LECTURE 18

Stochastic Volatility Models Continued:

A Variety of Approaches

Looking Ahead

Stochastic Volatility Models

Jump Diffusion Models

Guest Speakers

Michael Kamal - April 15
Jackie Rosner - April 20

If you have questions come to my office hours or see me some other time by appointment.

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A Qualitative Look at Stochastic Volatility Models Starting from a Black Scholes Point of View

Assume rates and dividends are zero. Add stochastic volatility to BS:

$$\begin{aligned}
dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma
\end{aligned}$$

Now suppose that we constructed a riskless hedge π that is long the call C and short just enough stock and enough volatility σ so that the hedged portfolio is instantaneously riskless.

Then

$$d\pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma$$

We don’t know the value of the partial derivatives in the above equation, since we haven’t applied the methods of risk-neutral valuation to determine the partial differential equation for the value of the option with both stochastic volatility and stock price. In order to proceed further we will replace the unknown partial derivatives by their values in the Black-Scholes model, hoping that these capture the approximate contribution to the P&L from the stochastic volatility.

Then

$$\frac{\partial C_{BS}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial S^2}\sigma^2 S^2 = 0$$

Then expected change in the value of the hedged portfolio from stochastic volatility is approximately

$$\frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma$$

volga,
butterfly spread

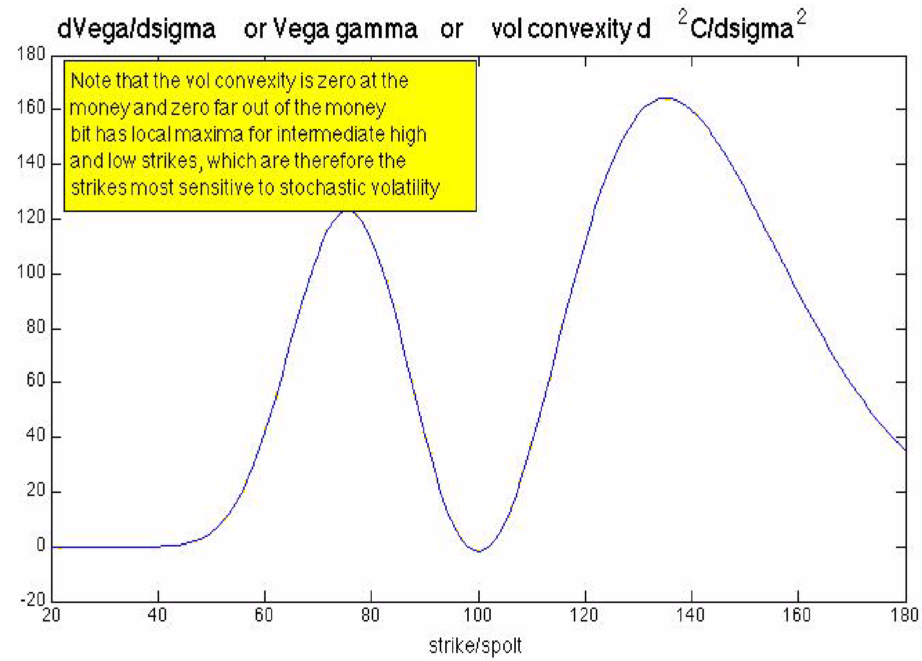
vanna
risk reversal

Cheat a little more by using the BS derivatives of $C_{BS}(S, t, K, T, r, \sigma)$ in the Ito expansion.

$$\frac{\partial C}{\partial \sigma} = \frac{Se^{-\frac{1}{2}d_1^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}}$$ vega is always positive Eq.18.1

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{S\sqrt{\tau}N'(d_1)}{\sqrt{2\pi}\sigma}(d_1 d_2) = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) = \frac{\partial C}{\partial \sigma}\frac{d_1 d_2}{\sigma}$$ volga is mostly positive

except atm



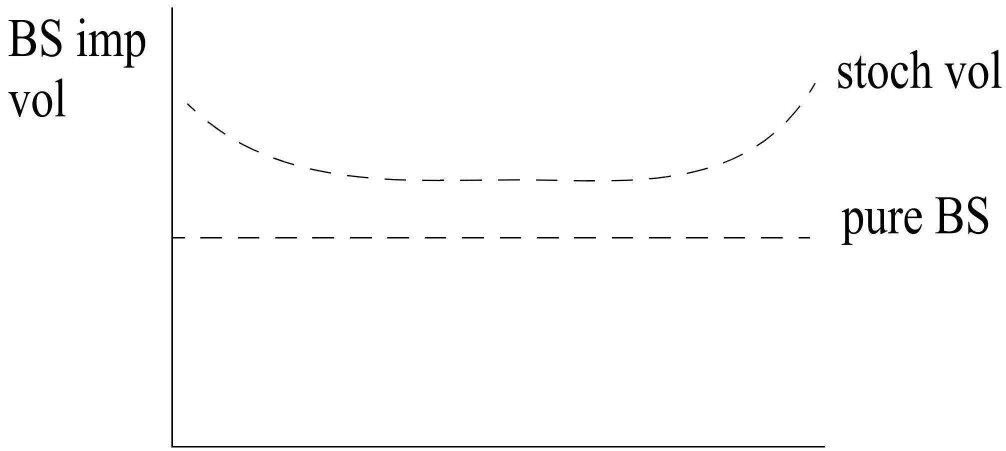
Mostly positive convexity, with peaks on either side.

$$\frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2} d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma} dS d\sigma$$

A hedged option is long gamma, long volatility and **long volatility of volatility**, esp out of money or deep in the money.

The Vanna Volga model/method for exotics involves approximately replicating an exotic option with vanillas that have the same vega, volga and vanna in a Black-Scholes no-smile world, and then turning on the smile to see the effect of moving away from Black-Scholes, *assuming* Black Scholes derivatives hold. (Later)

If volatility is volatile, then the convexity in volatility adds value to the option away from at-the-money, and adds value to out-of-the-money options relative to at-the-money options in a U-shaped smile.

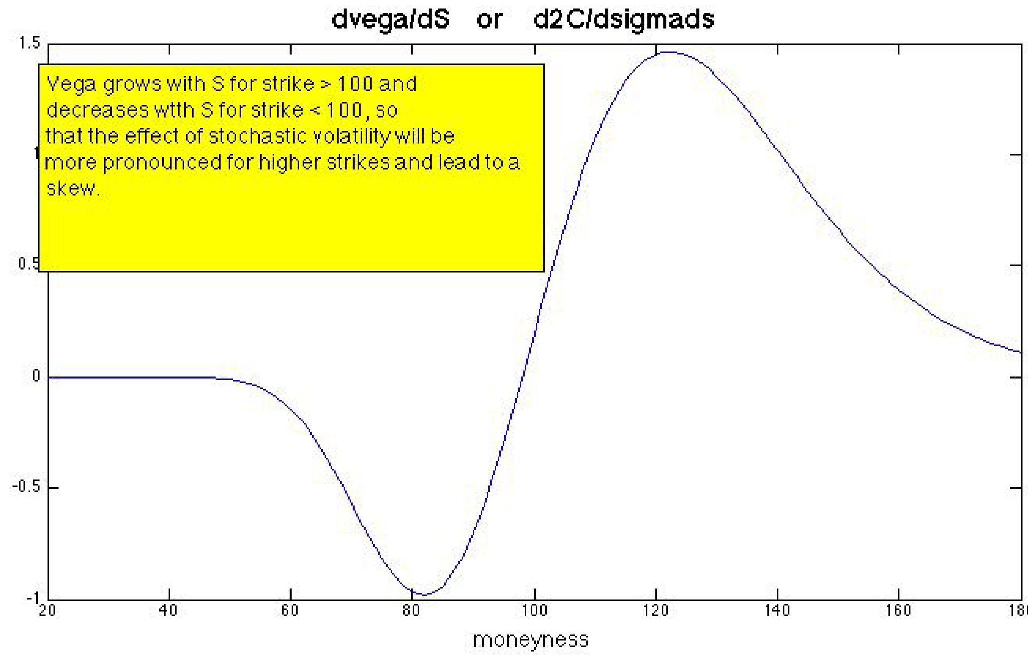


Similarly, one can plot the Black-Scholes vanna $\frac{\partial^2 C}{\partial\sigma\partial S}$ the rate of change of vega with spot S

$$\frac{\partial C}{\partial\sigma} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right)$$

$$\frac{\partial^2 C}{\partial\sigma\partial S} = \left[1 - Sd_1\frac{\partial d_1}{\partial S}\right]\frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right) = \left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]\frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right) = \left(\frac{\partial C}{\partial\sigma}\right)\frac{1}{S}\left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]$$

$$\underbrace{\frac{1}{2}\frac{\partial^2 C}{\partial\sigma^2}d\sigma^2}_{\text{volga}} + \underbrace{\frac{\partial^2 C}{\partial S\partial\sigma}dSd\sigma}_{\text{vanna}}$$



For typical values of σ and τ, vanna will be positive when the call option is out of the money (K > S) and negative when the call option is in the money (K < S). If E[dSd σ] is positive (if the stock price and its volatility are positively correlated), the vanna term will enhance the P&L and hence value of a Black-Scholes option at high strikes and reduce it at low strikes. The opposite is the case if the correlation is negative. Since the equity index skew is typically negative, with low strikes carrying greater implied volatility than high ones, we can guess that in a stochastic volatility model we will require a negative correlation between the index and its volatility in order to reflect the skew.

Crude usefully intuitive ways to understand the effect of stochastic volatility on the smile.

Another Approach: Start From Local Volatility with a Skew; Perturbatively Add Stochasticity: Stoch Local Vol

Add a stochastic element to a local volatility model.

$$
\begin{aligned}
\frac{dS}{S} &= \alpha S^{\beta - 1} dW \\
d\alpha &= \xi \alpha dZ \qquad \text{SABR model} \\
dZdW &= \rho dt
\end{aligned}
\qquad \text{Eq.18.2}
$$

For $\beta = 1$ this is geometric Brownian motion with no smile, else it's CEV with skew.
α is the stochastic part of the smile-inducing local volatility, and ξ is the volatility of volatility.

Assume $\rho = 0$ and β close to 1 (small skew): estimate the skew using our knowledge of local vol.

For $\xi = 0$ we know the implied volatility is roughly average of the local volatilities S to K:

$$\Sigma_{LV}(S, t, K, T, \alpha) = \frac{\alpha}{2}[S^{\beta - 1} + K^{\beta - 1}]$$

Taylor expansion in K for β close to 1: $\Sigma_{LV}(S, t, K, T, \alpha) \approx \dfrac{\alpha}{S^{1-\beta}}\left[1 + \dfrac{(\beta - 1)}{2}\ln\dfrac{K}{S}\right]$

a linear skew with negative slope, $\dfrac{\partial \Sigma}{\partial K} \approx \dfrac{\partial \Sigma}{\partial S}$ atm.

Now switch on the stochastic volatility $\xi \neq 0$ There is a range of possible α values. Estimate C_{SLV} in this Stochasticized Local Vol model as average of the BS prices over the range of α:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \alpha))\phi(\alpha)d\alpha$$

Taylor expand this about the mean $\bar{\alpha}$ for small volatility of volatility:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha} + (\alpha - \bar{\alpha})))\phi(\alpha)d\alpha$$

$$\approx \int \left\{ C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha})) + \frac{\partial C_{BS}}{\partial \alpha}(\alpha)\Big|_{\alpha = \bar{\alpha}}\cancel{(\alpha - \bar{\alpha})} + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\alpha)\Big|_{\alpha = \bar{\alpha}}(\alpha - \bar{\alpha})^2) \right\}\phi(\alpha)d\alpha \qquad \text{Eq.18.3}$$

$$\approx C_{BS}(\bar{\alpha}) + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\bar{\alpha}))var(\alpha)$$

Look at implied Black-Scholes volatility Σ_{SLV} deviation away from $\xi = 0$:

$$
\begin{aligned}
C_{SLV} \equiv C_{BS}(\Sigma_{SLV}) &\approx C_{BS}(\Sigma_{LV}(\bar{\alpha}) + \{\Sigma_{SLV} - \Sigma_{LV}\}) \\
&\approx C_{BS}(\Sigma_{LV} + \{\Sigma_{SLV} - \Sigma_{LV}\}) \\
&\approx C_{BS}(\bar{\alpha}) + \frac{\partial C_{BS}}{\partial \Sigma_{LV}}(\Sigma_{SLV} - \Sigma_{LV})
\end{aligned}
\qquad \text{Eq.18.4}
$$

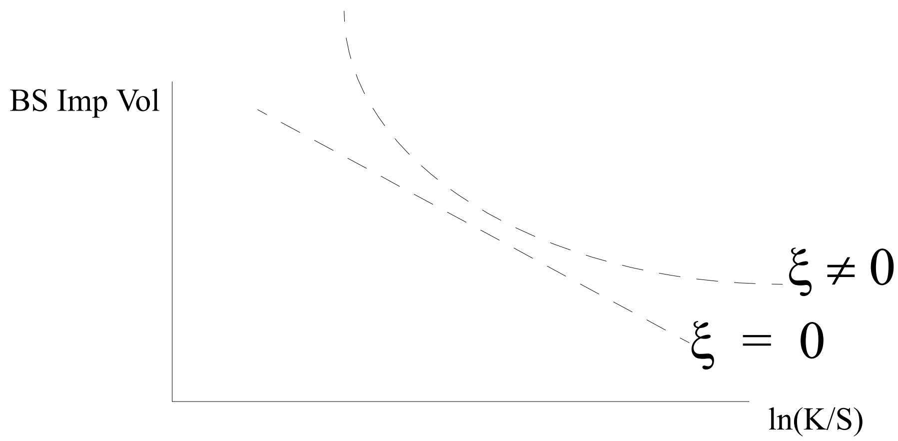
Comparing the above two equations, we obtain

$$
\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha}) + \frac{\left.\frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\alpha))\right|_{\alpha = \bar{\alpha}} var(\alpha)}{\frac{\partial C_{BS}}{\partial \Sigma_{LV}}} \qquad \text{Eq.18.5}
$$

Evaluate the BS derivatives above **for small times to expiration and close to at-the-money**:

$$
\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha})\left\{1 + \frac{1}{2}\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2\left[\ln\frac{S}{K}\right]^2\right\} \qquad \text{Eq.18.6}
$$

The local volatility smile is altered by the addition of a quadratic term in $\ln\frac{S}{K}$. (Homework)



No need for correlation between volatility and stock price in order to obtain a smile if we start from local volatility.

It's not hard to show that $\beta \approx 2$ in this model, as in local volatility.

Risk-neutral Valuation And Stochastic Volatility Models

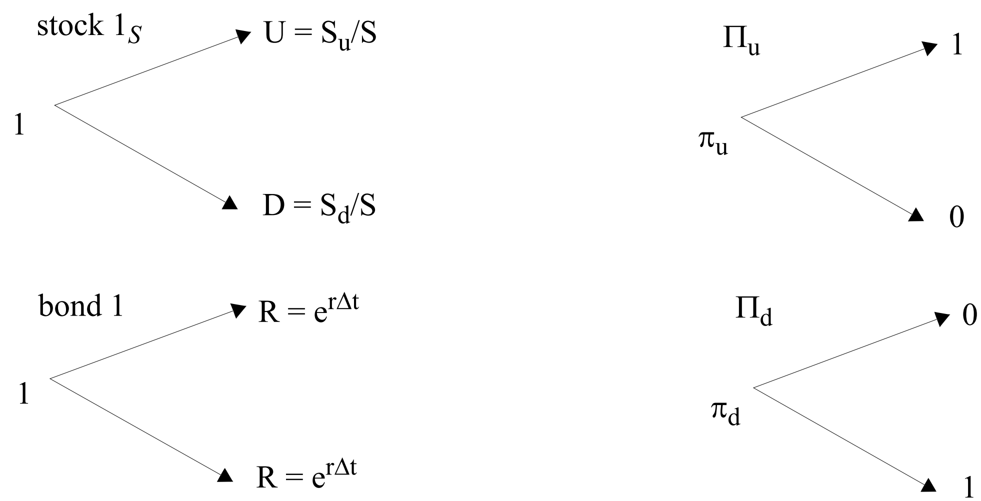
Arbitrage-free options valuation:

Hedge away all the risk instantaneously.
Need enough securities to span all the possible states of the world dynamically.
Then the riskless hedged portfolio must earn the risk-free rate.

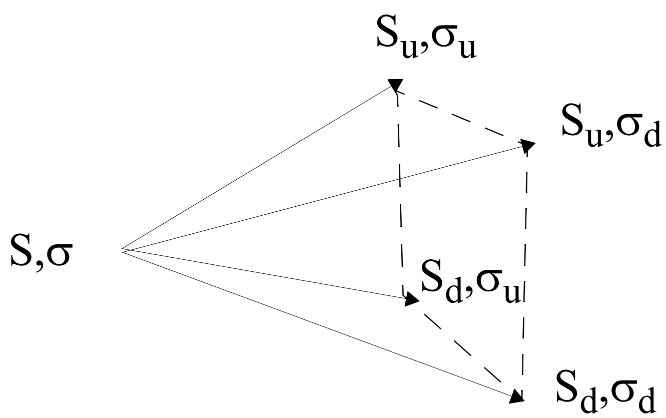
Recall how we derived Black-Scholes with a stochastic stock price:

Arrow-Debreu securities, Π_u and Π_d span the space of payoff states.

Two securities: two final states. Hence you can value any instrument irrespective of outcome or its probability.



Stochastic volatility: *S* **and** σ can vary: σ_u and σ_d differ; there is a correlation between *S* and σ



4 possible final states
Need 4 four Arrow-Debreu securities that pay $1.
But we know only two: S and B.
We would need to know the *price of the volatility* σ today in order to span the other states.

But volatility is not a security or a traded variable, it's a parameter.
Instead, we can only hedge options with other options to hedge the volatility sensitivity.

Cf: Vasiçek interest-rate model.
You cannot hedge the interest-rate exposure of a bond with "interest rates". You must hedge the interest-rate sensitivity of one bond with another.

If we hedge options only with shares of stock perfect replication is impossible. ***If*** you can also use other options, and if you *assume* you know the stochastic process for options as well as stock prices, then you can derive an arbitrage-free formula for options values. But do we? Nevertheless …

Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\
dWdZ &= \rho dt
\end{aligned}
\qquad \text{Eq.18.7}
$$

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

$$
\begin{aligned}
d\Pi = dt&\left[\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right] \\
&+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right)
\end{aligned}
$$

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma}\Big/\frac{\partial U}{\partial \sigma} \qquad \text{Eq.18.8}$$

$$\text{Then } d\Pi = dt\left[\begin{array}{c} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right] \qquad \text{Eq.18.9}$$

No riskless arbitrage: $d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho - rV$$

$$-\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho - rU\right)$$

$$+ r\Delta S = 0$$

But $\Delta = \frac{\partial V}{\partial S} - \delta \frac{\partial U}{\partial S}$ and so

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV$$

$$= \delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU\right)$$

and $\delta = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma}$

$$\frac{\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2+\dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2+\dfrac{\partial^2 V}{\partial S\partial\sigma}\sigma q S\rho+rS\dfrac{\partial V}{\partial S}-rV}{\dfrac{\partial V}{\partial \sigma}}$$

$$=\frac{\dfrac{\partial U}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2+\dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2+\dfrac{\partial^2 U}{\partial S\partial\sigma}\sigma q S\rho+rS\dfrac{\partial U}{\partial S}-rU}{\dfrac{\partial U}{\partial \sigma}} \qquad \text{Eq.18.10}$$

$$=\ \phi(S,\sigma,t) \qquad \text{separation of variables}$$

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2+\frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2+\frac{\partial^2 V}{\partial S\partial\sigma}\sigma q S\rho+rS\frac{\partial V}{\partial S}+\phi(S,\sigma,t)\frac{\partial V}{\partial \sigma}-rV\ =\ 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don't know the value of the function ϕ!

The Sharpe-ratio meaning of $\phi(S, \sigma, t)$ in terms of Sharpe ratios

See what PDE says about expected risk and return of the option V using Ito's lemma:

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dWdZ &= \rho dt \end{aligned}$$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt$$

$$= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + \frac{\partial V}{\partial \sigma}p dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \qquad \text{Eq.18.11}$$

$$+ \frac{\partial V}{\partial S}\sigma S dZ + \frac{\partial V}{\partial \sigma} q dW$$

$$\equiv \mu_V V dt + V\sigma_{V_S} dZ + V\sigma_{V_\sigma} dW$$

$$\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right] \qquad \text{Eq.18.12}$$

$$\sigma_{V_S} = \frac{S}{V}\frac{\partial V}{\partial S}\sigma \qquad \sigma_{V_\sigma} = \frac{1}{V}\frac{\partial V}{\partial \sigma}q \qquad \sigma_V \equiv \sqrt{\sigma_{V_S}^2 + \sigma_{V_\sigma}^2 + 2\rho\sigma_{V_S}\sigma_{V_\sigma}}$$

where σ_{V_S} and σ_{V_σ} are the partial volatilities of option V with total volatility σ_V.

The stochastic volatility PDE is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \text{ and}$$

$$\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right]$$

Thus in the LHS of the stochastic volatility PDE for the value of the option as

$$\frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right] \equiv \mu_V - \mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right) - p\left(\frac{\partial V}{\partial \sigma}\frac{1}{V}\right)$$

Substituting this into the stochastic vol PDE

$$\mu_V - \mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right) - p\left(\frac{\partial V}{\partial \sigma}\frac{1}{V}\right) + r\frac{S}{V}\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{1}{V}\frac{\partial V}{\partial \sigma} - r = 0$$

or $$(\mu_V - r) = \frac{S\partial V}{V\partial S}(\mu - r) + \frac{1}{V}\frac{\partial V}{\partial\sigma}(p - \phi)$$

or $$(\mu_V - r) = \sigma_{V_S}\frac{(\mu - r)}{\sigma} + \sigma_{V_\sigma}\frac{(p - \phi)}{q}$$

or $$\frac{(\mu_V - r)}{\sigma_V} = \frac{\sigma_{V_S}}{\sigma_V}\frac{(\mu - r)}{\sigma} + \frac{\sigma_{V_\sigma}}{\sigma_V}\frac{(p - \phi)}{q}$$

Excess return per unit of risk for the option = the excess return per unit of risk for the stock and the excess return per unit of risk for volatility.

ϕ plays the role for stochastic volatility that the riskless rate r plays for a stochastic stock price.

In the Black-Scholes case, r is the rate at which the stock price must grow in order that option payoffs can be discounted at the riskless rate.

Similarly, ϕ is the drift that volatility must undergo in order that option prices with stochastic volatility be discounted at the riskless rate.

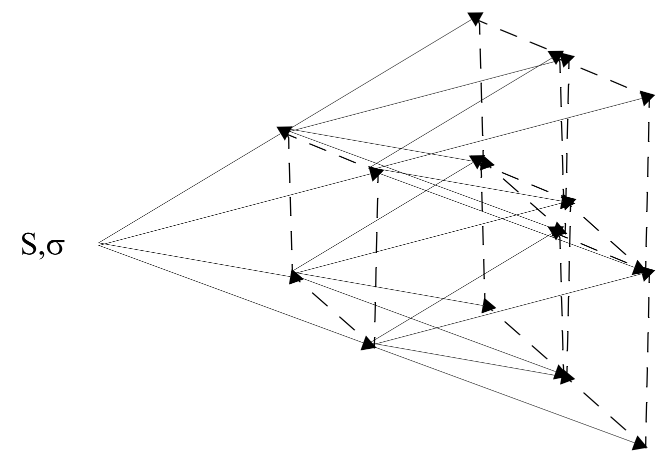
ϕ is not equal to r because σ is not traded. ϕ is the rate at which volatility σ must grow in order that the price of the option V grows at the rate r when you can hedge away all risk.

From a calibration point of view, ϕ must be chosen to make option prices grow at the riskless rate.

If we know the market price of just one option U, and we assume an evolution process for volatility, $d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the calculated value of U matches its market price.

Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.

Of course, it may be naive to assume that just one option can calibrate the entire volatility evolution process.)
Note that even though the payoffs of the option are the same as in the Black-Scholes world, the evolution process of the stock is different, and so the option price will be different too.