Lecture 2: Dynamic Replication: Realities and Myths of Options Pricing

Summary of Last Class

The smile: appeared after 1987; inconsistent with Black-Scholes; appears in all global index options markets; flattens at long expirations when plotted against K/S; has a negative correlation between slope of the smile and volatility level;

Modeling principles: financial engineering is constructive; financial science is reductive and unfortunately, has only a limited number of useful principles, mainly the law of one price: portfolios with the same payoff under all circumstances have the same value.

Models are (mostly) for comparison of related securities

Relative valuation works better than absolute valuation; models take you from liquid securities with known prices to the value of illiquid securities with unknown market prices;
relative valuation is for aritrageurs and manufactures of securities; dynamic replication is much more difficult than static replication.

It's hard to test models in finance, partly because of the dual nature of implied and realized variables.

> "My job, I believe, is to persuade others that my conclusions are sound. I will use an array of devices to do this: theory, stylized facts, time-series data, surveys, appeals to introspection and so on."
>
> "In the real world of research, conventional tests of [statistical] significance seem almost worthless."
> Fischer Black

Derivatives are not independent securities.

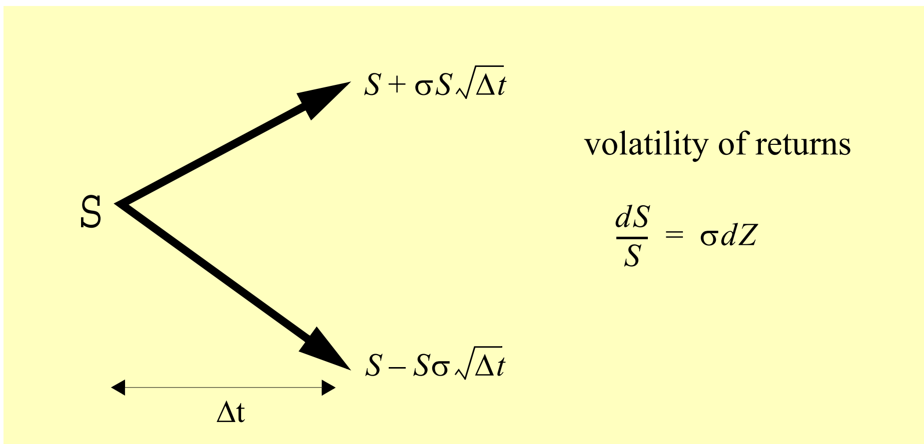
What We'll Cover in This One

Theory of dynamic replication.
Black-Scholes equation.
Trading Volatility.
Practical difficulties and limitations: discrete hedging, which hedge ratio, transactions costs.

2.1 Dynamic Replication and its Consequence: the Black-Scholes PDE

Classical options theory is based on the insight that, in an idealized and simplified world, options are not an independent asset, and can therefore be replicated perfectly in terms of simpler securities. Assume interest rates are zero here, to be simple.

2.1.1 The (assumed) stochastic behavior of stock prices: geometric Brownian motion

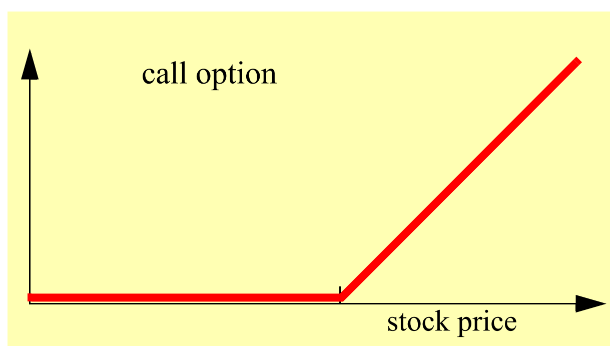


Therefore, over a small instant of time Δt, a stock with volatility σ will generate a move $\Delta S \approx \sigma S\sqrt{\Delta t}$.

A stock S is a primitive, linear underlying security If you own a stock worth S, you profit if it goes up, lose if it goes down. You have a linear position in ΔS.

If you are long an option, you can profit no matter whether the stock goes up or down! Look at a call with the payoff on the right. The call has curvature, or convexity.

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$$



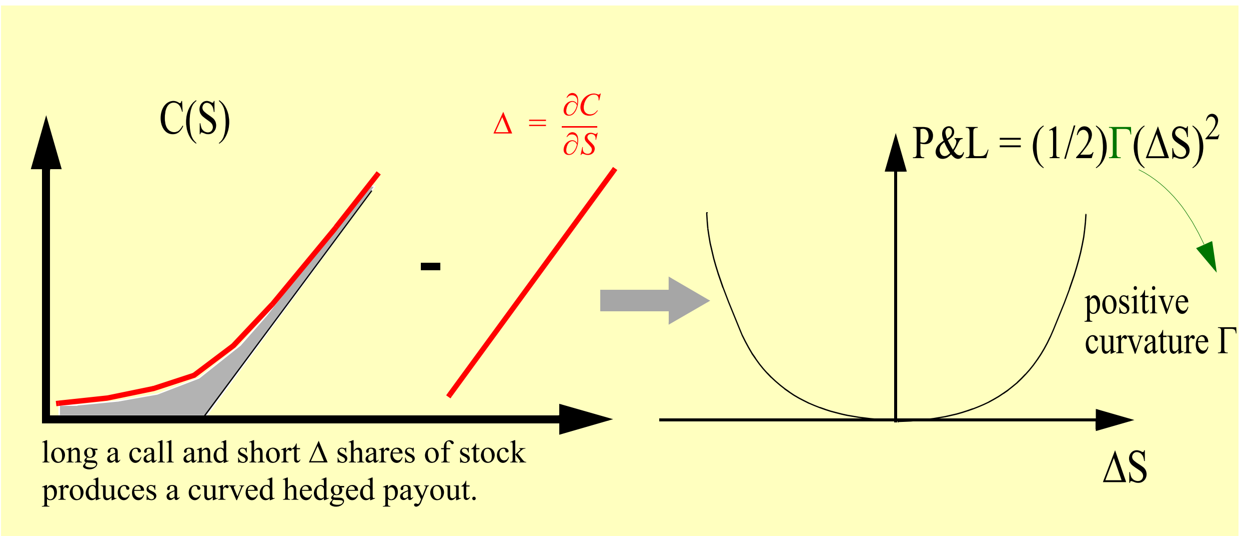
You profit on an up-move, lose nothing on a down move. What is the fair price for *C(S,K,t,T)*?

We can do a Taylor series expansion on the unknown price $C(\ )$ and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S+\Delta S, t+\Delta t) = C(S,t) + \left.\frac{\partial C}{\partial t}\right|_{S,t}\Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,t}\Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,t}\frac{(\Delta S)^2}{2} + .. \qquad \text{Eq.2.1}$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

If you were long the call C, you could hedge away the linear term in ΔS if you knew the derivative $\frac{\partial C}{\partial S}$ by shorting that many shares of the stock. ($\Delta = \frac{\partial C}{\partial S}$ is dimensionless, a number.) The the resultant profit and loss of the hedged option position looks like this:



Convexity generates a profit or loss that is quadratic in (ΔS). Positively curved or convex securities allow you to make a profit irrespective of whether the stock moves up or down. Similarly, negatively curved or concave securities generate a loss on any stock price move. To get the benefit of pure curvature you must delta-hedge away the linear part of the call option which would otherwise swamp it.

What Should You Pay for Convexity?

Suppose we think we know the future volatility of the stock, Σ. Over time Δt, a stock with volatility Σ is expected to move an amount $\Delta S = \pm\Sigma S Z(0,1)\sqrt{\Delta t}$, where $Z(0,1)$ represents a normal distribution with mean zero and standard deviation 1. In a binomial approximation to the continuous evolution of the stock price, you can represent the stock moves as $\Delta S = \pm\Sigma S\sqrt{\Delta t}$ with $(\Delta S)^2 = \Sigma^2 S^2 \Delta t$.

Change in value from the movement in stock price $= \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t)$

Change in value from passage of time $= \Theta(\Delta t)$ where $\Theta = \frac{\partial C}{\partial t}$

Then, from Equation 2.1, the total change in value of the hedged position is

$$dP\&L = d(C - \Delta S) = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t) + \Theta(\Delta t)$$

There is nothing stochastic about this; all dependence on the uncertain moves of S have cancelled out to leading order in the Taylor series. If we think we know the future volatility Σ, then we know exactly how far the stock will move over time Δt, even though we don't know its direction. The potential profit and loss is completely deterministic, irrespective of the direction of the stock price move.

Since the hedged portfolio is riskless, an investment in it behaves exactly like an investment in a short-term riskless bond. The principle of no riskless arbitrage dictates that hedged portfolio and the riskless bond should have the same current value. In this example with zero interest rates, the riskless bond generates no interest, so its final value and initial value are identical. Therefore the final value of the hedged portfolio must be identical to its initial value, and so

$$\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0 \qquad \text{Eq.2.2}$$

Written out in full, this is the Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \qquad \text{Eq.2.3}$$

The Black-Scholes equation relates the decay of an option's value over time to its curvature Γ and the expected stock volatility S. The solution is:

$$\begin{aligned} C_{BS}(S, K, \Sigma, t, T) &= SN(d_1) - KN(d_2) \\ d_{1,2} &= \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}} \\ N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy \end{aligned} \qquad \text{Eq.2.4}$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1) \qquad \text{Eq.2.5}$$

The option's Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate r is non-zero, we will show in a subsequent chapter that

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \qquad \text{Eq.2.6}$$

2.1.2 Hedging an Option Means Betting On Volatility

Σ is the implied volatility that we inserted in order to determine how to price the option based on our expectation of future volatility.

Suppose[1] the stock actually evolves with a realized volatility σ, so that in reality $\Delta S \equiv \sigma S Z(0, 1)\sqrt{\Delta t}$. Then the actual P&L is determined by

The gain from curvature is $\frac{1}{2}\Gamma\sigma^2 S^2 \Delta t$

The loss from time decay $\Theta\Delta t$ is $\frac{1}{2}\Gamma S^2 \Sigma^2 \Delta t$

The last identity above follows from Equation 2.2.

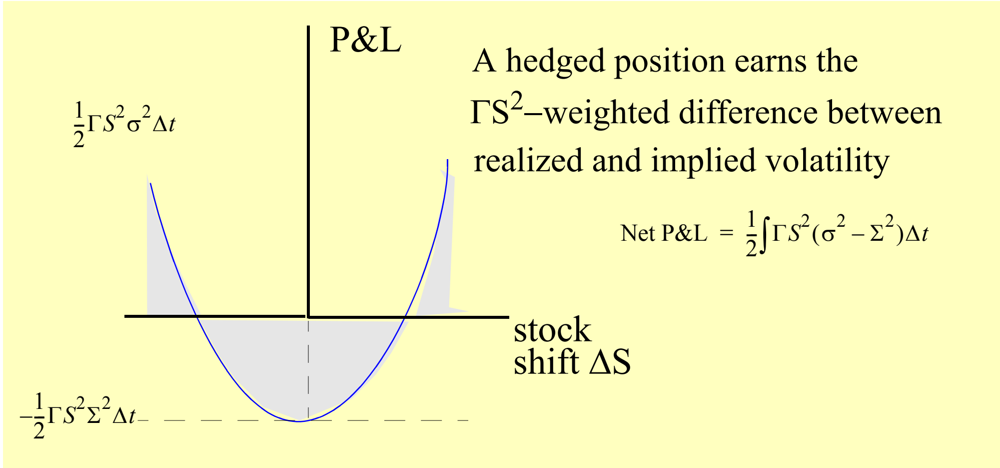
$$\text{The net P\&L during time } \Delta t \text{ is } \left[\frac{1}{2}\Gamma(\sigma^2 - \Sigma^2)S^2\Delta t\right]. \qquad \text{Eq.2.7}$$

This value is path dependent unless ΓS^2 is independent of the stock price, which is not the case for vanilla options.

After the Black-Scholes PDE and the Black-Scholes formula, Equation 2.7 is one of the most important equations in options valuation.

1. In these notes we (almost) always use σ to represent realized volatility and Σ to represent implied volatility.

Here is an illustration of the contributions to the P&L:



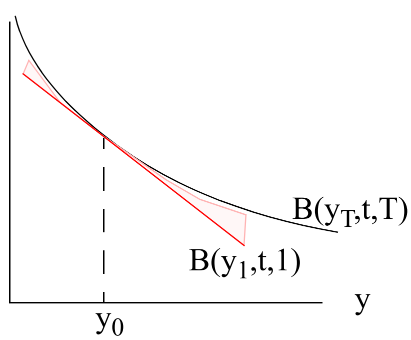
This illustration describes the bet you are making in taking a long position in a hedged option. To profit, you need the realized volatility to be greater than the implied volatility. A short position profits when the opposite is true.

Note: Black-Scholes uses a single unique volatility for all strikes *K* and expirations *T*, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of *K*, *t*, *T* and *S*.

2.1.3 An Aside: The Convexity of Bonds

Consider a zero-coupon *T*-maturity bond at time whose price at time *t* with yield to maturity y_T is represented by $B(y_T, t, T)$. You can think of bonds as nonlinear derivatives of interest rates.

The graph of bond price *B* vs. yield to maturity *y* looks like



If you hedge the long bond's price by buying a one-year bond with price $B(y_1, t, 1)$ whose behavior is approximately linear in the yield (because it's short term), then you again have a convex P&L for the hedged portfolio as a function of yields, assuming they move simultaneously and in parallel, and there are somewhat analogous PDEs for bond pricing that follow from the principle of no riskless arbitrage.

The tricky part which I'm ignoring here is that yields don't have to move in parallel, and so, at the minimum, you need a model which takes account of all yield moves in a consistent model, for example a one-factor short-rate model like Ho-Lee or BDT or Vasiçek. In these one-factor models all bond prices depend on the short rate as the only stochastic variable, and you can hedge any single bond with another bond of a different maturity, or indeed with an option on a bond. Of course, these models exclude the possibility of more than one factor, that is, of different bonds moving in independent ways.

2.1.4 The Two Kinds of Volatility

I will use Σ to denote implied volatility and σ to denote a stock's actual (realized) volatility.

Realized Volatility. The realized daily volatility σ_d of an index S_i over a period of N days is the square root of the variance of the daily log returns r_i:

$$r_i = \log\left(\frac{S_{i+1}}{S_i}\right) \approx \frac{\Delta S_i}{S_i}$$

$$\sigma_d^2 = \frac{1}{N-1}\sum_i (r_i - \bar{r})^2 = \overline{r^2} - \bar{r}^2$$

Therefore, over an instant of time Δt, an index with volatility σ will move roughly $\Delta S \approx \sigma_d S\sqrt{\Delta t}$ where Δt is the time elapsed in units of a fraction of a day.

The realized volatility of daily returns is $\sigma_d = \sqrt{\overline{r_d^2} - (\overline{r_d})^2}$.

The variance of one-year returns is $\sigma^2{}_a = \overline{r_a^2} - (\overline{r_a})^2$, where r_a is the annual log return.

If the stock evolution follows geometric Brownian motion, then $\sigma_{Nd}^2 = N\sigma_d^2$.

As a rough rule of thumb, a year has approximately 252 trading days, and so $\sigma_a \approx 16\sigma_d$.

Implied Volatility is the value of the volatility parameter in the Black-Scholes equation that makes the value of the option $C_{BS}(S, K, t, T, \Sigma)$ in that model match the market price C_{obs} of the option:

$$C_{BS}(S, K, t, T, \Sigma) = C_{obs} \qquad \text{Eq.2.8}$$

If different options imply different implied volatilities, then the implied Black-Scholes volatility Σ of a vanilla call or put can in general be a function $\Sigma(S, t, K, T)$ where K is the strike and T the expiration of the option when the underlyer value is S at time t. If in fact the market prices of vanilla calls and puts are such that Σ does vary with strike, then we have a volatility smile and the Black-Scholes model is inconsistent with market prices.

One way to think of implied volatility is as the market's guess at future average volatility, perhaps modified by a fear factor, supply and demand, and an understanding of how the Black-Scholes model fails to capture all the features of realized stock returns and their volatility. Implied volatility is often merely a quoting convention that incorporates into one number everything you need to know to get the dollar price from the Black-Scholes formula, and is closely linked to but not equal to expectations about realized volatility.

The value of a call option in the Black-Scholes model can never be worth more than the stock. If it were, an initial long position in the call and a short position in one share of stock requires a positive outlay of cash. However, at expiration, the long-call/short-stock position always has a negative payoff.) As long as a call price C is worth less than the current stock price S, there always exists a unique Black-Scholes implied volatility Σ, because C in the Black-Scholes model is a monotonic function of Σ whose derivative given by

$$\frac{\partial C}{\partial \Sigma} = \frac{Se^{\frac{-d_1^2}{2}}\sqrt{(T-t)}}{\sqrt{2\pi}} \qquad \text{Eq.2.9}$$

which is always positive.

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There is another way to derive Equation 2.9 that makes use of the way in which volatility and time are connected in the Black-Scholes partial differential equation. For interest rates set equal to zero, Equation 2.3 specifies that

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

where $\tau = T - t$ is the option's time to expiration.

For constant volatility Σ you can rewrite this as

$$\frac{\partial C}{\partial(\Sigma^2\tau)} - \frac{1}{2}S^2\frac{\partial^2 C}{\partial S^2} = 0$$

so that, since the partial differential equation depends on time τ only through the variable $\Sigma^2\tau$,

$$\frac{\partial C}{\partial \Sigma} = 2\Sigma\tau\frac{\partial C}{\partial(\Sigma^2\tau)} = \Sigma\tau S^2\frac{\partial^2 C}{\partial S^2}$$

You can differentiate Equation 2.5 easily to show that the RHS of the equation is equal to the RHS of Equation 2.9.

If the riskless rate r is non-zero, you can change variables in the Black-Scholes partial differential equation Equation 2.6 by writing the solution C as $C = \exp[-r(T-t)]X(F, K, t, T, \Sigma)$ where $F = S\exp[r(T-t)]$.

In terms of these variables, Equation 2.6 reduces to

$$\frac{\partial X}{\partial \tau} + \frac{1}{2}\Sigma^2 F^2\frac{\partial^2 X}{\partial F^2} = 0$$

and so the derivative with respect to τ is similarly related to the derivative with respect to the total variance $\Sigma^2\tau$.

A question to think about: To trade options efficiently, is it better to model implied volatility or realized volatility? What if they're not the same?

2.2 The Black-Scholes Equation and Sharpe Ratios

Valuation by perfect replication is the theoretical bedrock on which options pricing is based. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

Consider at time t a stock price S_t, a riskless bond B_t that earns interest rate r_t, and an option price C_t.

The stochastic evolution of the stock and bond prices are given by

$$\begin{aligned} dS_t &= \mu_S S_t dt + \sigma_t S_t dZ_t \\ dB_t &= B_t r_t dt \end{aligned} \qquad \text{Eq.2.10}$$

The option price is a function $C(S_t, t)$ whose evolution is given by

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2 dt \\ &= \left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2\right\}dt + \frac{\partial C_t}{\partial S}\sigma_t S_t dZ_t \\ &\equiv \mu_C C_t dt + \sigma_C C_t dZ_t \end{aligned}$$

where by definition

$$\begin{aligned} \mu_C &= \frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2\right\} \\ \sigma_C &= \frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_t S_t\right) \end{aligned} \qquad \text{Eq.2.11}$$

We can create a riskless portfolio out of S and C. Define $\pi = \alpha S + C$

Then

$$\begin{aligned} d\pi &= \alpha\{\mu_S S_t dt + \sigma_t S_t dZ_t\} + \{\mu_C C_t dt + \sigma_C C_t dZ_t\} \\ &= (\alpha\mu_S S_t + \mu_C C_t)dt + (\alpha\sigma_t S_t + \sigma_C C_t)dZ_t \end{aligned} \qquad \text{Eq.2.12}$$

That the portfolio be riskless necessitates

$$\alpha = -\frac{\sigma_C C}{\sigma_S S} \qquad \text{Eq.2.13}$$

That no riskless arbitrage is allowed means that a riskless portfolio must earn the riskless rate, so that $d\pi = \pi r dt$. From Equation 2.12 this leads to

$$\alpha\mu_S S + \mu_C C = (\alpha S + C)r$$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for α from Equation 2.13 leads to the relation

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S} \qquad \text{Eq.2.14}$$

Only if the stock and the option have equal Sharpe ratios, that is equal expected returns per unit of volatility, will the option and the stock allow no arbitrage. This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 2.11 into Equation 2.14 for μ_C and σ_C we obtain

$$\frac{\frac{1}{C_t}\left\{\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S}\mu_S S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S^2}(\sigma_t S_t)^2\right\} - r}{\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S}\sigma_t S_t\right)} = \frac{\mu_S - r}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC \qquad \text{Black-Scholes equation} \quad \text{Eq.2.15}$$

Note how the terms involving μ_S cancelled out of the equation, so that there is no dependence on the drift of the stock price.

We'll look at this solution and its derivatives in more details in subsequent lectures. It's good to get very familiar with manipulating this solution, its derivatives or "Greeks," and the approximations to it via Taylor expansions when the option is close to at-the money and $S \sim K$. The solution, the Black-Scholes formula and its implied volatility, is at the minimum the quoting currency for trading prices of vanilla options. You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$
\begin{aligned}
C(S, K, t, T, r, \sigma) &= e^{-r(T-t)} \times [S_F N(d_1) - KN(d_2)] \\
S_F &= e^{r(T-t)} S \\
d_{1,2} &= \frac{\ln(S_F/K) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy
\end{aligned}
\qquad \text{Eq.2.16}
$$

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma^2(T-t)$. If you rewrite the formula in terms of the prices of traded securities – the present value of the bond K_{PV} and the stock price S – then indeed time and volatility always appear together:

$$
\begin{aligned}
C(S, K, t, T, \sigma) &= [SN(d_1) - K_{PV}N(d_2)] \\
K_{PV} &= e^{-r(T-t)} K \\
d_{1,2} &= \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy
\end{aligned}
\qquad \text{Eq.2.17}
$$

Note that the time to expiration appears in the formulas in two different combinations, $r(T-t)$ the discount factor and $\sigma^2(T-t)$ the total variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.

Comment: Options traders get very familiar with the behavior of vanilla option prices and hedge ratios because they watch their movement and deltas all day long, and so get a feel for how prices vary. Theorists have to do the same, i.e. get familiar and gain intuition, but they have to do it by playing with the formula, manipulating, understanding and approximating it.

2.2.1 Some Useful Derivatives of the Black-Scholes Model

$$N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$N'(d_{1,2}) = \frac{1}{\sqrt{2\pi}}\exp\frac{-\left[\ln\frac{S_F}{K} \pm \frac{\sigma^2(T-t)}{2}\right]^2}{2\sigma^2(T-t)}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\frac{-\left[\ln\frac{S_F}{K}\right]^2}{2\sigma^2(T-t)}\exp\left[\frac{-\sigma^2(T-t)}{8}\right]\exp\mp\left[\frac{\ln(S_F/K)}{2}\right]$$

$$N'(d_2) = N'(d_1)\exp[\ln(S_F/K)] = \frac{S_F}{K}N'(d_1)$$

$$\frac{\partial d_{1,2}}{\partial K} = \frac{-1}{K\sigma\sqrt{T-t}}$$

$$\frac{\partial d_{1,2}}{\partial \sigma} = \frac{-1}{\sigma^2\sqrt{(T-t)}}\ln\left(\frac{S_F}{K}\right) \pm \frac{\sqrt{(T-t)}}{2}$$

$$\frac{\partial C}{\partial \sigma} = \frac{Se^{\frac{-d_1^2}{2}}\sqrt{(T-t)}}{\sqrt{2\pi}}$$

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)}N(d_2)$$

2.3 Another View: Option Valuation as Consistent Interpolation

Practitioners in the derivatives world tend to regard options models and their valuation formulas as interpolating functions for hybrid securities. A convertible bond, for example, is part stock, part bond: it becomes indistinguishable from the underlying stock when the stock price is sufficiently high, and equivalent to a corporate bond as when the stock price is sufficiently low. A convertible bond valuation model provides a rational formula for smoothly interpolating between these two extremes. In order to provide the correct limits, the model must be calibrated. A convertible model that didn't produce the market value of a pure corporate bond at asymptotically low stock prices would be fatally suspect.

One can view the Black-Scholes formula in a similar light. Assume that a stock S that pays no dividends has future returns that are lognormal with volatility σ. A plausible way to estimate the value at time *t* of a European call C with strike *K* expiring at time *T* is to calculate its expected discounted value, which is given by

$$\begin{aligned} C(S,t) &= e^{-R(T-t)}(E[S-K]_+) \\ &= e^{-R(T-t)}\{Se^{\mu(T-t)}N(d_1) - KN(d_2)\} \end{aligned} \qquad \text{Eq.2.18}$$

where R is the appropriate but unknown discount rate, still unspecified, and μ is the unknown expected growth rate for the stock.

The analogous formula for a put P is given by

$$\begin{aligned} P(S,t) &= e^{-R(T-t)}(E[K-S]_+) \\ &= e^{-R(T-t)}\{KN(-d_2) - Se^{\mu(T-t)}N(-d_1)\} \end{aligned} \qquad \text{Eq.2.19}$$

where

$$d_{1,2} = \frac{\ln\dfrac{Se^{\mu(T-t)}}{K} \pm \dfrac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} \qquad \text{Eq.2.20}$$

We have forced the discount rate R to be identical for both puts and calls, because we are taking the viewpoint of a dealer or market-maker who, for practical reasons, wants to use a single average discount rate for all securities in his portfolio with the same expiration.

A dealer or market-maker in options, however, has additional consistency constraints. As a manufacturer rather than a consumer of options, the market-maker must make prices consistent with the value of his raw supplies. He must notice that a portfolio $F = C - K$ consisting of a long position in a call and a short position in a put with the same strike *K* has exactly the same payoff as a forward contract with expiration time *T* and delivery price *K* whose fair current value is

$$F = S - Ke^{-r(T-t)} \qquad \text{Eq.2.21}$$

where *r* is the true riskless discount rate for time to expiration $(T-t)$.

The individual formulas of Equation 2.18 and Equation 2.19 must be consistent with the constraint that when they are combined to value the portfolio *F* which statically replicates a forward contract, they should produce the same value. If they are not calibrated to satisfy this, the market-maker will be valuing his options, stock and forwards inconsistently. What is necessary to satisfy this?

Combining Equation 2.18 and Equation 2.19 we obtain

$$F = C - P = e^{-R(T-t)}\{Se^{\mu(T-t)} - K\} \qquad \text{Eq.2.22}$$

The requirement that Equation 2.21 and Equation 2.22 be consistent dictates the appropriate average discount rate *R* in the options formula be the zero-coupon discount rate *R* and that the unknown expected growth rate μ for the stock be equated to the same value.

In this way the Black-Scholes options formula results from an assumed future volatility plus an expected- average-discounted-value approach to valuation, combined with the requirement that valuation be consistent with the static replication of forward contracts by portfolios of options.

A similar interpolation argument can be used to derive the values of more complex derivatives (quantos for example) that are dependent on a larger number of underlyers by requiring consistency with the values of all tradeable forwards contracts on those underlyers.

We now proceed to examining the behavior of the profit and loss of dynamic hedging strategies.

2.4 The P&L of Hedged Trading Strategies

Consider an initial position at time t_0 in an option C that is Δ-hedged with borrowed money which earns interest r, and then rehedged using in discrete steps at times t_i and corresponding stock prices S_i. We use the notation $C_n = C(S_n, t_n)$ $\Delta_n = \Delta(S_n, t_n)$.

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares Traded	Net Value of Position: Option + Stock + Cash
t_0, S_0	Buy C_0, short Δ_0 shares	$-\Delta_0$	$-\Delta_0 S_0$	$\Delta_0 S_0$	C_0
t_1, S_1	none	$-\Delta_0$	$-\Delta_0 S_1$	$\Delta_0 S_0 e^{r\Delta t}$	$C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\Delta t}$
	get short Δ_1 shares by shorting $\Delta_1 - \Delta_0$ shares	$-\Delta_1$	$-\Delta_1 S_1$	$\Delta_0 S_0 e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1$	$C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1$
t_2, S_2	none	$-\Delta_1$	$-\Delta_1 S_2$	$\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$	$C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$
t_2, S_2	get short Δ_2 shares by shorting $\Delta_2 - \Delta_1$ shares	$-\Delta_2$	$-\Delta_2 S_2$	$\Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$	$C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2$
etc.					
t_n, S_n	get short Δ_n shares by shorting $\Delta_n - \Delta_{n-1}$ shares	$-\Delta_n$	$-\Delta_n S_n$	$\Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\dots + (\Delta_n - \Delta_{n-1}) S_n$	$C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\Delta t}$ $\dots + (\Delta_n - \Delta_{n-1}) S_n$

Looking at the last line of the above table, you can see that the result of buying the initial call at a price C_0, shorting stock to hedge it and then investing the money in an interest-bearing account leads, after n steps of size Δt, where $T - t = n\Delta t$, to a final fair value

$$C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$$

where the subscript *b* at the end of the formula denotes a backwards Ito integral[1], and the initial value of the hedged portfolio was C_0. This amount could have been invested at the riskless interest and would have grown to $C_0 e^{r(T-t)}$. Therefore, the fair value of C_0 is given by equation these two quantities, so that

$$e^{r(T-t)} C_0 = C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b \qquad \text{Eq.2.23}$$

You can write this more transparently as

$$(C_0 - \Delta_0 S_0) e^{r(T-t)} = (C_T - \Delta_T S_T) + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b$$

Finally, you can integrate Equation 2.23 by parts using the relation

$$e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b = d[e^{r(T-\tau)} S_\tau \Delta_\tau] + r e^{r(T-\tau)} \Delta_\tau S_\tau d\tau - e^{r(T-\tau)} \Delta_\tau dS_\tau$$

to obtain

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[dS_\tau - S_\tau r d\tau] e^{-r(T-\tau)} \qquad \text{Eq.2.24}$$

Eq.2.23 and Eq.2.24 provide a way to calculate the value of the call in terms of its final payoff and the hedging strategy.

If you hedge perfectly and continuously, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration.

1. I'm grateful to Peter Carr for pointing out the backward nature of the integral to me.

2.5 The Effect of Different Hedging Strategies[1]

In the previous section, we hedge at each intermediate time between inception and expiration by shorting Δ_i shares of stock. How should we have calculated Δ_i – using implied volatility or realized volatility? How do the return profiles depend on the hedging strategy?

Realized (or actual) volatility is noisy, changing from moment to moment. Implied volatility is a parameter, the market's expected volatility plus some premium for other unknowns (hedging costs, inability to hedge perfectly, uncertainty of future volatility, the chance to make a profit, etc.). Implied volatility is usually greater than realized volatility.

2.5.1 Hedging with Realized (Known) Volatility

Consider the idealized case where we *know* that the future realized volatility σ will be greater than current implied volatility Σ. How can we make money as an options trader? We buy the option at its implied volatility and then hedge it at the realized volatility (which we hypothetically know) in order to replicate the option perfectly. Then the final P&L of this traded is

$$V(S, \tau, \sigma) - V(S, \tau, \Sigma)$$

where $V(S, \tau, \sigma)$ is the value obtained by perfectly replicating the option at the actual volatility, τ is the time to expiration, and for brevity we have suppressed displaying the dependence of non-essential variables such as interest rates and dividend yields. We will sometimes write $V(S, \tau, \sigma)$ as V_r (for realized) and $V(S, \tau, \Sigma)$ as V_i (for implied).

How Is This Known Profit Realized As The Stock Evolves Through Time?

Assume the stock evolves with drift μ and volatility σ, so that

$$dS = \mu S dt + \sigma S dZ. \qquad \text{Eq.2.25}$$

where μ is *not* the riskless rate *r*, and the stock S pays a continuous dividend yield *D*.

The Black-Scholes hedge ratio for a realized volatility σ_r is given by

1. Ahmad, R. and Paul Wilmott, "Which Free Lunch Would You Like Today, Sir?: Delta hedging, volatility arbitrage and optimal positions". Wilmott Magazine.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

$$d_{1,2} = \frac{\ln \frac{S_F}{K} \pm \frac{\sigma_r^2 (T-t)}{2}}{\sigma \sqrt{T-t}}$$

$$\Delta = N(d_1)$$

Here $S_F = S\exp[(r-D)t]$ is the forward price of the stock, and depends on the riskless interest rate r and the dividend yield D rather than μ.

In the table below we use the symbol $\vec{V}$ for a security with value V. Let's now figure out our moment-to-moment P&L over time when we borrow money to finance a long position in the security $\vec{V}$ and hedge it using the (assumed known) realized volatility, and show that we eventually capture $V_r - V_i$.

Table 1: Position Values when Hedging with Realized Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i$, V_i	$-\Delta_r\vec{S}$, $-\Delta_r S$	$\Delta_r S - V_i$	0
t + dt	$\vec{V}_i(t+dt, S+ds)$, $V_i + dV_i$	$-\Delta_r\vec{S}$, $-\Delta_r(S+dS)$	$(\Delta_r S - V_i)(1+rdt)$ $-\Delta_r DSdt$ dividends paid; interest received	$(V_i + dV_i - \Delta_r(S+dS))$ $(\Delta_r S - V_i)(1+rdt)$ $-\Delta_r DSdt$

Thus the change in the P&L in time dt is given by

$$\begin{aligned} d\text{P\&L} &= [V_i + dV_i - \Delta_r(S+dS)] + (\Delta_r S - V_i)(1+rdt) - \Delta_r DSdt \\ &= dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt \end{aligned} \qquad \text{Eq.2.26}$$

But the Black-Scholes equation is equivalent to the statement that if we had purchased the option at the fair future realized volatility σ_r, then the change in the value of the P&L while hedging with realized volatility would be zero, so that replacing the subscript i by r in Equation 2.26 gives

$$0 = dV_r - \Delta_r dS - rdt(V_r - \Delta_r S) - \Delta_r DSdt$$

which can be rewritten as

$$-\Delta_r dS - rdt(-\Delta_r S) - \Delta_r DSdt = rdtV_r - dV_r \qquad \text{Eq.2.27}$$

Substituting Equation 2.27 into the RHS of Equation 2.26 we obtain the P&L generated between times *t* and *t* + *dt*:

$$\begin{aligned} d\text{P\&L} &= dV_i - dV_r - rdt(V_i - V_r) \\ &= e^{rt} d[e^{-rt}(V_i - V_r)] \end{aligned}$$

The present value of this profit is obtained by discounting to the initial time t_0:

$$d\text{PV(P\&L)} = e^{-r(t-t_0)} e^{rt} d[e^{-rt}(V_i - V_r)] = e^{rt_0} d[e^{-rt}(V_i - V_r)]$$

By expressing the change in the P&L above in terms of total differential makes it easy to integrate over the total life of the option, which leads to

$$\begin{aligned} \text{PV(P\&L)} &= e^{rt_0} \int_{t_0}^{T} d[e^{-rt}(V_i - V_r)] \\ &= 0 - (V_i - V_r) = V_r - V_i \qquad \text{if T is expiration} \end{aligned} \qquad \text{Eq.2.28}$$

where the integrand at time T is zero because at expiration the payoff of the standard option is independent of volatility.

So we see that *the final P&L at the expiration of the option is known and deterministic* if we know the realized volatility, and is equal to the difference in value between the option valued at realized and implied volatility.

How is this known P&L realized over time? We show below that the P&L, while in sum total deterministic, has a stochastic component that vanishes only as we reach expiration. This is somewhat analogous to the value of bond, whose final payoff at expiration is known but whose present value varies with interest rates.

We showed in Equation 2.26 that the change in the P&L after hedging with implied volatility is given by

$$d\text{P\&L} = dV_i - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt$$

We can use Ito's Lemma to expand dV_i and the Black-Scholes equation to simplify the result. Applying Ito's Lemma to the above equation for the actual mark-to-market value of the P&L after a real move dS in time dt yields

$$\begin{aligned} d\text{P\&L} &= \left\{\Theta_i dt + \Delta_i dS + \frac{1}{2}\Gamma_i S^2\sigma^2 dt\right\} - \Delta_r dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt \\ &= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2\sigma^2\right\}dt + (\Delta_i - \Delta_r)dS - rdt(V_i - \Delta_r S) - \Delta_r DSdt \end{aligned} \qquad \text{Eq.2.29}$$

But the Black-Scholes equation for the option V valued at the implied volatility can be written as

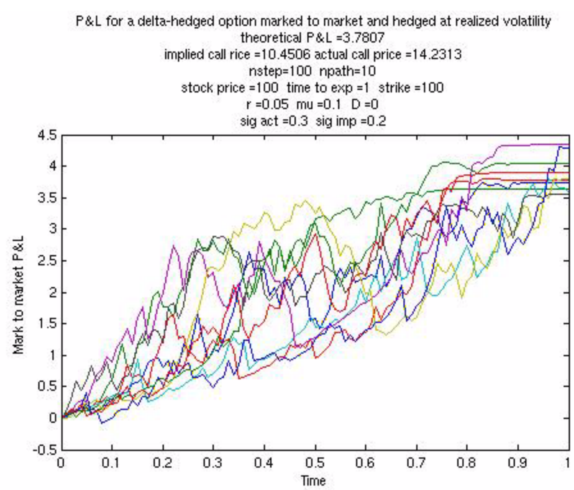
$$\Theta_i = -\frac{1}{2}\Gamma_i S^2\Sigma^2 + rV_i - (r-D)S\Delta_i$$

Substituting for Θ_i in Equation 2.29, we obtain

$$d\text{P\&L} = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt + (\Delta_i - \Delta_r)\{(\mu - r + D)Sdt + \sigma SdZ\} \qquad \text{Eq.2.30}$$

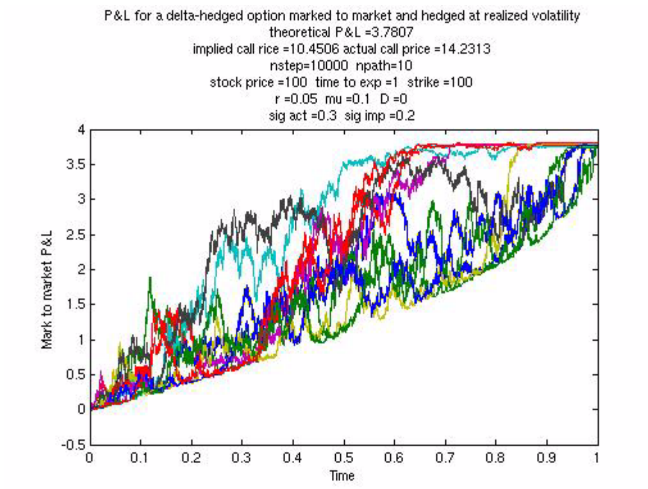
Thus, even though the total integrated P&L is deterministic, the increments in the P&L when you hedge with random volatility have a random component dZ. (Note that this is the non-discounted P&L, not its present value obtained by discounting each increment to the P&L by the appropriate discount factor. The total present value of the P&L should equal the difference in price between the option valued at implied volatility and valued at realized volatility.

To illustrate this, here is a plot of the cumulative P& along ten random stock paths, each generated with a realized volatility different from that of implied volatility. The accuracy with which the P&L converges to the known value depends on how continuous the hedging is, of course. Our example uses 100 hedging steps to expiration.



The final P&L is almost (but not quite) path-independent – almost, because 100 rehedgings per year is not quite the same continuous hedging.

Here is a similar plot of the cumulative P&L when we rehedge 10,000 times, i.e. almost continuously; then the final P&L is virtually independent of the stock path.



2.5.2 Hedging with Implied Volatility

When you hedge with implied volatility, the evolution of the P&L has no random *dZ* component, as we showed in our derivation of the hedged option's P&L in Section 2.1.2. In this case the final value of the P&L depends on the path taken, and is not deterministic.

Suppose we buy the option at the implied volatility, hedge it at implied volatility, and any cash received in the bank to earn the riskless rate. Let's now figure out our moment-to-moment P&L. We denote a security with price V by the symbol $\vec{V}$.

Table 2: Position Values when Hedging with Implied Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_i\vec{S}$	$\Delta_i S - V_i$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_r\vec{S}, -\Delta_i(S + dS)$	$(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$	$(V_i + dV_i - \Delta_i(S + dS))$ $(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$

The change in the P&L in time dt is given by

$$\begin{aligned} d\text{P\&L} &= [V_i + dV_i - \Delta_i(S + dS)] + (\Delta_i S - V_i)(1 + r dt) - \Delta_i D S dt \\ &= dV_i - \Delta_i dS - r(V_i - \Delta_i S)dt - \Delta_i D S dt \end{aligned} \qquad \text{Eq.2.31}$$

Using Ito's lemma for dV_i we obtain

$$\begin{aligned} d\text{P\&L} &= \Theta_i dt + \cancel{\Delta_i dS} + \frac{1}{2}\Gamma_i S^2 \sigma^2 dt - \cancel{\Delta_i dS} - r(V_i - \Delta_i S)dt - \Delta_i D S dt \\ &= \left\{ \Theta_i + \frac{1}{2}\Gamma_i S^2 \sigma^2 + (r - D)\Delta_i S - r V_i \right\} dt \end{aligned}$$

But the Black-Scholes equation for the option valued at implied volatility states that

$$\Theta_i + \frac{1}{2}\Gamma_i S^2 \Sigma^2 + (r - D)\Delta_i S - r V_i = 0$$

Substituting for Θ_i from the last equation into the previous one, we obtain

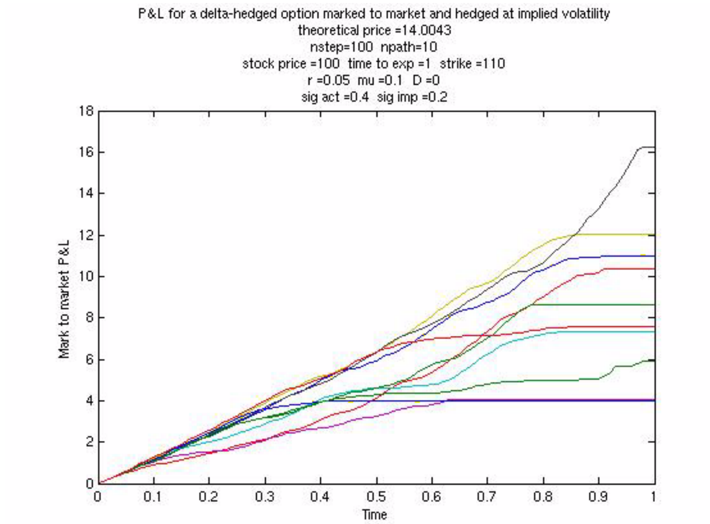
$$d\text{P\&L} = \frac{1}{2}\Gamma_i S^2 (\sigma^2 - \Sigma^2) dt \qquad \text{Eq.2.32}$$

The present value of this profit is obtained by discounting to the initial time t_0, and then integrating, we obtain

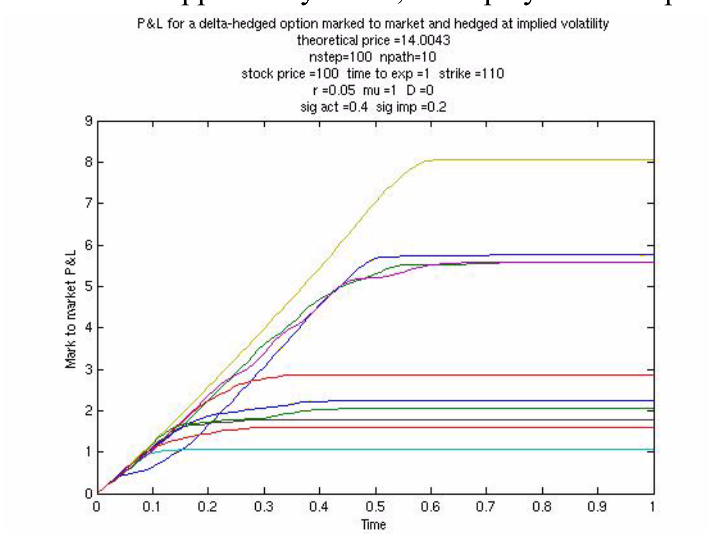
$$\text{P\&L} = \frac{1}{2}\int_{t_0}^{T} \Gamma_i S^2 (\sigma^2 - \Sigma^2) e^{-r(t - t_0)} dt \qquad \text{Eq.2.33}$$

The P&L depends upon the value of $\Gamma_i S^2$ along the path to expiration, and this factor, especially Γ_i which varies exponentially with $\ln S/K$, is highly path-dependent. Although the hedging strategy captures a value proportional to $(\sigma^2 - \Sigma^2)$, the coefficient will be close to zero if the option is far in or out of the money when the proportionality constant is small, and therefore the hedging strategy will be insensitive to volatility in those regions.

Below is a plot of the cumulative P&L along ten random stock paths generated with a realized volatility different from that of implied volatility. Because we hedge using implied volatility, the P&L depends upon the path taken. Our example uses 100 hedging steps to expiration and a stock drift of 10%.



Here is a similar example where the stock growth rate is much larger (100%); with this drift the stock price is much more likely to move out of the money, with the Γ-factor on average becoming much smaller. The average cumulative P&L captured is therefore appreciably lower, as displayed on the plot.



The above hedging strategies, using realized or implied volatilities, are of course somewhat idealized. In practice, realized volatility isn't know and keeps changing, and so you cannot hedge at the known realized volatility. A trading desk would most likely hedge at the constantly varying implied volatility which would move in synchronization with (but not be exactly equal to) the recent realized volatility. One could simulate this case too.

2.5.3 Hedging at an Arbitrary Constant Volatility

Suppose, for the more general case, we buy an option at an implied volatility Σ and hedge it to expiration at a volatility σ_h, the chosen hedge volatility, while realized volatility remains constant at σ_r. The present value of the P&L is thengiven by

$$PV(\text{P\&L}) = V_h - V_i + \frac{1}{2}\int_{t_0}^{t} e^{-r(t-t_0)}\Gamma_h S^2(\sigma_r^{\ 2} - \sigma_h^2)dt \qquad \text{Eq.2.34}$$

Note that in the limit that the hedge volatility σ_h is set equal to either the realized volatility σ_r or the implied volatility σ_i, Equation 2.34 reduces to our previous results.

2.5.4 .The Maximum P&L When Hedging With An Arbitrary Volatility

What can we deduce about the behavior of the P&L in Equation 2.34? The minimum is clearly $V_h - V_i$. The maximum occurs when the path-dependent term $S^2\Gamma_h$ is a maximum along the entire path. Now

$$S^2\Gamma_h = \frac{SN'(d_1)e^{-D\tau}}{\sigma_h\sqrt{\tau}} \qquad \text{Eq.2.35}$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

and d_1 depends on σ_h.

The maximum occurs when

$$\frac{\partial}{\partial S}(S^2\Gamma) = \frac{N'(d_1)e^{-D\tau}}{\sigma_h\sqrt{\tau}} - \frac{SN'(d_1)e^{-D\tau}}{\sigma_h\sqrt{\tau}}\frac{d_1}{S\sigma_h\sqrt{\tau}} = 0$$

or

$$\frac{1}{\sigma_h\sqrt{\tau}} = \frac{d_1}{\sigma_h{}^2\tau} \equiv \frac{\ln(S/K) + (r-D)\tau + \sigma_h{}^2\tau}{(\sigma_h{}^2\tau)(\sigma_h\sqrt{\tau})}$$

which has the solution

$$S = Ke^{-(r-D-\sigma_h^2/2)\tau} \qquad \text{Eq.2.36}$$

At this value of S, in Equation 2.35

$$S^2\Gamma_h = \frac{Ke^{-r\tau}}{\sigma_h\sqrt{\tau}\sqrt{2\pi}} \qquad \text{Eq.2.37}$$

Taking the path for the evolution of the stock that moves along this value of S, we maximize the P&L. By combining Equation 2.34 and Equation 2.37, we obtain

$$\begin{aligned} PV(\text{P\&L}) &= V_h - V_i + \frac{1}{2}(\sigma_r{}^2 - \sigma_h^2)\int_{t_0}^{T} e^{-r(t-t_0)}\frac{Ke^{-r(T-t)}}{\sigma_h\sqrt{T-t}\sqrt{2\pi}}dt \\ &= V_h - V_i + \frac{K(\sigma_r{}^2 - \sigma_h^2)e^{-r(T-t_0)}\sqrt{T-t_0}}{\sqrt{2\pi}\sigma_h} \end{aligned} \qquad \text{Eq.2.38}$$

2.5.5 Expected Profit after Hedging at Implied Volatility

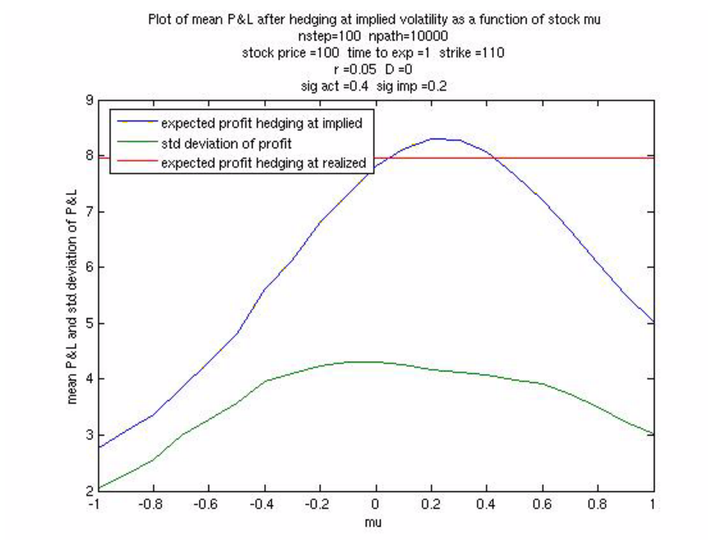
From Equation 2.33 we have the P&L when hedging at implied volatility:

$$\text{P\&L} = \frac{1}{2}\int_{t_0}^{T}\Gamma_i S^2(\sigma^2 - \Sigma^2)e^{-r(t-t_0)}dt$$

where we denote the realized volatility by σ.

This integral is path-dependent, so let’s find the mean P&L over all paths. It’s possible with some difficulty to take this average analytically over all stochastic paths for the stock price; this is similar to valuing a path-dependent option, for example an option on the average of the stock price. You can write down a PDE for it and try to solve it, as illustrated in Wilmott’s paper.

It’s easier, however, to write a Monte-Carlo program to evaluate the average P&L over all paths. Shown below is the expected P&L (blue line) over all paths for an option bought and hedged at an implied volatility of 0.2 when realized volatility is 0.4, for a range of different stock growth rates μ. The green line shows the standard deviation of the P&L. You can see, roughly, that the expected P&L is a maximum when the growth rate μ is such that the stock is close to at-the-money at expiration, so that its Γ is largest. From Equation 2.35 this occurs when $\mu = r - D - 0.5\sigma_i^2 = 0.05 - 0.5(0.2^2) = 0.03$, which corresponds closely the value of μ at the maximum in this illustration. The red line shows the P&L obtained by hedging at realized volatility, and is just equal to the difference between the price of the option at realized volatility and the price of the option at implied volatility. This value is fairly close to the maximum expected P&L when the option is hedged at implied volatility.

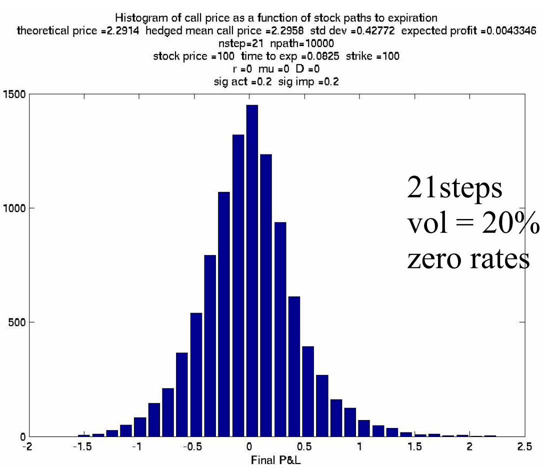


2.6 Hedging Errors from Discrete Hedging

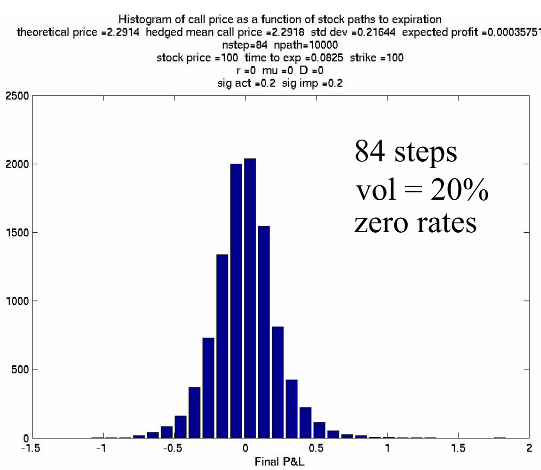
2.6.1 A Simulation Approach

In the real world you cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss hedging at regular time intervals.

Below we show the results of carrying out a Monte Carlo simulation, re-hedging or rebalancing to zero net delta at equally spaced intervals. Consider the case where the time to expiration is 1 month, the realized volatility is 20%, with the growth rate of the non-dividend-paying stock equal to the riskless interest rate, so that $\mu = r = 0.05$. Now let's examine an at-the- money option hedged at an implied volatility of 20% equal to the realized volatility.



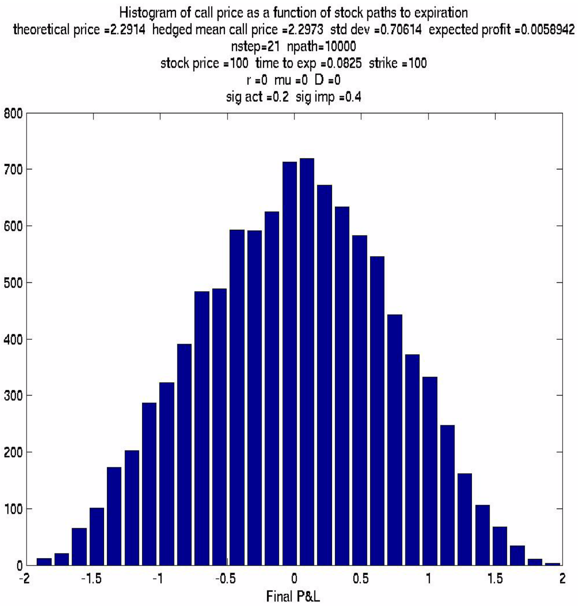
21 Rehedgings, Std. deviation. = 0.42

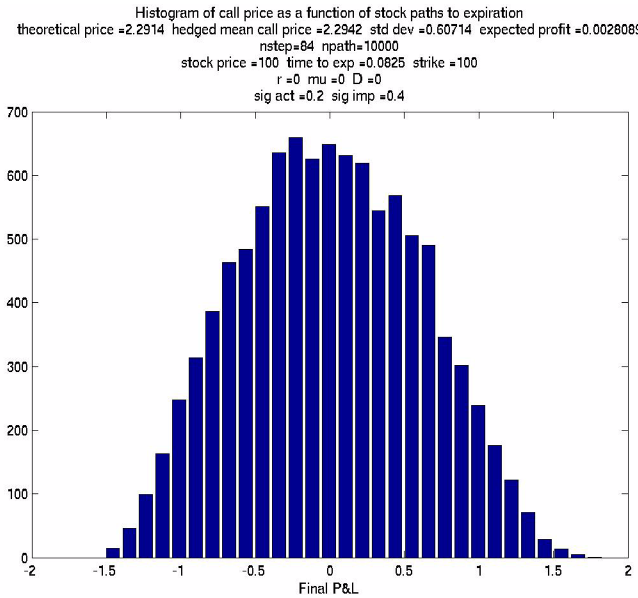


84 Rehedgings, Std. deviation. = 0.21

Note that the mean P&L is zero and that when we quadruple the number of hedgings, the standard deviation of the P&L halves. We will find the explanation for this a little later.

Now let's see what happens if the implied volatility differs from realized volatility. Choose an implied volatility of 40% as the hedging volatility, that is, as the volatility used to calculate the value of Δ.

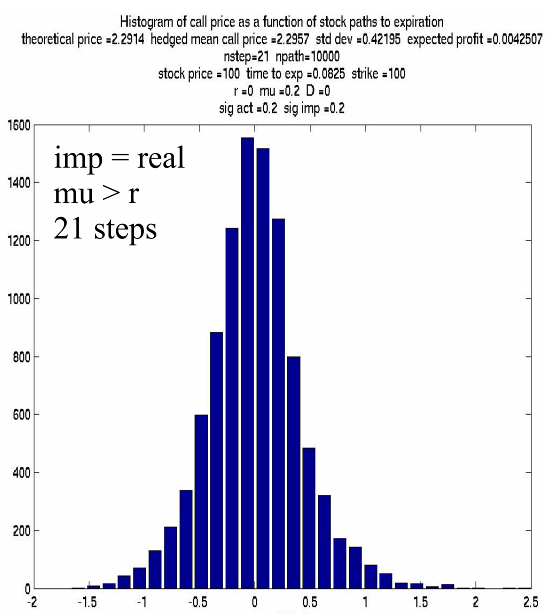


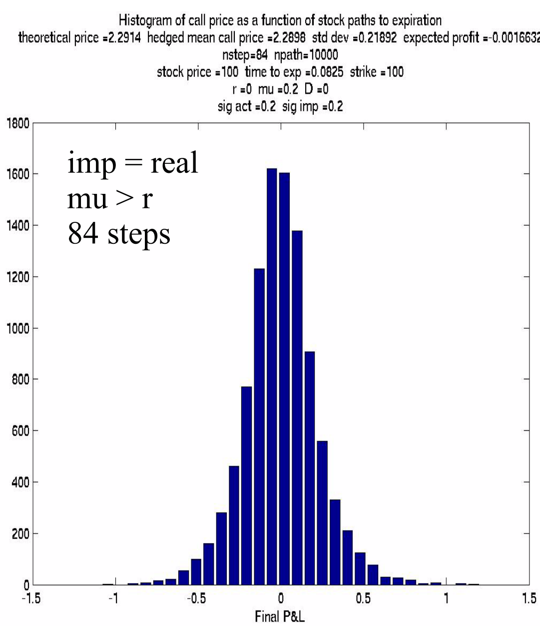


no reduction in variance with increasing rehedges unless hedge vol = realized vol

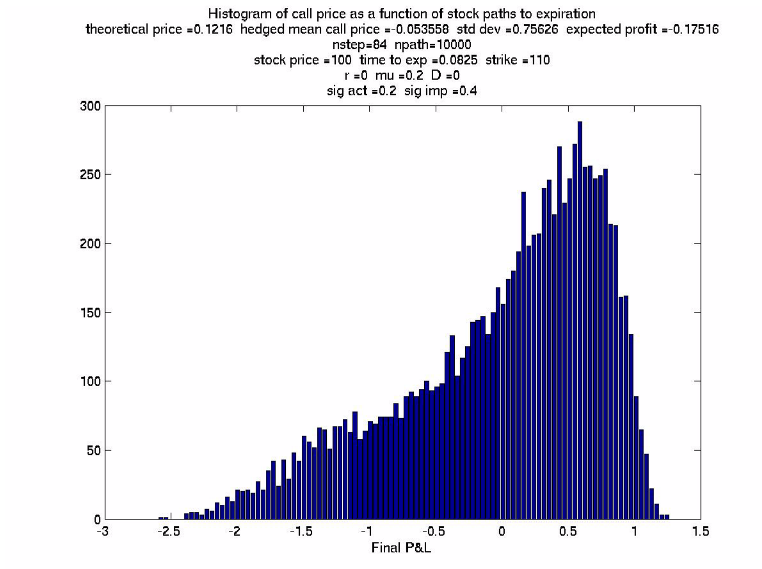
In the figure above we now see that the distribution looks similar, but there is no longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

Finally let's see what happens when the drift μ is not the same as the riskless rate, even though implied/hedging) and the realized volatility are both set equal to 0.2. Here we see that the standard deviation of the P&L still halves as the number of rehedgings doubles.





Finally, for completeness, we look at the where implied is not equal to realized volatility *and* $\mu \neq r$. In this case the distribution is very asymmetric.



2.6.2 Understanding Hedging Error Analytically

Here we assume that implied and realized volatility are identical.

Suppose in discrete time $\frac{\Delta S}{S} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}$

$$\varepsilon \in N(0, 1)$$

The delta-hedged option portfolio is given by

$$\pi = C - \left(\frac{\partial C}{\partial S}\right)S \qquad \text{Eq.2.39}$$

The hedging error accumulated over time Δt due to the mismatch between a continuous hedge ratio and discrete time step is given by

$$HE = \pi + d\pi - \pi e^{r\Delta t}$$

$$\approx \begin{array}{l} -r\Delta t\left[C - \left(\frac{\partial C}{\partial S}\right)S\right] + \left[C_t\Delta t + C_S dS + C_{SS}\frac{\sigma^2 S^2 \varepsilon^2}{2}\Delta t - C_S dS\right] \end{array}$$

$$\approx \left(C_t + C_{SS}\frac{\sigma^2 S^2 \varepsilon^2}{2} - r\left[C - \left(\frac{\partial C}{\partial S}\right)S\right]\right)\Delta t$$

Now from the Black-Scholes equation, the last term in the square brackets is given by

$$r\left[C - \left(\frac{\partial C}{\partial S}\right)S\right] = C_t + C_{SS}\frac{\sigma^2 S^2}{2}$$

and so the C_t term cancels, and for one step Δt

$$HE = \frac{1}{2}C_{SS}\sigma^2 S^2(\varepsilon^2 - 1)\Delta t \qquad \text{Eq.2.40}$$

Now for a normal variable $E(\varepsilon^2) = 1$ and so the expected value of the hedging error is zero, with a χ^2 distribution.

Over n steps to expiration, the total HE is

$$HE = \sum_{i=1}^{n}\frac{1}{2}\Gamma_i S_i^2\sigma_i^2(\varepsilon_i^2 - 1)\Delta t \qquad \text{Eq.2.41}$$

The variance of the hedging error can be approximately calculated and shown to be

$$\sigma_{HE}^2 = E\sum_{i=1}^{n}\frac{1}{2}[\Gamma_i S_i^2]^2(\sigma_i^2\Delta t)^2 \qquad \text{Eq.2.42}$$

One can show by integration that contingent on an initial stock price S_0 the expected value when the option is close to at the money is roughly given by

$$E[\Gamma_i S_i^2]^2 = S_0^4\Gamma_0^2\sqrt{\frac{T^2}{T^2-t_i^2}}$$

Thus for constant volatility

$$\begin{aligned}\sigma_{HE}^2 &= \frac{1}{2}\sum_{i=1}^{n} S_0^4\Gamma_0^2\sqrt{\frac{T^2}{T^2-t_i^2}}(\sigma^2\Delta t)^2 \\ &= (S_0^4\Gamma_0^2)(\sigma^2\Delta t)^2\frac{1}{2\Delta t}\int_t^T\sqrt{\frac{T^2}{T^2-\tau^2}}d\tau \\ &= (S_0^4\Gamma_0^2)(\sigma^2\Delta t)^2\frac{\pi T}{4\Delta t} \\ &= \frac{\pi}{4}n(S_0^2\Gamma_0\sigma^2\Delta t)^2\end{aligned}$$

Now $S_0^2\Gamma_0 = \frac{1}{\sigma(T-\mathrm{t})}\frac{\partial C}{\partial\sigma}$ from Black-Scholes, where $T-t$ is the time to expiration, so that we can write

$$\sigma_{HE}^2 = \frac{\pi}{4}n\left(\frac{1}{\sigma(T-t)}\frac{\partial C}{\partial\sigma}\sigma^2\Delta t\right)^2 = \frac{\pi}{4}n\left(\frac{1}{n}\frac{\partial C}{\partial\sigma}\sigma\right)^2 = \frac{\pi}{4n}\left(\frac{\partial C}{\partial\sigma}\sigma\right)^2$$

since $(T-t)/(\Delta t) = n$, so that

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial\sigma}\frac{\sigma}{\sqrt{n}} \qquad \text{Eq.2.43}$$

Thus, the hedging error is approximately $\frac{\partial C}{\partial\sigma}\times\frac{\sigma}{\sqrt{n}}$. What does this mean?

2.6.3 Understanding The Results Intuitively

Hedging discretely rather than continuously introduces uncertainty in the hedging outcome but does not bias the final profit/loss -- the expected value is zero.

Simple analytic rule for the standard deviation of P&L

$$\frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$$

For an option struck close to spot, there is a simpler version of the rule

$$\frac{\sigma_{\mathrm{P\&L}}}{\text{fair option value}} \sim \sqrt{\frac{\pi}{4n}}$$

Thus, approximately, quadrupling the number of hedges halves the hedging error, as we saw in the simulation results in the previous section.

The way to understand this formula is to realize that volatility itself, when measured or sampled discretely, is uncertain.

The standard deviation of a constant volatility σ measured discretely is $\frac{\sigma}{\sqrt{2N}}$

You can think of this as being due to statistical sampling error.

This is quite a large error, and here we have assumed we know the future volatility with certainty. Imagine the error when you don't even know future volatility and therefore your hedge ratio is incorrect not just because it is discontinuous, but because you don't know the appropriate volatility to use.