic* function of a *stochastic variable* S.

One-factor model so replication and risk-neutral valuation still works. But is it true?
Calibration: how to choose $\sigma(S, t)$ to match market values of $\Sigma(S, t, K, T)$?
Provide great intuition. People make use of them for trading and as a proxy for other models.

What might account for local volatility being a function $\sigma(S, t)$?
The leverage effect: leverage makes volatility increase as the stock price moves lower:

$$S = A - B \qquad \text{assets - liabilities}$$

$$\frac{dA}{A} = \sigma dZ$$

$$\frac{dS}{S} = \frac{dA}{S} = \frac{A\sigma dZ}{S} = \frac{(S+B)}{S}\sigma dZ$$

$$\sigma_S = \sigma(1 + B/S)$$

Constant Elasticity of Variance (CEV) models:

$$dS = \mu(S, t)dt + \sigma S^{\beta} dZ$$

$\beta = 1$ lognormal; $\beta = 0$ normal evolution. β needs to be large and negative, but then model has problems.

CEV is a parametric models and cannot fit an arbitrary smile; local volatility models are non-parametric and $\sigma(S, t)$ can be calibrated numerically.

Stochastic Volatility Models

Volatility is random too.

$$
\begin{aligned}
dS &= \mu_S(S, V, t)dt + \sigma_S(S, V, t)dZ_t \\
dV &= \mu_V(S, V, t)dt + \sigma_V(S, V, t)dW_t \\
V &= \sigma^2 \\
E[dWdZ] &= \rho dt
\end{aligned}
$$

Perfect replication is impossible if you can hedge only with the stock. Unpleasant but may be true.

If you can hedge with options, and ***assuming (!?)***you know the stochastic process for volatility, then you can hedge one option's exposure to volatility with another option and derive an arbitrage-free formula for options values.

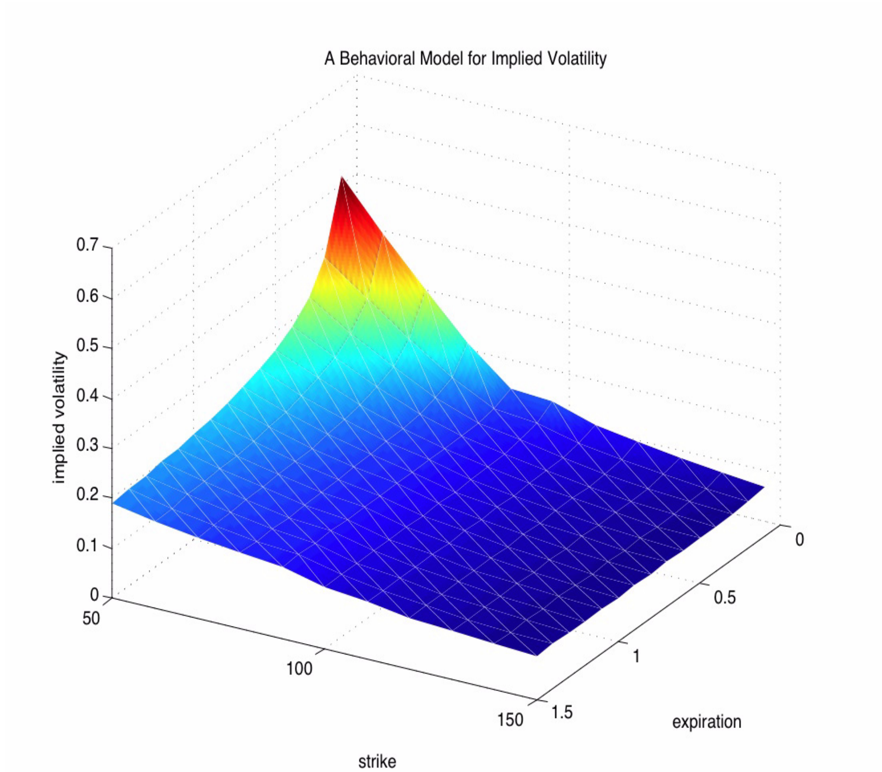
Stochastic volatility models assume that the correlation ρ is constant but that is stochastic too.

Jump-Diffusion Models

Black-Scholes ignores discontinuous jumps. Merton model allows an arbitrary number of jumps plus diffusion.

With a finite number of jumps of known size in the model, one can replicate any payoff perfectly by dynamic trading in a finite number of options, the stock and the bond, and so achieve risk-neutral pricing.

With arbitrary number of jumps, one cannot, but people use risk-neutral pricing anyway.



A Plenitude of Other Models

There are many other smile models too, which we may discuss later: mixing models, variance gamma models, stochastic volatility models of other types, stochastic implied volatility models …

In practise, one has to see which model best describes the market one is working in.

In the real world there is indeed diffusion, jumps and stochastic volatility!

There are too many different ways of fitting the observed smile that the model is non-parsimonious and offers too many choices.

In the end, you want to model the market with reasonable (but not perfect) accuracy via a fairly simple model that captures most of the important behavior of the asset.

A model is only a model, not the real thing.

Problems Caused By The Smile

There cannot be many GBM volatilities for the same stock.

Black-Scholes is often simply being used as a quoting mechanism, rather than a pure valuation model. Right price, wrong model, wrong hedge ratio.

Note: yield to maturity is used in quoting rather than calculating bond or mortgage prices.

What problems does this cause?

SPX One-Year Options on Feb 17 2015

S&P 500 2095.02

Jan 2016 Implied Volatility

	Call	Put
1700	23.43%	23.27%
1750	22.56%	22.35%
1800	21.54%	21.42%
1850	20.54%	20.50%
1900	19.65%	19.58%
1950	18.78%	18.64%
2000	17.89%	17.75%
2050	16.94%	16.82%
2100	16.01%	15.93%
2150	15.18%	15.08%
2200	14.31%	14.20%
2250	13.48%	13.40%
2300	12.80%	12.62%
2350	12.14%	11.84%
2400	11.44%	11.14%
2450	11.19%	10.52%
2500	10.67%	9.93%
2550	10.75%	8.89%
2600	10.87%	

Fluctuations in the P&L from incorrect hedging of standard options

If we have the wrong model, then, even if liquid vanilla options prices are forced to be correct, the hedge ratio is wrong. This causes a variance in the P&L.

Estimate using the chain rule,

$$\Delta = \frac{dC_{BS}(S, t, K, T, \Sigma)}{dS} = \frac{\partial C_{BS}}{\partial S} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \qquad \text{Eq.10.1}$$

At the money, the vega for the S&P 500 index assuming $S \sim 2000$ and $T = 1$ year is given by

$$\frac{\partial C}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 800$$

Estimate $\frac{\partial \Sigma}{\partial S}$ of order $\frac{\partial \Sigma}{\partial K}$ on dimensional grounds: $\frac{\partial \Sigma}{\partial S} \sim \frac{\partial \Sigma}{\partial K} \sim \frac{0.02}{100} \sim 0.0002$ currently

$$\Delta - \frac{\partial C_{BS}}{\partial S} = \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \sim 800 \times 0.0002 = 0.16$$

Daily index move of 1% or 20 S&P points:

P&L fluctuation from 0.16 of a share when the index moves 20 points is about 3.2 index points.

The incremental P&L from hedging with a volatility of 0.2 when the index moves 20 points is order

$$\Gamma \times \frac{\delta S^2}{2} \sim \frac{1}{S\Sigma\sqrt{T}} \frac{\delta S^2}{2} \sim \frac{1}{400}(200) = 0.50 \quad \text{points}$$

The mismatch in Δ can cause a large distortion in the incremental P&L from hedging at each step.

Errors in the Valuation of Exotic Options

Values of exotics need models.

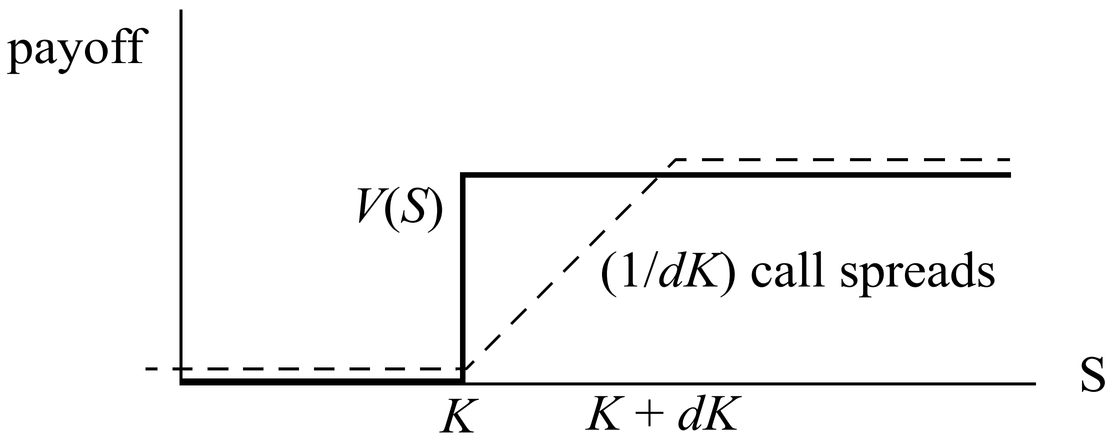
European-style pseudo-exotic option V which pays \$1 if $S \geq K$ at time T, and zero otherwise.

This serves as insurance against a fixed loss above the strike K, but not against a proportional loss as in the case of a vanilla call.

It is very hard to hedge this because the payoff oscillates between 0 and 1.

Approximately replicate V with a call spread with strikes separated by dK.

In the limit as $dK \to 0$ the call spread's payoff converges to that of the exotic option. (In practice, this is often how exotic traders hedge themselves, choosing a small value of dK.)



The value of the call spread at stock price S and time to time t is

$$\frac{-C_{BS}(S, K+dK, t, T, \Sigma(K)) + C_{BS}(K, S, t, T, \Sigma(K))}{dK} \approx -\frac{d}{dK}C_{BS}(S, K, t, T, \Sigma(K))$$

The total derivative with respect to K includes the change of all variables with K, including that of the implied volatility.

We can estimate the current value $V(S, K, t, T, \Sigma(K))$ if we know how call prices vary with strike K:

$$V(S, K, t, T) = -\frac{d}{dK}C_{BS}(S, K, t, T, \Sigma(K)) = -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \Sigma} \times \frac{\partial \Sigma}{\partial K}$$

For $r = 0$, $\Sigma = 20\%$, $T - t = 1$ year, $K = S = 2000$, and a skew slope 0.0002,

$$-\frac{\partial C_{BS}}{\partial K} = N(d_2) = N\left(-\frac{\Sigma}{2}\right) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\Sigma}{2} = 0.46$$

$$\frac{\partial C_{BS}}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 800$$

$$\begin{aligned} V(100, 100, 0, 1, 0.2) &\approx (0.46) + (800 \times 0.0002) \\ &= 0.46 + 0.16 = 0.62 \end{aligned}$$

The non-zero slope of the skew adds about 30% to the value of the option. This is a significant difference.

Why does the skew *add* to the value of the derivative V?

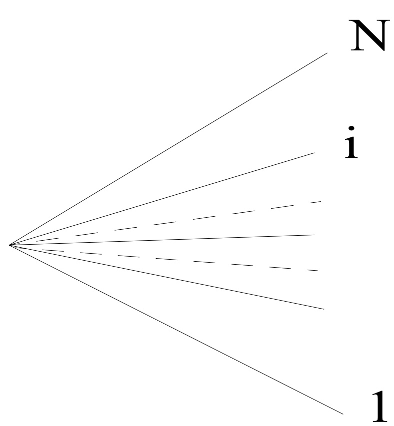
How can we “fix” it or extend Black-Scholes to match the skew and allow us to calculate all these quantities correctly? What changes can we make? Or, how, as we did in the above example, can we tread carefully and so avoid our lack of knowledge about the right model and still get reasonable estimates of value? Those are the questions we will tackle later.

Static Hedging and Implied Distributions

The Black-Scholes formula calculates options prices as the expected discounted value of the payoff over a lognormal stock distribution in a risk-neutral world, and produces a flat smile.

The inverse question: for a fixed expiration, what risk-neutral stock distribution (the so-called ***implied distribution***) matches the observed smile when options prices are computed as expected risk-neutrally discounted payoffs?

State security with price π_i that pays \$1 only when the stock is in state i with price S_i at time T, zero otherwise. Suppose you know the current market price π_i for each of these securities.



Sum of all π_i is a riskless bond because it pays off \$1 in every future state: $\sum_1^N \pi_i = \exp[-r(T-t)] \equiv \frac{1}{R}$

Pseudo-probabilities $p_i \equiv R\pi_i$ and we can write $\pi_i = \frac{p_i}{R}$, $\sum p_i = 1$

Complete Market Constraint

If there is one state-contingent security π_i for each state *i* then these securities provide a complete basis that span the space of future payoffs, and the market is "complete".

For any payoff at time *T*: $V = \sum V_i \pi_i$ with value $V = \sum_i \frac{p_i}{R} V_i$.

In continuous-state notation $$V(S, t) = e^{-r(T-t)} \int_0^\infty p(S, t, S', T) V(S', T) dS'$$

Here $p(S, t, S', T)$ is the risk-neutral (pseudo-) probability density.

Define $$\pi(S, t, S', T) = e^{-r(T-t)} p(S, t, S', T)$$

$\pi(S, t, S', T)dS'$ is the price at time *t* of a state-contingent security that pays \$1 if the stock price at time *T* lies between S' and $S' + dS'$: $\int_0^\infty \pi(S, t, S', T)dS' = e^{-r(T-t)}$ and $\int_0^\infty p(S, t, S', T)dS' = 1$

$p(S,t,S',T)$ Determines the Value all European-style payoffs at time T.

For a standard call option C with strike K,

$$C(S', T) = [S' - K]_+ = max(S' - K, 0) = [S' - K]\theta(S' - K)$$

where $\theta(x)$ is the Heaviside/indicator function, equal to 1 when x is greater than 0 and 0 otherwise.

$$\begin{aligned} C_K(S, t) &= e^{-r(T-t)}\int_K^{\infty} p(S, t, S', T)(S' - K)dS' \\ &= e^{-r(T-t)}\int_0^{\infty} dS'(S' - K)\theta(S' - K)p(S, t, S', T) \end{aligned} \qquad \text{Eq.10.2}$$

We'll see a knowledge of call prices (or put prices) for all strikes K at expiration time T are enough to determine the density $p(S, t, S', T)$ for all S'. One can statically replicate any known European-style payoff at time T through a combination of zero-coupon bonds, forwards, calls and puts.

The risk-neutral distribution $p(S, t, S', T)$ is insufficient for valuing all options on the underlier. The risk-neutral distribution at expiration tells you nothing about the evolution of the stock price on its way to expiration. Hence, implied distributions are not useful in determining dynamic hedges.

The Heaviside and Dirac Delta functions

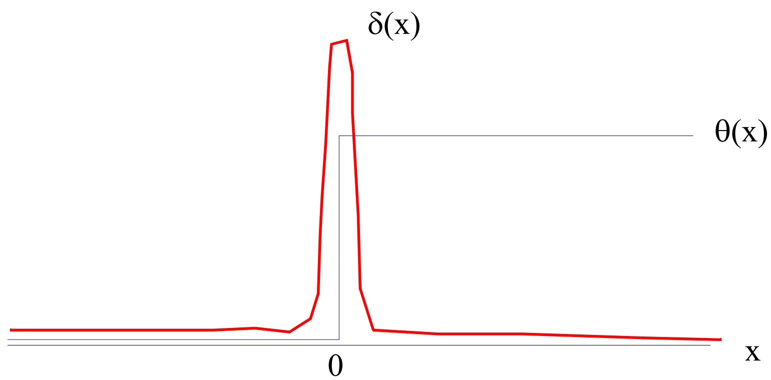
The derivative of the Heaviside function is the Dirac *delta function*: $\frac{\partial}{\partial x}\theta(x) = \delta(x)$

$\delta(x)$ is a distribution, a very singular function that only makes sense when used within an integral. $\delta(x)$ is zero everywhere except at $x = 0$, where its value is infinite. Its integral over all x is 1.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

$$x\delta(x) = 0 \qquad \forall x$$



Finding the risk-neutral probability density from call prices: The Breeden-Litzenberger formula

$$\exp(r\tau) \times C(S, t, K, T) = \int_K^\infty dS'(S' - K)p(S, t, S', T)$$

$$\equiv \int_0^\infty dS'(S' - K)\theta(S' - K)p(S, t, S', T)$$

$$\exp(r\tau) \times \frac{\partial C}{\partial K} = -\int_K^\infty p(S, t, S', T)dS'$$

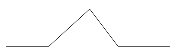
$$\exp(r\tau) \times \frac{\partial^2 C}{\partial K^2} = p(S, t, K, T) \qquad \text{Eq.10.3}$$

The second derivative with respect to *K* of call prices is the risk-neutral probability distribution.

You can do the same for puts.

Butterfly Spread is a State Contingent Security

$\frac{\partial^2 C}{\partial K^2}$ is a butterfly spread, proportional to $C_{K+dK} - 2C_K + C_{K-dK}$ with terminal payoff ~

 with dK and area $(dK)^2$.In the limit that $dK \to 0$, $\frac{\partial^2 C}{\partial K^2}$ has a payoff with area 1 if

$S = K$ and zero otherwise; it behaves like a state-contingent security.

Note that at any time *t:*

$$\int_0^\infty p(S, t, K, T)dK = e^{r\tau}\int_0^\infty \frac{\partial^2 C}{\partial K^2}dK = e^{r\tau}\left[\left.\frac{\partial C}{\partial K}\right|_\infty - \left.\frac{\partial C}{\partial K}\right|_0\right] \equiv 1$$

- $\left.\frac{\partial C}{\partial K}\right|_\infty = 0$ as the strike gets very large and calls become worthless; and

- for $K \to 0$ the call becomes a forward with value $S - Ke^{-r\tau}$, so that $\left.\frac{\partial C}{\partial K}\right|_0 = -e^{-r\tau}$.

You can get the probability density from put prices too

$$\exp(r\tau) \times P = \int_0^{\infty} dS'(K - S')\theta(K - S')p(S, t, S', T)$$

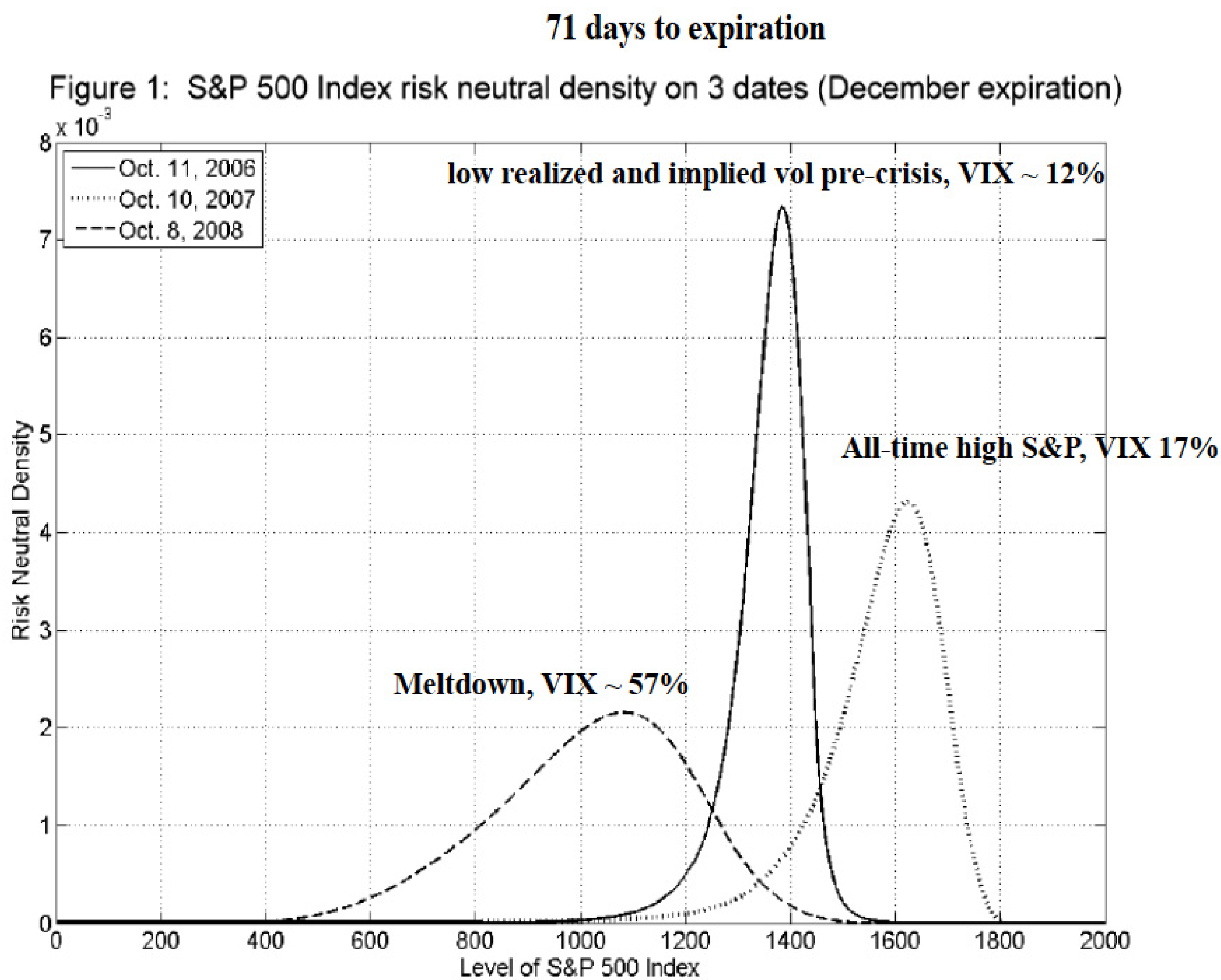
Now differentiate under the integral

$$\exp(r\tau) \times \frac{\partial P}{\partial K} = \int_0^K p(S, t, S', T)dS'$$

Differentiate again to obtain the Breeden-Litzenberger result:

$$\exp(r\tau) \times \frac{\partial^2 P}{\partial K^2} = p(S, t, K, T) \tag{Eq.10.4}$$

During Great Financial Crisis:



Static Replication: Valuing arbitrary payoffs at a fixed expiration

For any W: $$W(S,t) = \int_0^{\infty} \frac{\partial^2 C}{\partial K^2}(S,t,K,T)W(K,T)dK$$

strike terminal stock price

If we know call prices (or put prices) and their derivatives for all strikes at a fixed expiration, we can find the value of any other European-style derivative security at that expiration.

This **involves no use of option theory at all, and no use of the Black-Scholes equation**. It works even if there is a smile or skew or jumps. As long as the options are honored.

Replicating by standard options

Integration by parts to get V as the sum of portfolios of zero coupon bonds, forwards, puts & calls.

European payoff $W(K, T)$. K represents **the terminal stock price**.
Use puts below strike A and for calls above strike A.

$$
\begin{aligned}
W(S,t) &= e^{-r\tau}\int_0^{\infty}\rho(S,t,K,T)W(K,T)\,dK \\
&= e^{-r\tau}\left[\int_0^{A}\rho(S,t,K,T)W(K,T)\,dK + \int_A^{\infty}\rho(S,t,K,T)W(K,T)\,dK\right] \\
&= \int_0^{A}\frac{\partial^2 P}{\partial K^2}W(K,T)\,dK + \int_A^{\infty}\frac{\partial^2 C}{\partial K^2}W(K,T)\,dK
\end{aligned}
$$

Integrate by parts twice to get

$$
W(S,t) = \begin{array}{l}
\displaystyle\int_0^{A}\frac{\partial^2}{\partial K^2}W(K,T)P(S,K)\,dK + \int_A^{\infty}\frac{\partial^2 W}{\partial K^2}C(S,K)\,dK \\
\displaystyle + \left(\left(W\frac{\partial P}{\partial K} - P\frac{\partial W}{\partial K}\right)\Bigg|_{K=0}^{K=\mathrm{A}} + \left(W\frac{\partial C}{\partial K} - C\frac{\partial W}{\partial K}\right)\Bigg|_{K=A}^{K=\infty}\right)
\end{array}
$$

where $P(S, K)$ and $C(S, K)$ are the current values at time *t* and stock price *S* of a put and call with strike *K* and expiration *T*.

Use the following conditions for the current call and put prices.

$$P[S, 0] = 0$$

$$\frac{\partial}{\partial K} P[S, 0] = 0$$

$$C[S, \infty] = 0$$

$$\frac{\partial}{\partial K} C[S, \infty] = 0$$

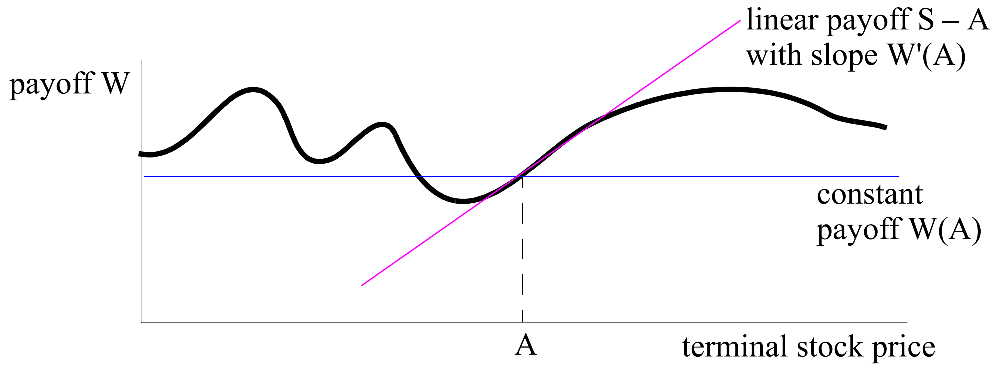
$$P[S, K] - C[S, K] = Ke^{-r\tau} - S$$

$$\frac{\partial}{\partial K} P[S, K] - \frac{\partial}{\partial K} C[S, K] = e^{-r\tau}$$

Then

$$W(S, t) = W(A)e^{-r\tau} + W'(A)[S - Ae^{-r\tau}] + \int_0^A P(K)W''(K)dK + \int_A^\infty C(K)W''(K)dK. \qquad \text{Eq.10.5}$$

zero coupon forward calls and puts



Two views of static replication.

• If you know the risk-neutral density ρ then you W(S,t) is an integral over the terminal payoff.

• Alternatively, W(S,t) as an integral over call and put prices with different strikes.

If you can buy every option in the continuum you need from someone who will never default on their payoff, then you have a perfect static hedge. No math involved.

If you cannot buy every single option, then you have only an approximate replicating portfolio whose value will deviate from the value of the target option's payoff. Picking a reasonable or tolerable replicating portfolio is up to you.

This works even if there is volatility skew.

A Static Replication Example in the Presence of a Skew

Consider an exotic option, strike B, quadratic payoff:

$$V(s) = s \times max[s - B, 0] = s \times (s - B)\theta(s - B)$$

Replicate by adding together a collection of vanilla calls with strikes starting at B, and then adding successively more of them to create a quadratic payoff, as illustrated below.

$$\mathrm{V(S)} = \int_0^{\infty} q(K)\theta(K - B)\mathrm{C(S,K)dK}$$

where $q(K)$ is the unknown density of calls with strike *K*. A in the formula is chosen to be 0.

$$\begin{aligned}\frac{\partial V}{\partial s}(s) &= \frac{\partial}{\partial s}[s \times (s - B)\theta(s - B)] \\ &= (s - B)\theta(s - B) + s\theta(s - B) + s(s - B)\delta(s - B) \\ &= (s - B)\theta(s - B) + s\theta(s - B)\end{aligned}$$

Second derivative:

$$\begin{aligned}\frac{\partial^2 V}{\partial s^2} &= (s - B)\delta(s - B) + 2\theta(s - B) + s\delta(s - B) \\ &= 2\theta(s - B) + s\delta(s - B)\end{aligned}$$

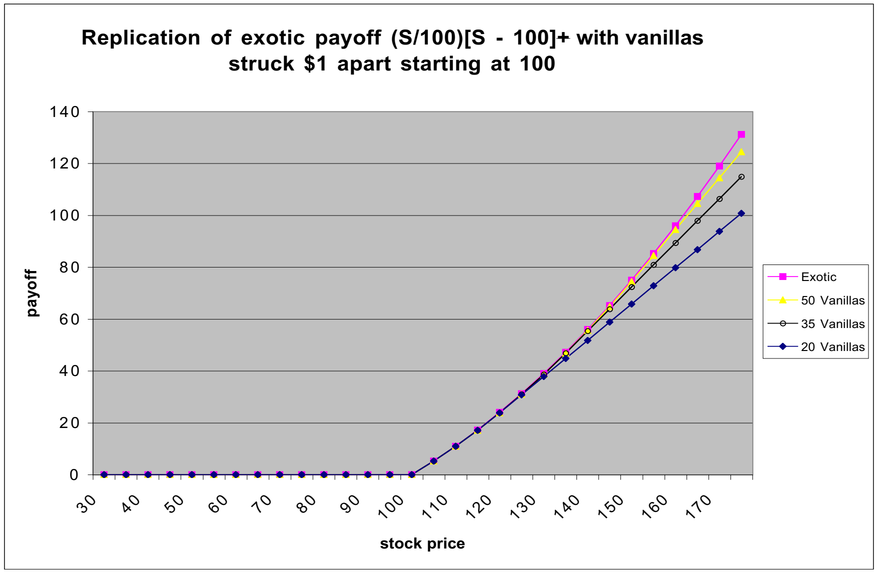
Integrate over calls with this density: $V(S, t) = \int_A^{\infty} \frac{\partial^2 V}{\partial K^2} C(S, K) dK$

S***ecurity* V** in terms of call options $C(K)$ of various strikes K:

Security: $\boldsymbol{V} = B\mathbf{C}(\mathbf{B}) + \int_B^{\infty} 2\mathbf{C}(\mathrm{K})\mathrm{dK}$

Value: $V(S, t) = BC(S, t, B, T) + 2\int_B^{\infty} C(S, t, K, T) dK$

Payoff of 50 calls with strikes equally spaced and \$1 apart between 100 and 150.



Convergence of the value of the replicating formula to the correct no-arbitrage value for two different smiles.

$$\Sigma(K) = 0.2\left(\frac{K}{100}\right)^{\beta}$$

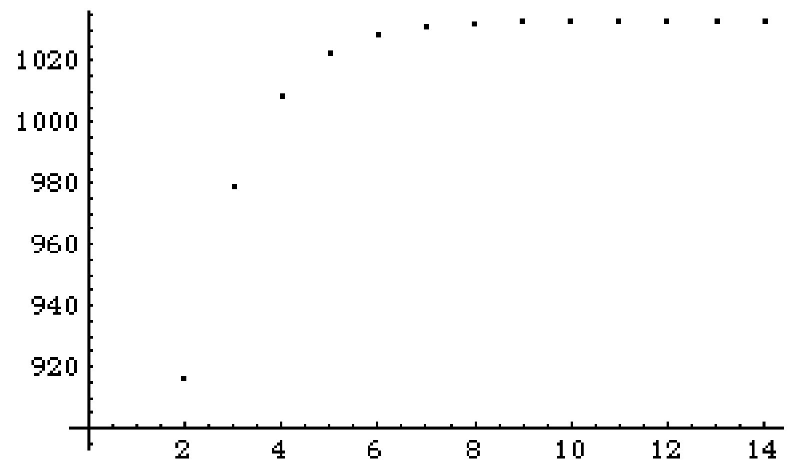
$\beta = -0.5$ "negative" skew. Implied volatility increases with decreasing strike.

$\beta = 0$ corresponds to no skew at all.

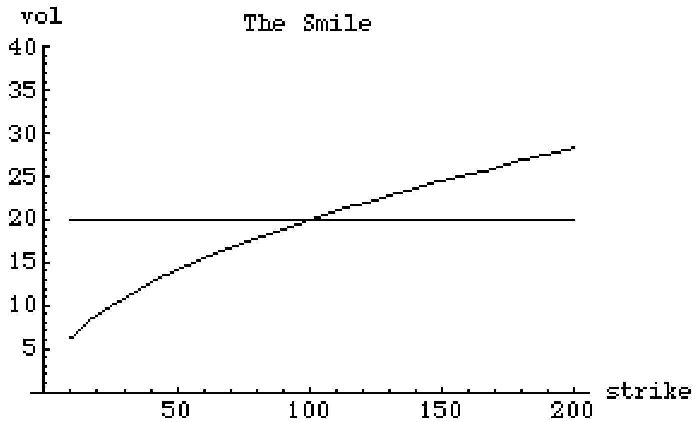
$\beta = 0.5$ corresponds to a positive skew.

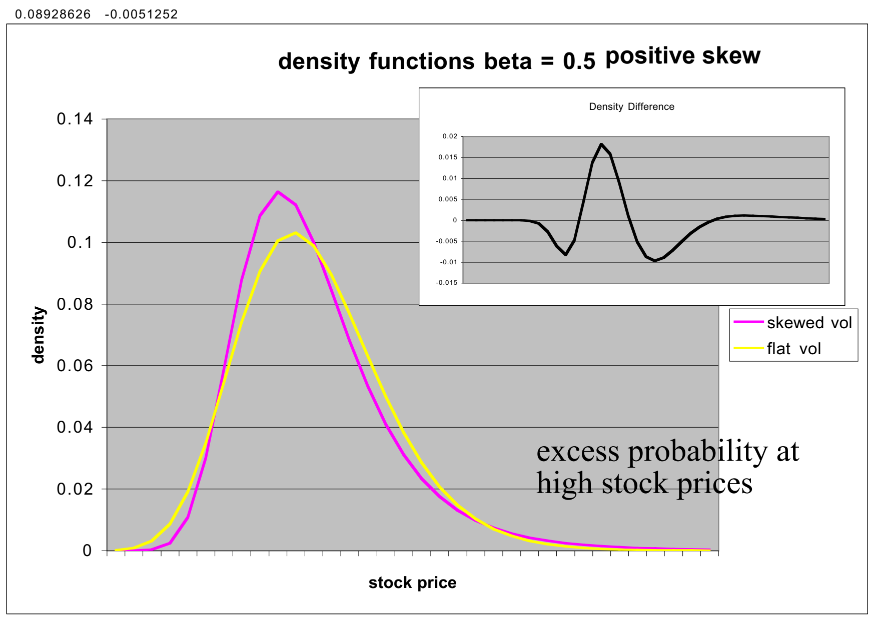
For $\beta = 0$ the fair value of V when replicated by an infinite number of calls is 1033. With 10 strikes the value has virtually converged.

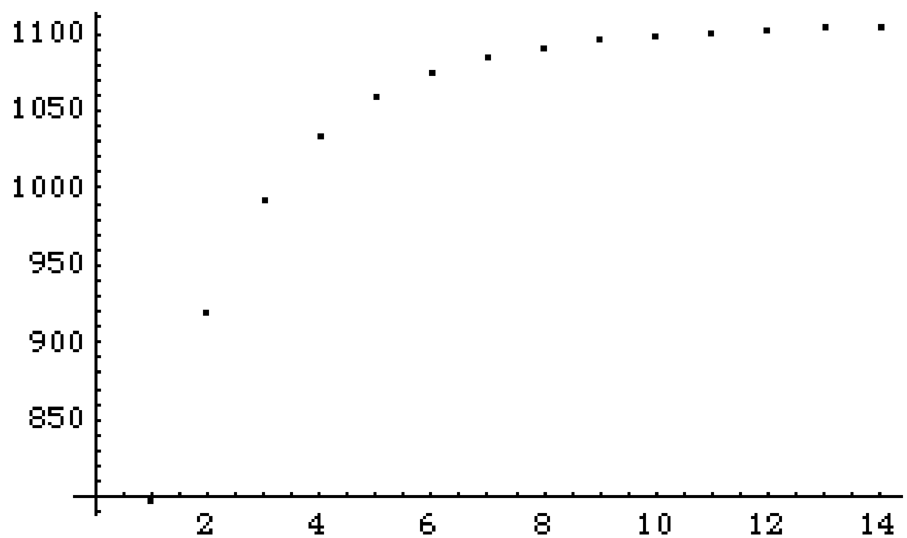
<u>Convergence as we increase number of strikes for flat 20% volatility</u>



Positive skew $\beta = 0.5$

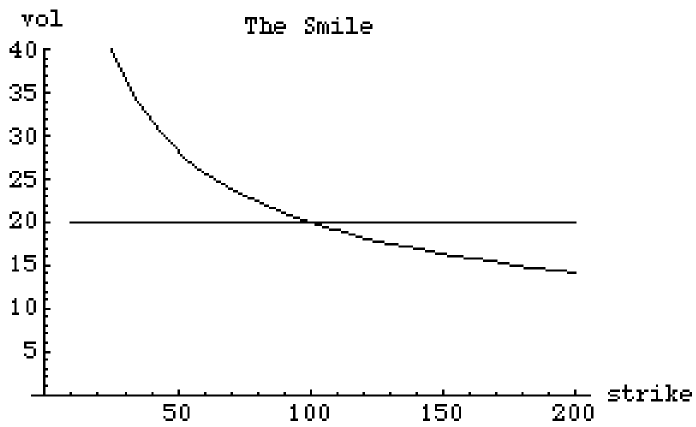


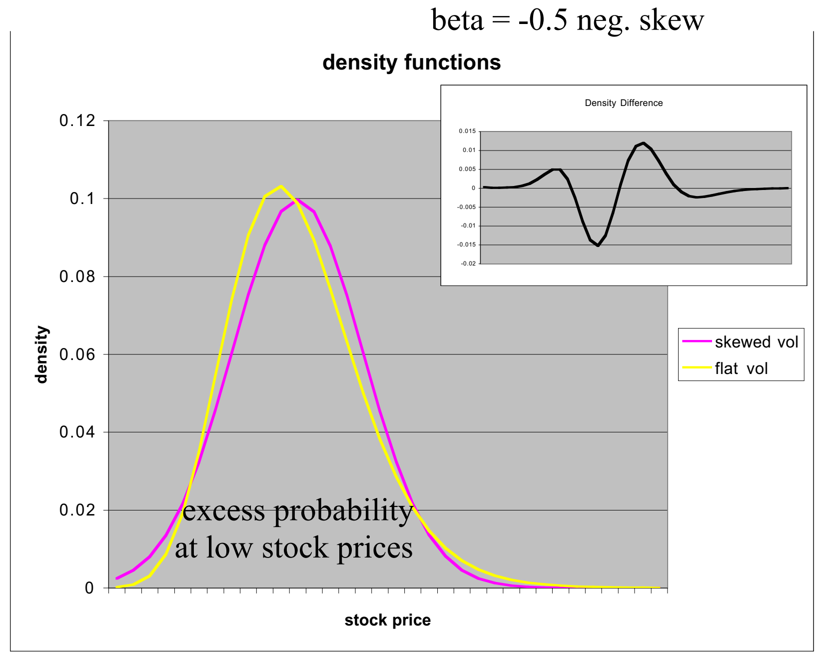


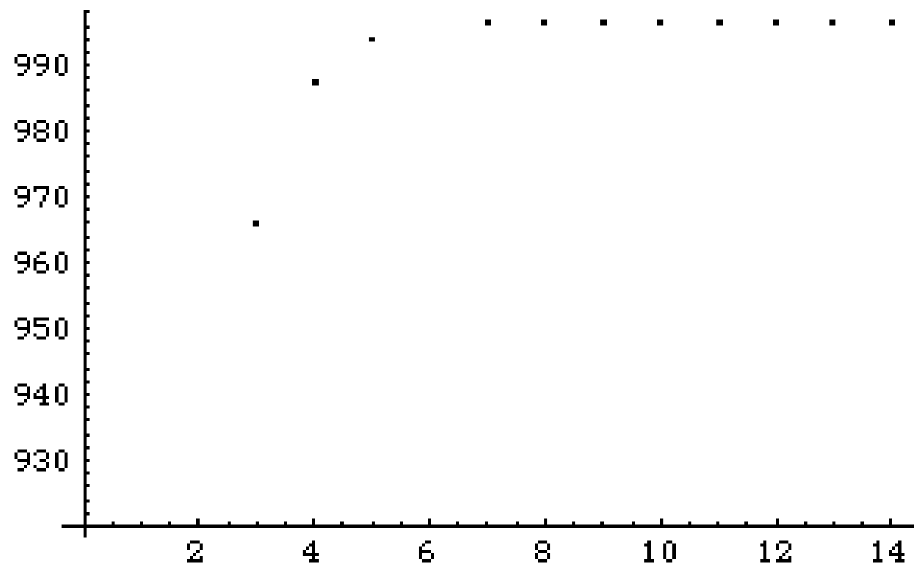


Convergence for a positive skew to a fair value of 1100 is slower and requires more strikes

Negative skew $\beta = -0.5$







Convergence for a negative skew to a fair value 996 is faster and requires fewer strikes. GS.

Static Replication of Non-European Options

- **Strong static replication:** Replication is independent of model.

  What we just did.

- **Weak static replication:** Weak replication needs a model and an assumption about the future smile. The method relies on the assumptions behind the Black-Scholes theory, or any other theory you used to replace it. The more and more liquid options you use to replicate the target portfolio, the better you can do.The costs of replication and transaction are embedded in the market prices of the standard options employed in the replication.

  What we are about to do.

Note: The Black-Scholes risk-neutral probability density

In the BS evolution, returns $\ln S_T / S_t$ are normal with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$.

Therefore,

$$x = \frac{\ln S_T / S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}} \qquad \text{Eq.10.6}$$

is normally distributed with mean 0 and standard deviation 1, with a probability density $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The returns $\ln S_T / S_t$ can range from $-\infty$ to ∞. From Eq.10.6,

$$\frac{dS_T}{S_T} = \sigma\sqrt{\tau}\,dx$$

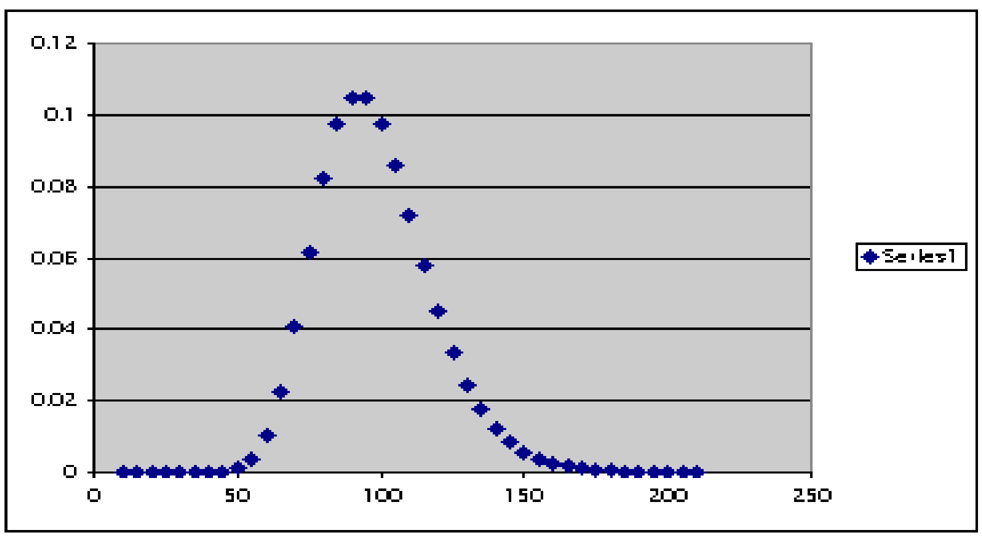
The risk-neutral value of the option is given by

$$e^{r\tau}C = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{K}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)\frac{dS_T}{S_T}$$

where

$$\frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi\tau}\sigma S_T}$$

is the risk-neutral density function to be used in integrating payoffs over S_T, plotted below

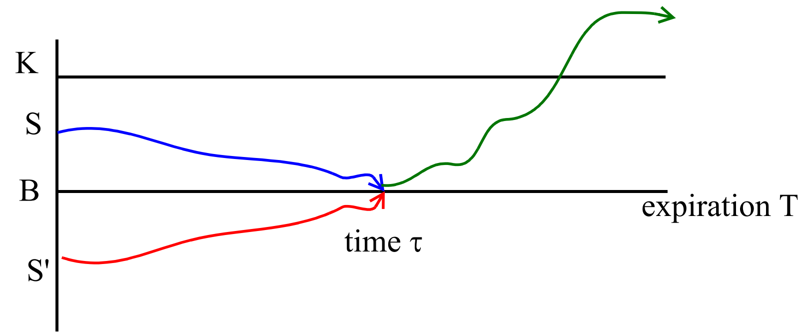


Valuing Barrier Options

This is important because the valuation suggests a method of replication.

Valuing a Barrier Option for GBM with Zero Risk-Neutral Stock Drift

A **down-and-out** option with strike K and barrier B.



- Choose a “reflected” imaginary stock S': The blue trajectory from S and the red trajectory from S' have equal probability to get to any point on B at time τ,.
- From there, they have equal probability of taking the future green trajectory that finishes in the money.
- For any green trajectory finishing in the money, the paths beginning at S and S' have the same probability of producing the green trajectory.
- Subtract the two probability densities and then above the barrier B, the contribution from every path emanating from S that touched the barrier at any time τ will be cancelled by a similar path emanating from S'.

Where is S'?

The probability to get from S to S' in a GBM world depends only on $\ln S/S'$; the reflection S' of S in the barrier B must be a log reflection, that is

$$\ln\frac{S}{B} = \ln\frac{B}{S'} \text{ or } S' = \frac{B^2}{S}$$

The density for getting from S to a stock price S_τ a time τ later is therefore

$$n' = n\left(\frac{\ln S_\tau/S + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \qquad \text{Eq.10.7}$$

for some coefficient α, where $n(x)$ is a normal distribution with mean 0 and standard deviation 1, and we want this density to vanish when $S_\tau = B$, which requires $\alpha = \left(\frac{S}{B}\right)$ independent of τ

Thus the option price is $C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K\right)$

C_{DO} vanishes on boundary S = B at any time. If S > K > B at expiration, $B^2/S < K^2/S < K$ and second option finishes out of the money. Thus C_{DO} has the correct boundary conditions.

Valuation for non-zero risk-neutral drift $\mu = r - 0.5\sigma^2$

When the drift is non-zero then probabilities for reaching B from both S and S' differ, since the drift distorts the symmetry. Pick a superposition of densities and S and the same reflection $S' = B^2/S$.

Trial down-and-out density for reaching a stock price S_τ a time τ later is

$$n' = n\left(\frac{\ln S_\tau/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln(S_\tau S)/B^2 - \mu\tau}{\sigma\sqrt{\tau}}\right)$$

$$n\left(\frac{\ln B/S - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha n\left(\frac{\ln S/B - \mu\tau}{\sigma\sqrt{\tau}}\right) = 0 \quad \text{implies} \quad \alpha = \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} \quad \text{independent of } \tau$$

$$C_{DO} = C_{BS}(S, t, \sigma, K) - \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} C_{BS}\left(\frac{B^2}{S}, t, \sigma, K\right)$$

A superposition of solutions. Method of images in electrostatics.