LECTURE 3

PRINCIPLES OF VALUATION CONTINUED

The One Commandment of Quantitative Finance

If you want to know the value of a security, use the price of another security that's as similar to it as possible.

The *law of one price*, or *the principle of no riskless arbitrage:*

Any two securities with identical future payoffs, no matter how the future turns out, should have identical current prices.

Almost everything in finance follows from this.

Valuation by Replication

Target security
Replicating portfolio, a collection of more *liquid* securities that, collectively, has the same future payoffs as the target *no matter how the future turns out.*

The target's value is then simply that value of the replicating portfolio.

> No matter how the future turns out: **the science of markets: a model**
>
> Replication: **engineering**

Styles of Replication

Static: rarely possible: put-call parity and variance swaps

Dynamic: BSM

Implied Variables and Realized Variables

Physics models start from today and **predict the future**.

Financial models think about the future and **predict values today**.

What matters is not only what will happen, but what people *think* will happen.

What people think will happen affects what happens today.

Realized variables describe what actually happens.

Financial models calibrate the future to current known prices to produce implied variables about the future that match known prices today. One then has to compare these implied values to the future values that are actually realized as time passes.

Implied variables describe what people think will happen to the variables in a model, deduced form current market prices.

MODELING MARKETS:

First, Modeling A Share of Stock: Return and Volatility

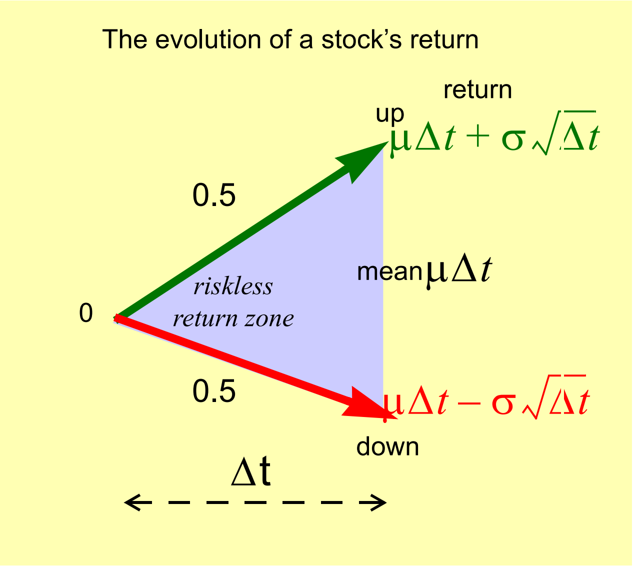
•

A stock's most important feature is the uncertainty of its return.

No Arbitrage means the riskless return is a convex combination of up and down.

This naive either-up-or-down model captures much of the inherent risk of owning a stock and many other securities. But not all.

The Efficient Market Models often uses Geometric Brownian Motion to describe return distributions.



This means that all we care about in a stock is its volatility and return; these parameters specify the entire distribution of returns.

Obviously not strictly true, but let's see where it takes us.

What return should we expect for a given volatility?

The law of one price will give us the answer.

The Law of One Price Relates Risk to Return

We can extend the law of one price (identical payoffs have identical prices*)* to demand that *identical expected risks have identical expected returns*.

But some risks can be avoided. Therefore the principle: *identical* ***unavoidable*** *expected risks have identical expected returns*.

How can you avoid risk? There are three ways to reduce or avoid risk:

(i) dilution with a riskless bond

(ii) diversification

(iii) hedging

We'll combine these with the law of one price to derive everything we can use.

Aside: Finance For Future Generations

- Feynman: One sentence about physics to guide future civilizations

- Feynman: One sentence about biology to guide future civilizations:

- One sentence about finance to guide future civilizations:

  If you can hedge away all correlated risk, and then diversify over all uncorrelated risk, you should expect to earn the riskless return.

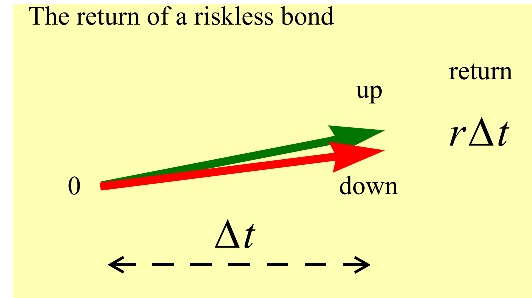
- This is a sensible principle.

  The trouble is that correlation and diversification can’t really be carried out because risk is not really purely statistical. It can’t be specified for all time by a stochastic pde or Monte Carlo.

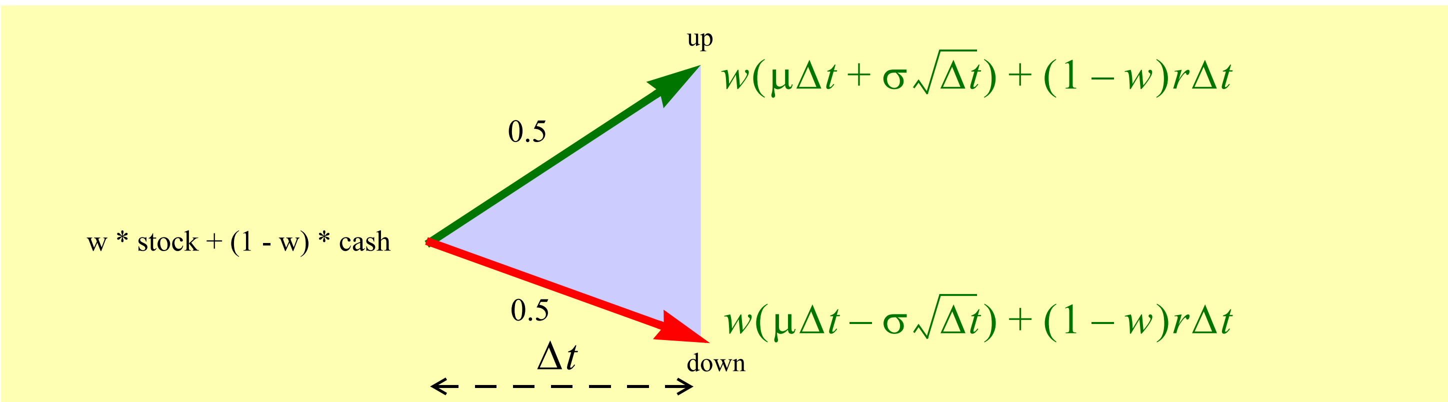
- Bedazzled

(i) Risk Reduction by Dilution Means Risk and Return are Proportional

By adding a riskless bond with zero volatility to the stock of volatility σ and expected return μ, you reduce both the risk and return of your investment.



Consider a mixture of w% risky stock with volatility σ and (1 - w)% riskless bond.



The expected return for this mixture is $w\mu\Delta t + (1-w)r\Delta t \;=\; r\Delta t + w(\mu - r)\Delta t$.
The volatility of returns is $w\sigma$.

Thus extra risk of magnitude $w\sigma$ must generate extra return $w(\mu - r)\Delta t$

$$\frac{\mu - r}{\sigma} = \lambda \qquad\qquad \mu - r = \lambda\sigma$$

Excess return is proportional to risk. By law of one price, must be true for all securities. A similar result holds for options values, and is equivalent to the Black-Scholes equation.

(ii) Risk Reduction by Diversification Means λ is Zero

If you can accumulate a portfolio of so many uncorrelated unavoidable risks that they cancel in the limit as the number of stocks become large, the portfolio's net volatility σ approaches zero.

Then, by the law of one price, it must produce an excess return of zero for the entire portfolio.

But the excess return of the entire portfolio is the weighted sum of the excess returns of each individual member of the portfolio, each of which is proportional to their individual non-zero volatility via the Sharpe ratio.

Hence the Sharpe ratio in the equations above for this portfolio must be zero. But the Sharpe ratio is the same for all portfolios of stocks, so that $\lambda = 0$ in general. Thus,

$$\mu = r$$

All stocks must be expected to earn the riskless rate if you can diversify.

(iii) Risk Reduction by Hedging

You can't always diversify because stocks are sensitive to the entire market M.

Let ρ be the correlation of the returns between stock S with volatility σ and the market M with volatility σ_M

$$\frac{dS}{S} = \mu dt + \sigma\left(\sqrt{1-\rho^2}\,dZ + \rho dZ_M\right)$$

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

You can *hedge away* the M-related risk of any stock to create an M-neutral portfolio:

$dS_M = dS - \Delta dM$ has no market risk if $\Delta = \rho(\sigma/\sigma_M)\frac{S}{M} \equiv \beta\frac{S}{M}$.

This M-neutral stock has expected return $\frac{\mu - \beta\mu_M}{1-\beta}$ sensitive only to the volatility of the stock.

By diluting it, we can show that its excess return of the M-neutral stock must then be proportional to its residual volatility. Furthermore, by diversifying over many M-neutral stocks we can show that the M-neutral stock can expect only the riskless rate r, so that[1]

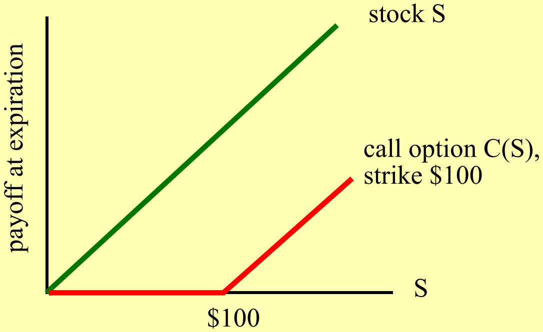
$$(\mu - r) = \beta(\mu_M - r) \qquad \text{CAPM in “Efficient Markets”}$$

Do you believe this?

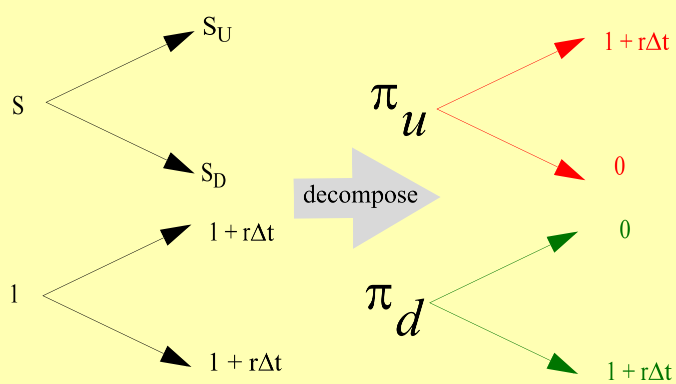
1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://www.ederman.com/new/docs/qf-market_bubbles.pdf

Derivative Valuation by Replication

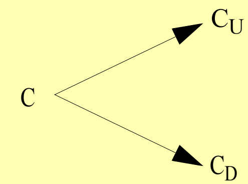
A stock is straight, an option curved.



Binary price trees for a stock S and a $1 investment in a riskless bond. The stock and bond can be decomposed into a security $\pi_{\mathbf{u}}$ that pays off only in the up state and a security $\pi_{\mathbf{d}}$ that pays off only in the down state.



You can replicate an option's nonlinear payoff over each instant by suitable investments in the elemental securities τ_u and τ_d.



$(1+r\Delta t)C = (C_U).\pi_{\mathbf{u}} + (C_D).\,\pi_{\mathbf{d}}$

A derivative is a contract whose payoff depends on the price of a "simpler" *underlier*. The most relevant characteristic is the *curvature* of its payoff *C(S)*, as illustrated for a simple call option. **What is the value of curvature?**

You can use linear algebra to decompose the stock and bond into a basis of two more elemental securities π_u and π_d, each respectively paying $\$(1+r\Delta t)$ in only one of the final states.

Then you can replicate the payoff of any non-linear function *C(S)* over the next instant of time Δt, no matter into which state the stock evolves. Note that the portfolio consisting of both π_u and π_d is riskless and is therefore worth $1.

The value of the option is the price of the mixture of stock and bond that replicates it. The coefficients depend on the difference between the up-return and the down-return at each node, that is, on the stock's volatility σ.

The choice-of-currency/numeraire trick

You can use any currency to value a security if markets are efficient.

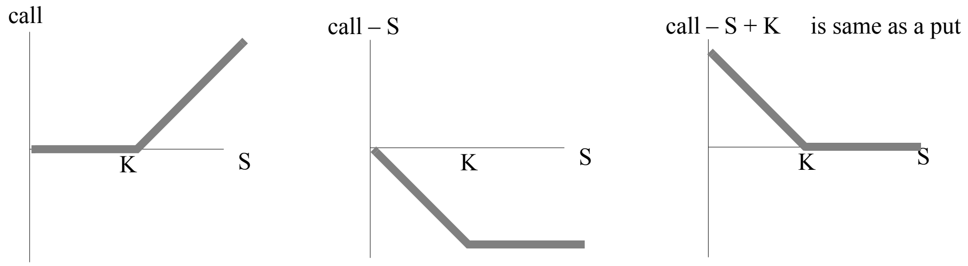
A convenient choice of currency can greatly simplify thinking about a problem, and sometimes reduce its dimensionality.

Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modeled by choosing a bond itself as the natural valuation currency.

Static Replication

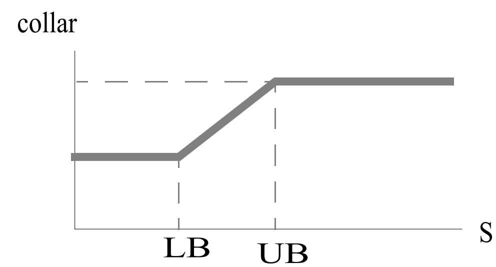
If you can create a static replicating portfolio for your payoff, you have very little model risk.

European put from a call: Put-Call Parity



Thus price of put = price of call - price of stock + PV(K).

A collar is a very popular instrument for portfolio managers who have made some gains during the year and now want to make sure they keep some upside but don't lose too much downside.



You can write the payoff as

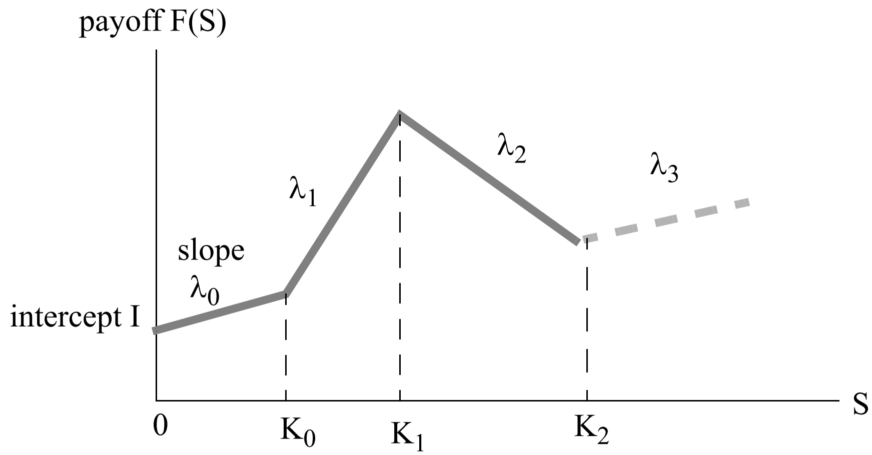
$$LB + call(S, LB) - call(S, UB)$$

Using put-call parity: $S + put(S, LB) - call(S, UB)$.

Its popularity forces dealers to be short puts and long calls.

Generalized European payoffs:

Piecewise-linear function of the terminal stock price S



Replicating portfolio consisting of a zero-coupon bond ZCB(I) plus a series of calls $C(K_i)$:

$$ZCB(I) + \lambda_0 S + (\lambda_1 - \lambda_0) C(K_0) + (\lambda_2 - \lambda_1) C(K_1) + \ldots$$

whose value can be determined from market prices.

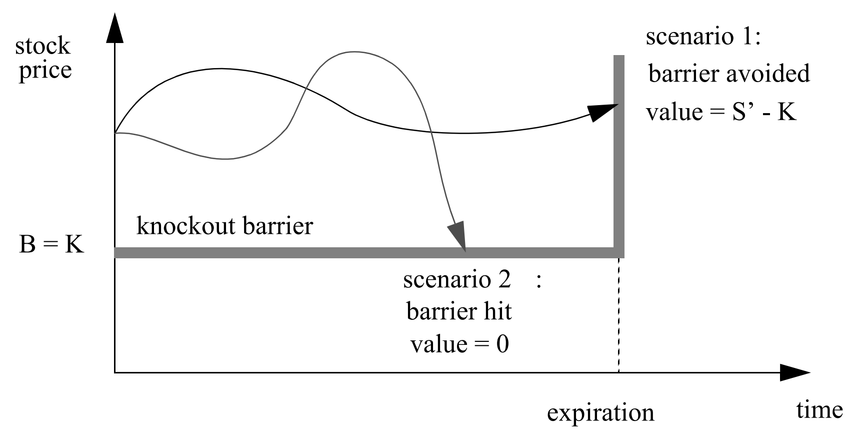
Static Hedge for a Down-and-Out Call with Strike = Barrier

This option has a time-dependent boundary.

Stock price *S* and dividend yield *d,* strike *K* and out-barrier $B = K$.

Scenario 1 in which the barrier is avoided and the option finishes in-the-money.

Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



In scenario 1 the call pays out $S'-K$, the payoff of a forward contract with delivery price *K* worth

$$F = Se^{-dt} - Ke^{-rt}$$

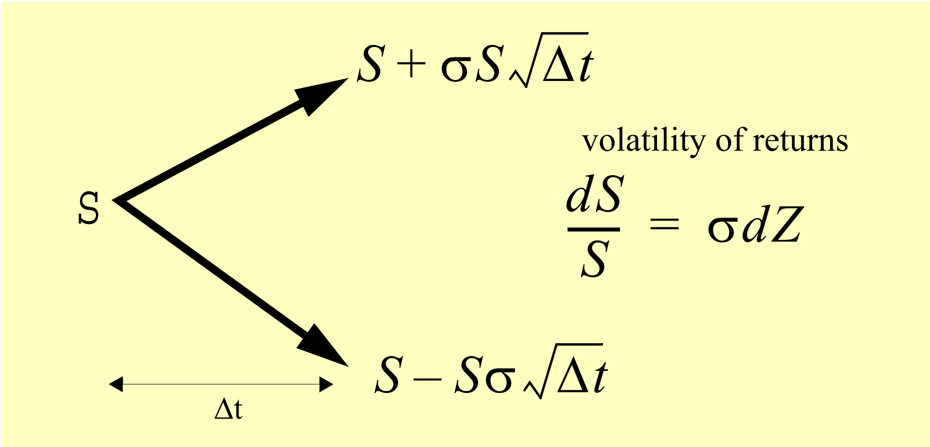
For paths in scenario 2, the down-and-out call immediately expires with zero value. In that case, the above forward *F* that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$. This is close to zero.

When the stock hits the barrier you must sell the forward to end the trade.

DYNAMIC REPLICATION

Quick Derivation of the Black-Scholes PDE

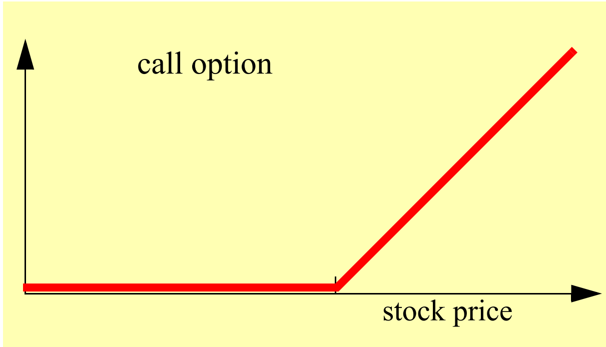
Assume GBM with zero rates for simplicity.



In time Δt, $\Delta S \approx \sigma S\sqrt{\Delta t}$.

The stock S is a primitive, linear underlying security that provides a linear position in ΔS.

If you are long an option, you profit whether the stock goes up or down! The call has curvature, or convexity. $\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$



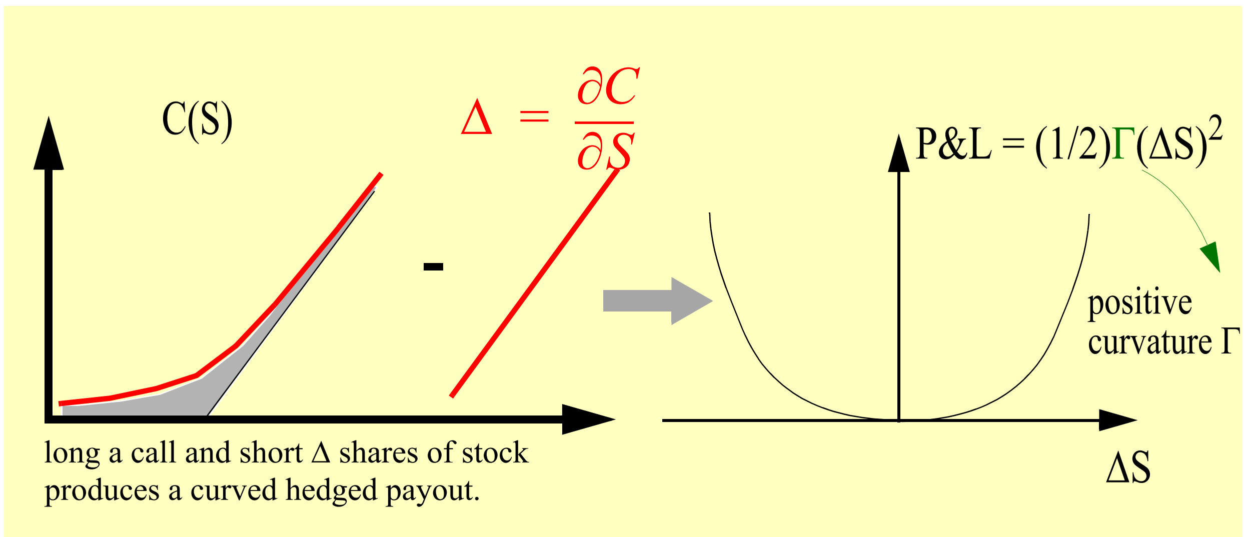
What is the fair price for *C(S,K,t,T)*?

We can do a Taylor series expansion on the unknown price *C( )* and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S+\Delta S,\, t+\Delta t) \;=\; C(S,t) + \left.\frac{\partial C}{\partial t}\right|_{S,\,t}\Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,\,t}\Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,\,t}\frac{(\Delta S)^2}{2} + \ldots$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

If you hedge away the linear term in ΔS by shorting $\Delta \;=\; \dfrac{\partial C}{\partial S}$ shares the profit and loss of the hedged option position looks like this:



Positive convexity generates a profit or loss that is quadratic in (ΔS).

What Should You Pay for Convexity?

Suppose we think we know the future volatility of the stock, Σ.

Over time Δt, the stock should move an amount $\Delta S = \pm\Sigma S Z(0, 1)\sqrt{\Delta t}$.

Binomially, this corresponds to $\Delta S = \pm\Sigma S\sqrt{\Delta t}$ with $(\Delta S)^2 = \Sigma^2 S^2 \Delta t$.

Change in value from the movement in stock price $= \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t)$

Change in value from passage of time $= \Theta(\Delta t)$ where $\Theta = \frac{\partial C}{\partial t}$

Total change in value of the hedged position is $d\text{P\&L} = d(C - \Delta S) = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t) + \Theta(\Delta t)$

If we know Σ, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it behaves like a riskless bond and must earn zero interest: $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

The Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

time decay and curvature are linked

$$C_{BS}(S, K, \Sigma, t, T) = SN(d_1) - KN(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}\, dy$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

The option's Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate r is non-zero, we will show in a subsequent chapter that

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

Hedging an Option Means Betting On Volatility

Σ is the implied volatility that we inserted, our expectation of future volatility.

Suppose the stock actually evolves with a realized volatility σ Then the actual P&L is:

The gain from curvature is $\frac{1}{2}\Gamma\sigma^2 S^2 \Delta t$

The loss from time decay $\Theta\Delta t$ is $\frac{1}{2}\Gamma S^2 \Sigma^2 \Delta t$ because BS equation is $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

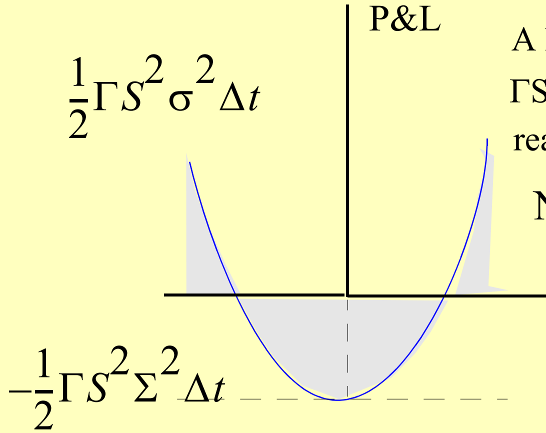
The net P&L during time Δt is $\left[\frac{1}{2}\Gamma(\sigma^2 - \Sigma^2) S^2 \Delta t\right]$.

This is path dependent unless ΓS^2 is independent of the stock price, which is not the usual case.

Here is an illustration of the contributions to the P&L:

To profit, you need the realized volatility to be greater than the implied volatility. A short position profits when the opposite is true.

Note: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S.



A hedged position earns the
ΓS^2–weighted difference between
realized and implied volatility

$$\text{Net P\&L} = \frac{1}{2}\int \Gamma S^2 (\sigma^2 - \Sigma^2)\Delta t$$

Betting on Pure Volatility

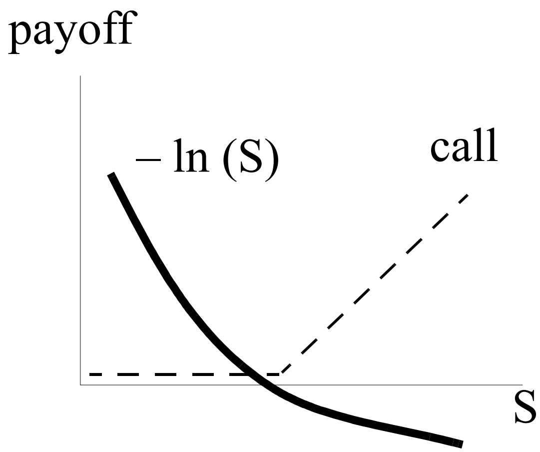
$$\text{Net P\&L} = \frac{1}{2}\int \Gamma S^2(\sigma^2 - \Sigma^2)\Delta t$$

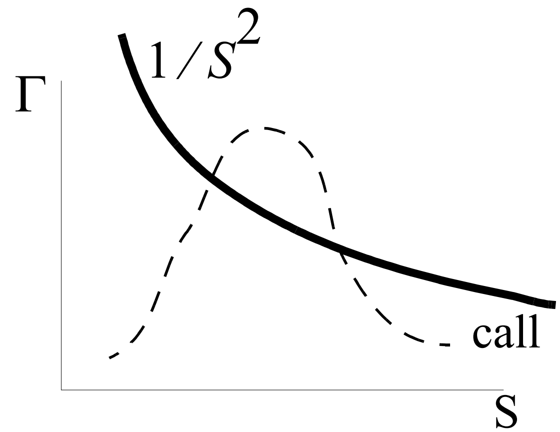
In a BS world, you can capture pure volatility if you own a derivative O whose curvature satisfies

$$\Gamma_O = 1/S^2 \quad \text{P\&L(O)} = \int \frac{1}{2}(\sigma^2 - \Sigma^2)\Delta t$$

The security with this gamma is the "log contract" with value $O = -\ln S$ and a hedge ratio $\Delta = -1/S$, **independent** of volatility! You hedge it by owning $1 worth of stock always.

A log contract, hedged, will capture realized variance.


payoff
– ln (S)
call
S


Γ
$1/S^2$
call
S

VARIANCE SWAPS

Volatility and Variance Swap Contracts

A Volatility swap is a forward contract on realized volatility. At expiration it pays the difference in dollars between the actual return volatility realized by the index over the lifetime of the contract σ_R and some previously agreed upon "delivery" volatility K_{vol}:

$$(\sigma_R - K_{vol}) \times N \text{ where } N \text{ is the notional amount.}$$

Similarly, a **variance swap** is a forward contract on realized variance. It pays

$$\left(\sigma_R^2 - K_{var}\right) \times N$$

Note: $N\left(\sigma_R^2 - K_{var}\right) \approx 2N\sqrt{K_{var}}(\sigma_R - \sqrt{K_{var}})$, therefore notional vega is approximately $2\sqrt{K_{var}}$ times notional variance. But variance is a derivative of volatility.

The contract must also specify the precise method for calculating at expiration the realized volatility, including the source and observation frequency of prices, the annualization factor and whether the sample mean is subtracted from each return.

Exhibit 1.1.1 — Variance Swap on S&P 500 : sample terms and conditions

VARIANCE SWAP ON S&P500

SPX INDICATIVE TERMS AND CONDITIONS

Instrument:	Swap
Trade Date:	TBD
Observation Start Date:	TBD
Observation End Date:	TBD
Variance Buyer:	TBD (e.g. JPMorganChase)
Variance Seller:	TBD (e.g. Investor)
Denominated Currency:	USD ("USD")
Vega Amount:	100,000
Variance Amount:	3,125 (determined as Vega Amount/(Strike*2))
Underlying:	S&P500 (Bloomberg Ticker: SPX Index)
Strike Price:	**16**
Currency:	USD
Equity Amount:	**T+3** after the Observation End Date, the Equity Amount will be calculated and paid in accordance with the following formula:

*Final Equity payment = Variance Amount * (Final Realized Volatility² – Strike Price²)*

If the Equity Amount is positive the Variance Seller will pay the Variance Buyer the Equity Amount.
If the Equity Amount is negative the Variance Buyer will pay the Variance Seller an amount equal to the absolute value of the Equity Amount.

where

$$\textbf{\textit{Final Realised Volatility}} = \sqrt{\frac{252 \times \sum_{t=1}^{t=N} \left(\ln \frac{P_t}{P_{t-1}} \right)^2}{\text{Expected\_N}}} \times 100$$

Expected_N = [number of days], being the number of days which, as of the Trade Date, are expected to be Scheduled Trading Days in the Observation Period

P_0 = The Official Closing of the underlying at the Observation Start Date

P_t = Either the Official Closing of the underlying in any observation date t or, at Observation End Date, the Official Settlement Price of the Exchange-Traded Contract

Calculation Agent:	JP Morgan Securities Ltd.
Documentation:	ISDA

1.1 Valuing Variance Swaps

What's the fair value of a variance or volatility swap, the K that makes it worth zero at inception?

Fair values are found (as always) by replication!

Variance swaps are more naturally replicated via options than volatility swaps. Therefore people are more willing to trade them.

Volatility should be thought of as the square root of variance, and hence as a derivative contract on variance.

You can replicate a variance swap by owning and hedging a log contract. But where do you get a log contract?

You have to replicate that.

1.2 Intuitive Approach to Log Replication

Zero interest rates for simplicity, so $C = C(S, K, \nu)$ where $v = \sigma\sqrt{\tau}$.

$$C_{BS} = SN(d_1) - KN(d_2) \qquad d_{1,2} = \frac{\ln S/K \pm v^2/2}{v}$$

Then the exposure to volatility is given by

$$\kappa \equiv \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

You can see that the option has sensitivity to S and σ, and is therefore not a good way to make a clean bet on volatility. What we want is a portfolio whose exposure κ to volatility is independent of the stock price S, so that we can bet on volatility no matter what the stock price does.

Construct a portfolio $\pi(S) = \int_0^{\infty} \rho(K) C(S, K, v) dK$ such that $\kappa = \frac{\partial \pi}{\partial \sigma^2}$ is independent of S.

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(K) \frac{S\sqrt{\tau}}{2\sigma}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}} dK \sim \int_0^{\infty} \rho(K) S f\left(\frac{K}{S}, v\right) dK$$

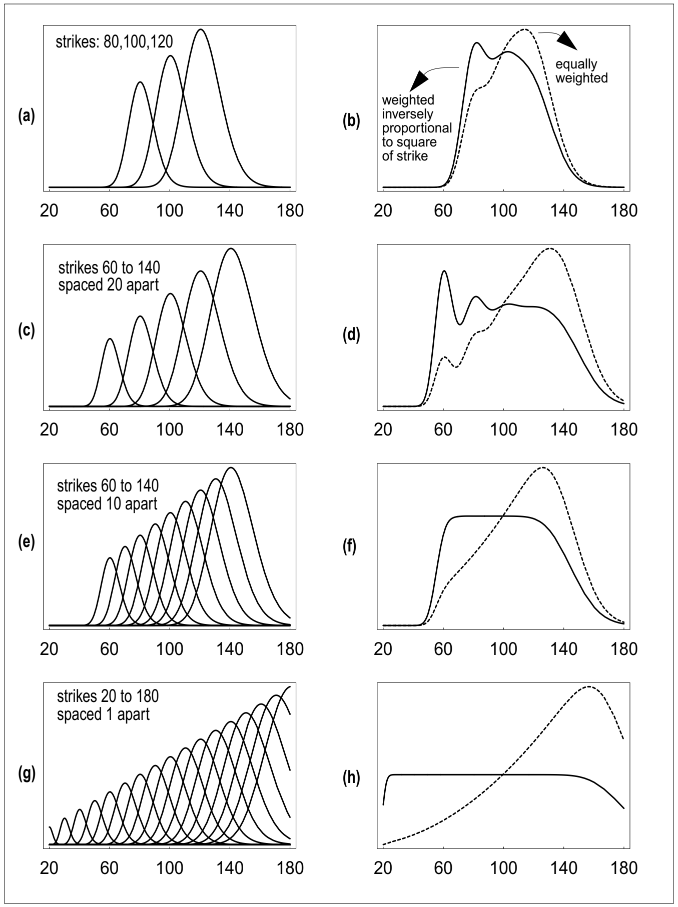
We can make the S-dependence of this explicit by changing variable to $x = K/S$ so that

$$\frac{\partial\pi}{\partial\sigma^2} = \int_0^\infty \rho(xS)S^2 f(x, v)dx$$

In order for this to be independent of S, we require that $\rho(K) \sim 1/K^2$

A density of options whose weights decrease as K^{-2} will give the correct volatility dependence.

FIGURE 1. The variance exposure, V, of portfolios of call options of different strikes as a function of stock price S. Each figure on the left shows the individual V_i contributions for each option of strike K_i. The corresponding figure on the right shows the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally-weighted sum of options; the solid line corresponds to weights inversely proportional to K^2, and becomes totally independent of stock price S inside the strike range



This $1/K^2$ weighted sum of options is equivalent to a short position in the payoff of a log contract L, long the payoff a forward

Use liquid puts below some strike S^* and use calls with strikes above S^*. The payoff at expiration is

$$\pi(S, S^*, v) = \int_{(K > S^*)} C(S, K, v)\frac{dK}{K^2} + \int_{(K < S^*)} P(S, K, v)\frac{dK}{K^2}$$

What does this payoff look like at expiration when $\tau = 0$: Call has S > K; Put has S < K

$$\pi(S, S^*, v) = \left(\int_{S^*}^{S} (S-K)\frac{dK}{K^2}\right) \text{ for S>S* and } \int_{S}^{S^*} (K-S)\frac{dK}{K^2} \text{ for S<S*}$$

$$= -\ln\frac{S}{S^*} + \left(\frac{S-S^*}{S^*}\right)$$

We need to short a log contract L and own a forward contract with delivery price S^*, which has no volatility dependence and can be replicated **statically**.

1.3 Log Contract in a Black-Scholes World

We can simply solve the Black-Scholes equation $\frac{\sigma^2 S^2}{2}\frac{\partial L}{\partial S^2} + \frac{\partial L}{\partial t} = 0$ for r =0, with the boundary condition for the log payoff $L(S, S^*, 0) = -\ln\frac{S}{S^*}$.

The very simple solution is $L(S, S^*, \tau) = -\ln S/S^* + (\sigma^2\tau)/2$. with volatility exposure $\kappa = \tau/2$.

The delta of the contract is $-1/S$.

Going long 1/S shares at any instant – i.e. by owning exactly \$1 worth of shares at any instant – you have exactly a $\Gamma = 1/S^2$ and the right vol. exposure.

At the start of the trade, when $\tau = T$, you need to buy 2/T contracts to have $\kappa = 1$, a variance exposure of \$1 for the whole trade.

$\Pi = \frac{2}{T}\left[\frac{S - S_*}{S_*} - \ln\frac{S}{S_*}\right] + \frac{\tau}{T}\sigma^2$, after hedging, captures $\sigma_R^2 - \sigma_I^2$

Proof that the price of a log contract with S^* is actually the variance. (Assume r = 0)

Consider a log contract that pays out $\log(S_T/S_0)$ at expiration time T. Let its value today be denoted by L_0. Look at the trading strategy below that starts with a short position in one log contract and long \$1 worth of shares, and then maintains this dollar value of shares by rehedging as below.

time	stock	#shares	value	Dollars	log contracts	Total value
t_0	S_0	$1/S_0$	1	0	-1 worth L_0	$-L_0 + 1$
t_1	S_1	$1/S_0$	S_1/S_0	0	-1 worth L_1	$-L_1 + S_1/S_0$

Now rebalance to own \$1 worth of shares:. buy $(1/S_1 - 1/S_0)$ shares by borrowing $(1/S_1 - 1/S_0)S_1 =$ $(S_0 - S_1)/S_o$ dollars. You then own $1/S_1$ shares worth \$1, and you have borrowed (that is, you are short) $(S_0 - S_1)/S_o$ dollars. Then, after rebalancing,

time	stock	#shares	value	Dollars	log contracts	Total value
t_1	S_1	$1/S_1$	1	$-(S_0 - S_1) \div S_o$	-1 worth L_1	$-L_1 + 1 + (S_1 - S_0) \div S_0$

Now move to time t_2 and rebalance again, to get

time	stock	#shares	value	Dollars	log contracts	Total value
t_2	S_2	$1/S_2$	1	$-(S_0 - S_1)/S_o$ $-(S_1 - S_2)/S_1$	-1 worth L_2	$-L_2 + 1 + \begin{array}{l} -(S_0 - S_1) \div S_o \\ -(S_1 - S_2) \div S_1 \end{array}$

Repeat rehedging N times to expiration:

$$1 - L_T + \frac{S_1 - S_0}{S_0} + \frac{S_2 - S_1}{S_1} + \ldots + \frac{S_N - S_{N-1}}{S_{N-1}} = 1 - L_T + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \log\frac{S_N}{S_{N-1}}\frac{S_{N-1}}{S_{N-2}}\ldots\frac{S_1}{S_0} + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$= 1 - \sum_{i=0}^{N-1}\left(\log\frac{S_{i+1}}{S_i}\right) + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$

$$\approx 1 - \sum_{i=0}^{N-1}\left[\frac{\Delta S_i}{S_i} - \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2\right] + \sum_0^{N-1} \frac{\Delta S_i}{S_i}$$ in a Taylor expansion to second order

$$= 1 + \sum_{i=0}^{N-1} \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 = 1 + \sum \frac{\sigma_i^2 \Delta t_i}{2}$$

Thus, if you assume zero interest rates, we've shown that an initial investment at time t = 0 of value $-L_0 + 1$, by dynamic rehedging, leads to a final value at time t = T of $1 + \sum \frac{\sigma_i^2 \Delta t_i}{2}$.

Therefore, the fair value of L_0 at the beginning must be $L_0 = -\sum_i \frac{\sigma_i^2 \Delta t_i}{2}$.

That is, the fair value of the log contract struck at the initial stock price S_0 is minus half the variance of the underlier over the life of the contract.

Problems with Replication

If you could buy a log contract you'd have exactly what you want. Instead you have to buy a continuum of calls and puts, which doesn't exist. You can only buy a discrete number in a discrete range, so you have no sensitivity to volatility outside the strike range.