LECTURE 15

LOCAL VOLATILITY MODELS:

DUPIRE EQUATION & RULES OF THUMB
HEDGE RATIOS
EXOTIC OPTIONS

Looking Ahead

Dupire Equation and Justification of Rules of Thumb

Hedge Ratios of Vanillas

Values of Exotics

Hedging Rules

Stochastic Volatility Models

Jump Diffusion Models

Guest Speakers

Michael Kamal - April 15
Jackie Rosner - April 20

If you have questions come to my office hours or see me some other time by appointment.

An Exact Relationship Between Local and Implied Volatilities and Its Consequences (Homework Problem)

For zero interest rates and dividend yields, we derived $\sigma^2(K, T) = \left(2\left.\frac{\partial C}{\partial T}\right|_K\right) \Big/ \left(K^2 \left.\frac{\partial^2 C}{\partial K^2}\right|_T\right)$

Quoting in terms of BS implied vols: $C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T))$

By carefully using the chain rule for differentiation and the formulas for the Black-Scholes Greeks:

$$\sigma^2(K, T) = \frac{2\frac{\partial \Sigma}{\partial T} + \frac{\Sigma}{T-t}}{K^2\left(\frac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{T-t}} + d_1\frac{\partial \Sigma}{\partial K}\right\}^2\right)}$$

where $d_1 = \frac{\ln(S/K)}{\Sigma\sqrt{T-t}} + \frac{\Sigma\sqrt{T-t}}{2}$, and $\Sigma = \Sigma(S, t, K, T)$ is a function of S, t, K, T.

This formula is the generalization of the notion of forward volatilities in a no-skew world to local volatilities in a skewed world.

We can now prove rigorously the previous relations we intuited between implied local volatility.

Rule of Thumb 1: Implied variance is average of local variance if there no skew.

$\Sigma(S, t, K, T)$ is independent of strike K, $\frac{\partial\Sigma}{\partial K} = 0$ with no skew at all. Then, writing $\tau = T - t$

$$\frac{1}{2}\sigma^2(K, T) = \frac{\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{2\tau}}{K^2\frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}}\right\}^2} = \tau\Sigma\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma^2}{2}$$

$$\sigma(\tau)^2 = \frac{\partial}{\partial\tau}(\Sigma^2\tau)$$

$$\tau\Sigma^2(\tau) = \int_0^\tau \sigma^2(u)du$$

the standard result that expresses the total variance as an average of forward variances.

Rule of Thumb 2: Near the money, for no term structure, the slope of the skew w.r.t strike is 1/2 the slope of the local volatility w.r.t. spot

$\Sigma = \Sigma(K)$ alone, independent of expiration, and $\dfrac{\partial\Sigma}{\partial\tau} = 0$.

Assume a *weak linear* dependence of the skew on K, so that we keep only terms of order $\dfrac{\partial\Sigma}{\partial K}$,

assuming $\left(\dfrac{\partial\Sigma}{\partial K}\right)^2$ and $\dfrac{\partial^2\Sigma}{\partial K^2}$ are negligible.

Then

$$\sigma^2(K, T) = \frac{2\dfrac{\partial\Sigma}{\partial T} + \dfrac{\Sigma}{T-t}}{K^2\left(\dfrac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\dfrac{\partial\Sigma}{\partial K}\right)^2 + \dfrac{1}{\Sigma}\left\{\dfrac{1}{K\sqrt{T-t}} + d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)} \approx \frac{\dfrac{\Sigma}{\tau}}{\dfrac{K^2}{\Sigma}\left(\left\{\dfrac{1}{K\sqrt{\tau}} + d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + d_1 K\sqrt{\tau}\dfrac{\partial\Sigma}{\partial K}\right\}^2}$$

and so

$$\sigma(K) = \frac{\Sigma(K)}{1 + d_1 K\sqrt{\tau}\dfrac{\partial\Sigma}{\partial K}}$$

Close to at-the-money, $K = S + \Delta K$. Then

$$d_1 \approx \frac{\ln S/K}{\Sigma\sqrt{\tau}} \approx -\frac{(\Delta K)}{S(\Sigma\sqrt{\tau})} \approx -\frac{(\Delta K)}{K(\Sigma\sqrt{\tau})}$$

so that

$$\sigma(K) \approx \frac{\Sigma(K)}{1 - \frac{(\Delta K)}{\Sigma}\frac{\partial \Sigma}{\partial K}} \approx \Sigma(K)\left(1 + \frac{(\Delta K)}{\Sigma}\frac{\partial \Sigma}{\partial K}\right) \approx \Sigma(K) + (\Delta K)\frac{\partial \Sigma}{\partial K}$$

Therefore since, $K = S + \Delta K$ $\qquad \sigma(S + \Delta K) \approx \Sigma(S + \Delta K) + (\Delta K)\frac{\partial \Sigma}{\partial K}$

and so, since $\sigma(S + \Delta K) \approx \sigma(S) + (\Delta K)\frac{\partial \sigma}{\partial S} = \Sigma(S) + (\Delta K)\frac{\partial \sigma}{\partial S}$ and $\Sigma(S + \Delta K) \approx \Sigma(S) + (\Delta K)\frac{\partial \Sigma}{\partial K}$

we obtain $\qquad \frac{\partial}{\partial S}\sigma(S) = 2\left(\frac{\partial \Sigma}{\partial K}\right)\Bigg|_{K = S}$

The local volatility $\sigma(S)$ grows twice as fast with stock price *S* as the implied volatility $\Sigma(K)$ grows with strike!

Rule of Thumb 3: Implied volatility is an harmonic average over local volatility at short expirations.

For zero rates and dividends: $$\sigma^2(K,T) = \frac{2\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{\tau}}{K^2\left(\frac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

Multiplying top and bottom by τ: $$\sigma^2(K,T) = \frac{2\tau\frac{\partial\Sigma}{\partial\tau} + \Sigma}{K^2\left(\tau\frac{\partial^2}{\partial K^2}(\Sigma) - d_1\tau\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

As $\tau \to 0$, this becomes the o.d.e. $$\sigma^2(K,T) = \frac{\Sigma}{K^2\left(\frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{d\Sigma}{dK}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + \sqrt{\tau}Kd_1\frac{d\Sigma}{dK}\right\}^2}$$

Now $\sqrt{\tau}Kd_1 \to \frac{K\ln(S/K)}{\Sigma}$ as $\tau \to 0$, and we obtain $$\sigma = \frac{\Sigma}{1 + \frac{K}{\Sigma}\frac{d\Sigma}{dK}\ln(S/K)}$$

Transforming from K into the new variable $x = \ln(S/K)$ we can rewrite this as the o.d.e.

$$\frac{\Sigma}{1-\frac{x}{\Sigma}\frac{d\Sigma}{dx}} = \sigma$$

In the homework you are asked to show that the solution is $\frac{1}{\Sigma(x)} = \frac{1}{x}\int_{o}^{x}\frac{1}{\sigma(y)}dy$

In other words, at very short times to expiration, the implied volatility is the harmonic mean of the local volatility as a function of $\ln S/K$ between spot and strike.

This is intuitively reasonable, more sensible than an arithmetic mean.

Suppose that $\sigma(y)$ falls to zero above a certain level S, so that the stock price can never diffuse higher. Then the implied volatility of any option with a strike above that level should be zero.

If $\Sigma(x) = \frac{1}{x}\int_{o}^{x}\sigma(y)dy$, an ordinary arithmetic mean, then its value would be non-zero, which is impossible if the stock can never reach the strike. In contrast, for the harmonic mean in Equation , if $\sigma(y)$ becomes zero anywhere in the range between spot and strike, then the implied volatility for that strike, $\Sigma(x)$, becomes zero too, which is as to be expected.

There is an intuitive way to understand the harmonic mean

If the stock's volatility were infinite, it would be transparent to diffusion. Think of it as a medium.

$1/\sigma^2(\ln S)$ is the time taken to diffuse through the medium for $\ln S$.

$\int_{\ln S}^{\ln K} 1/\sigma^2$ is the total diffusion time. Or $\int_{\ln S}^{\ln K} 1/\sigma$ is the square root of the total diffusion time.

$\int_o^x \frac{1}{\sigma(y)} dy$ is roughly the total $\sqrt{\text{diffusion time}}$ computed from the sum of local $\sqrt{\text{diffusion times}}$.

But the Total $\sqrt{\text{time}}$ = total distance / Average Volatility = $\dfrac{\ln S/K}{\Sigma}$

$\frac{x}{\Sigma(x)} = \int_o^x \frac{1}{\sigma(y)} dy$ The average volatility is found from the total time and total distance.

This is similar to the statement that, for a car with a velocity that varies with position, the **total time** for the trip is the sum of the local times. The average velocity is *not* the average of the local velocities. The **average velocity is the total distance divided by the total time** and is therefore the harmonic average of the local velocities.

$$T = \int_0^D \frac{dx}{v(x)} \equiv \frac{D}{\overline{\overline{V}}}$$

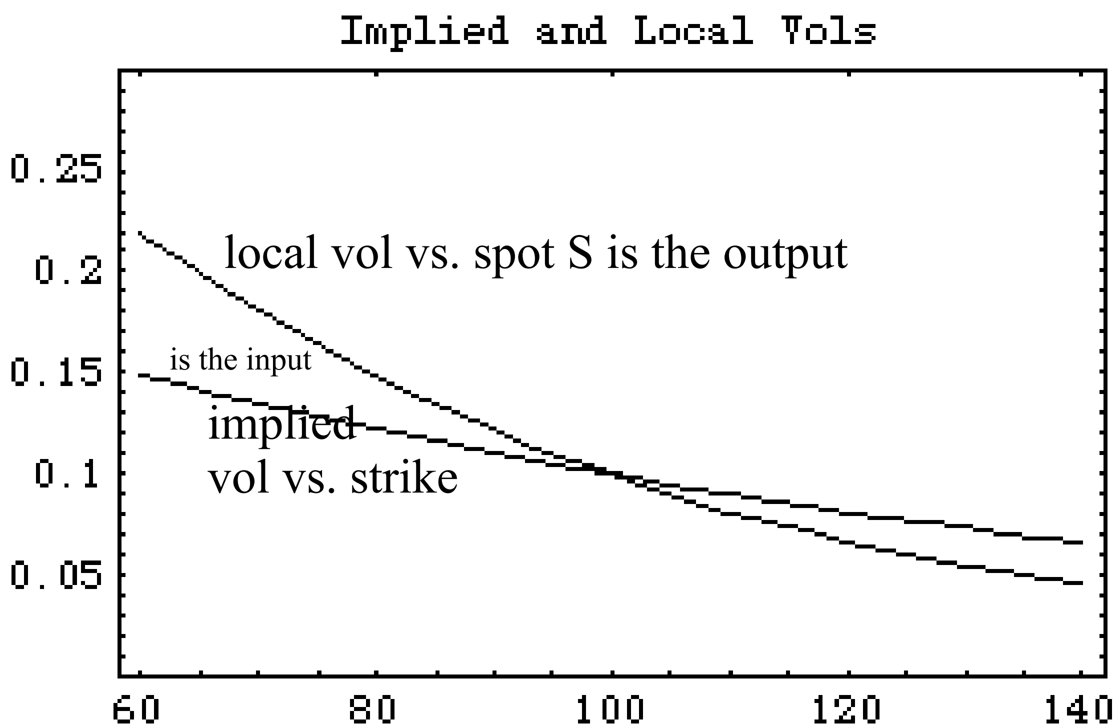
$$\frac{1}{\overline{\overline{V}}} = \frac{1}{D}\int_0^D \frac{dx}{v(x)}$$

An Example Of Using The Dupire Equation To Calibrate the Local Volatilities

These graphs are produced inversely. Given a skew, we compute the local volatilities, i.e. we solve the inverse problem. In the previous lectures, we proceeded from local volatilities to implieds.

Assume $\Sigma(K, T) = 0.1 \exp\left[-\left(\frac{K}{100} - 1\right)\right]$ with no term structure.

I used Mathematica to evaluate the derivatives of the Black-Scholes options prices using this skew to calculate the local volatility.

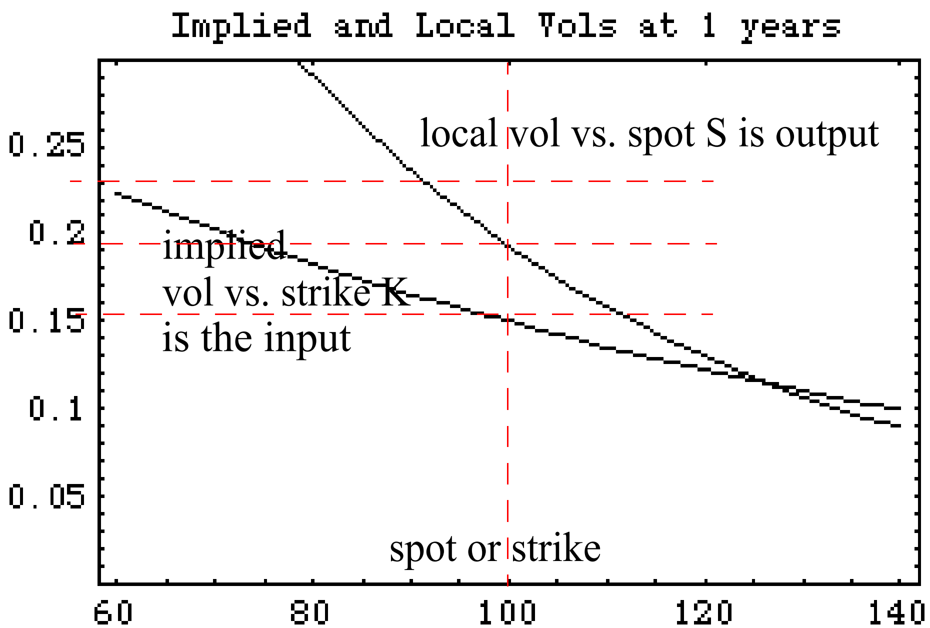


Local volatility does increase roughly twice as fast with spot as implied volatility varies with strike.

Example where volatility varies with strike and expiration, increasing with expiration T but decreasing with strike K according to the formula

$$\Sigma(K, T) = (0.1 + 0.5T)\exp\left[-\left(\frac{K}{100} - 1\right)\right]$$

This smile has both term structure and skew.



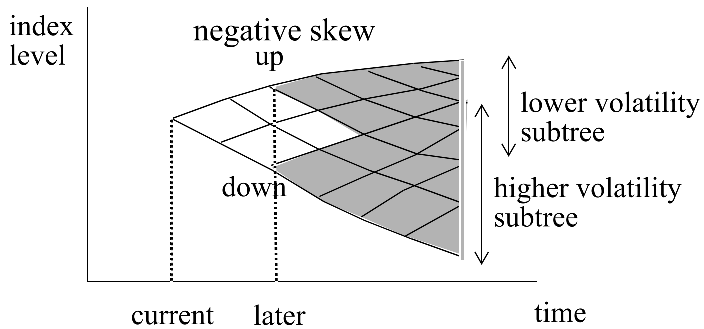
At one year in the future, the local volatility still has the characteristic variation with spot, but its value is higher. This illustrates our previous argument that if the term structure of implied volatilities is increasing, then local volatilities must grow about twice as fast with time as well as decrease twice as fast with spot.

Hedge Ratios in Local Volatility Models

Black-Scholes in inconsistent with the smile. It makes no sense to hedge or value an exotic in a model that cannot value vanillas correctly.

Local volatility models are consistent. The **Rules of Thumb** make it easier to get the results of the local volatility model in the framework of Black Scholes.

$\frac{\partial}{\partial S}\Sigma(S, t, K, T) \approx \frac{\partial}{\partial K}\Sigma(S, t, K, T)$. The change in implied volatility of a given option for a change in market level is about the same as the change in implied volatility for a change in strike level



Implied volatility is average of local volatility between spot and strike.

Changing strike is same as changing spot.

Effect on hedge ratios?

$$\Sigma(S, K) \approx \sigma_0 - \beta(S + K) + 2\beta S_0$$

approximate symmetry of local volatility

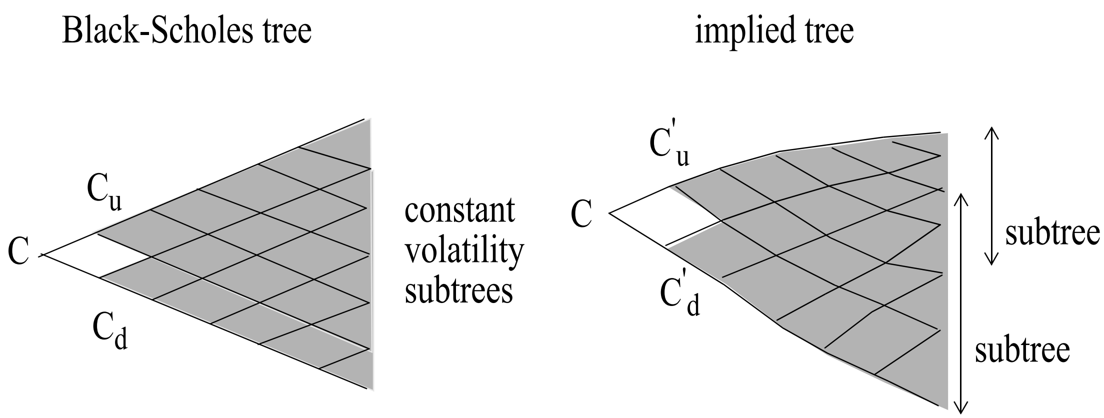
Follows from averaging:

The correct exposure Δ of an option is approximately given by the chain rule formula

$$\Delta = \Delta_{BS} + Vega_{BS} \times \beta \qquad \text{Eq.1.1}$$

One-year S&P option with a B-S hedge ratio of 60% probably has a true hedge ratio of 50%, because volatility moves down as the market moves up.

S = 2000; $Vega_{BS}$ = 800 dollars; β = -0.0001 vol point per strike pt.: $Vega_{BS}\beta \sim -0.1$

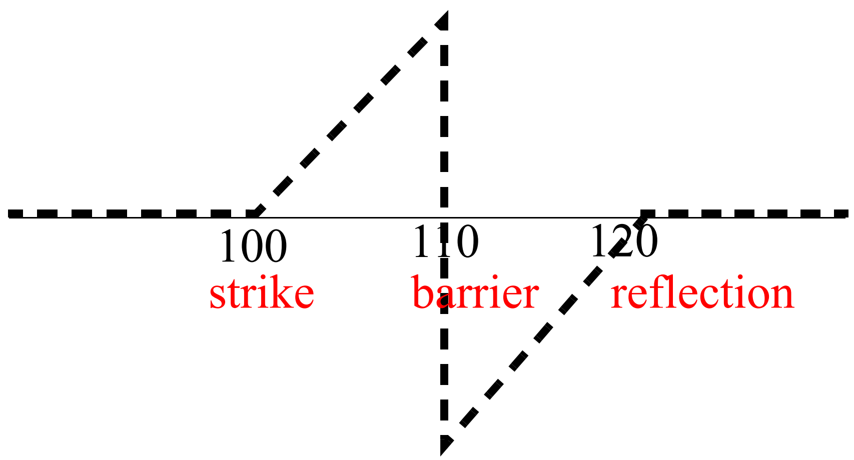


Exotics: The Theoretical Value of Barrier Options in Local Volatility Models

Barrier options depend on the risk-neutral probability of index remaining in the region between the strike and the barrier, and hence on the local volatility in this region.

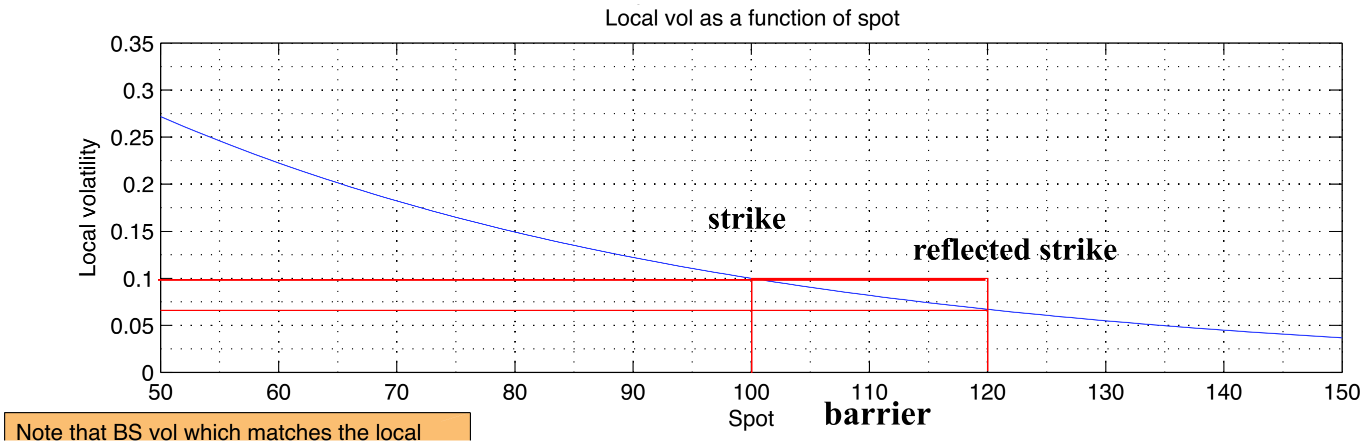
Example 1: An Up-and-Out Call. with Strike 100 and Barrier 110

• You can approximately replicate an up and out call by means of a payoff like this:.



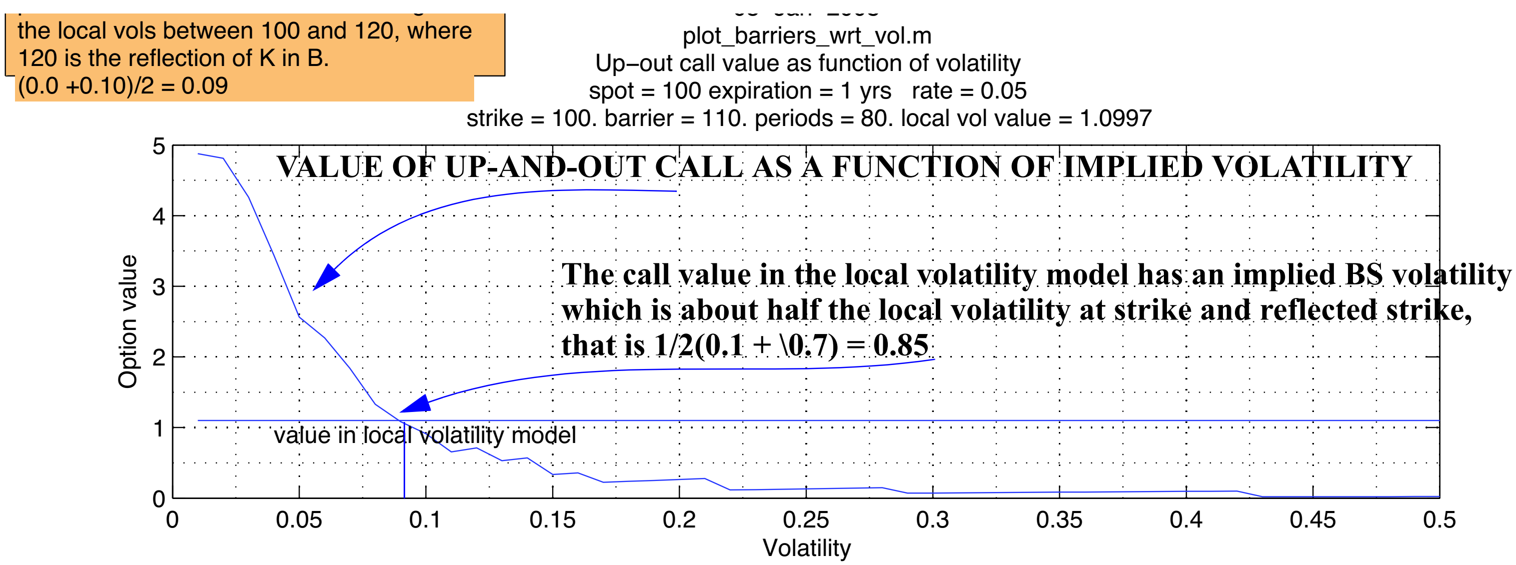
• In a skewed world, we estimate the implied BS volatilikty as approximately the average of the local volatilities between spot and strike.

• Thus the approximate value of the Black-Scholes implied volatility for the up-and-out call is the average of the local volatilities between 100 and 120.



• Local volatility varies between 0.1 and 0.07 in this range, with an average of a about 0.085.

.VALUE OF UP-AND-OUT CALL AS A FUNCTION OF IMPLIED VOLATILITY IN BLACK-SCHOLES

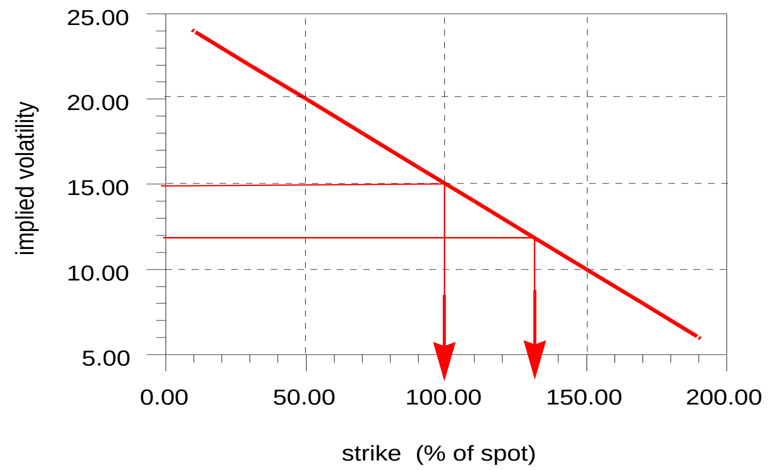


The value of the up and out option in the local volatility model is about 1.1, which corresponds to a Black-Scholes implied volatility of about 0.09, so this intuition about averaging works reasonably

Example 2. An Up-and Out Call that has no Black-Scholes Implied Volatility

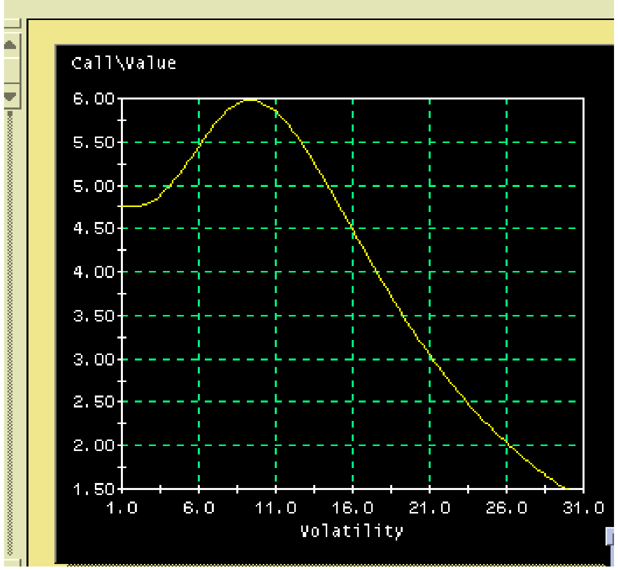
- In some cases, the local volatilities can produce options values that cannot be matched by *any* Black-Scholes implied volatility. No amount of intuition can get you the exactly correct value.
- Up and out call, spot and strike at 100, and the barrier at 130, and the skew as shown below.
- The value of the barrier option in this local volatility model is 6.46.

A hypothetical volatility skew for options of any expiration. We assume r = 5% d = 0%



- The maximum Black-Scholes value in a no-skew world is 6.00 corresponding to a 9.5% implied volatility.
- The BS value is smaller than the “correct” value in the local volatility model.
- There is NO Black-Scholes implied volatility which gives the local-volatility “correct” option value.
- The implied volatility that comes closest to it is about 10%. Why?

No skew: Up-and-Out call value as a function of Black-Scholes Implied Volatility



- The slope of the skew is 1 vol pt. per 10 strike points. The rule of 2 then indicates that the slope of the local volatilities will be about 1 vol pt. per 5 strike points.
- An up-and-out call with strike 100 and barrier 130 as being roughly replicated by an ordinary call with strike 100 and a reflected put with strike 160.
- The local volatility that is relevant to valuation ranges between spot prices 100 and 160 with a slope of approximately 1 vol pt. per 5 strike points, that is from values of 15% to $15 - (60/5) = 3\%$. The average local volatility in this range is about 9%, which substantiates the approximate claim the implied volatility is the average of the local volatilities between spot and strike

Exotics in Local Volatility Models

Lookback Call Options With A Smile

Path-dependent options (averages, lookbacks) are sensitive to local volatility in multiple regions. No single constant volatility is correct for valuing a path-dependent option in a skew. You can simulate the index evolution over all future market levels and their corresponding local volatilities to calculate fair value.

A lookback call pays out $[S_T - S_{min}]$

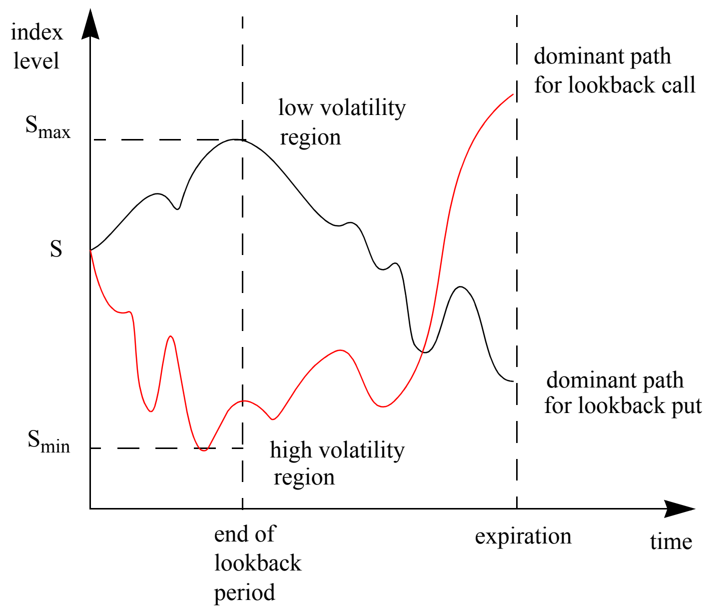
One-year lookback call or put with a initial three-month lookback period on the strike.

Lookback Call: $max[S_T - S_{min}]$

Lookback Put: $max[S_{max} - S_T]$

Value the option by simulating index paths whose local volatilities are extracted from the relevant implied volatility smile.

Dominant paths contributing to the value of a lookback call and put. The local volatilities are negatively skewed.



The **most important path** for a lookback call sets a low strike S_{min} during the first three months, and then rises to achieve a high payoff.

In a negatively skewed world, low S_{min} has higher index local volatility from then on, and therefore the option is worth more than in a flat world.

Conversely, the dominant path for a lookback put is more likely to have a high strike S_{max}, and a low subsequent volatility.

Therefore, in a negatively skewed world, compared to Black-Scholes, lookback puts are worth less and lookback calls are worth more.

Lookback calls will have higher implied volatilities than lookback puts. In a Black-Scholes world with zero drift they would have the same implied volatilities.

Example: Index = 100, d = 2.5%, r = 6% per year.
Negative skew: ATM implied vol = 15%, decreases by 3 percentage points for each increase of 10 index strike points.

Monte Carlo simulation:

Lookback call value = 10.8 ($\Sigma_{BS} = 15.6\%$), Lookback put value = 5.8 ($\Sigma_{BS} = 13.0\%$).

Hedging: Lookback Call Deltas With A Smile are Different from Black-Scholes

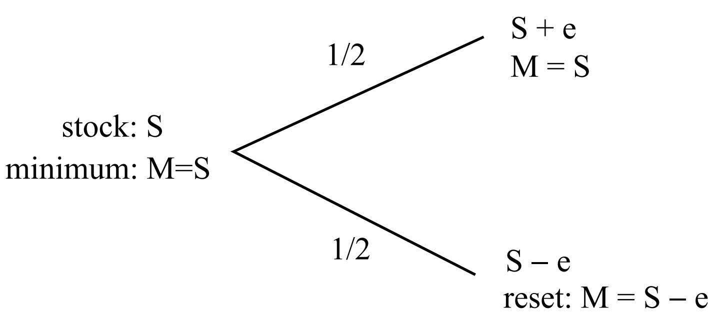
Claim: In the Black-Scholes model at inception, a lookback call has a delta of approximately zero:

Indicative Proof: Let $V(S, M, \tau)$ be the value of the lookback call when the stock is S and the minimum value of the index so far is M, and τ is the time to expiration.

First: intuitively, when $M = S$, a little increase in S is the same as a correspondingly small decrease in M, so that

$$\left.\frac{\partial V}{\partial S}\right|_{M = S} = -\left.\frac{\partial V}{\partial M}\right|_{M = S}$$

Second: consider the next infinitesimal move $e = \sigma\sqrt{\tau}$ in the stock price:



$$V(S, S, \tau) = \frac{1}{2}V(S+e, S, \tau-d\tau) + \frac{1}{2}V(S-e, S-e, \tau-d\tau)$$

$$\approx V(S, S, \tau) - \frac{\partial V}{\partial M}\frac{e}{2} + O(\tau)$$

$$\frac{\partial V}{\partial M} \approx 0$$

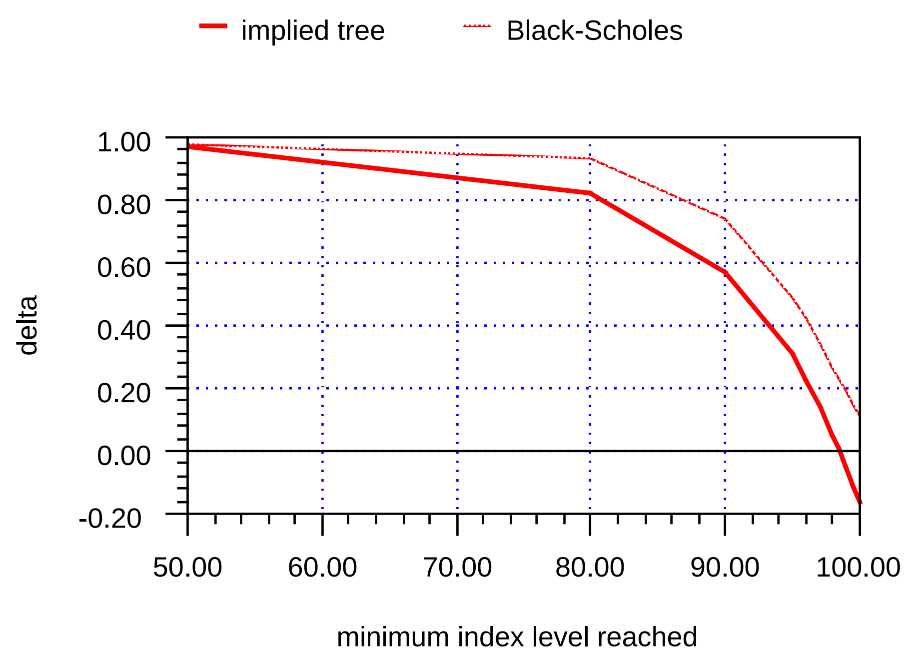
Then by backwards induction in the risk neutral world with zero interest rates, to leading order.

Thus for M = S, $\frac{\partial V}{\partial S} \approx \frac{\partial V}{\partial M} \approx 0$ and so the delta of the lookback is approximately zero under these conditions. Cf. the value 0.5 for ordinary call.

Local volatility model deltas compared to Black-Scholes deltas for the one-year lookback call described above, for a range of minima M.

- Black-Scholes deltas are calculated at the Black-Scholes implied volatility of 15.6%.
- The local-volatility-model delta of an at-the-money lookback call is negative – to hedge a long call position you must actually go long the index! Increasing the stock price decreases the value of the option! It's advantageous for the stock price to first drop.
- The delta of the lookback call is always lower than in Black-Scholes.

FIGURE 1.1. The delta of a one-year call with a three-month lookback period that has identical prices in the implied tree model and the Black-Scholes world with no skew. The current market level is assumed to be 100.



The mismatch is greatest where volatility sensitivity is largest, where M = S.
The mismatch is smallest when the lowest level previously reached is much lower than the current index level, since M is unlikely to change and the lookback is effectively a forward contract with zero volatility sensitivity.

Practical Calibration of Local Volatility Models (Brief)

In practice we are given implied volatilities $\Sigma(K_i, T_i)$ and must calibrate a smooth local volatility function. To use Dupire's equation, we need a smooth implied volatility surface that is at least twice differentiable in the strike direction and once differentiable in the time direction, and that doesn't violate arbitrage.

But all we have is discrete points with noise. They have to be smoothed.

.The most straightforward way to do this is to write down a smooth parametric form for the implied volatilities, and then compute the parameters that minimize the distance between computed and observed standard options prices.

One can then calculate the local volatilities by taking the appropriate derivatives of the implieds. One difficulty with this method is how to determine a realistic form of the parametrization, particularly on the wings where prices are hard to obtain.

Other methods involve splines (nonparametric) or semiparametric interpolations.

There are many papers on this.

SVI (Stochastic Volatility Inspired) Parametrization

In its original formulation (Gatheral, 2004), SVI model is defined at each maturity T in terms of the 5 parameters a, b, ρ, m, σ such that the square of the implied volatility $\theta(K,T)$ is

$$\begin{aligned} \theta^2(K,T) &= v(x,T) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2}\right) \\ x &= \ln(K/F(T)) \end{aligned} \tag{1}$$

where $F(T)$ is the forward and the parameters lie in the following definition domain

$$b > 0 \tag{2}$$

$$\sigma \geq 0 \tag{3}$$

$$\rho \in [-1, 1] \tag{4}$$

$$a \geq -b\sigma\sqrt{1-\rho^2}. \tag{5}$$

(Rogers & Tehranchi, 2008) derived the necessary condition for no-strike arbitrage

$$\forall x,\ \forall T, \quad |\frac{\partial v(x,T)}{\partial x}| \leq 4, \tag{6}$$

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

Heuristic Rules & Models for Variation of Implied Volatility Σ: Skew Relates Statics to Dynamics

Traders like heuristic rules for the B-S *quoting parameter* Σ rather than models of stochastic evolution.

Specify what **doesn't change** rather than what changes. Invariance principles or symmetries.

The Sticky Strike Rule.
The Sticky Delta Rule.
The Sticky Local Volatility Rule.

The Sticky Strike Rule

Each option of a definite strike maintains its initial implied volatility – hence the "sticky strike" appellation. This is the simplest "model" of implied volatility:

$$\Sigma(S, K, t) = \Sigma_0 - b(K - S_0) \qquad \text{Sticky Strike Rule, independent of } S \text{ for all } t$$

(We have *assumed* $b(t) \equiv b$, independent of t. b can change, especially during crisis periods.)

Intuitively, "sticky strike" is a poor man's inconsistent attempt to preserve the BS model.
As S moves, each option keeps the exactly the same constant future instantaneous volatility in its evolution, inconsistently different for different options.

Implied volatility for an option of strike K is independent of S, and therefore $\Delta = \Delta_{BS}$.

TABLE 1. Volatility behavior using the sticky-strike rule.l

Quantity	Behavior
Fixed-strike volatility:	is independent of index level
At-the-money volatility $\Sigma_{atm}(S)$:	$\Sigma_{atm}(S, t) = \Sigma_0 - b(S - S_0)$ which decreases as index level increases
Exposure Δ:	$= \Delta_{BS}$

You can think of this model as representing Irrational Exuberance. Σ_{atm} decreases as S increases.

The Sticky Delta/Moneyness Rule

It's easier to start by explaining the related concept of sticky moneyness.

Sticky moneyness means that an option's volatility depends only on its moneyness K/S

$$\Sigma(S, K, t) = \Sigma_0 - b(K/S - 1)S_0 \qquad \text{Sticky Moneyness Rule}$$

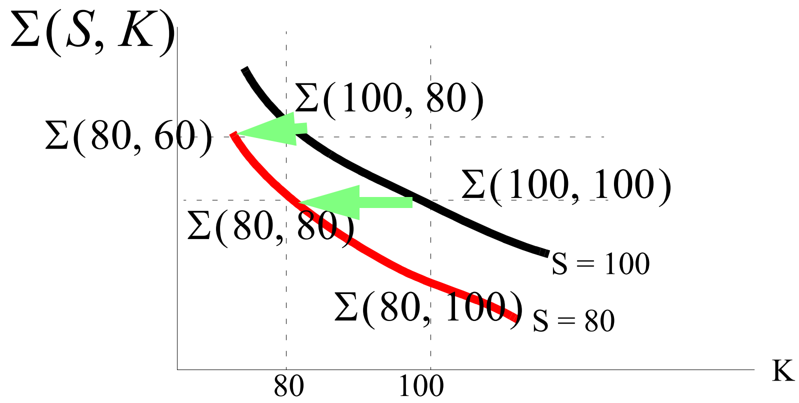
Approximately, close to atm: $\Sigma(S, K, t) = \Sigma_0 - b(K - S)$

Intuition: the volatility of the most liquid option, should stay constant as the index moves. Similarly, a 10% out-of-the-money should always have same volatility.

It's a scale-invariant model of common sense and moderation.

For a roughly linear skew $\Sigma \approx \Sigma(S - K)$

Therefore implied volatility must rise when S rises.



In the Black-Scholes model, Δ_{BS} depends on K and S through the moneyness *K/S,* so that "sticky moneyness" is equivalent to "sticky delta," with an at-the-money being $\Delta_{BS} \approx 0.5$.

Sticky delta means that the implied volatility must be purely a function of Δ_{BS}, i.e. $\frac{\ln S/K}{\Sigma\sqrt{\tau}}$.

Therefore, if Σ decreases with K, it must increases with S.
Therefore, the delta of the option is greater than the Black-Scholes delta.

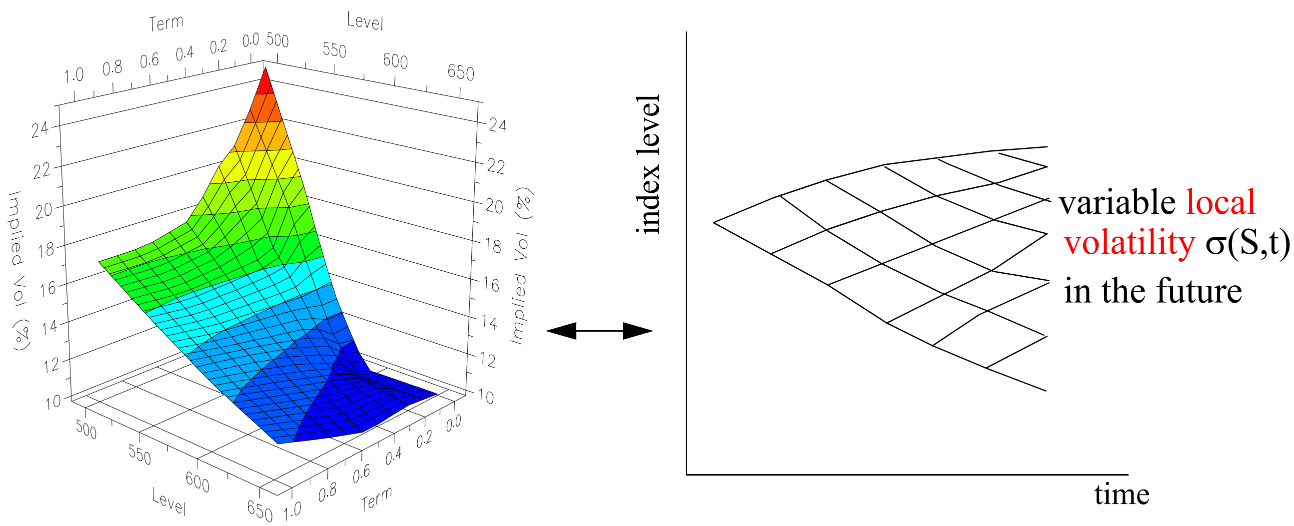
TABLE 2. Index Volatility behavior using the sticky-delta/moneyness rule.

Quantity	Behavior
Fixed-strike volatility:	increases as index level increases
At-the-money volatility:	is independent of index level
Exposure Δ:	$> \Delta_{BS}$

The (Sticky) Local Volatility Model

All current index options prices determine a single **consistent** unique set of local volatilities.

The implied tree corresponding to a given implied volatility surface.



The implied tree/local volatility model attributes the implied volatility skew to the market's expectation of higher realized volatilities and higher implied volatilities if the index moves down.

Index = 100			
Strike	**BS Vol. (%)**	**Index**	**Local Vol. (%)**
100	20%	100	20%
99	21%	99	22%
98	22%	98	24%
97	23%	97	26%

Implied Volatility In The Sticky Implied Tree Model

As the index level within the tree rises, you can see that the local volatilities decline, monotonically and (roughly) linearly, in order to match the linear strike dependence of the negative skew.

$$\Sigma(S, K, t) = \Sigma_0 - b(K + S - 2S_0) \qquad \text{Local Volatility Model, symmetric in K,S}$$

At-the-money volatility is given by

$$\Sigma_{atm}(S, t) = \Sigma_0 - 2b(S - S_0)$$

Implied volatilities decrease as K or S increases.
At-the-money implied volatility decreases twice as fast.
Δ is smaller than the Black-Scholes delta of an option with the same implied volatility.

In the linear approximation $\Sigma \approx f(K+S)$ which relates the skew at different strike and spot levels.

In order to satisfy $\Sigma(100, 80) = \Sigma(80, 100)$ with a negative skew, you can see that implied volatility must rise as the stock price decreases.

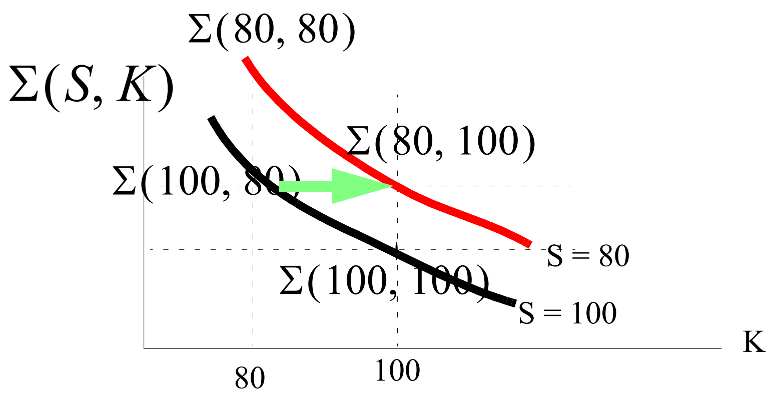


TABLE 3. Equity index volatility behavior in the sticky implied tree model.

Quantity	Behavior
Fixed-strike volatility:	decreases as index level increases
At-the-money volatility:	decreases twice as rapidly as index level increases
Exposure Δ:	$< \Delta_{BS}$

In this regime the options market experiences fear. The implied tree model implicitly assumes the skew arises from a fear of higher market volatility in the event of a fall, and assumes that after the fall, atm market volatility will rise twice as fast.

Problems and Benefits of Local Volatility Models

Inadequacy of the Short-Term Skew

For equity indexes, future short-term local volatilities have less skew than current short-term implied volatilities.Therefore the short-term future skew in a local volatility model is too flat. One needs the threat of jumps in the near future to produce a short-term skew.

A good model would look more or less time-invariant.

On the other hand, all financial models need recalibration; even in Black-Scholes, the implied volatility changes from day to day. Local volatility models, like Black-Scholes, must be recalibrated regularly; they allow the valuation of exotic options consistent with the volatility surface for vanilla options, and are widely used as a means of valuing exotics.

The question is: to what extent do they mirror the behavior of realized volatility?

Local Vol May Provide Better Hedge Ratios During Volatility Regimes

The best hedge is the one that minimizes the variance of the P&L of the hedged portfolio. If the replication were exact, the variance of the P&L would vanish.

Compare local volatility hedge ratios and P&L vs. Black-Scholes hedge ratios and P&L.

Call $C(S, t, K, T, \Sigma(S, t, K, T))$

Hedged portfolio $\Pi = C - \Delta S$

$$\Pi_{BS} = C - \Delta_{BS} S$$
$$\Pi_{loc} = C - \Delta_{loc} S$$

The difference between the BS-hedged P&L and the local-volatility-hedged P&L for a move δS:

$$\delta\Pi_{loc} - \delta\Pi_{BS} = (-\Delta_{loc} + \Delta_{BS})\delta S \equiv \varepsilon \times \delta S$$

In a negatively skewed market,

$$\Delta_{loc} = \Delta_{BS} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \approx \Delta_{BS} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \le \Delta_{BS}$$

and so $\varepsilon = (\Delta_{BS} - \Delta_{loc}) \ge 0$.

From the theory of hedging in Lecture 2, the individual P&L's are:

$$\delta\Pi_{BS} = \frac{1}{2}\Gamma_{BS}S^2\left[\sigma_R^2 - \Sigma^2\right]\delta t$$

$$\delta\Pi_{loc} = \frac{1}{2}\Gamma_{loc}S^2\left[\sigma_R^2 - \sigma(S, \delta t)^2\right]\delta t$$

since the p.d.e. for local volatility is the Black-Scholes equation with Σ replaced by $\sigma(S, \delta t)$, and if the hedging is perfect, $\delta\Pi_{loc}$ should vanish.

$\delta\Pi_{BS}$ is positive or negative depending on whether σ_R is greater or less than Σ.
$\delta\Pi_{loc}$ is positive or negative depending on whether σ_R is greater or less than $\sigma(S, \delta t)$.

What happens to $\delta\Pi_{BS}$ in actual markets?

Combining the equation, we see that the BS P&L in a world with negative skew is

$$\delta\Pi_{BS} = \delta\Pi_{loc} - \varepsilon \times \delta S = \frac{1}{2}\Gamma_{loc}S^2\left[\sigma_R^2 - \sigma(S, \delta t)^2\right]\delta t - \varepsilon \times \delta S$$

The gamma term is quadratic and non-directional, and depends on the volatility mismatch, and the δS hedging mismatch is linear and directional.

Crepey: Four different market regimes: indexes can move up or down, with high or low realized volatility compared to instantaneous local volatility.

TABLE 4. Types of Markets: Equity index markets have the characteristics of the yellow cells.

Volatility / Direction	volatile	non-volatile
up	$\sigma_R > \sigma(S, \delta t), \delta S > 0$	$\sigma_R < \sigma(S, \delta t), \delta S > 0$
down	$\sigma_R > \sigma(S, \delta t), \delta S < 0$	$\sigma_R < \sigma(S, \delta t), \delta S < 0$

For volatile down markets (a fast sell-off), both terms for $\delta\Pi_{BS}$ are positive, and the errors to $\delta\Pi_{BS}$ are additive. The Black-Scholes P&L differs from zero (the perfect hedge value) due to two additive contributions.

For non-volatile up markets (slow rise), both terms are negative and the same is true.

For slow sell-offs or fast rises, the two terms have opposite signs, and the hedging errors tend to cancel. Therefore, the Black-Scholes hedging strategy will perform worst in fast sell-offs or slow rises, which is just what characterizes negatively skewed equity index markets.

Crepey has also backtested the hedging of actual options to show that the P&L of a hedged portfolio has less variance under the local volatility hedging strategy.