LECTURE 19

Stochastic Volatility Models Continued Continued:

Solution to the PDE

Looking Ahead

Stochastic Volatility Models

Jump Diffusion Models

Guest Speakers
Michael Kamal - April 15
Jackie Rosner - April 20

If you have questions come to my office hours or see me some other time by appointment.

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Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\
dWdZ &= \rho dt
\end{aligned}
\qquad \text{Eq.19.1}
$$

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta \frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta \frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta \frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma} \qquad \text{Eq.19.2}$$

$$\text{Then } d\Pi = dt\left[\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right] \qquad \text{Eq.19.3}$$

No riskless arbitrage: $d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don’t know the value of the function φ!

The Sharpe-ratio meaning of $\phi(S, \sigma, t)$ in terms of Sharpe ratios

See what PDE says about expected risk and return of the option V using Ito's lemma:

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dWdZ &= \rho dt \end{aligned}$$

$$dV \equiv \mu_V V dt + V\sigma_{V_S} dZ + V\sigma_{V_\sigma} dW \qquad \text{Eq.19.4}$$

$$\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right] \qquad \text{Eq.19.5}$$

$$\sigma_{V_S} = \frac{S}{V}\frac{\partial V}{\partial S}\sigma \qquad \sigma_{V_\sigma} = \frac{1}{V}\frac{\partial V}{\partial \sigma}q \qquad \sigma_V \equiv \sqrt{\sigma_{V_S}^2 + \sigma_{V_\sigma}^2 + 2\rho\sigma_{V_S}\sigma_{V_\sigma}}$$

The stochastic volatility PDE is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0$$ which is equivalent to

$$\frac{(\mu_V - r)}{\sigma_V} = \frac{\sigma_{V_S}}{\sigma_V}\frac{(\mu - r)}{\sigma} + \frac{\sigma_{V_\sigma}}{\sigma_V}\frac{(p - \phi)}{q}$$

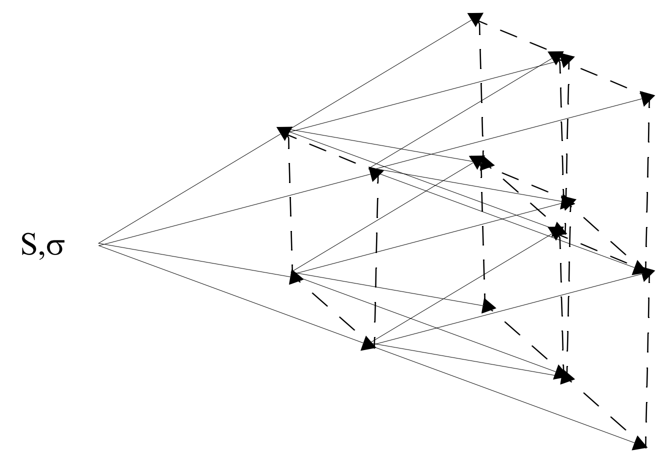
ϕ plays the role for stochastic volatility that the riskless rate r plays for a stochastic stock price.

From a calibration point of view, ϕ must be chosen to make option prices grow at the riskless rate.

If we know the market price of just one option U, and we assume an evolution process for volatility, $d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the calculated value of U matches its market price.

Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.
Of course, it may be naive to assume that just one option can calibrate the entire volatility evolution process.)

The Characteristic Solution to the Stochastic Volatility Model with Zero Correlation

First consider a time-dependent stochastic volatility:

$$dS = rSdt + \sigma(t)SdW_t \text{ in the risk-neutral world.}$$

$$d\ln S = \frac{dS}{S} - \frac{1}{2}\frac{1}{S^2}\sigma^2(t)S^2dt = rdt + \sigma(t)dW - \frac{1}{2}\sigma^2(t)dt$$

$$\ln S(t) = \ln S(0) + rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_s$$

Thus at time t the mean of the distribution of $\ln S$ is at

$$\left(r - \frac{\bar{\sigma}^2}{2}\right)t$$

where $\bar{\sigma}^2 = \frac{1}{t}\int_0^t \sigma^2(s)ds$ is the **path variance**

The distribution is given by a sum of normals, which itself is normal, with total variance given by

$$E\int_0^t\int_0^t \sigma(u)\sigma(s)dW_u dW_s = \int_0^t\int_0^t \sigma(u)\sigma(s)E[dW_u dW_s]$$

$$= \int_0^t\int_0^t \sigma(u)\sigma(s)duds\delta(u-s) = \int_0^t \sigma^2(u)du = t\bar{\sigma}^2(t)$$

Thus $\log S(t)$ is distributed normally too, as follows.

$$\log S_t/S_0 \sim \mathcal{N}\left(\left(r - \frac{\bar{\sigma}^2}{2}\right)t, \bar{\sigma}^2 t\right)$$

Thus, calculating the value of the option as an expected value of the payoff in a risk-neutral world, we find

$$C = C_{BS}(S, t, K, T, r, \bar{\sigma}^2(t))$$

Now let's look at stochastic volatility rather than deterministic volatility

The price of the hedged European option is given by the expected risk-neutral value of the terminal payoff where the stock price $S(t)$ and the variance $v(t) = \sigma^2(t)$ are both stochastic:

$$C_{SV}(S, v, t) = e^{-r\tau}\int f(S_T)p(S_T | S_t, v_t)dS_T$$

the integral over the payoff conditional on the two diffusions.

Now we use the rule of iterated expectations:

$$p(x|y) = \int g(x|z)h(z,y)dz$$

and apply this to the integral above where $z = \bar{\sigma}^2$, the **path variance** from time t to T.

$$C_{SV}(S_t, v_t, t) = e^{-r(T-t)} \int \int f(S_T) g(S_T | S_t, \bar{\sigma}^2) h(\bar{\sigma}^2 | S_t, v_t) dS_T d\bar{\sigma}^2$$

Now rearrange the integrations:

$$C_{SV}(S_t, v_t, t) = \int \left(e^{-r(T-t)} \int f(S_T) g(S_T | S_t, \bar{\sigma}^2) dS_T \right) h(\bar{\sigma}^2 | S_t, v_t) d\bar{\sigma}^2$$

If the correlation ρ between the stock S_t and the volatility v_t is zero, then Hull and White (1987) show that the inner bracket is just the **Black-Scholes price** with a path variance $\overline{\sigma_T}^2 = \frac{1}{T}\int_0^T \sigma_u^2 du$

That is, if you characterize all the stock paths by their variance along the path and the final stock price, then all the paths with a given variance are distributed lognormally.

Then

$$C_{SV}(S_t, v_t, t) = \int C_{BS}(S(T), T, \bar{\sigma}^2) h(\bar{\sigma}^2 | S_t, v_t) d\bar{\sigma}^2$$

Mixing Theorem

The price of an option in a stochastic volatility model with zero correlation is the weighted integral/sum over BS prices over the distribution of path volatilities.

$$V = \sum_{\sigma_T} p(\sigma_T) \times BS(S, K, r, \sigma_T, T)$$

It doesn't matter what order the stochastic volatilities occur in -- as long as the variance along the path is the same, all paths with that variance have the same BS terminal distribution of the stock price.

What is the advantage of this?

When we do a Monte Carlo evaluation of the option in a stochastic volatility model, we have to do a double simulation over S and the volatility.

The mixing theorem reduces this to a one-dimensional simulation or integration in the model.

IF the correlation is different from zero, then all paths conditional on a variance $\overline{\sigma_T}^2 = \frac{1}{T}\int_0^T \sigma_u^2 du$ have a normal distribution, BUT that variance depends on the stock price path, not just on time.

For non-zero correlation there are similar formulas $V = E[BS(S'(\sigma_T, \rho), K, r, \overline{\sigma_T}'(\rho), T)]$

where the stock price in the Black-Scholes formula is shifted to $S'(\sigma_T, \rho)$ and the volatility $\overline{\sigma_T}$ is shifted to $\overline{\sigma_T}'(\rho)$ so it's not quite as useful or intuitive.

[Refs: Fouque, Papanicolaou and Sircar book, and Roger Lee, *Implied and Local Volatilities under Stochastic Volatility*, International Journal of Theoretical and Applied Finance, 4(1), 45-89 (2001).]

The Smile That Results From Stochastic Volatility

The zero-correlation smile depends on moneyness

Mixing: average BS solutions over the volatility distribution to get the stochastic volatility solution.
Example: path volatility can be one of two values, either high or low, with equal probability:

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)] \qquad \text{Eq.19.6}$$

Homogeneity:
$$C_{SV} = \frac{1}{2}\left[SC_{BS}\left(1, \frac{K}{S}, \sigma_H\right) + SC_{BS}\left(1, \frac{K}{S}, \sigma_L\right)\right] = Sf\left(\frac{K}{S}\right)$$

Now, defining BS Σ by
$$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}\left(1, \frac{K}{S}, \Sigma\right)$$

and so
$$\Sigma = g\left(\frac{K}{S}\right)$$

Implied volatility is a function of moneyness in stochastic volatility models with zero correlation (conditional on not knowing the future volatility).

Deriving Euler's equation:
$$\frac{\partial \Sigma}{\partial S} = \left(-\frac{K}{S^2}\right)g' \qquad \frac{\partial \Sigma}{\partial K} = \frac{1}{S}g' \qquad S\frac{\partial \Sigma}{\partial S} + K\frac{\partial \Sigma}{\partial K} = 0$$

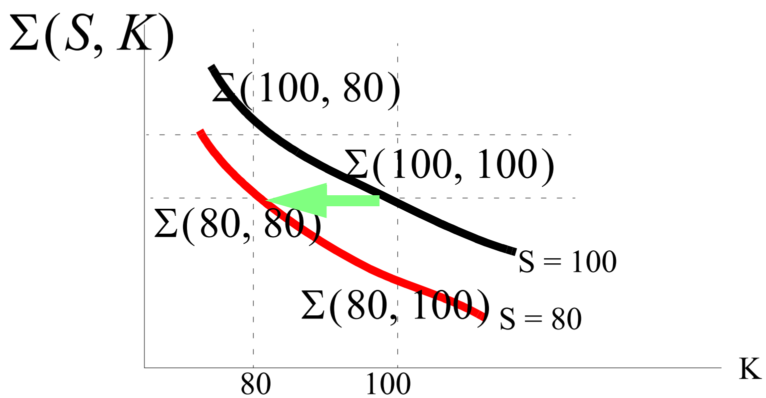
Close to at-the-money, $$\frac{\partial\Sigma}{\partial S} \approx -\frac{\partial\Sigma}{\partial K}$$

just the opposite of what we got with local volatility models.

Close to at-the-money

$$\Sigma \approx \Sigma(S-K)$$

In stochastic volatility models, *conditioned on not knowing the future volatility,*



Note that the volatility of all options drops when the stock price drops. Of course if the volatility itself changes, then the whole curve can move.

The zero correlation smile is symmetric

The mixing theorem: $$C_{SV} = \int_0^{\infty} C_{BS}(\sigma_T)\phi(\sigma_T)d\sigma_T$$

Taylor expansion about the average value $\overline{\sigma_T}$ of the path volatility, dropping subscript *T*.

$$C_{SV} = \int_0^{\infty} C_{BS}(\bar{\sigma} + \sigma - \bar{\sigma})\phi(\sigma)d\sigma$$

$$= \int \left\{ C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma}C_{BS}(\bar{\sigma})\right](\sigma - \bar{\sigma}) + \frac{1}{2}\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})(\sigma - \bar{\sigma})^2 + \dots \right\}\phi(\sigma)d\sigma$$

$$= C_{BS}(\bar{\sigma}) + 0 + \frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})\right]var[\sigma] + \dots$$

where $var[\sigma]$ is the *variance of the path volatility of the stock over the life* τ *of the option*.

Define BS implied volatility Σ by

$$C_{SV} \equiv C_{BS}(\Sigma) = C_{BS}(\bar{\sigma} + \Sigma - \bar{\sigma})$$

$$= C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma} C_{BS}(\bar{\sigma})\right](\Sigma - \bar{\sigma}) + \ldots \qquad \text{Eq.19.7}$$

Then equating two expressions

$$\Sigma \approx \bar{\sigma} + \frac{\frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2} C_{BS}(\bar{\sigma})\right] var[\sigma]}{\frac{\partial}{\partial\sigma} C_{BS}(\bar{\sigma})} \qquad \text{Eq.19.8}$$

Use BS derivatives to find the functional form of $\Sigma(S, K)$.

$$\frac{\partial C}{\partial\sigma} = \frac{Se^{-\frac{1}{2}d_1^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}} \qquad \text{vega is always positive} \qquad \text{Eq.19.9}$$

$$\frac{\partial^2 C}{\partial\sigma^2} = \frac{S\sqrt{\tau}N'(d_1)}{\sqrt{2\pi}\sigma}(d_1 d_2) = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) = \frac{\partial C}{\partial\sigma}\frac{d_1 d_2}{\sigma}$$

volga is mostly positive except atm

$$\frac{C_{\sigma\sigma}}{C_{\sigma}} = \frac{1}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) \qquad \text{Eq.19.10}$$

So

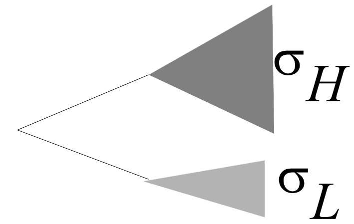
$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}var[\sigma]\left[\frac{(\ln S_F/K)^2 - \left(\bar{\sigma}^4\tau^2\right)/4}{\bar{\sigma}^3\tau}\right] \qquad \text{Eq.19.11}$$

and note that $\bar{\sigma}$ is the average of the path volatility over the life of the option and $var[\sigma]$ is the *variance of the path volatility of the stock over the life of the option.*

Quadratic function of $\ln S_F/K$, parabolically shaped smile that varies as $(\ln S_F/K)^2$ or $\left(K - S_F^2\right)$ close to atm. Sticky moneyness smile, no scale, a function of K/S_F.

A Simple Two-State Stochastic Volatility Model

Mixing two path volatilities $$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)]$$

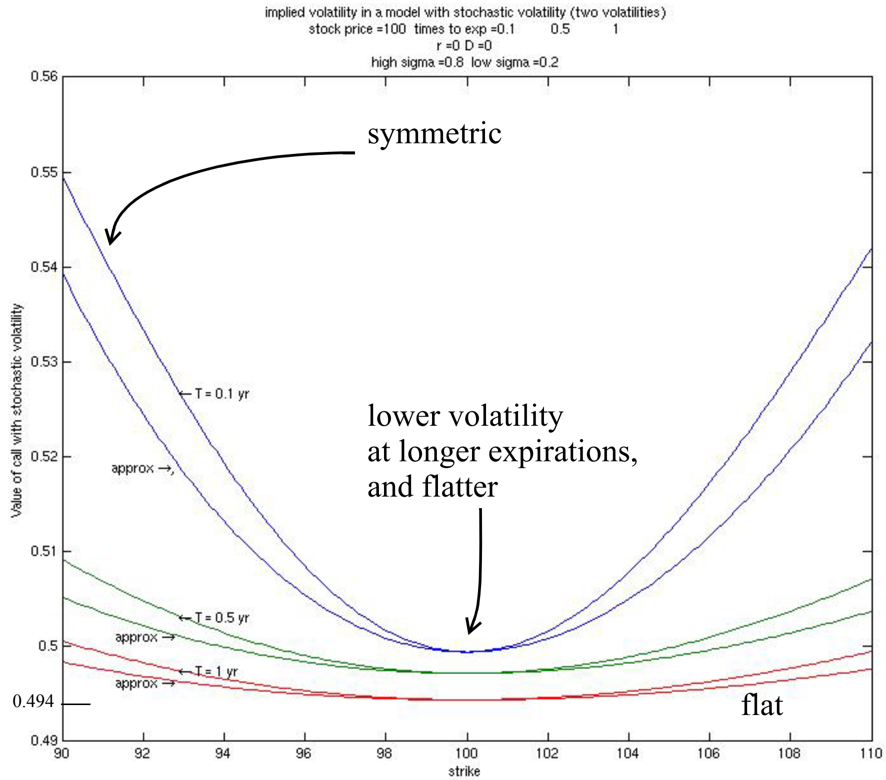


Low volatility be 20% and the high volatility 80% with a mean volatility of 50%.

Variance of the volatility is $0.5(0.8-0.5)^2 + 0.5(0.5-0.2)^2 = 0.09$ per year.

In the figure below we show the smile corresponding to the exact mixing formula together with the approximation $\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma] \left[\frac{(\ln S_F/K)^2 - \left(\bar{\sigma}^4 \tau^2\right)/4}{\bar{\sigma}^3 \tau} \right]$

The Volatility Smile in a Two-Volatility Model With Zero Correlation



- The smile with zero correlation is symmetric;
- the long-expiration smile is relatively flat, while the short expiration skew is more curved (note the τ^{-1} coefficient of $(\ln S/K)^2$ in the formula; and
- at the forward price of the stock, the at-the-money implied volatility decreases monotonically with time to expiration, and lies below the mean volatility of 0.5, because of the negative convexity of the Black-Scholes options price at the money.

The approximate solution works quite well.

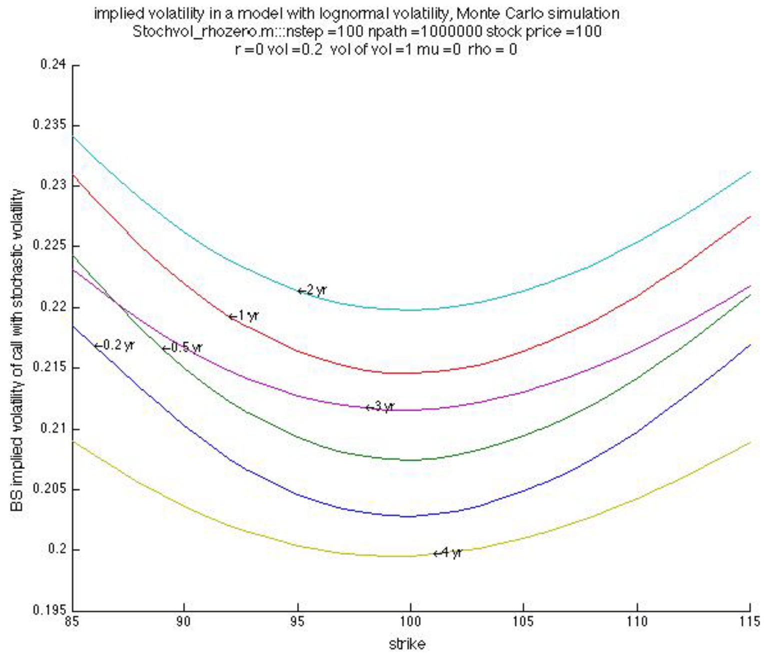
At-the-money, with these parameters, the approximation reduces to

$$\Sigma_{SV}^{ATM} \approx \bar{\sigma} + \frac{1}{2} var[\sigma]\left[\frac{-\left(\bar{\sigma}^4\tau^2\right)/4}{\bar{\sigma}^3\tau}\right] \approx \bar{\sigma} - \frac{1}{8} var[\sigma][-\bar{\sigma}\tau] \approx 0.5 - \frac{1}{8}(0.09)\bar{\sigma}\tau \approx 0.5 - 0.0056\tau$$

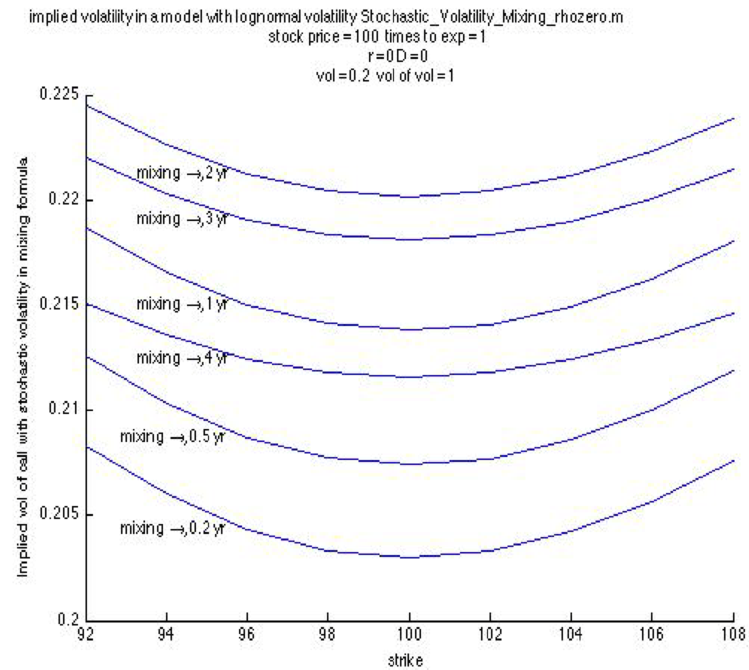
For $\tau = 1$, the at-the-money volatility is 0.494, which agrees well with the figure above.

The Smile for GBM Stochastic Volatility with No Mean Reversion (cf Hull White)

A more sophisticated continuous distribution of stochastic volatilities. $d\sigma = a\sigma dt + b\sigma dZ$ $\rho = 0$, an initial volatility of 0.2 and a volatility of volatility of 1.0, straightforward Monte Carlo simulation of stock paths.



Symmetric smile. At-the-money volatility is no longer monotonic in τ. Skew flattens as τ increases.



Same via mixing formula.

$$\Sigma_{SV}^{ATM} \approx \overline{\sigma_T} - \frac{1}{8} var[\sigma_T] \overline{\sigma_T} \tau \sim \overline{\sigma_T} \left\{ 1 - \frac{1}{8} var[\sigma_T] \tau \right\} \textbf{ Equation GBM}$$

Let's estimate the time-to-expiration dependence of these path-volatility quantities in a geometric Brownian motion model for the *instantaneous* volatility σ. Note that the subscript T on sigma indicates a path volatility.

$$d\sigma = a\sigma dt + b\sigma dZ$$

σ^2, therefore, satisfies a similar stochastic differential equation with drift **(2a+b^2)** and volatility **2b**, that is, roughly double the drift and exactly double the volatility. The extra b^2 term in the drift arises from Ito's Lemma for the square of a Wiener process.

Now let's consider the path variance σ_T^2 which is relevant to the mixing formula. The path variance is an arithmetic average of the instantaneous variances out to time T, but the variance itself evolves geometrically, and so there is no closed-form expression for its value. Nevertheless, it is well known that the average has approximately 1/2 the drift and $1/\sqrt{3}$ the volatility of the non-averaged variable. Thus, approximately, the drift of σ_T^2 is $a + \frac{1}{2}b^2$ and the volatility of σ_T^2 is $2b/\sqrt{3}$.

But Equation GBM involves the square root, σ_T, and the drift of the path volatility is roughly ½ the drift of the path variance, that is $\frac{1}{2}(a+\frac{1}{2}b^2)-\frac{1}{8}(\frac{2b}{\sqrt{3}})^2 = a/2+\frac{1}{12}b^2$, where the second parentheses on the LHS follows, again, from Ito's Lemma for the square root of a variable undergoing geometric Brownian motion. The volatility of the path volatility is ½ the log volatility of the path variance, i.e. $b/\sqrt{3}$. The variance of the path volatility is therefore $var[\sigma_T] = \frac{b^2\sigma^2\tau}{3}$

Thus

$$\Sigma_{i,atm} \approx \sigma[1+(a/2+\frac{b^2}{12})T+\frac{1}{2}(\frac{a}{2}+\frac{b^2}{12})^2T^2][1-\frac{1}{8}b^2\!\!\left/_{3}\right.\sigma^2T^2] \approx \sigma[1+(a/2+b^2\!\!\left/_{12}\right.)T+\{\frac{1}{2}(\frac{a}{2}+\frac{b^2}{12})^2-\frac{b^2}{24}\sigma^2\}T^2]$$

For a = 0 in numerical example

$$\Sigma_{i,atm} \approx \sigma[1+(b^2\!\!\left/_{12}\right.)T+b^2\!\!\left/_{24}\right.\{(b^2\!\!\left/_{12}\right.)-\sigma^2\}T^2]$$

The τ^2 terms has a negative coefficient and explains the non-monotonicity.

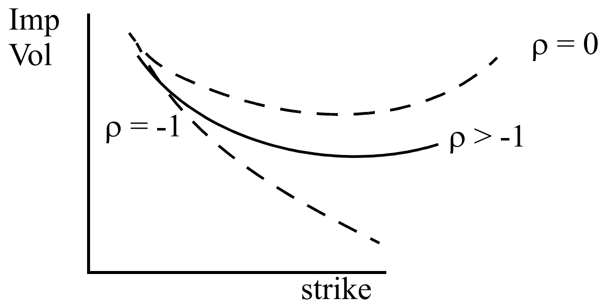
You can also (more or less) understand the decreasing curvature of the smile with increasing τ since the term in $(\ln S/K)^2$ is $\frac{1}{2}var[\sigma_T]\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3\tau}\right]^2$

For GBM, $var[\sigma_T] \sim \sigma_T^2 \tau$ and $\bar{\sigma}$ in the denominator, the time average of the instantaneous volatility, can also increase with τ, so that the curvature term tends to decrease as τ increases.

Non-zero correlation ρ in stochastic volatility models

No correlation lead to a symmetric smile.

With correlation the smile still depends on (K/S_F) but the dependence is not quadratic.:

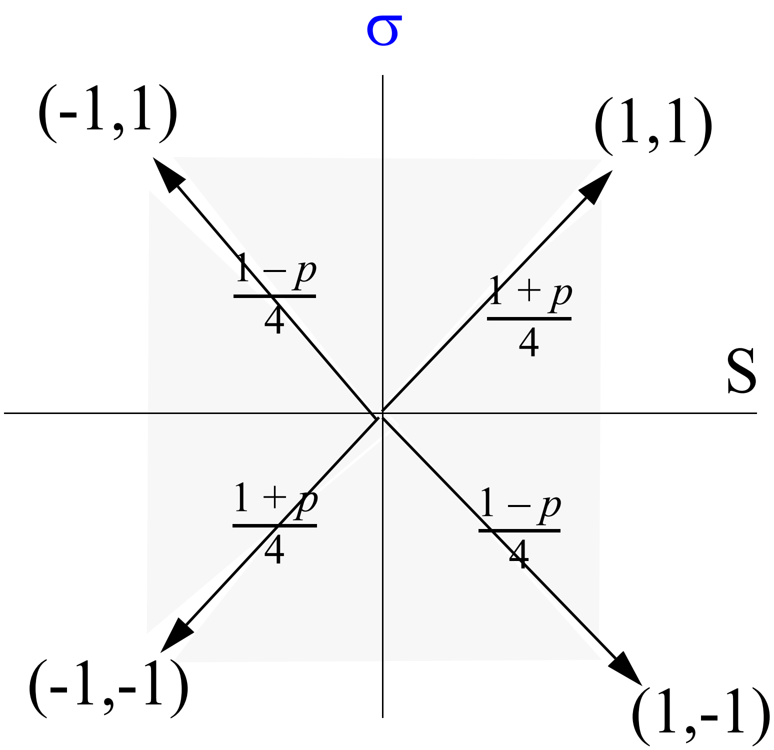


A very steep short-term skew is difficult in these models; since volatility diffuses continuously in these models, at short expirations volatility cannot have diffused too far. A very high volatility of volatility and very high mean reversion are needed to account for steep short-expiration smiles. (There is more on this in Fouque, Papanicolaou and Sircar's book.)

Approximate Imperfect Analytic Intuitive Approximation $\rho \neq 0$

Use the convexity approach: Estimate value of the option with stochastic volatility as an average over the four states with different stock prices and volatilities:

Correlated moves in stock price and volatility with correlation *p*



$$E(S) = \frac{1}{4}[(1+p)1 + (1-p)(-1) + (1+p)(-1) + (1-p)1] = 0$$

$$E(\sigma) = \frac{1}{4}[(1+p)1 + (1-p)(1) + (1+p)(-1) + (1-p)(-1)] = 0$$

$$\mathrm{var}(S) = \frac{1}{4}[(1+p)1^2 + (1-p)1^2 + (1+p)(-1)^2 + (1-p)(-1)^2] = 1$$

$$\mathrm{var}(\sigma) = 1$$

$$\mathrm{cov}(S, \sigma) = \frac{1}{4}[(1+p)1 \times 1 + (1-p)(-1) \times 1 + (1+p)(-1) \times (-1) + (1-p)$$

$$= \frac{1}{4}[(1+p) - (1-p) + (1+p) - (1-p)] = p$$

Let the four nodes correspond to $S \pm \sigma S$, $\sigma \pm \xi\sigma$ and average over BS prices at these ranges of stock prices and path volatilities.

$$
\begin{aligned}
C_{SV} \approx\ & \frac{(1+\rho)}{4} C_{BS}(S + \sigma S, \sigma + \xi\sigma) \\
& + \frac{(1-\rho)}{4} C_{BS}(S + \sigma S, \sigma - \xi\sigma) \\
& + \frac{(1+\rho)}{4} C_{BS}(S - \sigma S, \sigma - \xi\sigma) \\
& + \frac{(1-\rho)}{4} C_{BS}(S - \sigma S, \sigma + \xi\sigma)
\end{aligned}
$$

Do Taylor expansion for small vol and small vol of vol.

We then find that all additional terms cancel out except for the volga and vanna terms, to second order in Taylor series:

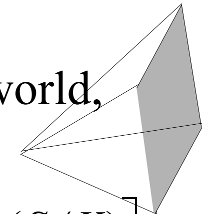
$$
C_{SV} = C_{BS}(S, \sigma) + C_{\sigma\sigma}\frac{var[\sigma]}{2} + \rho C_{s\sigma} S var^{\frac{1}{2}}[\sigma]
$$

But in implied volatility terms

$$
C_{SV} \equiv C_{BS}(S, \sigma + \Sigma - \sigma) \approx C_{BS}(S, \sigma) + C_{\sigma}(\Sigma - \sigma)
$$

Thus $\Sigma \approx \sigma + \frac{C_{\sigma\sigma}}{C_\sigma}\frac{(\xi\sigma)^2}{2} + \frac{C_{s\sigma}}{C_\sigma}\rho S\sigma^2\xi$

Now from what we know about vanna and volga in a Black Scholes world,



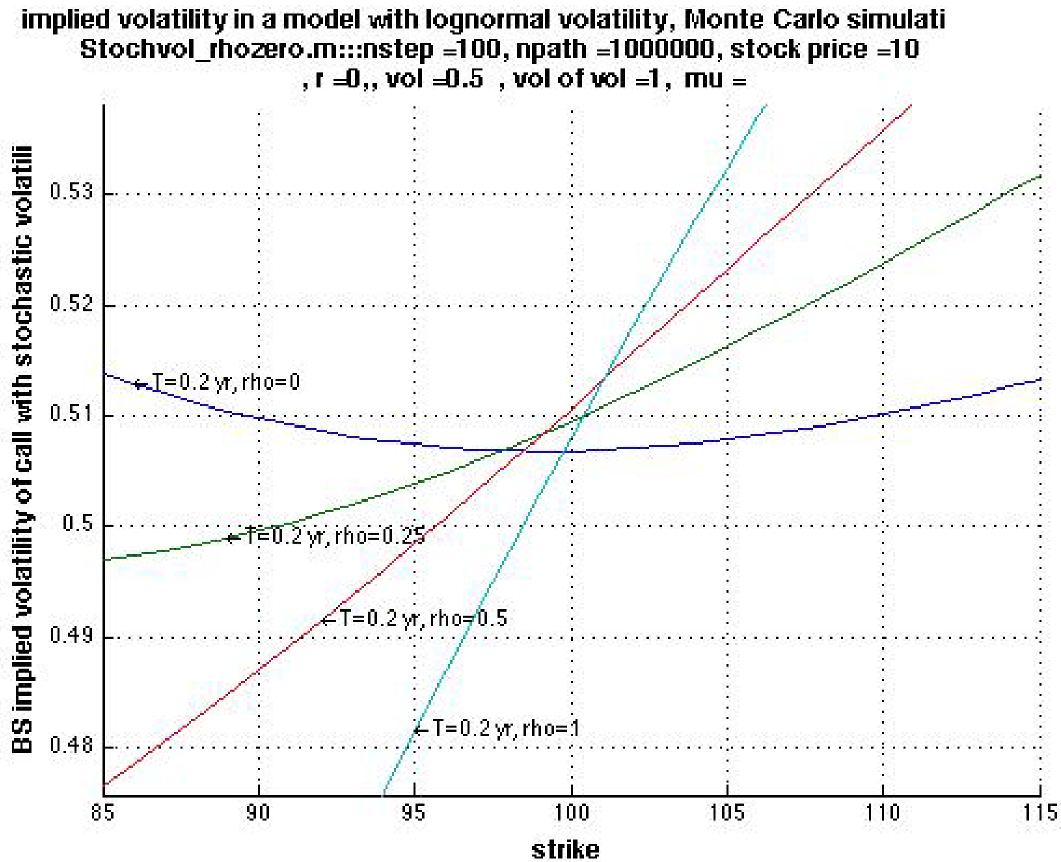
$$\frac{C_{\sigma\sigma}}{C_\sigma} = \frac{1}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) \qquad \frac{C_{s\sigma}}{C_\sigma} = \frac{1}{S}\left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]$$

So

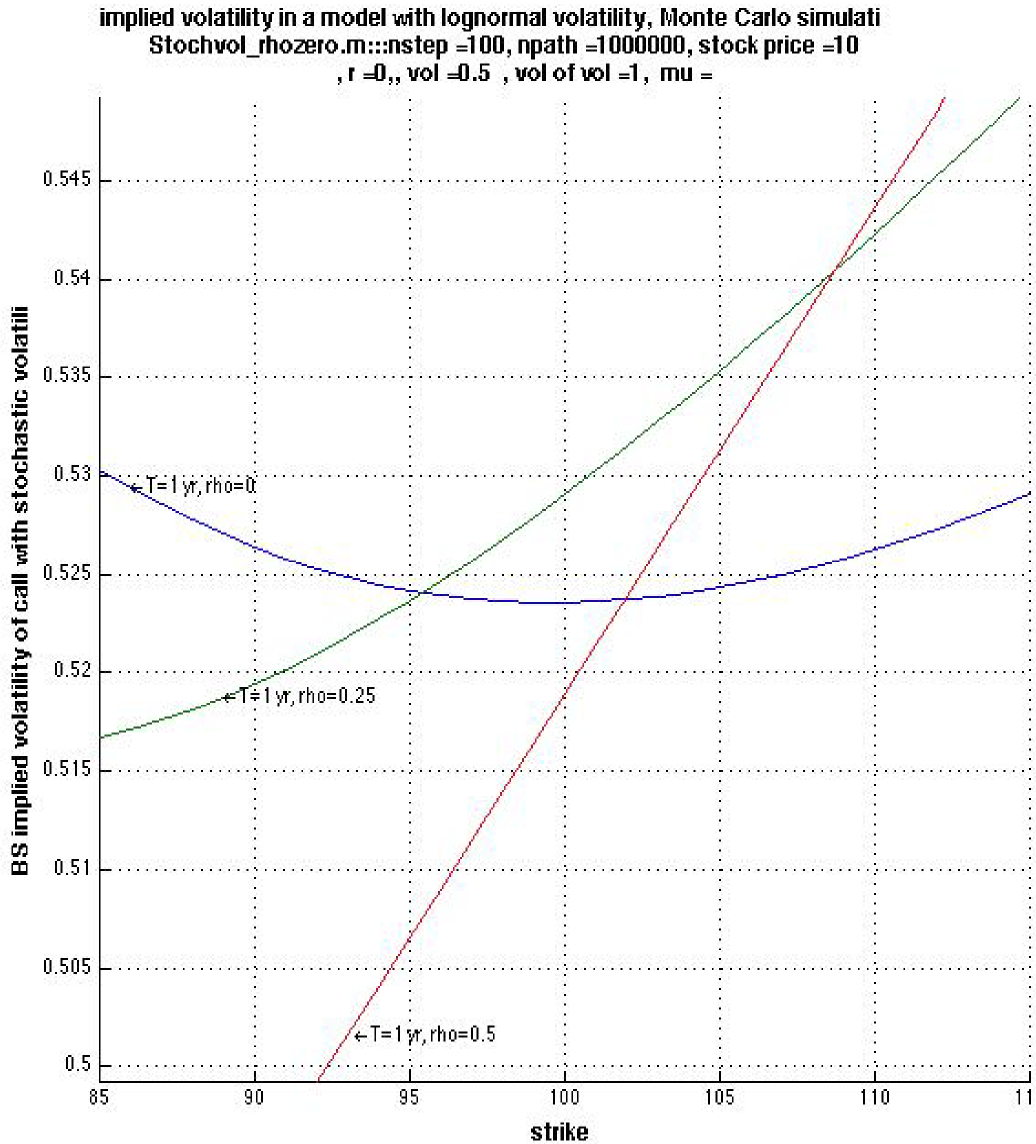
$$\Sigma \approx \sigma + \left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right)\frac{\xi^2\sigma}{2} + \left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]\rho\sigma^2\xi$$

We see a quadratic and a linear term, depending on correlation.

Monte Carlo simulation for $\tau = 0.2\text{yrs}$ with non-zero ρ. You can see that increasing the value of the correlation steepens the slope of the smile.



$\tau = 1yr.$



implied volatility in a model with lognormal volatility, Monte Carlo simulati
Stochvol_rhozero.m:::nstep =100, npath =1000000, stock price =10
, r =0,, vol =0.5 , vol of vol =1, mu =
BS implied volatility of call with stochastic volatili
strike
←T=1 yr, rho=0
←T=1 yr, rho=0.25
←T=1 yr, rho=0.5

The Smile in Mean-Reverting Stochastic Volatility Models

Finally, we explore the smile when volatility mean reverts:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad d\sigma = \alpha(m-\sigma)dt + \beta\sigma dW \qquad dZdW = \rho dt$$

The following pages show the results of a Monte Carlo for BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation.

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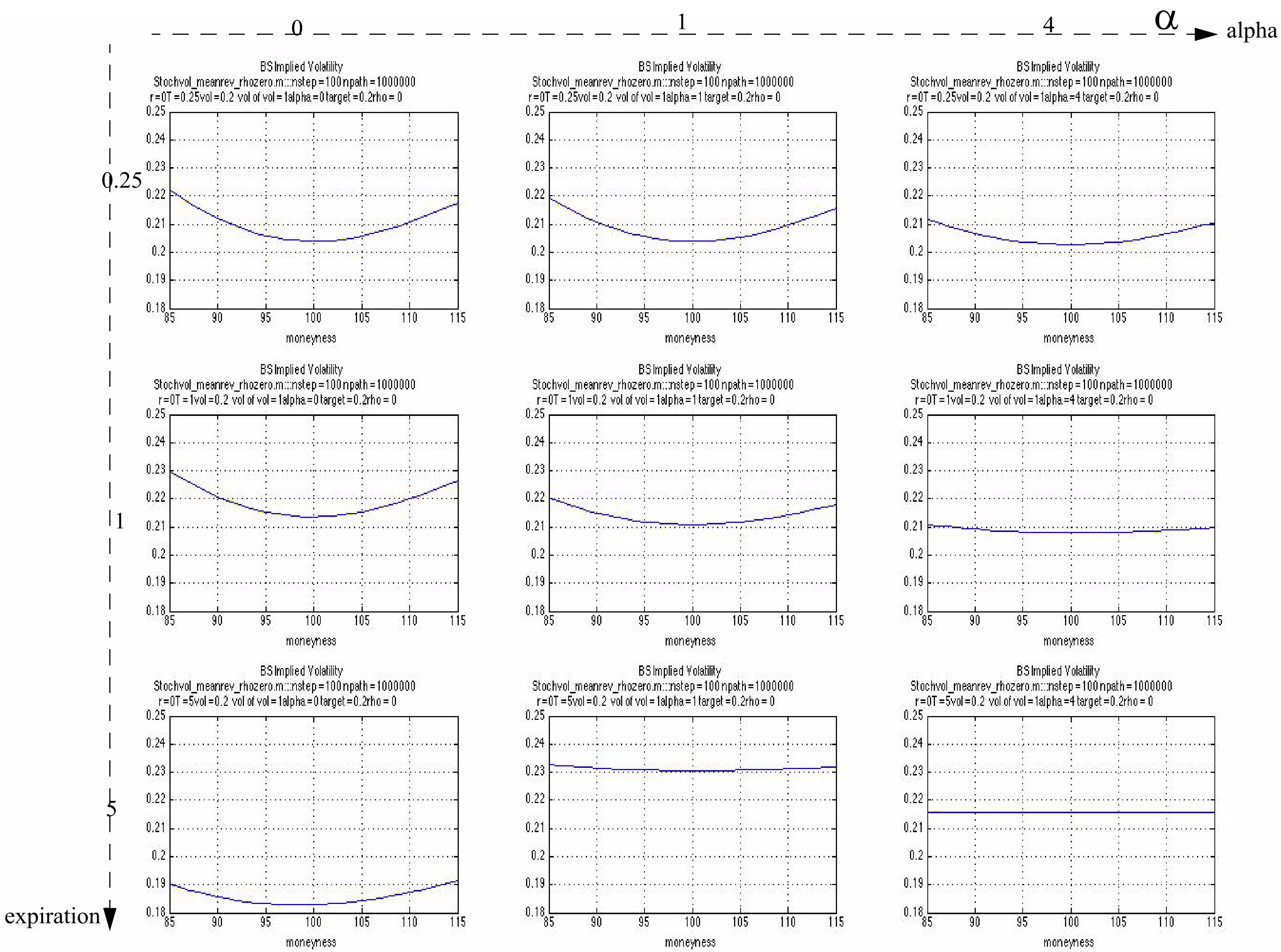
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.The target volatility and the initial volatility are both 0.2



BS Implied Volatility as a function of mean reversion s and expiration for correlation -0.1. The target and the initial volatility are both 0.2

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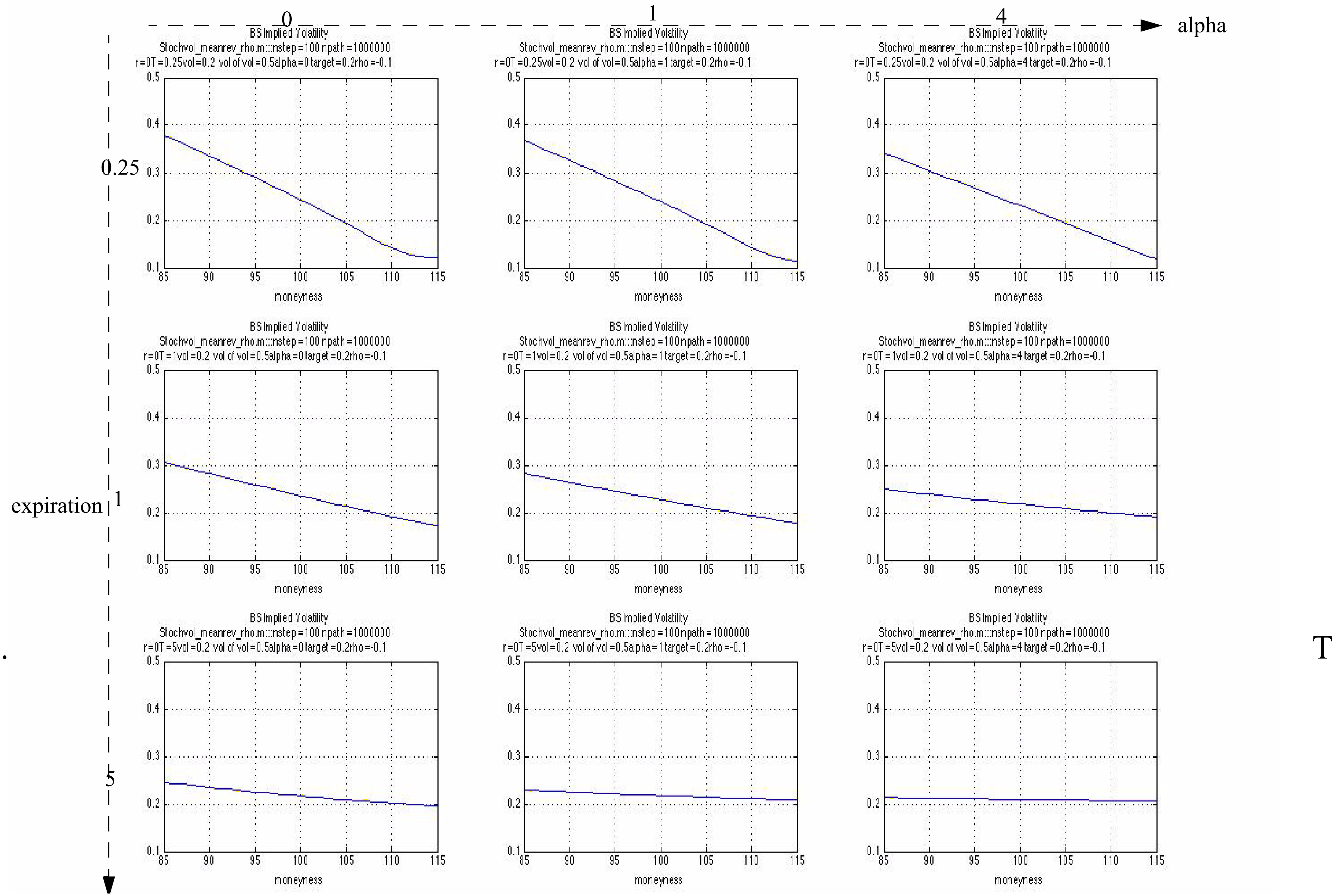
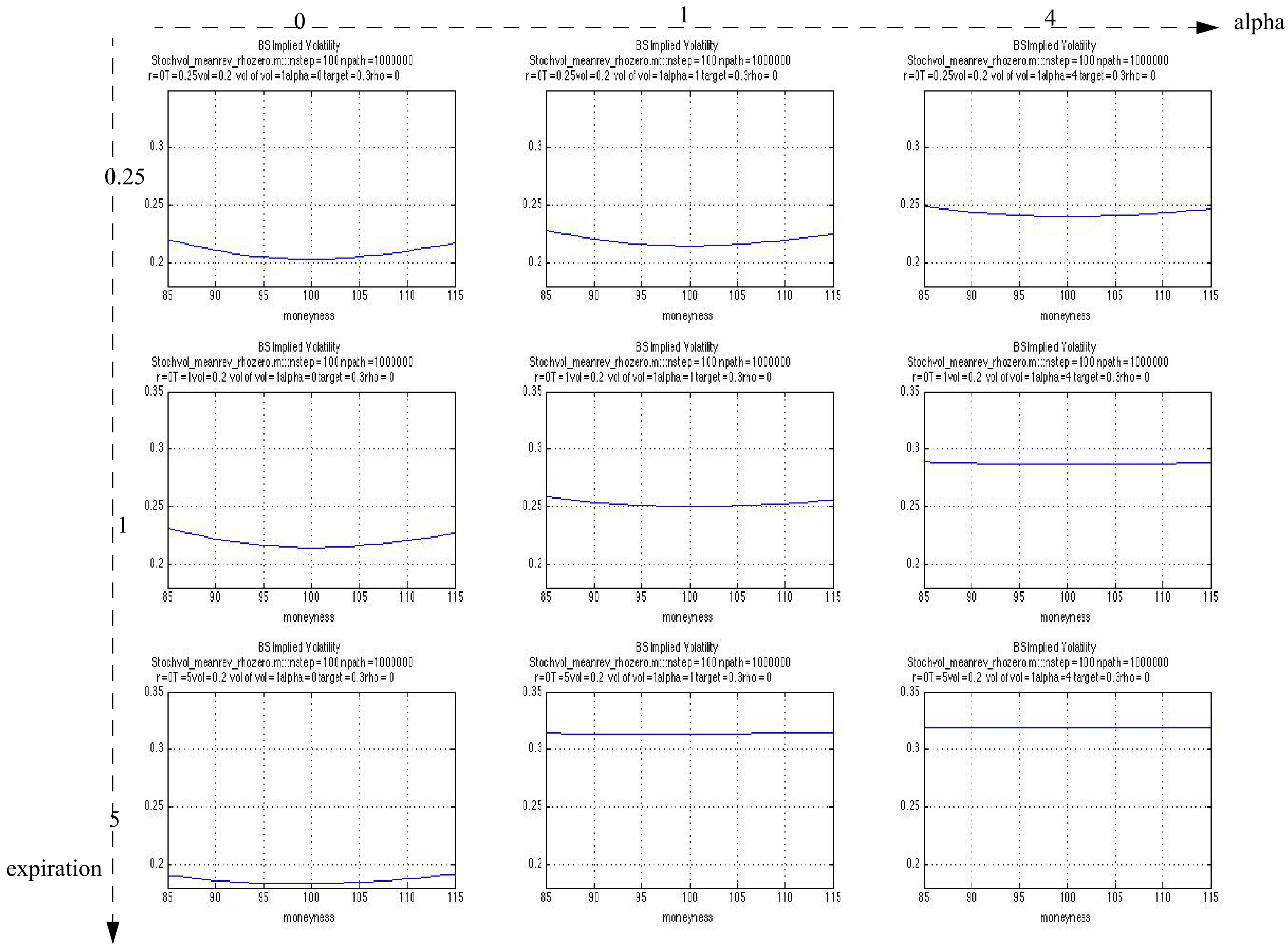


Note the flattening of the smile with both expiration and mean-reversion strength α

BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.



Mean-Reverting Stochastic Volatility and the Asymptotic Behavior of the Smile.

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}var[\sigma]\left[\frac{(\ln S_F/K)^2 - \left(\bar{\sigma}^4\tau^2\right)/4}{\bar{\sigma}^3\tau}\right] \qquad \text{Eq.19.12}$$

and insert intuition about mean reversion for σ.

Short Expirations, Zero Correlation

In the limit that $\tau \to 0$

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}var[\sigma]\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3\tau}\right]$$

$var[\bar{\sigma}] = \beta\tau$. Substituting this relation into Equation leads to the expression

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}\beta\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3}\right] \qquad \tau \to 0 \text{ limit} \qquad \text{Eq.19.13}$$

Smile is quadratic and finite as $\tau \to 0$ for short expirations.

Long Expirations

As $\tau \to \infty$ $$\Sigma_{SV} \approx \bar{\sigma} - \frac{1}{2} var[\sigma]\left[\frac{\bar{\sigma}\tau}{4}\right]$$

where $\bar{\sigma}$ is the path volatility over the life of the option and is itself a function of the time to expiration due to the stochastic nature of the instantaneous volatility.

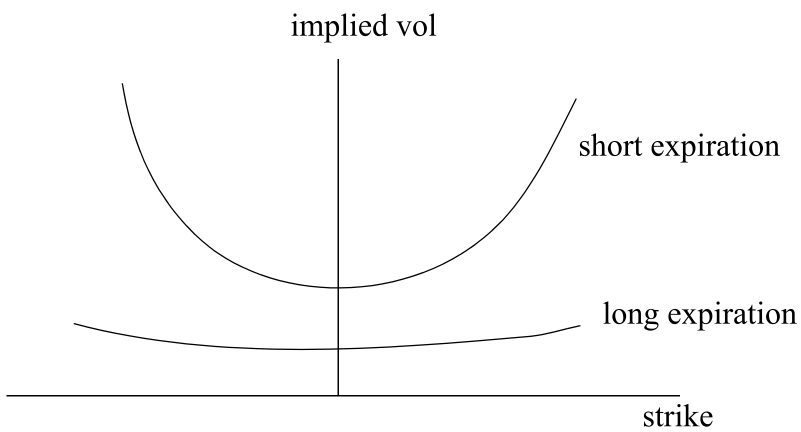
For Ornstein-Uhlenbeck the path volatility to expiration $\bar{\sigma}$ converges to a constant along all paths as $\tau \to \infty$, and so $\bar{\sigma}$ has zero variance as $\tau \to \infty$, $var[\bar{\sigma}] \to \text{const}/\tau$.

$$\Sigma_{SV} \approx \bar{\sigma} - \frac{\text{const}}{8}\bar{\sigma} \qquad \text{Eq.19.14}$$

NO smile at large expirations.

Why is the correction term negative? The option price $C_{BS}(\sigma)$ has negative convexity, and for a concave function $f(x)$, the average of the function $\overline{f(x)}$ is less than the function $f(\bar{x})$ of the average.

Thus, for zero correlation, we expect to see stochastic volatility smiles that look like this:



We can understand this intuitively as follows. In the long run, all paths will have the same volatility if it mean reverts, and so the long-term skew is flat. In the short run, bursts of high volatility act almost like jumps, and induce fat tails