Equity Volatility Modeling: Progress, Challenges, and (Simple) Problems That Endure.

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Presentation to Prof. Derman's class
Columbia University
April 15, 2015

Themes

- Exotic options on the underlyer are less popular than they were in the late 90’s, early 00’s.
  But is the world simpler now?
  - Spreads on vanilla options are tight: more precision is required.
  - Volatility is much more of a true asset class: Simple payoffs, but subtle valuation. The introduction of new products creates interesting pricing and risk management linkages across vol. derivatives.
- Volatility is Fundamental : Pricing Models, Empirical Studies formulated entirely in Vol Terms.

Outline

- Two simple but subtle aspects of implied vol.:
    - Put/call parity
    - Fitting volatility surfaces: Not as simple as it sounds, even for global indices. Remains challenging and perhaps always will be!
        - The long wings of SPX
        - SX5E and its vanishing quotes.
- Key empirical insights for index vols and why they matter.
- Vol. Derivatives:
    - Variance, VIX futures and VIX options
    - Corridor variance swaps.

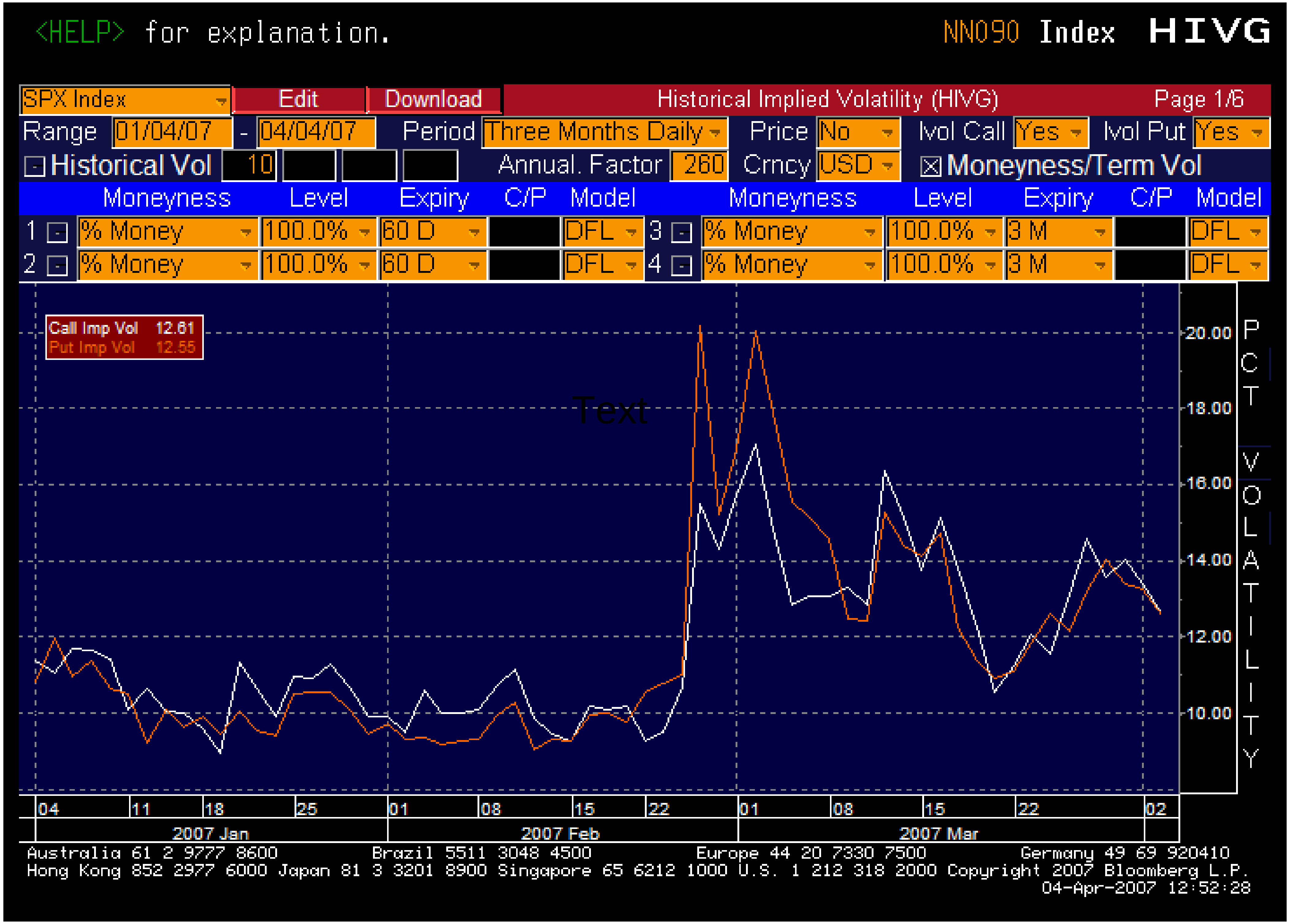
1. Simple but subtle: Put/Call Parity

Current (2015) Bloomberg Implied Vols SPX

ATM Call vol = put Vol

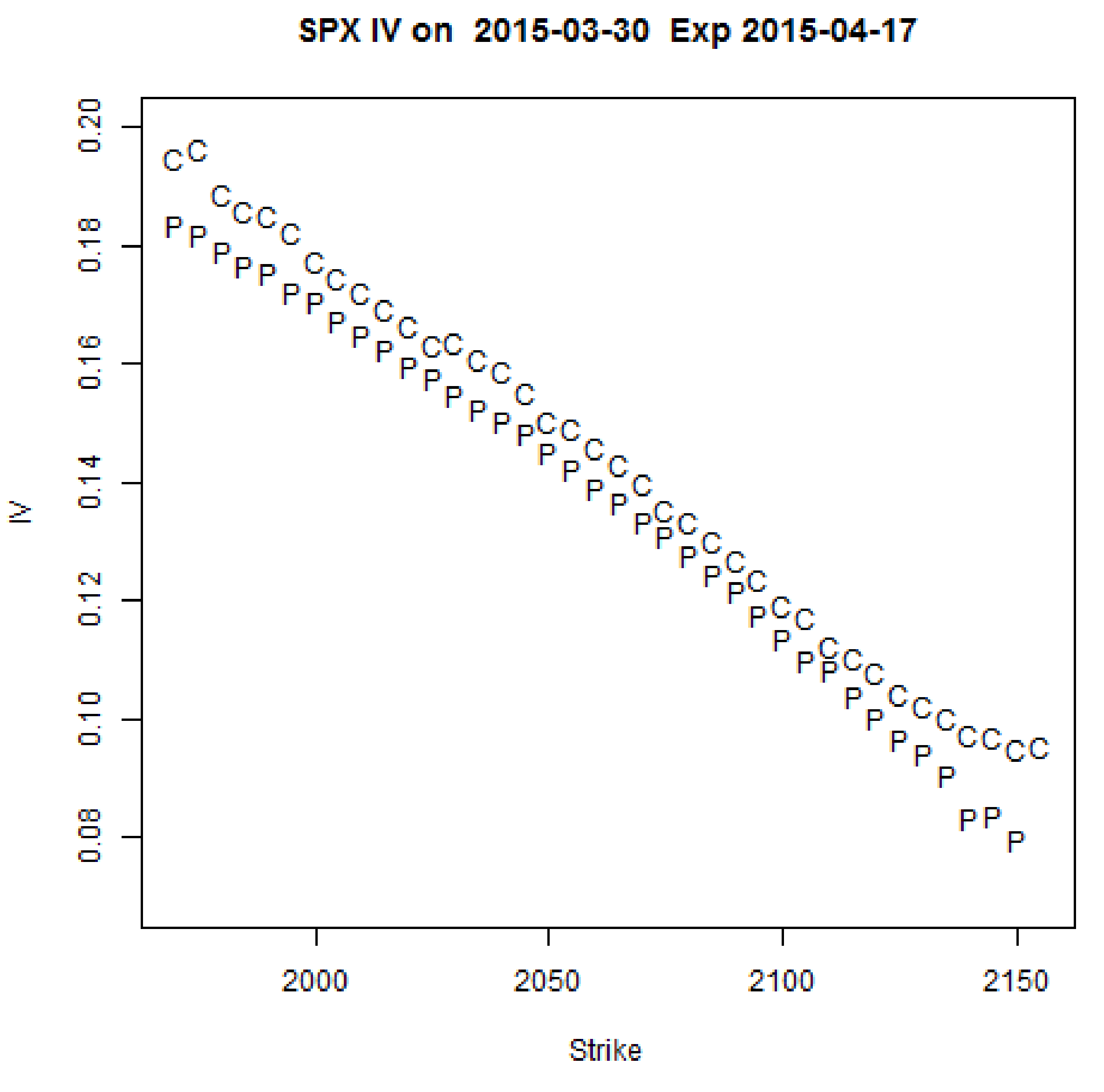
Old (2007) Bloomberg Implied Vols SPX

Call
Put



Why distinct vols for call/puts?

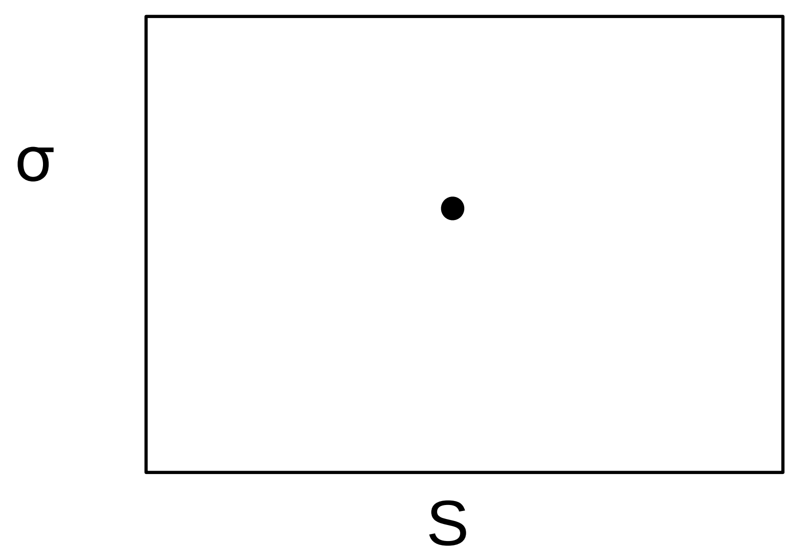
Current Vendor Implied Vols.



Same issue: Separate call/put vol surfaces

Implied Vols Derived From Prices

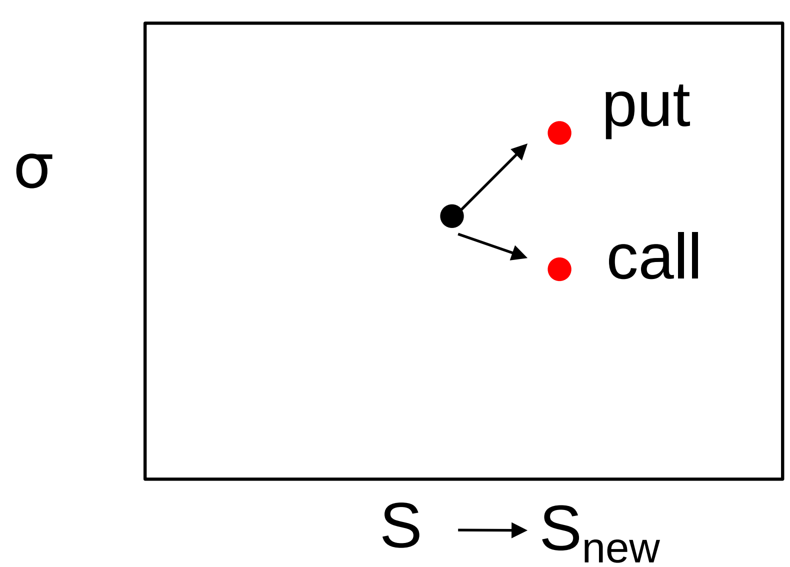
- Market price: $C = Cbs(S,K,T,r,y,\sigma)$



$\sigma_{call} = \sigma_{put}$

One way $\sigma_{call} \neq \sigma_{put}$

- Suppose spot increases but option prices are not updated
- $S \uparrow$ If vol is unchanged, expect : $C \uparrow$, $P \downarrow$ (BS formula)
- But if prices are not updated : $\sigma_{call} \downarrow$, $\sigma_{put} \uparrow$



But typically: Inconsistent Forward

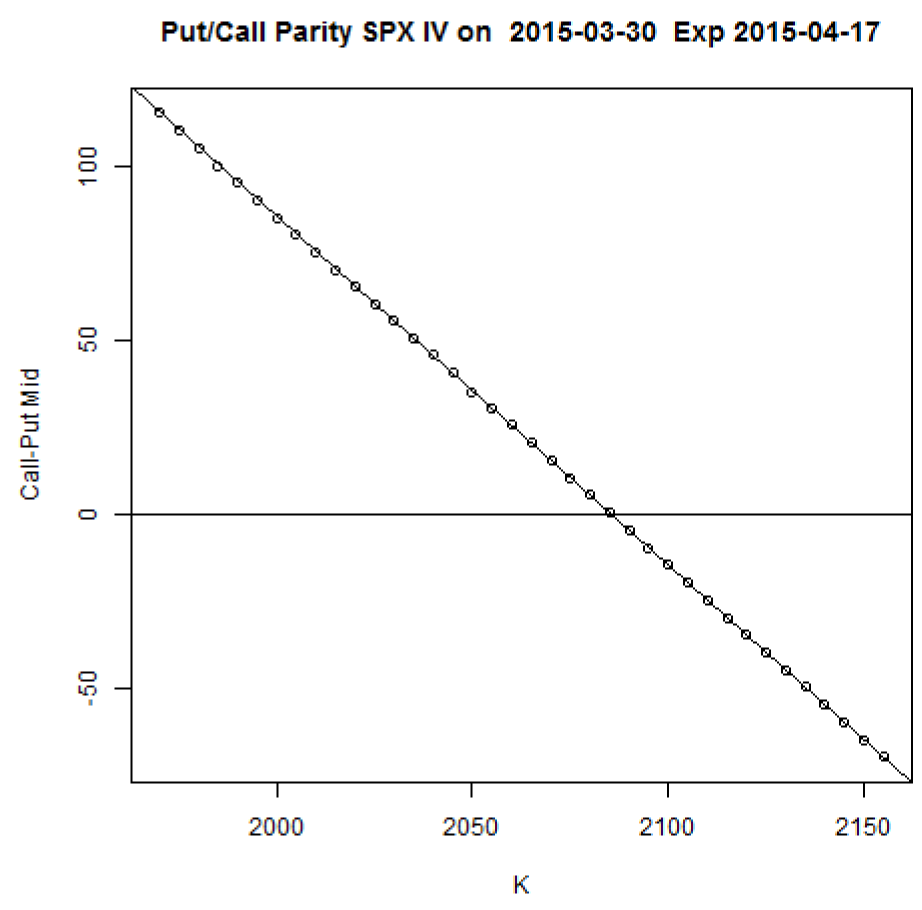
- Implied Forward (Put/Call parity)
- vs.
- Forward based on your opinion/reality of dividends, interest rates, borrow cost.

Broader lesson: Forwards, “delta-one” issues (funding, dividends) can’t be entirely decoupled from the calibration of the volatility surface.

Volatility “quants”, especially those involved in flow businesses, or options market making spend a lot of time worrying about these issues

(one of those enduring, simple problems!).

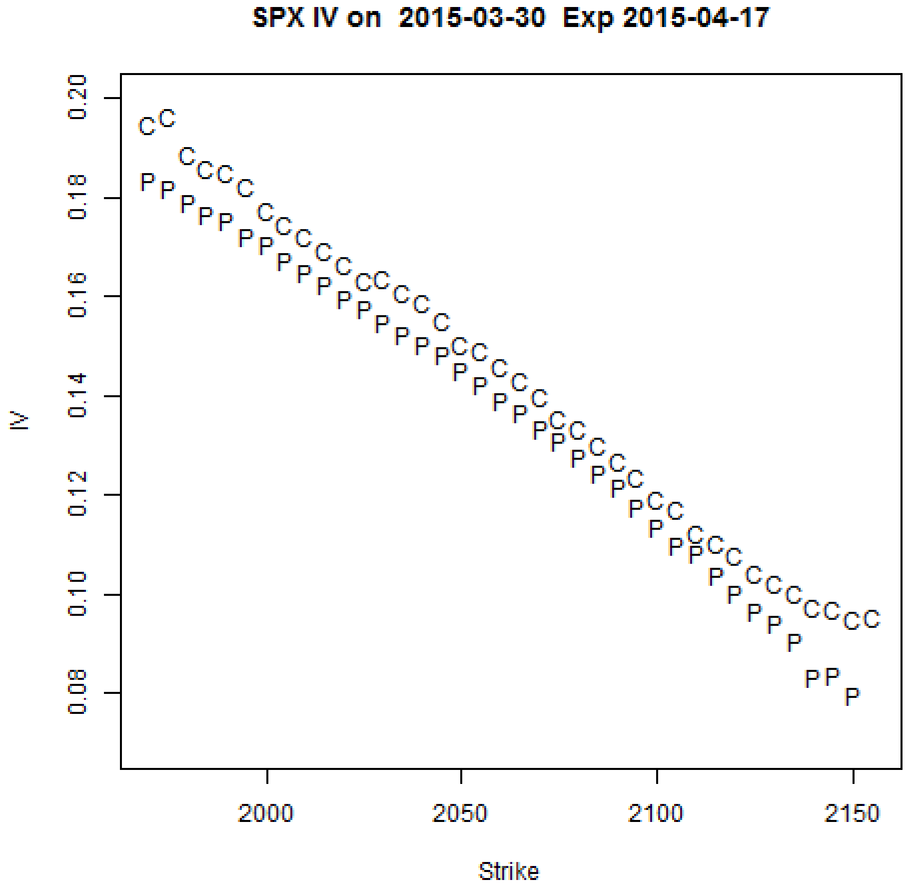
Put/Call Parity Graph
From Vendor Option Prices



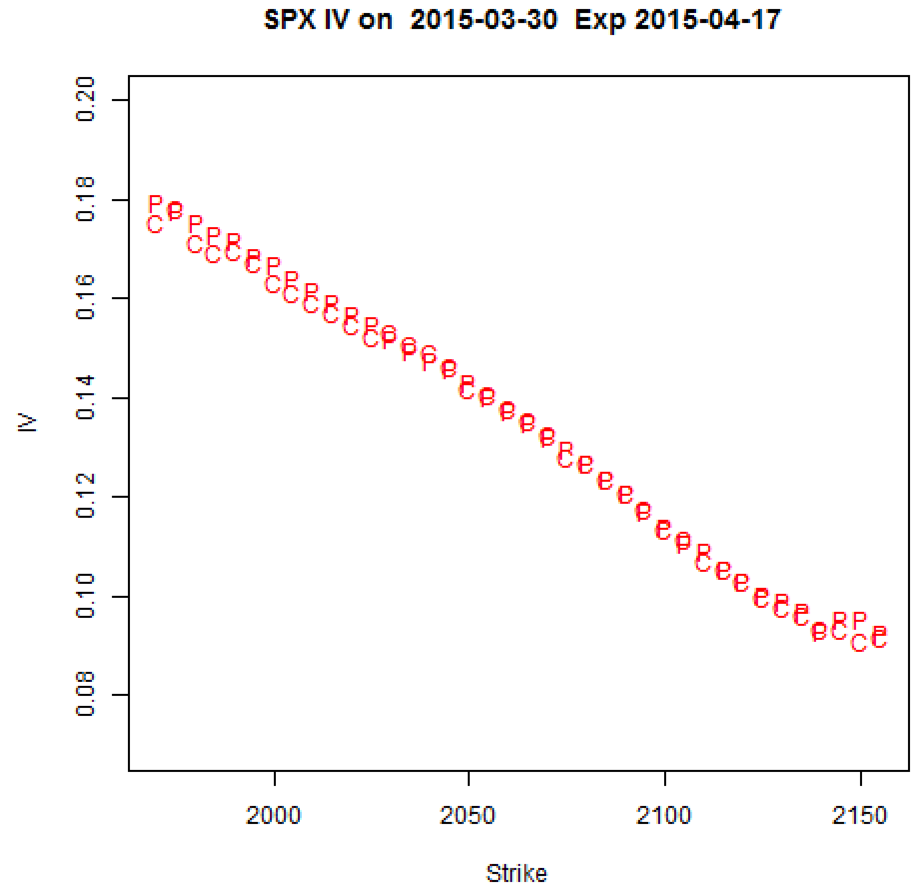
Graph of C-P vs Strike :
 Regression estimate of *implied* forward= 2085.63.
Quality of regression suggests options prices are consistent with a single, global forward value.

Corrected Vendor Implied Vols

Original vendor vols



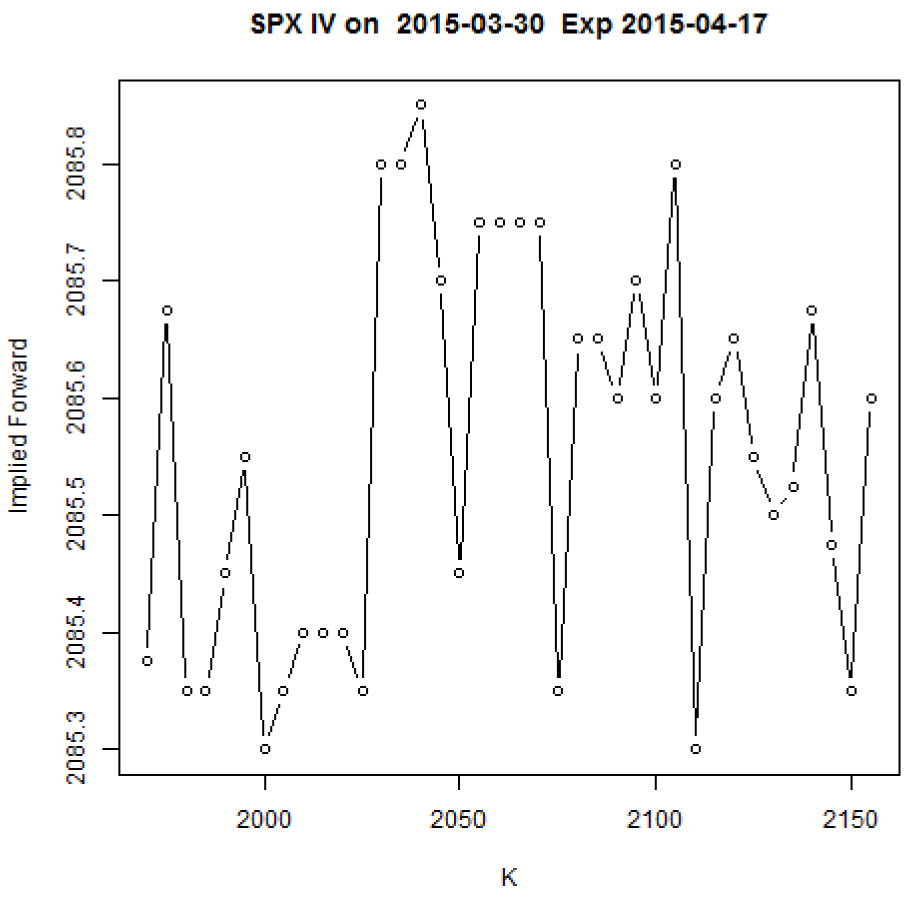
IV estimated using the implied forward



Call/put vols move closer together when the implied forward is used t o estimate

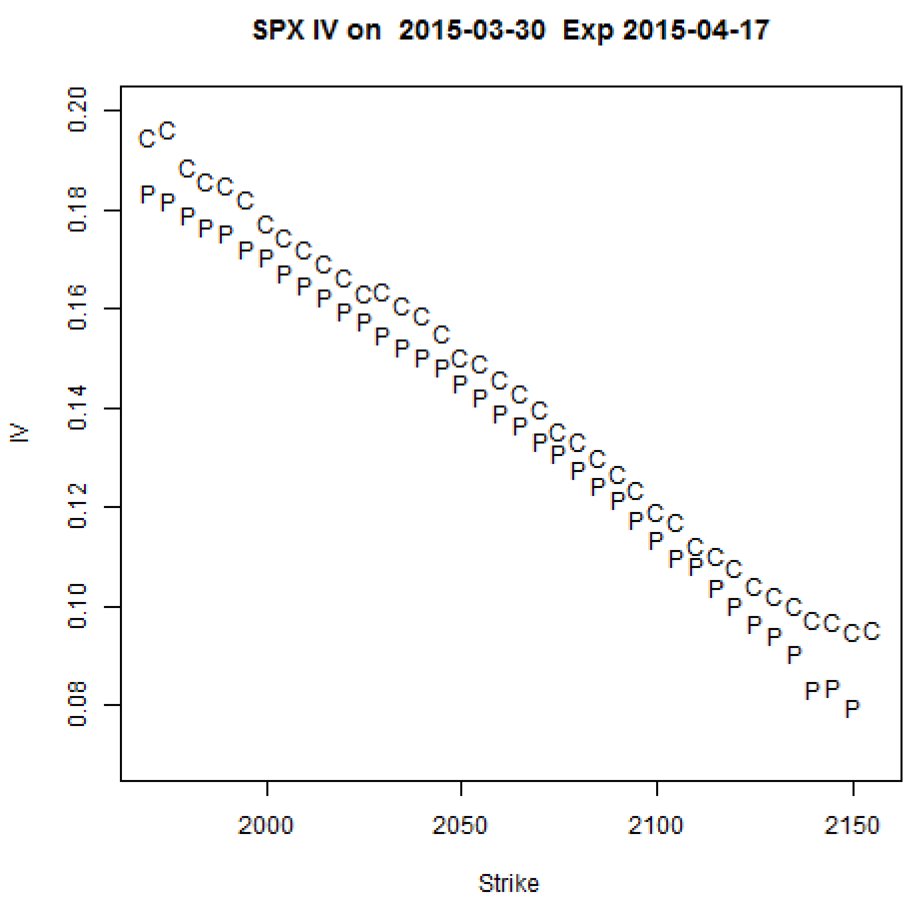
Implied Forward for Each Strike

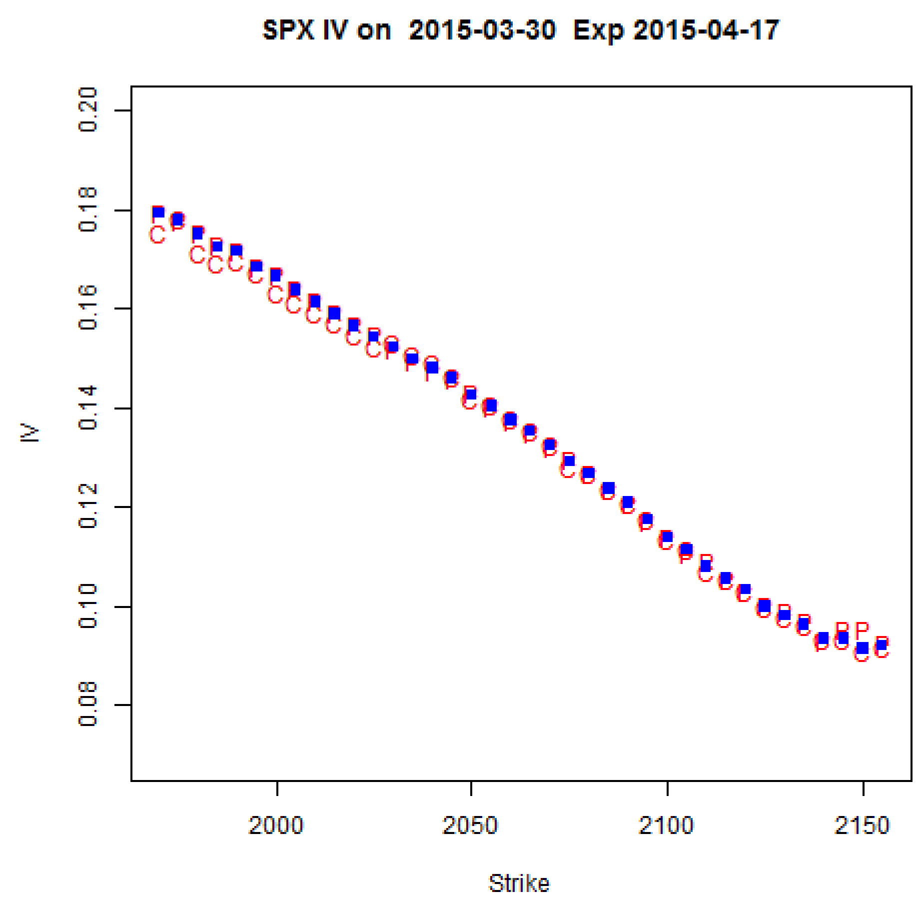
Alternative approach:
For each strike, imply a forward and implied vol from the two option prices



Range of variation is within the bid-ask of ATM option price

Corrected Vendor Implied Vols





Blue dots : IV estimated using implied forward for that strike

Simple approximation for correcting 1

- True spot is S, but you mistakenly assume spot is S_1.
- You infer call vol by matching call, put prices to BS formulas and discover different call,put vols because you used the wrong spot.

$$C = C_{BS}(S_1, K, \tau, \sigma_C)$$
$$P = P_{BS}(S_1, K, \tau, \sigma_P)$$

- Suppose that S_1 is not too far from S; C,P are in sync:

$$C \approx C_{BS}(S, K, \tau, \sigma) + \frac{\partial C}{\partial S}(S_1 - S) + \frac{\partial C}{\partial \sigma}(\sigma_C - \sigma)$$
$$P \approx P_{PS}(S, K, \tau, \sigma) + \frac{\partial P}{\partial S}(S_1 - S) + \frac{\partial P}{\partial \sigma}(\sigma_P - \sigma)$$

(The greeks should be evaluated at the unknown S, σ, we approximate by the greeks evaluated at S_1, σ_c, σ_p)

Simple approximation for correcting 2

- Simplest, intuitive correction:

$$\sigma = \frac{(\Delta_C \sigma_P - \Delta_P \sigma_C)}{\Delta_C - \Delta_P}$$

True vol is the delta-weighted average of the call and put vols, each weighted by the other's delta.

You need to assume same vega for put, call (not true when vols are different for calls, puts)

Comments:

-- You can derive a better correction keeping vega dependence.

-- This is sort of like applying numerical Newton-Raphson root finding

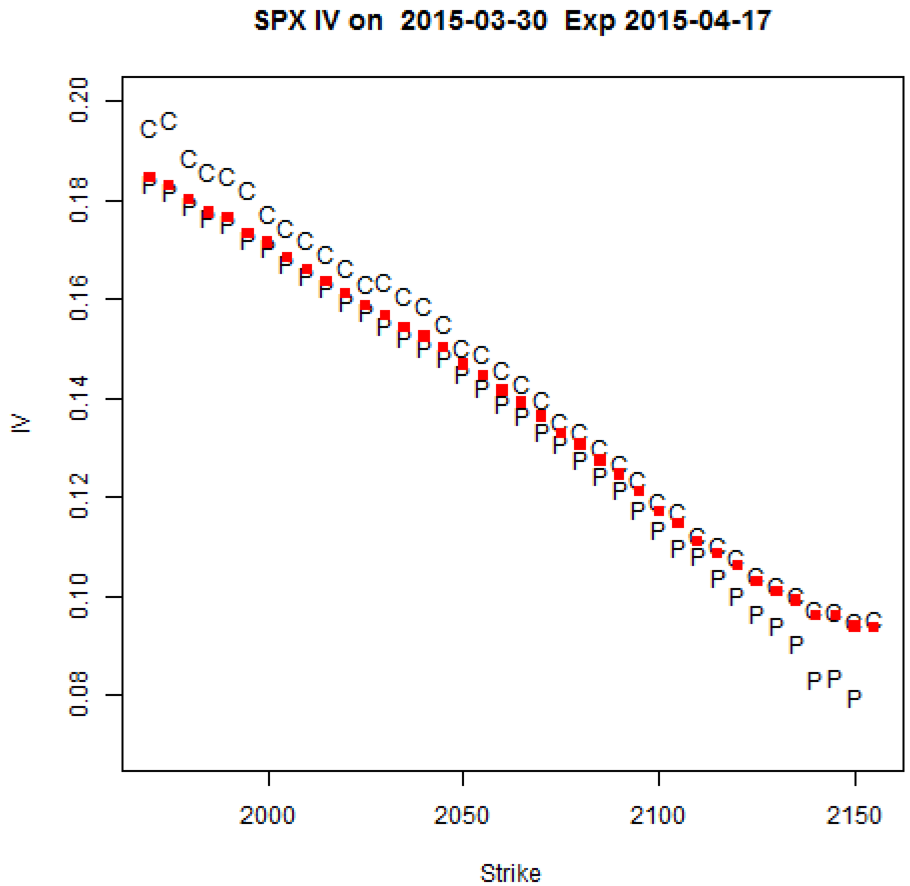
by hand and stopping at the first step

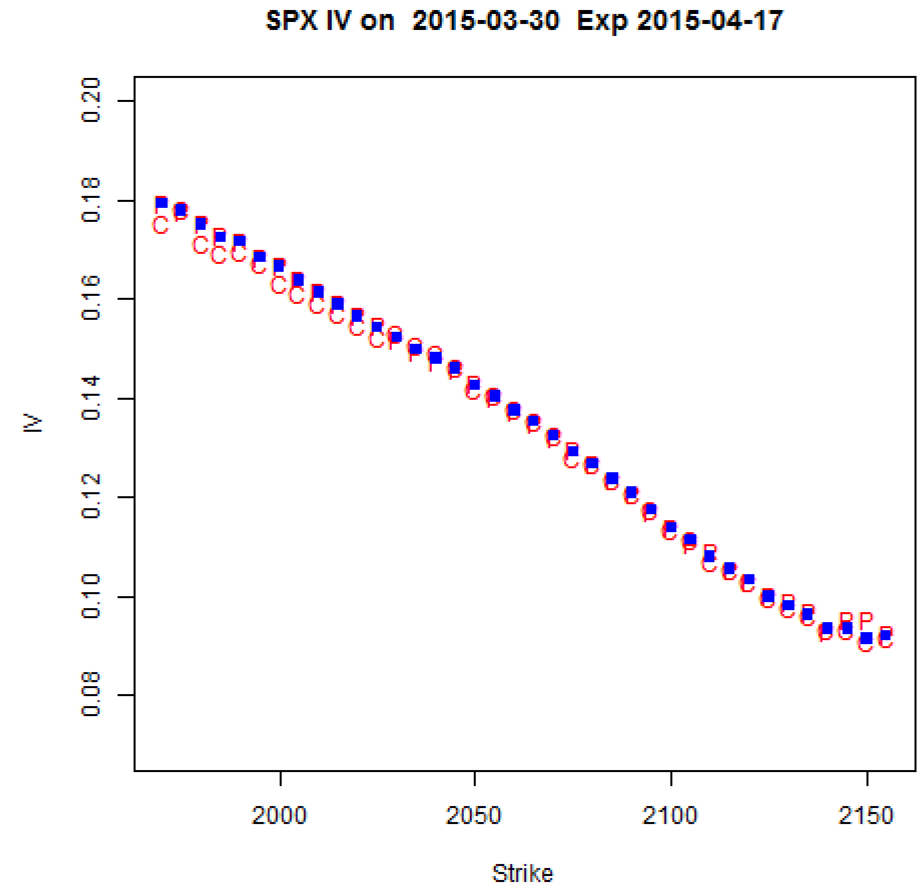
-- Insight: Put vols count more for very OTM puts, ITM calls.

Example of Vol Correction

- S=100, K=90, term=1.0 year , true vol=0.15
- TV call=12.02173, put= 2.021727
- You think Spot = 101, and infer σ_c=0.1199 (less), σ_p=0.1573.
- Using these vols and your spot, you find delta call=0.8466, delta put=-0.2085.
- You find a corrected vol of 0.1499 – closer to σp as it should be.

Blended Vendor vs Refit Vols





Red dots : delta-blended vendor IVs

Recommended Homework 1

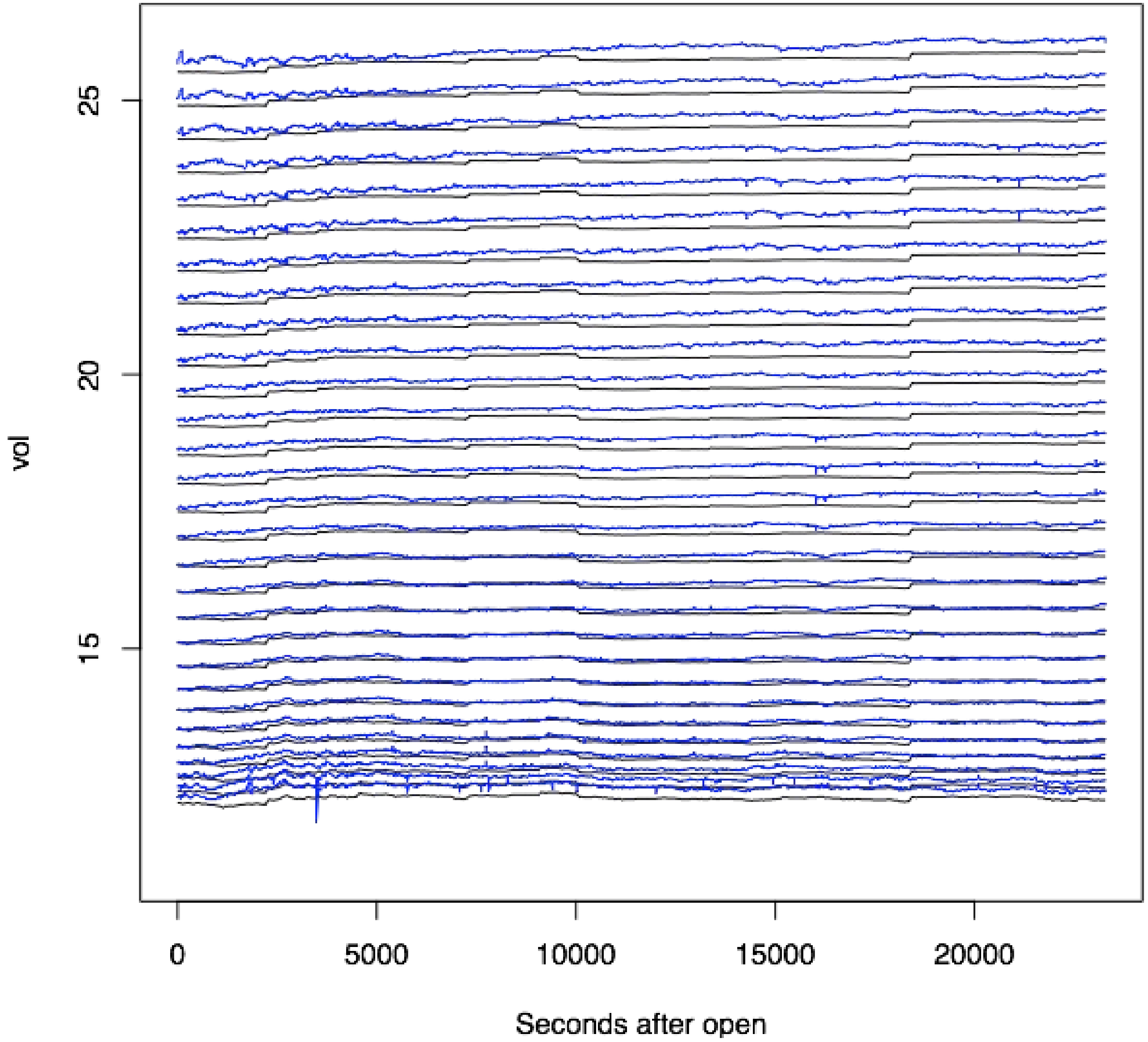
- Convince yourself that if put/call parity holds exactly for European option:
  $\sigma_{call} = \sigma_{put}$

- What does one learn from slope, intercept from P/C parity graph? How do you use that information to extract implied vols from option prices? In particular, how do you estimate the implied forward?
- We derived a formula for inferring the correct vol underlying stale quotes for an option pair. Derive the companion formula for the implied spot and show that it works using the same example.

Related (Deeper) Issues

- How often do you update vols ? Less frequently than option price itself, but how to decide
- How often do you update your implied forwards?
- Do you refit implied vol surface if just one option changes? update slice or surface?
- Is volatility even that useful? Can you work mostly with prices directly?

Digression –
Man vs Machine: Intraday Vols



Intraday time series:

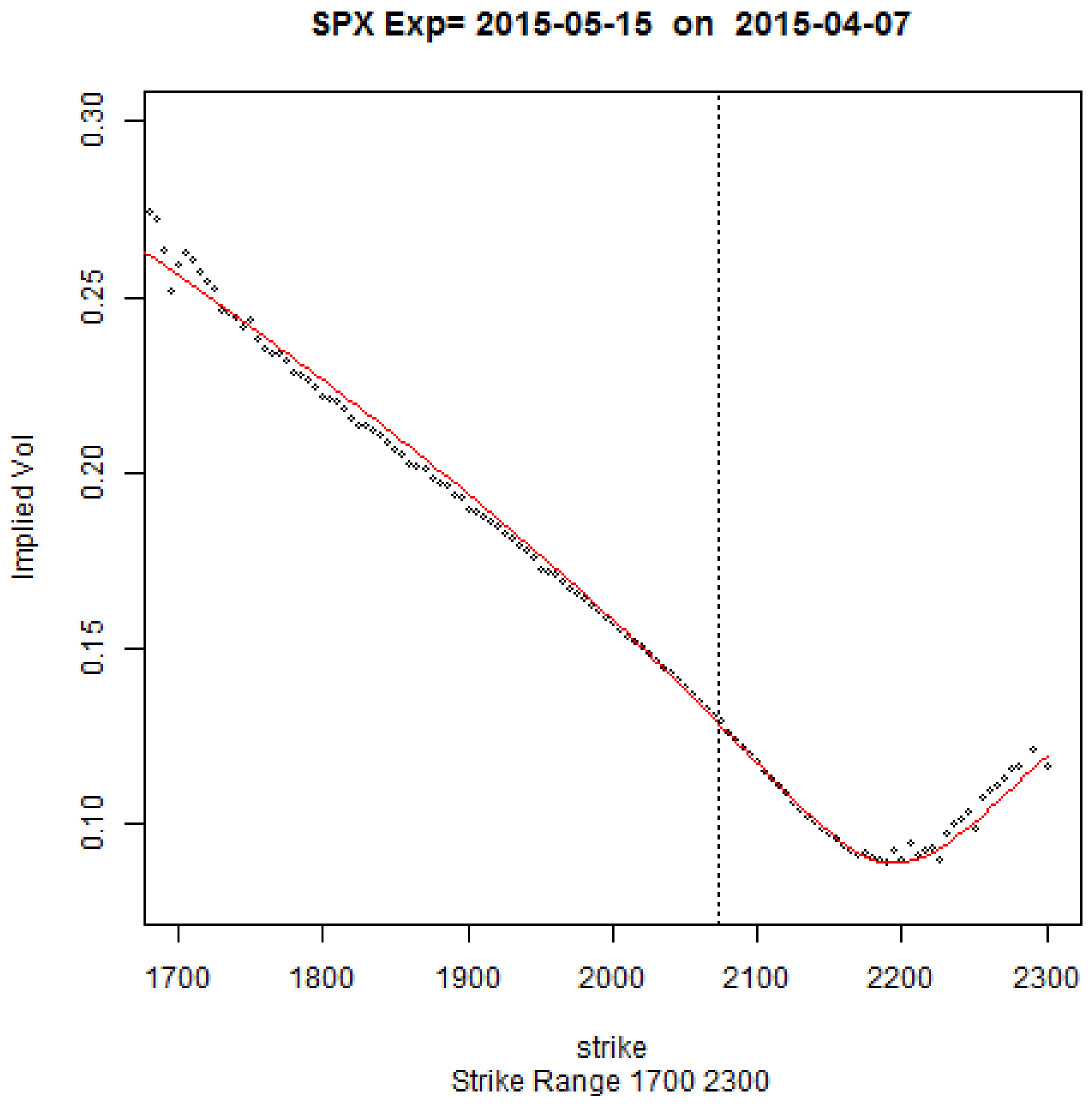
Automated Vols.

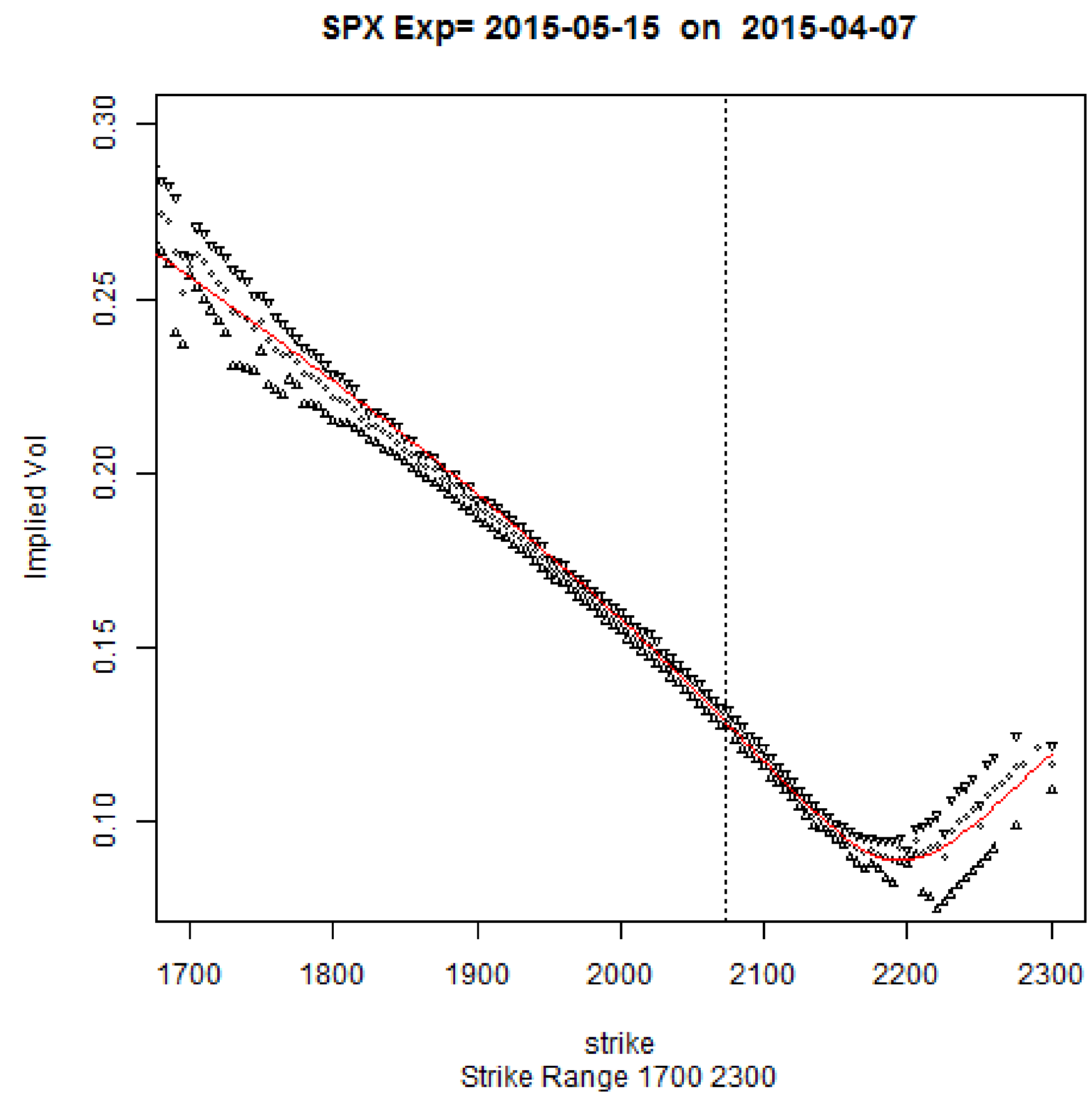
Trader Vols.

Each line is the vol for a traded option (fixed strike)

2. Fitting the Vol. Surface:
The Long Wings of SPX

Recent SPX Vols for May15 Fit By SVI



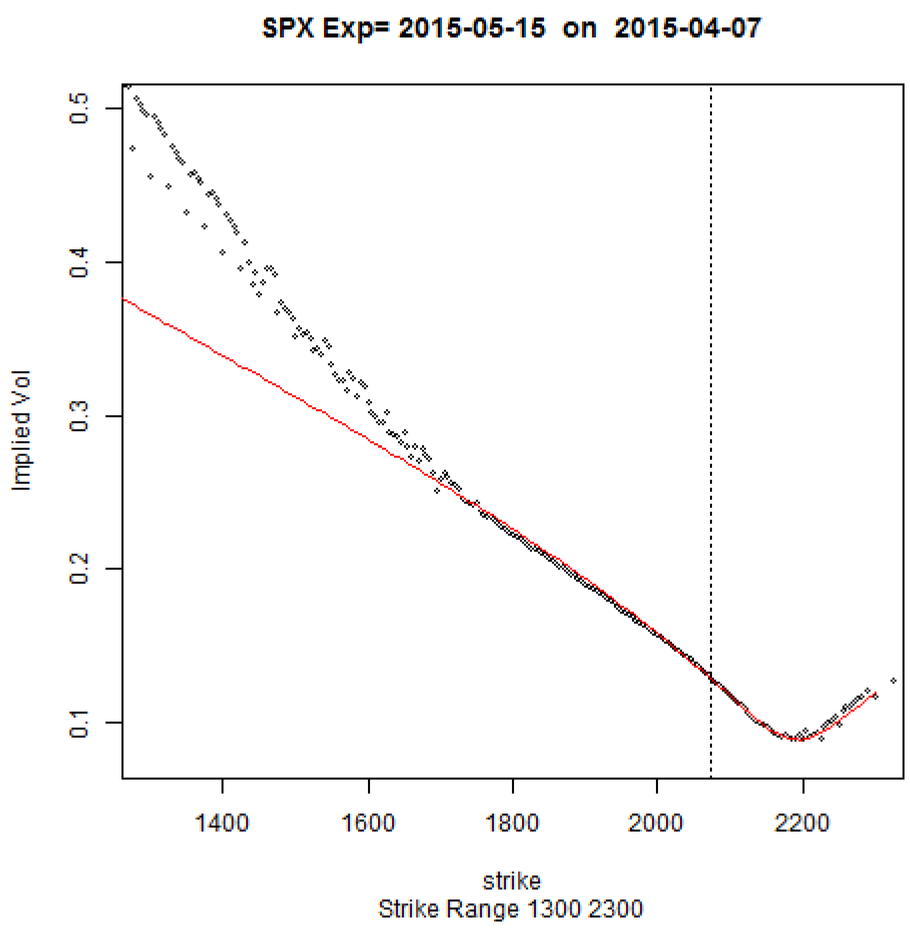


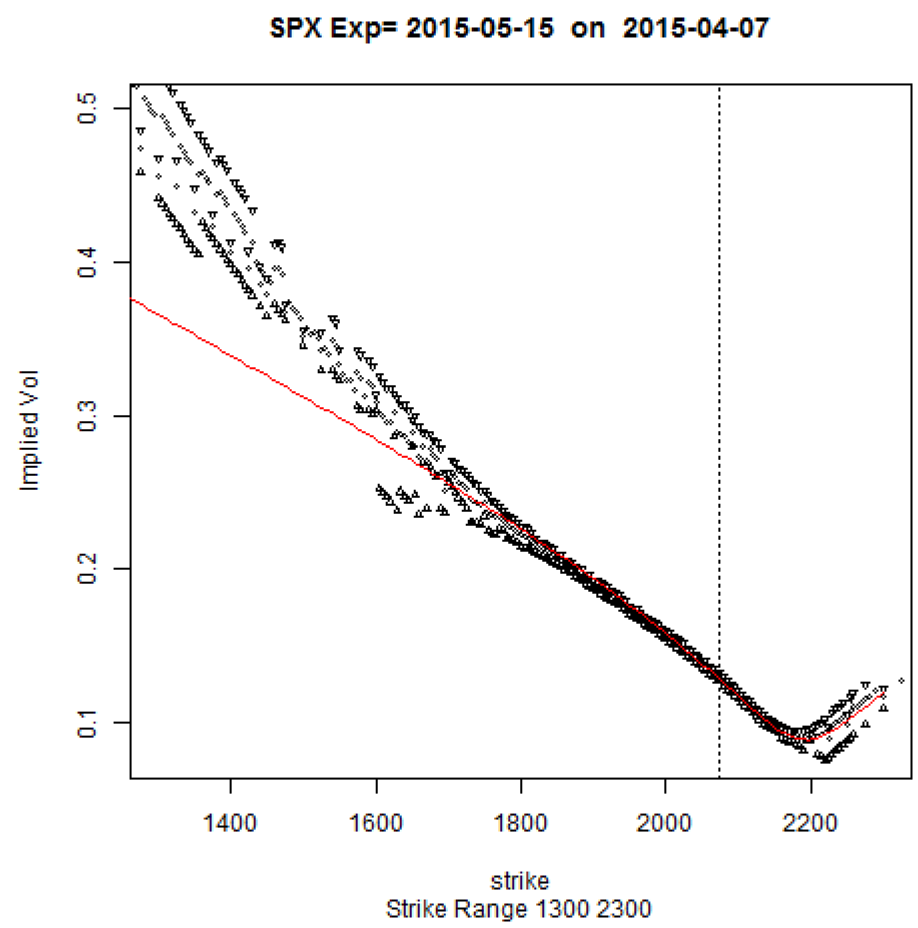
Points show option IV for mid price

Points show option IV for bid, mid and ask price

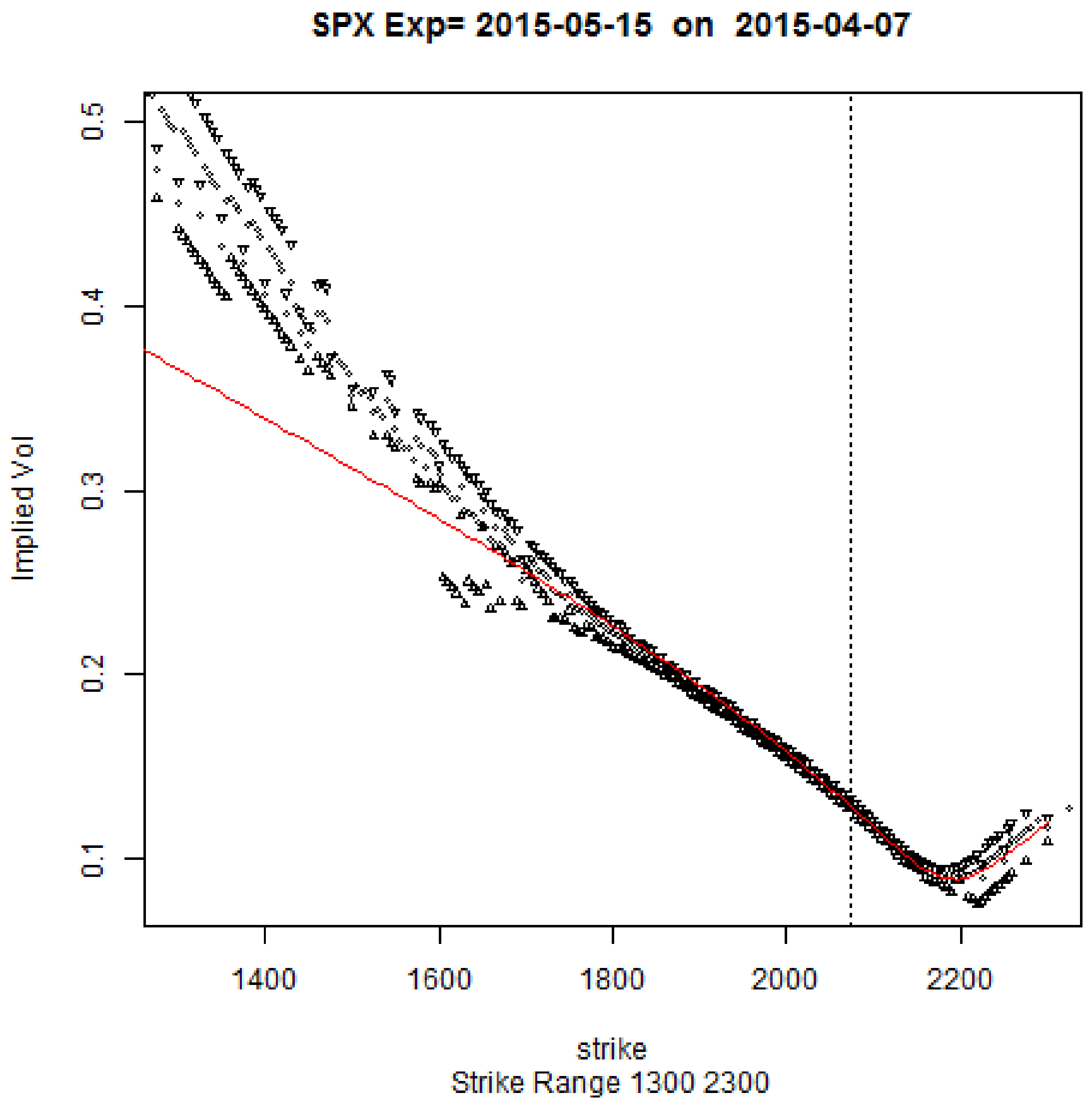
Great fit – SVI fit is within bid/ask, close to mid vols.

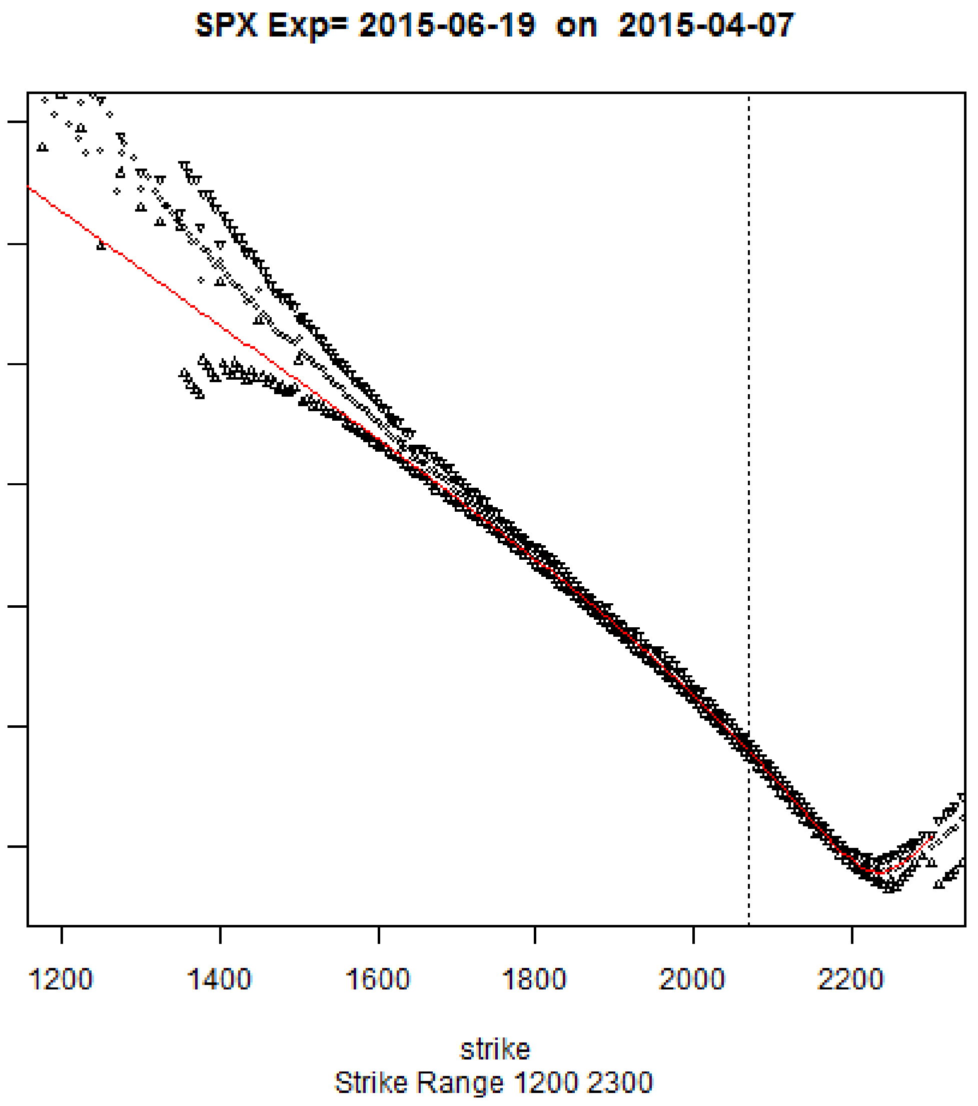
SVI Fit Is Worse For Very OTM Puts



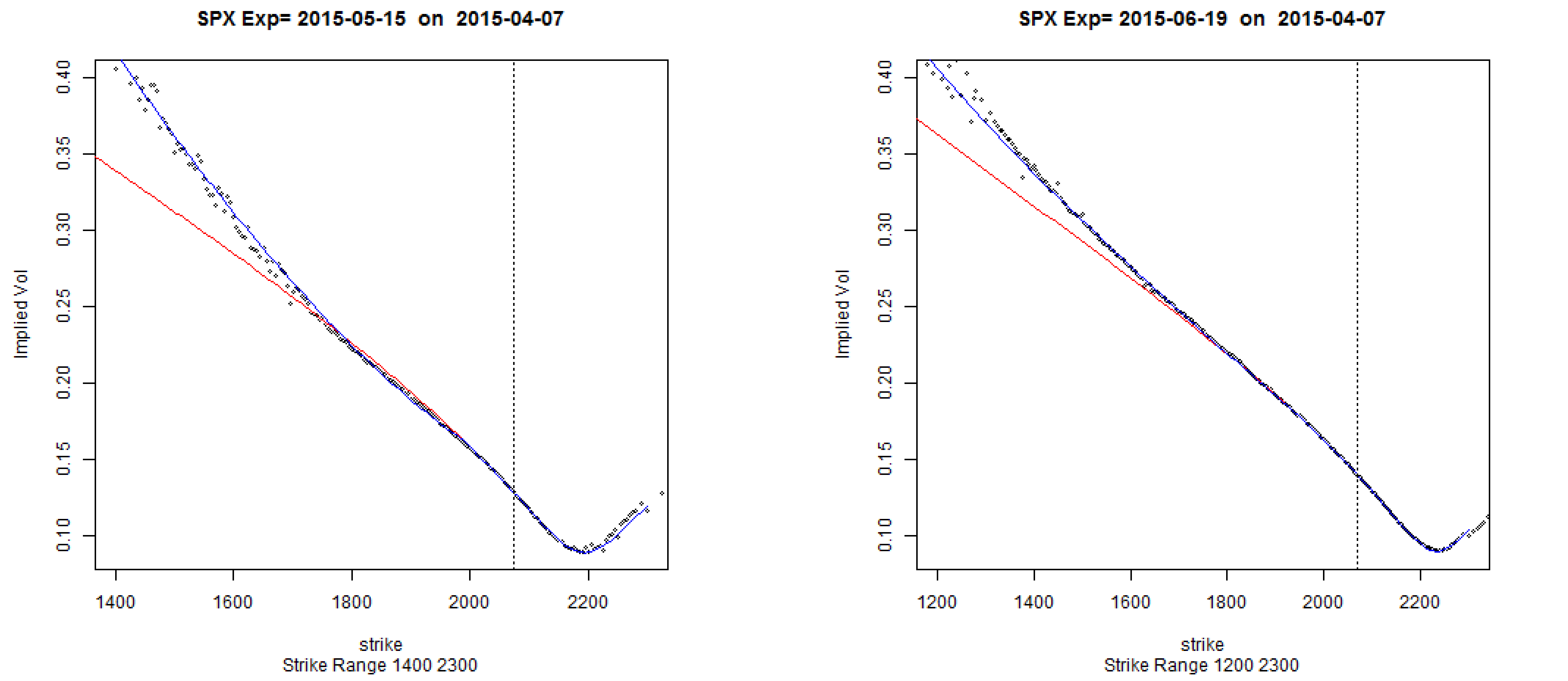


Fit Bias Is Less for Jun15 Exp.





Wing Model



WingModel fits the tails: Allows vols (not variance) to depend linearly on log (K/F). In contrast, SVI variance is linear in log(K/F) in the wings.

But theoretical fair variance is infinity!

Should we care?

Why does Theoretical Fair Var. Diverge?

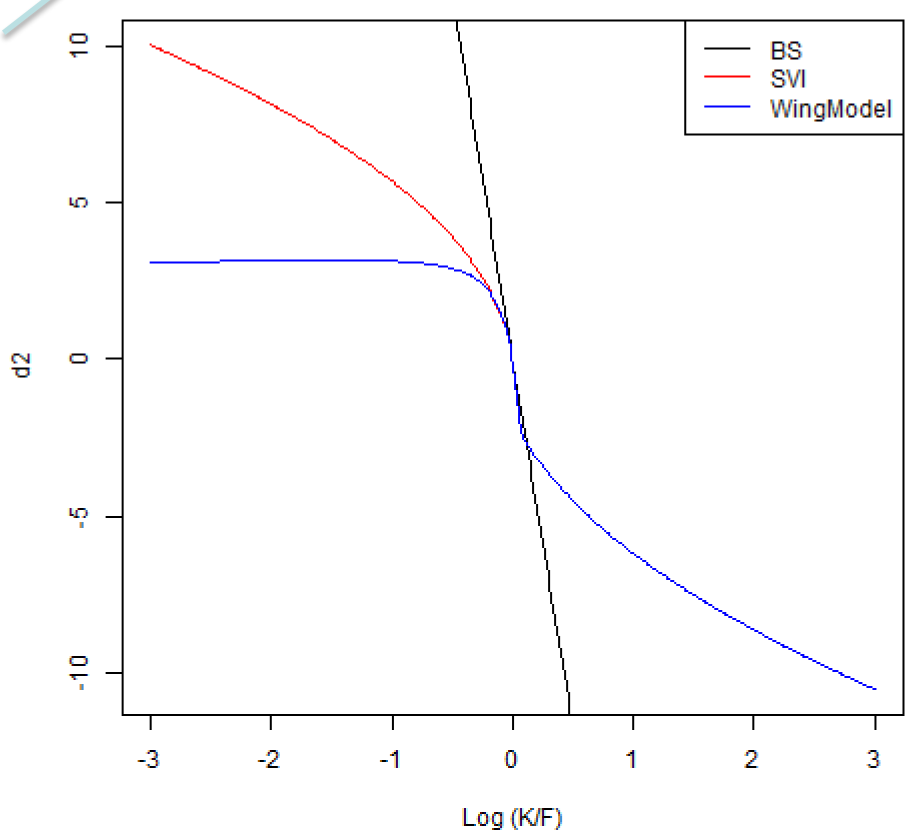
$$\text{FairVar} = \frac{2}{T}\left[\int_0^F dK\frac{P(K)}{K^2} + \int_F^\infty dK\frac{C(K)}{K^2}\right]$$

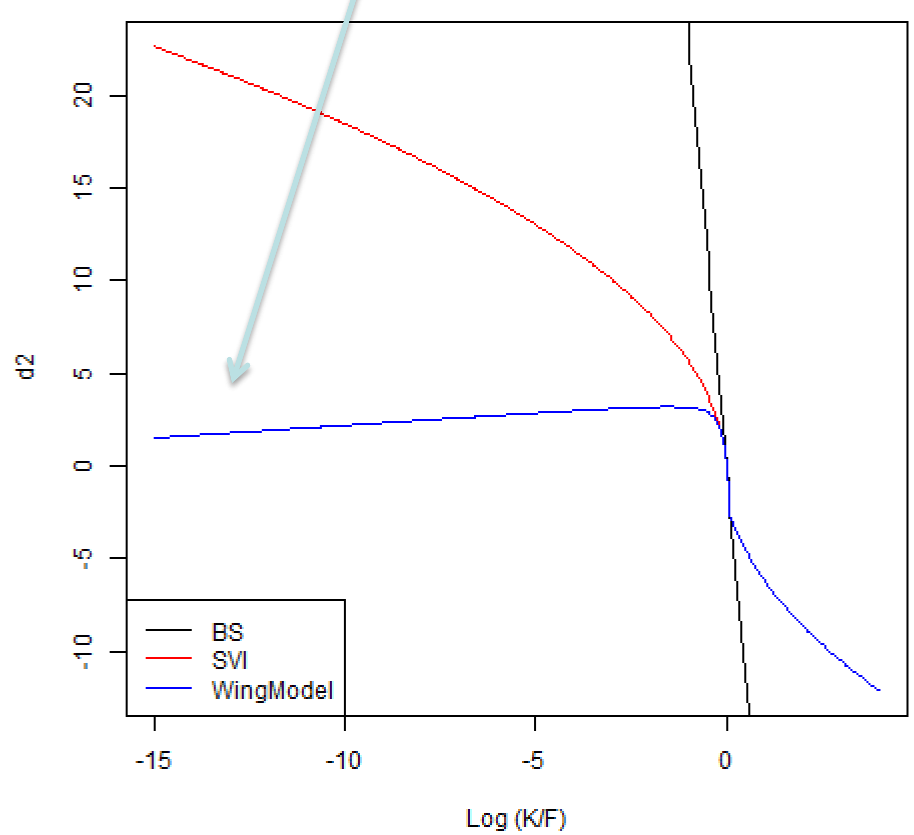
$$P = KN(-d_2) - FN(-d_1)$$

Ultimately N(-d2) starts increasing, not decreasing!
For d2 less than say -15, N(-d2) > ½, so P(K) ~K,
Integral diverges!
(fair var ~log(lower cutoff)

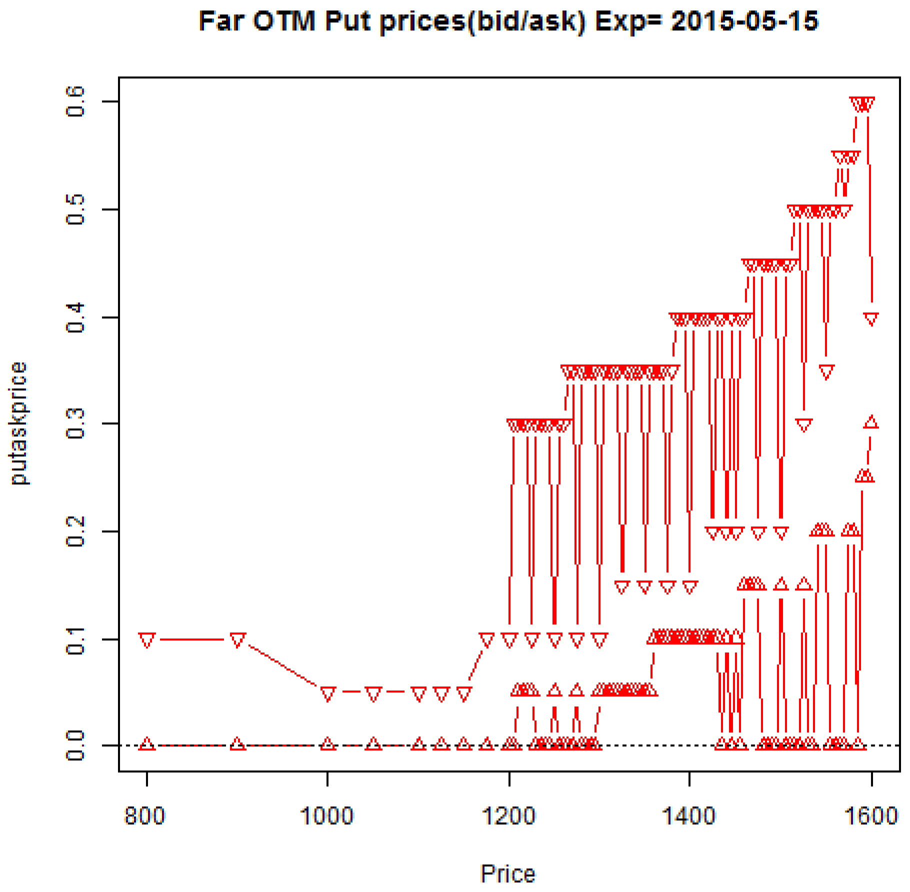
$$d_2 = \frac{-\log\frac{K}{F} - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \qquad d_1 = \frac{-\log\frac{K}{F} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

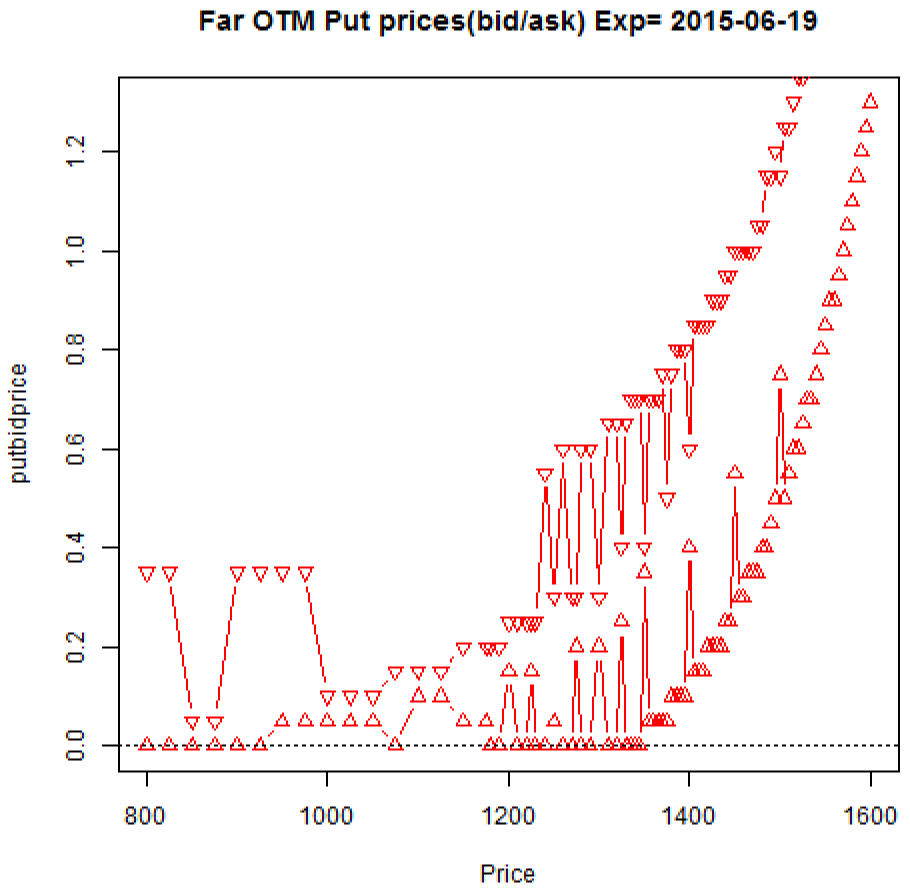
Competes with log-strike term for OTM puts





How do we pick a sensible lower cutoff?





- As is done in the VIX index: Stop when you have two zero bids in a row
- Or: When TV falls below a price threshold.

Fair Variance Strongly Dependent on Cutoff

Target Price	Min Strike	Fair Variance
0.100	1224	0.1451
0.050	769	0.1471
0.025	166	0.1574
0.020	90	0.1651

Slow price decay (long tails) leads to strong dependence of FV on the choice of cutoff!

Wing model is not able to produce a well-defined fair variance estimate without exogenous constraints, but clearly provides the best empirical fits

Fair Variance In Terms of Vol

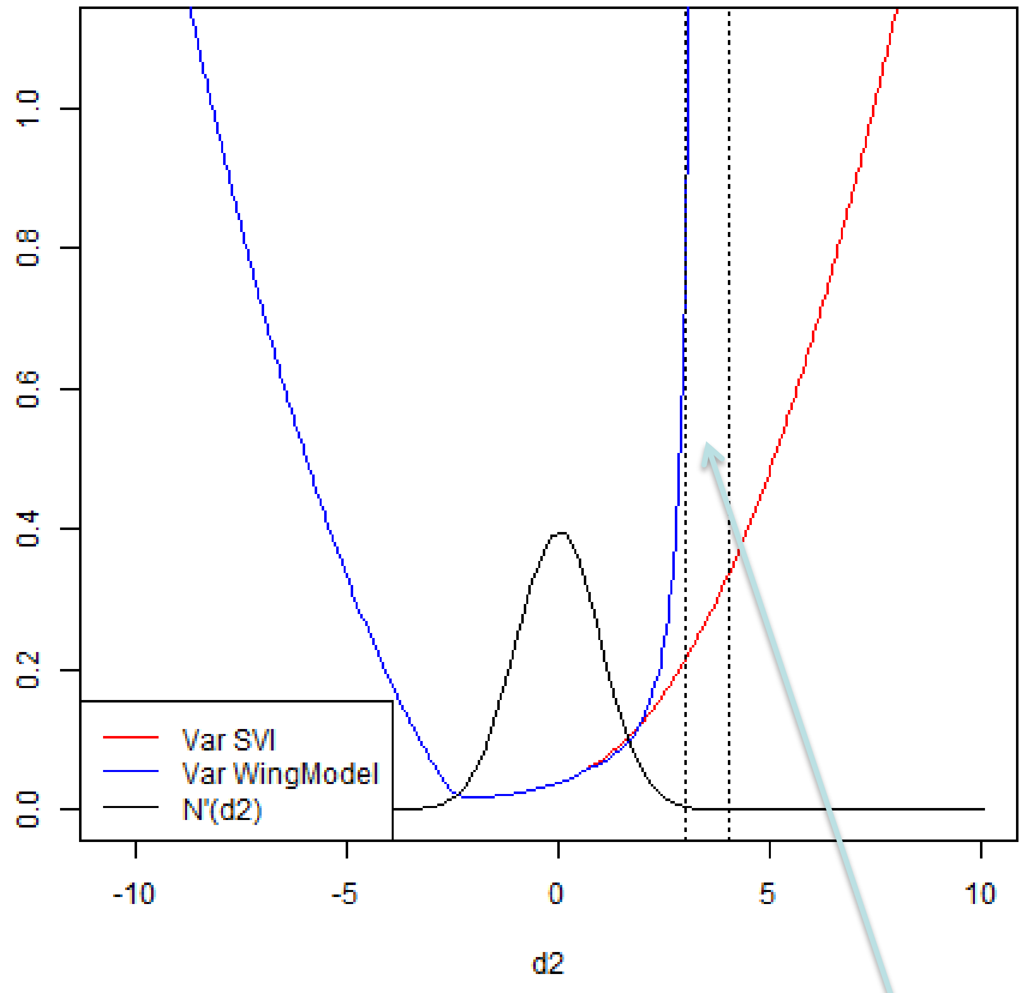
- You can express as weighted average of implied vol^2

$$FV^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} d(d_2) \exp\left(-d_2^2/2\right) \sigma^2(d_2)$$

$$d_2(K) = -\frac{\log \frac{K}{F}}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}$$

- Better suited to think about the effect of the shape of the vol. surface on Fair Variance than price formulation (portfolio of puts/calls)

FV as Vol. Integral: Divergence makes sense



Rapidly increasing around d2=3,
Consistent with D2 vs Log(K/F) graph
we showed earlier

FV as Vol. Integral: Application 2

- Can use to derive approximate formulas for FV in terms of atmvol, skew
  - –GS formula for linear

$$\sigma(K,T) \approx \sigma_{ATM} + b\frac{K-F}{F}$$
$$FV \approx \sigma^2_{ATM}\left(1 + 3b^2T\right)$$

  - –JPM formula for linear in log(K/F);

$$\sigma(K,T) \approx \sigma_{ATM} + b\log K/F$$
$$FV \approx \sigma^2_{ATM}\left(1 + 3b^2T - b\sigma_{ATM}T + \frac{5}{4}(b^2T)(\sigma^2_{ATM}T)\right)$$

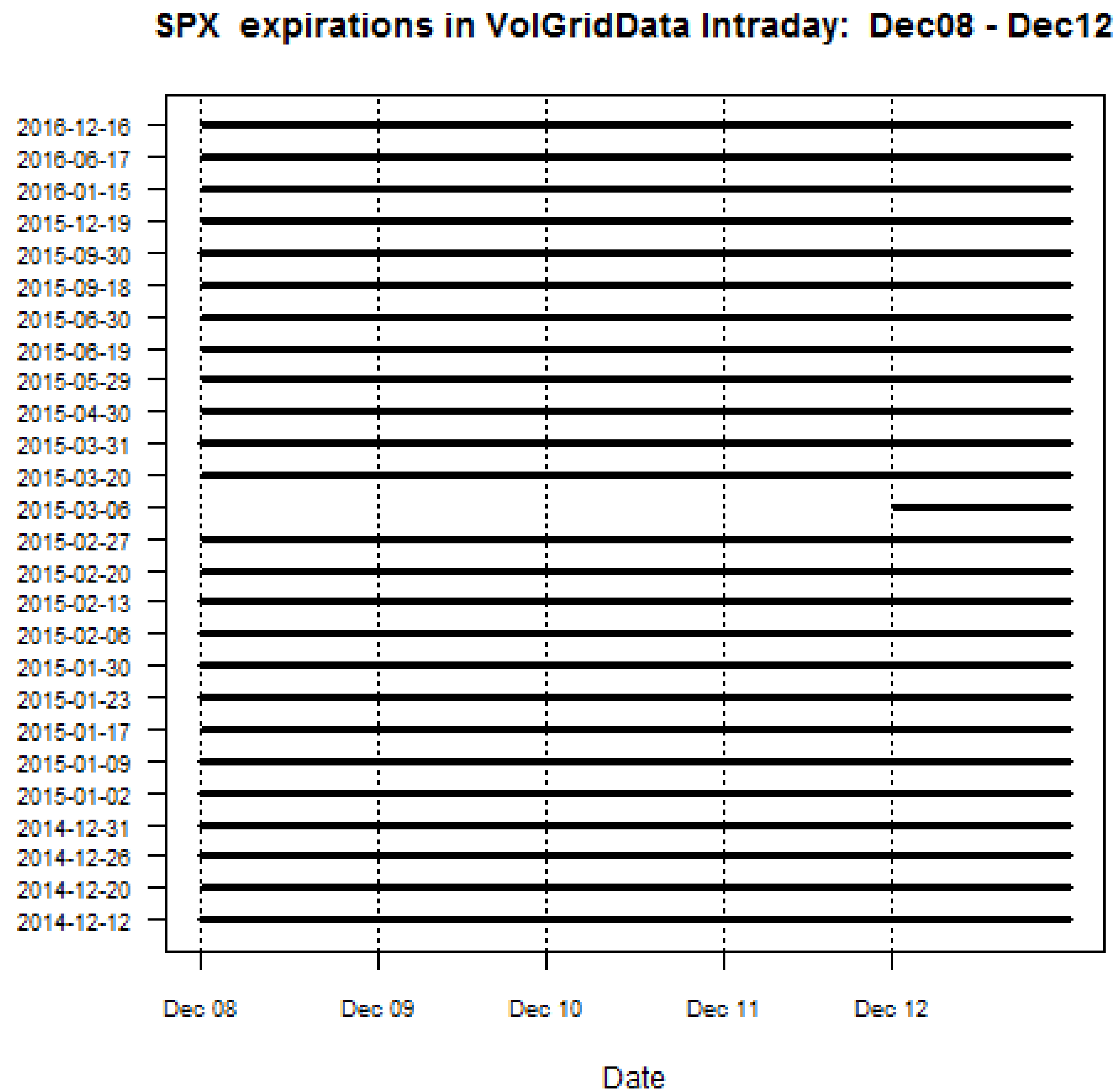
2. Fitting the Vol. Surface:
The Vanishing Quotes of SX5E

SPX: Intraday Presence of Quotes

Snapshots captured every minute

If expiration is present, plot a dot

Boring graph: SPX always has data.

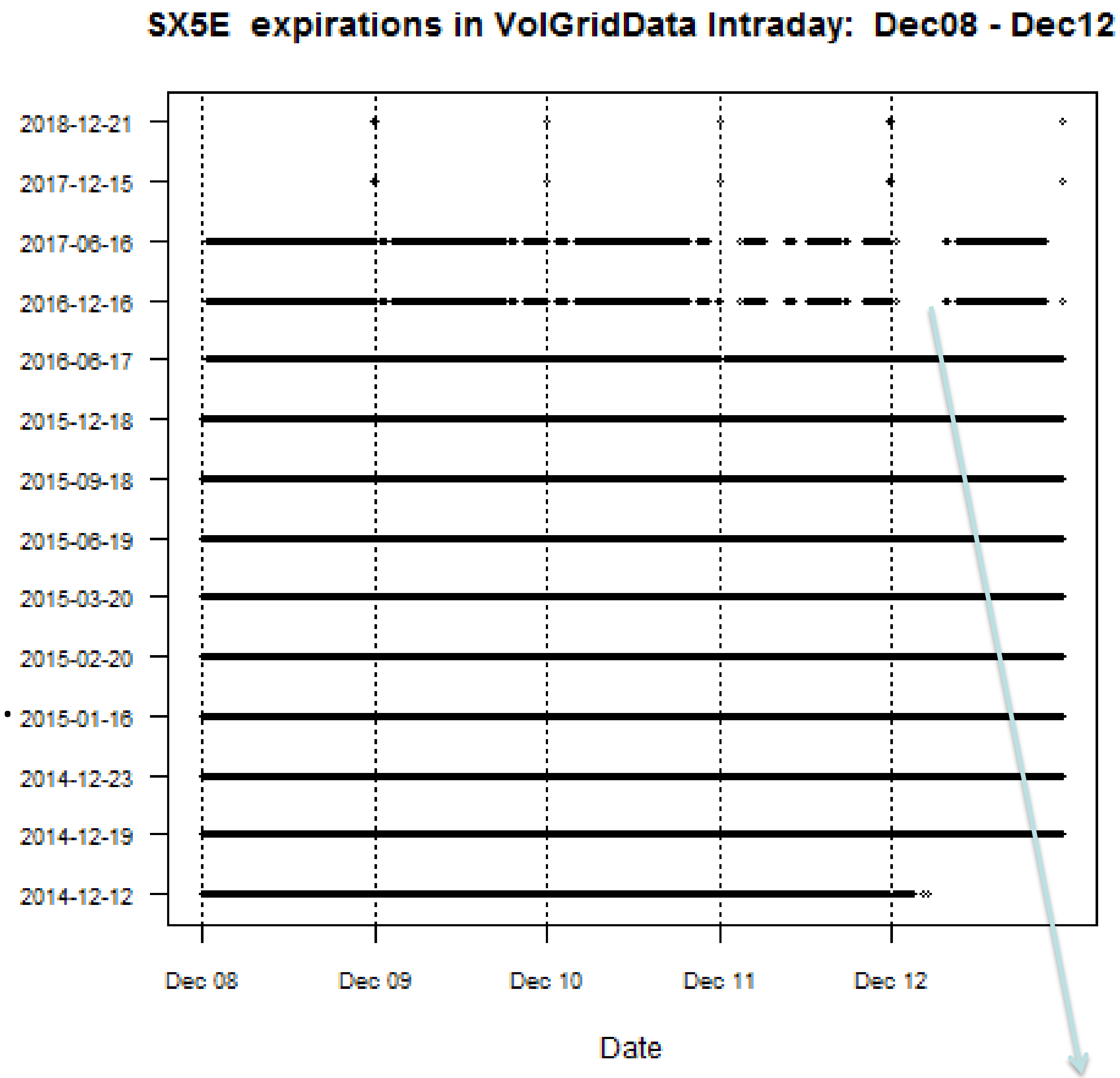


SX5E: Intraday Presence of Quotes

Snapshots captured every minute

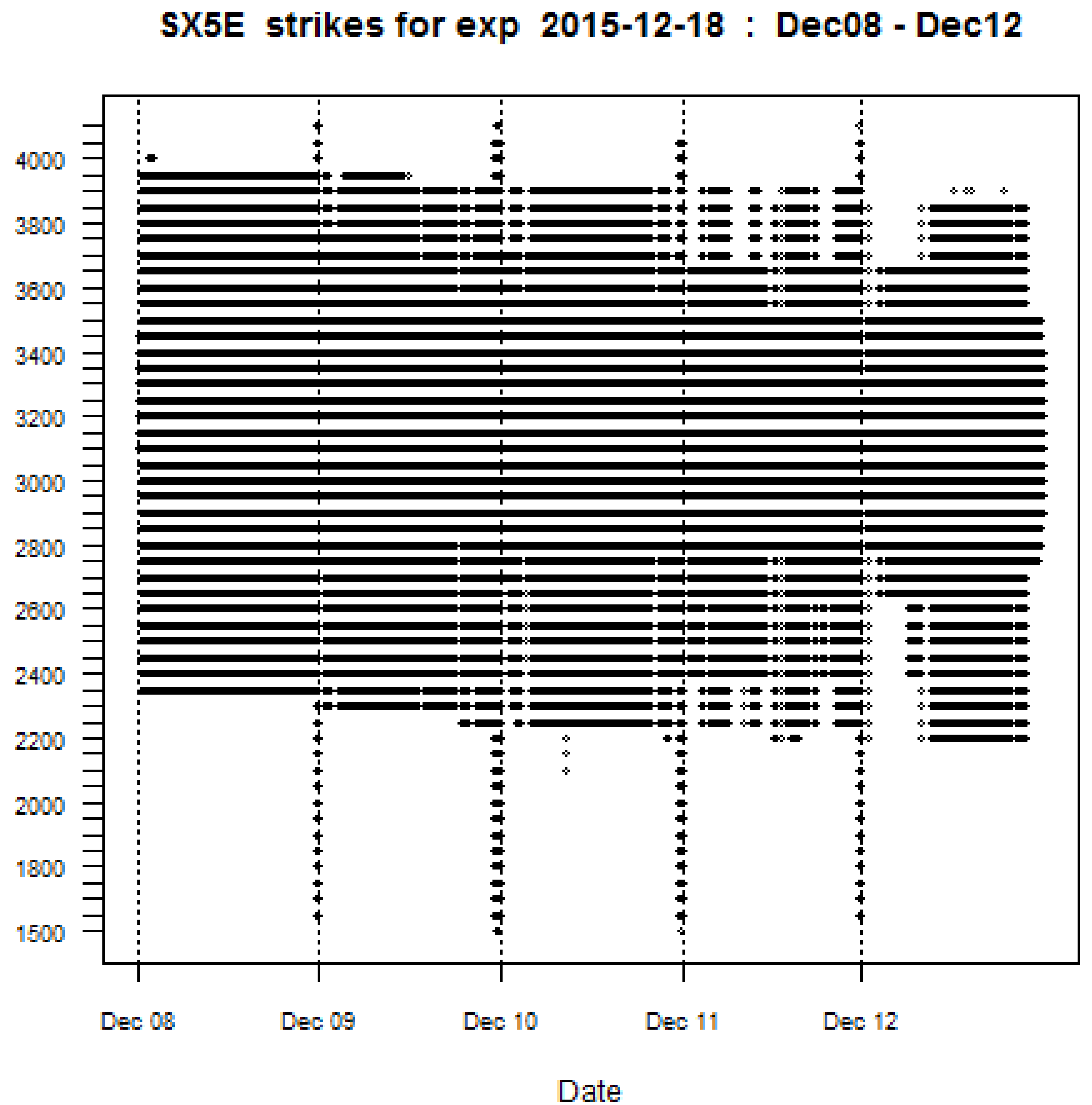
If expiration is present, plot a dot.

SX5E quotes are much less consistent.



Quotes vanished!

Presence of Strikes For Dec15 expritaion

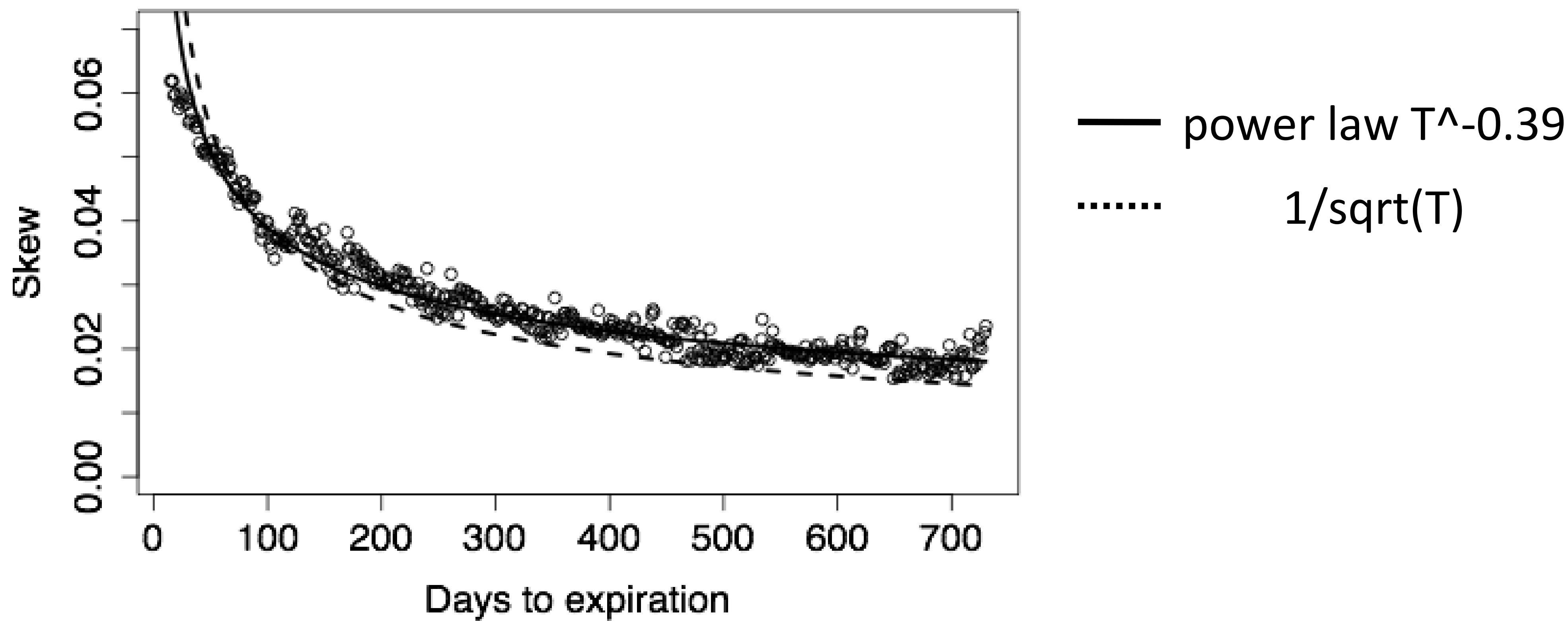


Great range of strikes
near closing time

Variability in the data can compromise the stability of parametric fits to the volatility surface.

3. Key Empirical Results for Index Vol

Skew decays as sqrt(T) (sort of)

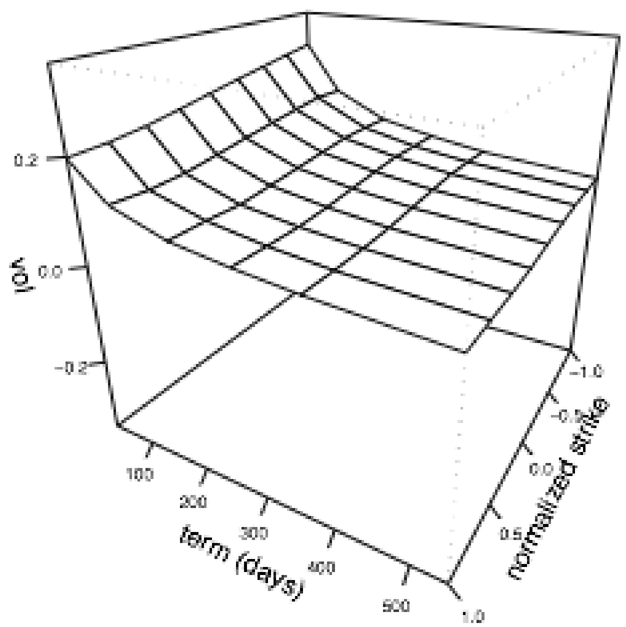


- S&P500 2001-Feb 2009 closing surfaces
- skew= vol(95%F)-vol(105% F)
- Normalize 1 year ATMF vol to be median historical value (18.80%) (mutiply all vols by a constant)

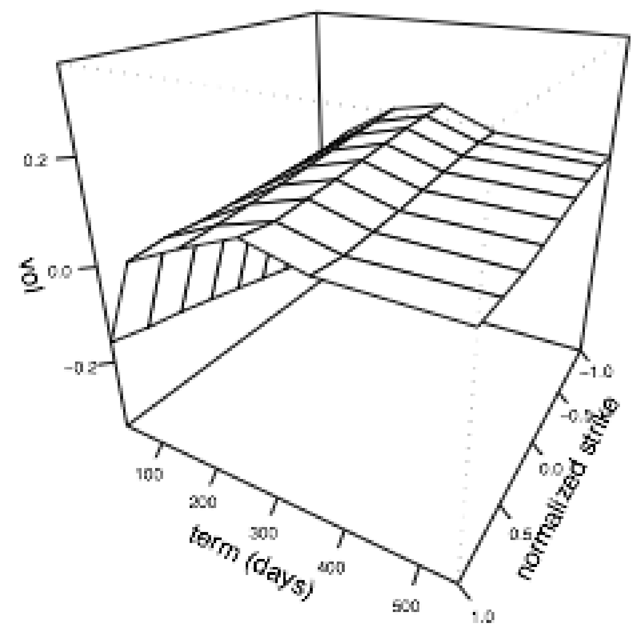
Modeling challenges:

- Stochastic vol: 1/T (unless multiple time scales)
- SV+ jumps? could help make short term more steep
- Local vol can fit, but surface fixed, no vol of vol
- General result: Vol. skew is related to the skewness of the distribution of log(S_T) : If changes in d(logS) are ultimately uncorrelated (for large T) then vol skew ~ 1/T.

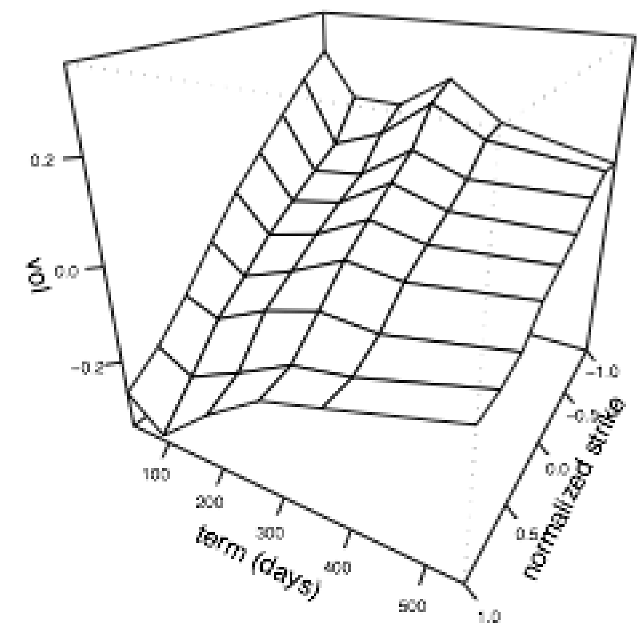
Surface changes: Simple, intuitive modes



Level



Term structure



Skew

PCA modes: Level explains 95.%, top 3: 98.2%

Changes in vol surface are simpler that the shape of volsurface itself

Vol, spot changes: strong negative corr.

	2001-Feb	Post Lehman '08 (Sept 15 -Dec 31, 2008)
Level	-0.87	-0.92
TS	-0.11	~0
Skew	~0	-0.55

Even during extremely high vol period (post Lehman) spot-vol corr remained high

Similar conclusions from other PCA studies

Table 1: PCA studies of the volatility surface. GS= Goldman Sachs study [9]; ML= Merrill Lynch proprietary data

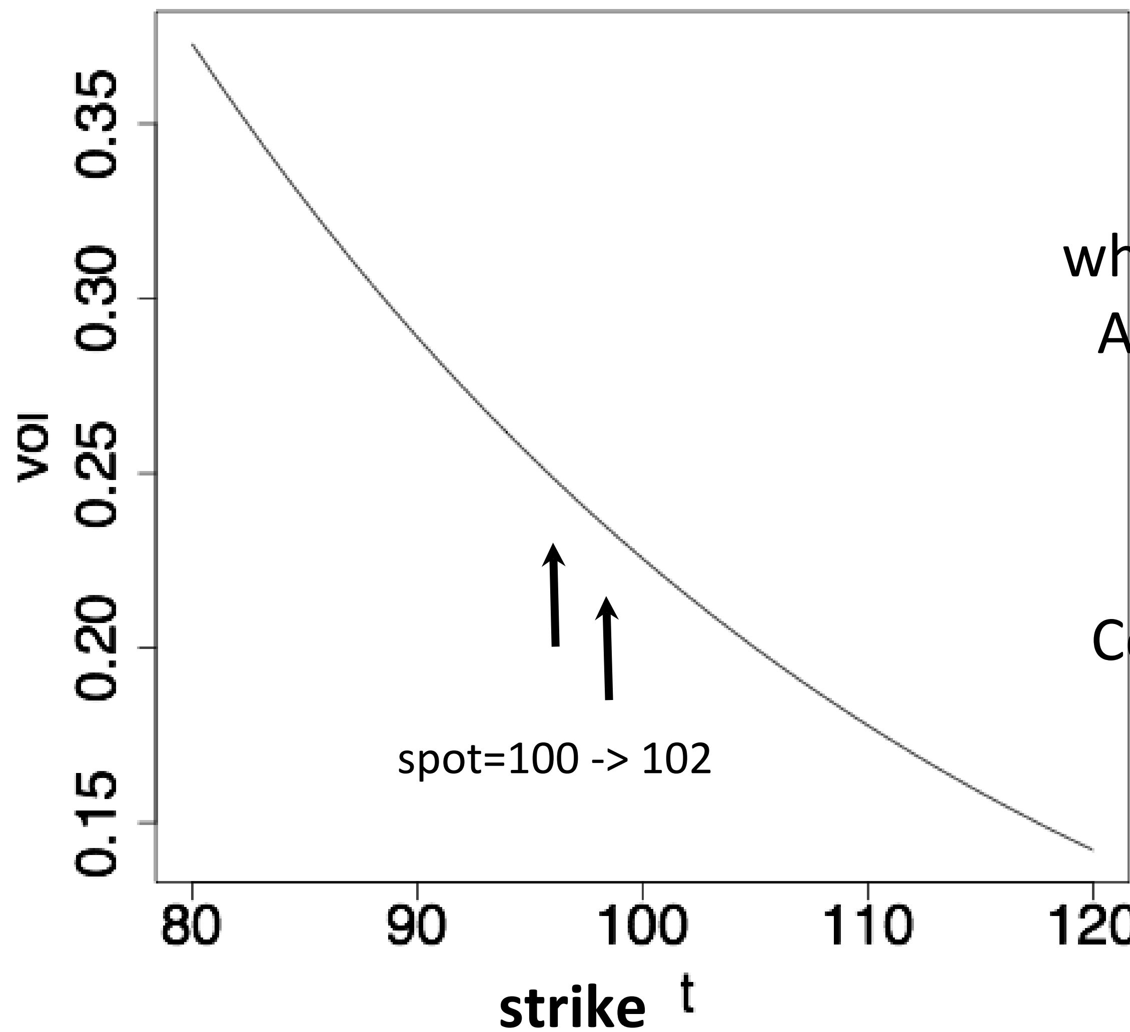
Source	Market	Top 3 modes	Var. explained by First mode	Var. explained by Top 3	Correlation of 3 modes with spot
GS	S&P500, weekly, '94-'97	Level, TS, Skew	81.6%	90.7%	-0.61, -0.07, 0.07
GS	Nikkei, daily, '94-'97	Level, TS, Skew	85.6%	95.9%	-0.67, -0.05, 0.04
Cont, et. al.	S&P500, daily, '00-'01	Level, Skew, Curvature	94%	97.8%	-0.66, $\sim$0, 0.27
Cont, et. al.	FTSE100, daily, '99-'01	Level, Skew, Curvature	96%	98.8%	-0.70,0.08, 0.7
Daglish et. al,	S&P500, monthy, '98-'02	Level, TS, Skew	92.6%	99.3%	n.a
ML	S&P500, daily, '01-'09	Level, TS, Skew	95.3%	98.2%	-0.87,-0.11, $\sim$0

Level mode dominates

Level: strong, stable negative corr
Other modes: weak, not as stable

Skew relates statics to dynamics

$$\Delta\sigma_{ATMF}(T) = \beta_T \frac{d\sigma}{d(\log K)} \frac{\Delta S}{S}$$



t

β=1 : surface fixed by strike,
when spot changes, move along curve,
ATM vol found by looking up K=F on
the same strike

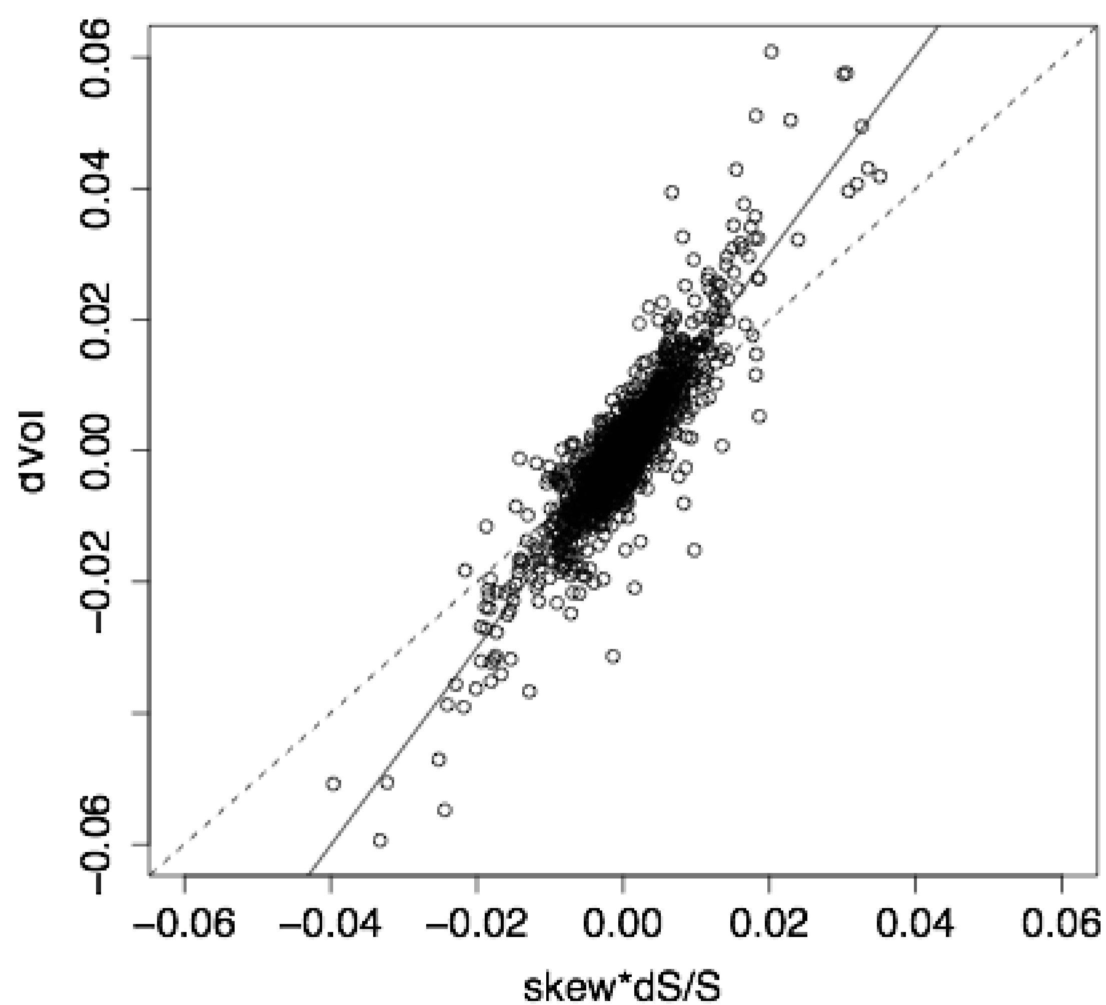
Could be called the “skewMultiplier”

Empirical β= skewMultiplier

Table 2: Historical estimates of β_T

T	β_T (std. err.)	R^2
30	1.55 (0.02)	0.774
91	1.50 (0.02)	0.825
182	1.48 (0.02)	0.818
365	1.49 (0.02)	0.791

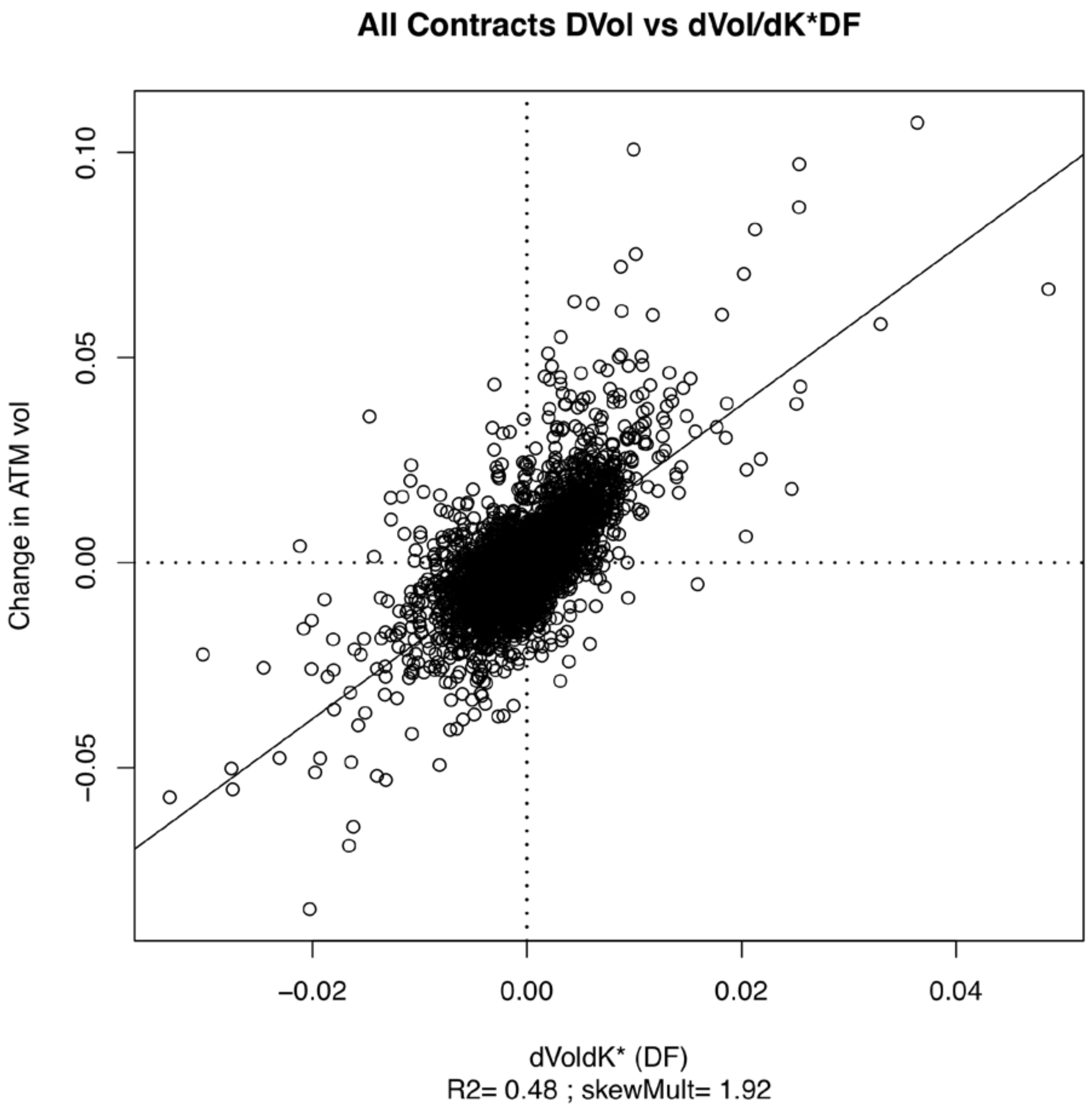
Figure 4: Regression of 91-day volatility changes vs spot returns. A zero-intercept least squares fit to model (1) leads to $\beta_{91} = 1.50$ (solid lines). The $\beta = 1$ ("sticky-strike") prediction (dashed line) clearly doesn't fit.



Skew underestimates ATMF vol by roughly same factor for all maturities (β~1.5)

Skew (static) plays a natural role in dynamics

Contrast SPX and Oil Vol Dynamics



- History: Feb09 to Dec11
- Take all 2010, 2011 and 2012 CL contracts.
- Consider 1 year of data per contract, no closer than 2 weeks to expiration.

- Overall R^2=0.48,

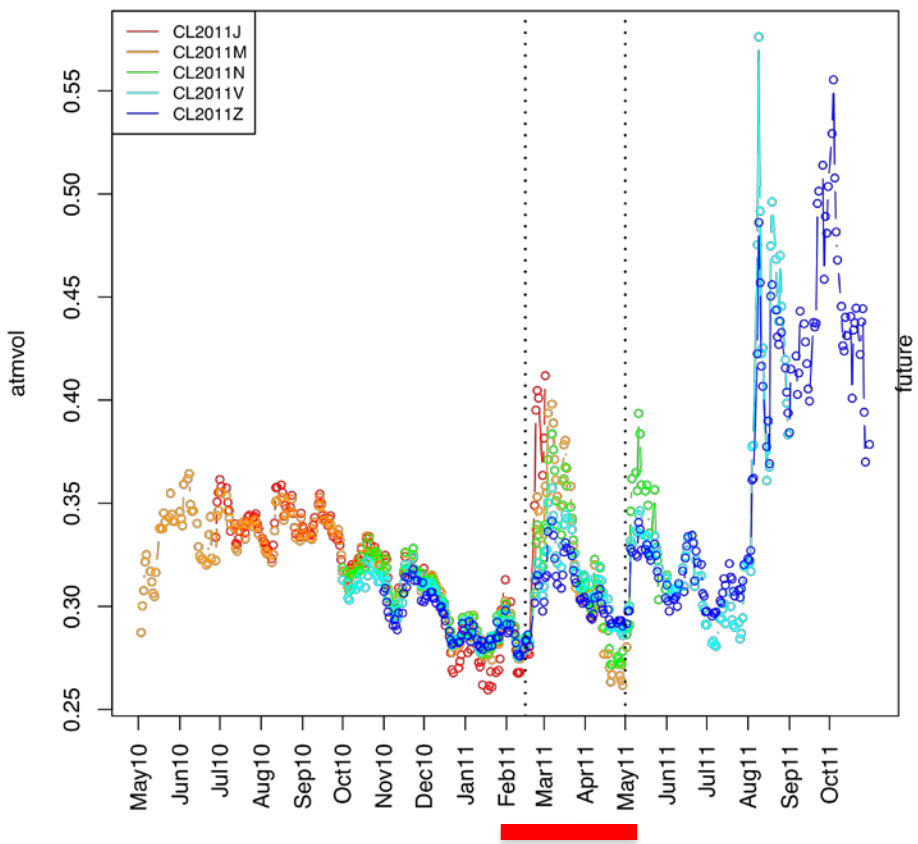
  β ~ 1.9

BUT depends on historical

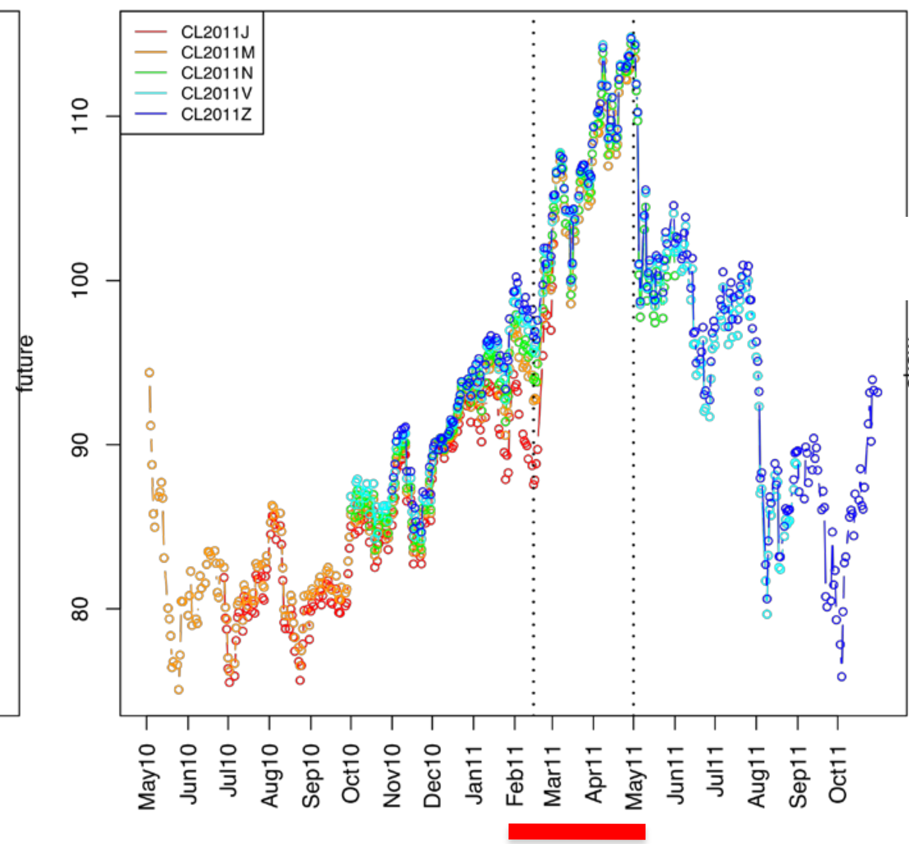
sub-period.

Breakdown of Sliding: Libya

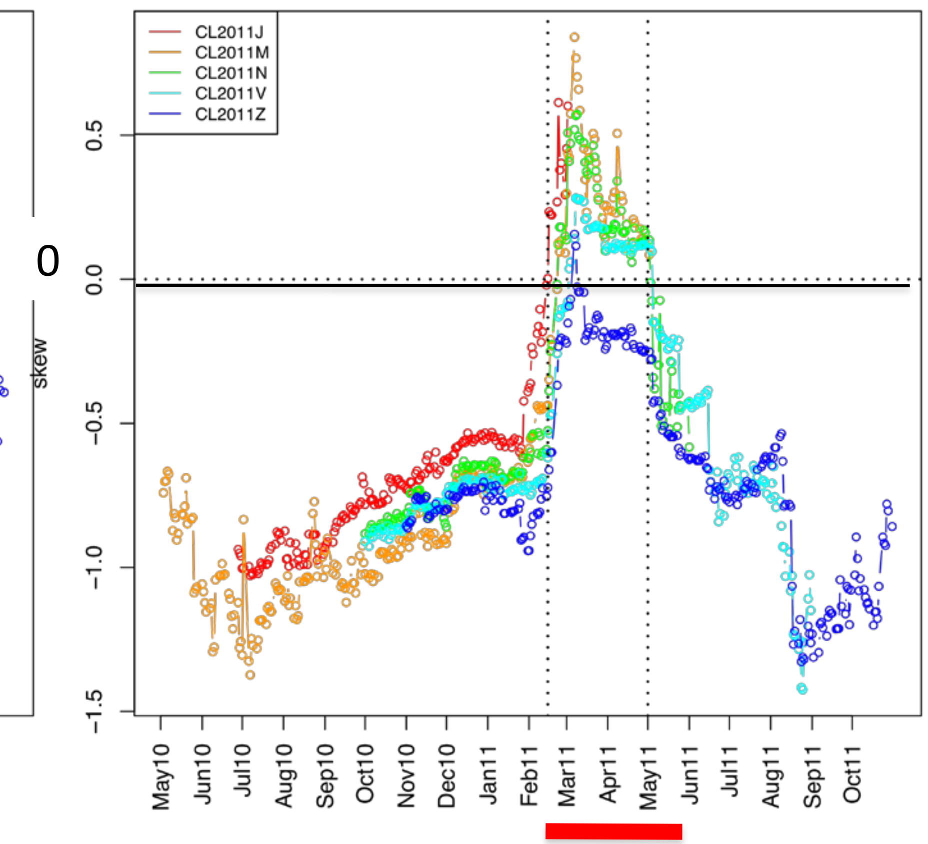
ATM Vol



Futures Price

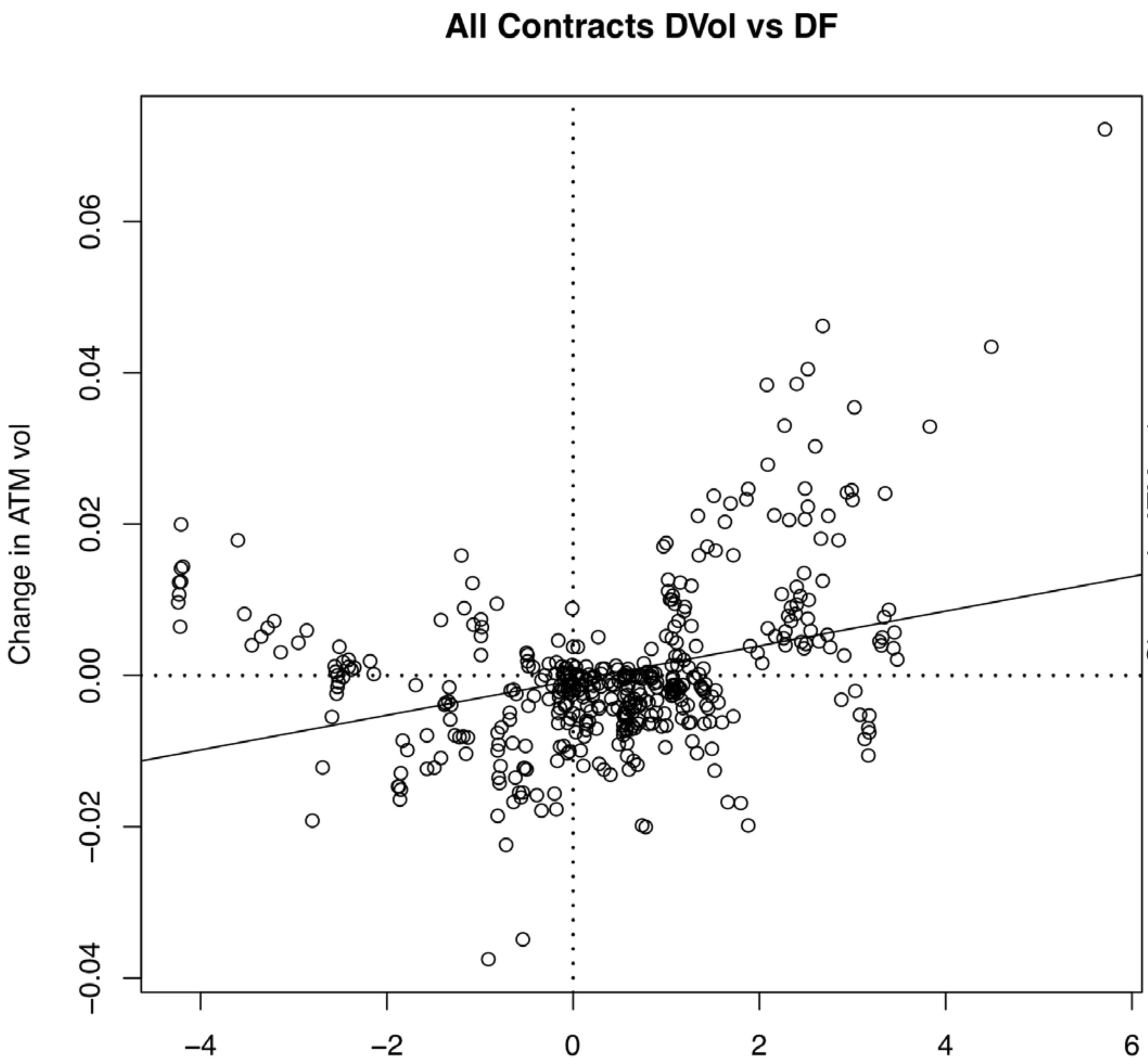


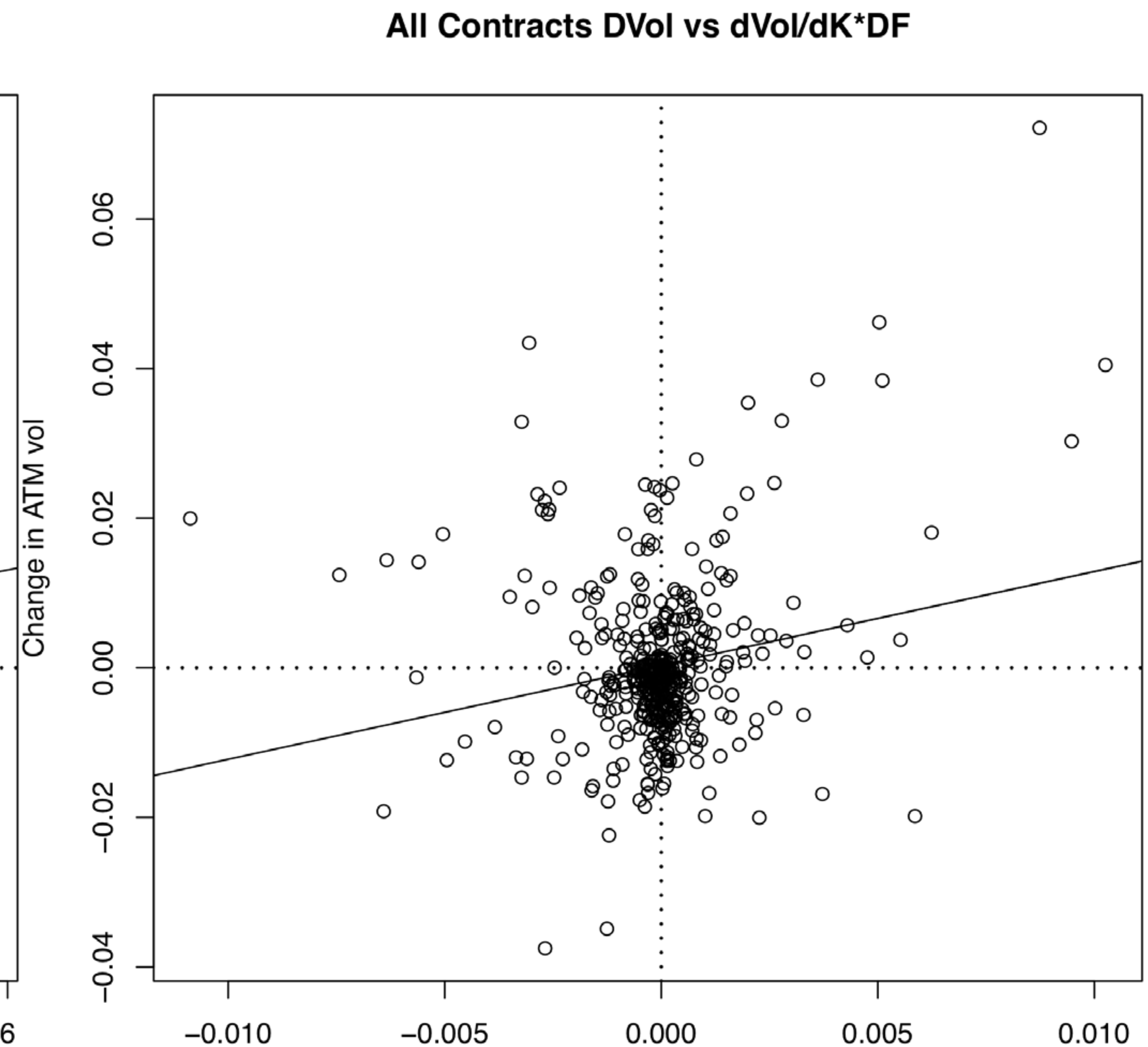
Skew



- Libyan unrest: Feb 15 –May 1,2011
- Spike in ATMVol, but also skew changed sign: skew to the call side, instead of put side.
- How do we model, anticipate these events? (can we?)

Bad regressions for this period





dF

R2~0.11

dvol/dK*(dF)

R2~0.04

Theoretical Links Between Statics and Dynamics

Recognition that β depends on the type of microscopic model, not its details:

- Local volatility models : $\beta \approx 2$ (exactly as T->0 for well-behaved)
- Jumps uncorrelated with spot: $\beta = 0$.
- Stochastic volatility models (Bergomi)
  - $\beta = 2$ as T->0 (look like local vol.)
  - As $T \to \infty$
    - $\beta = 1$ when skew decays with term to exp. as $1/T$ (Heston)
    - $\beta = 2 - \gamma$ when skew decays more slowly: as $1/T^{\gamma}$, $\gamma < 1$
  - Spot-vol correlation is key.

Different combinations of microscopic models can lead to $\beta = 1.5$

Do we need microscopic models?

Intuition From SABR: Why is β~2?

Recall the SABR model:
Local Vol (Power-law) + Lognormal stochastic vol

$$dF = \alpha F^{\beta-1} F dZ$$
$$d\alpha = \nu \alpha dW$$
$$E[dZdW] = \rho dt$$

Simplified, approximation near the forward:

$$\sigma_{BS}(K,F) = \frac{\alpha}{F^{1-\beta}} \left\{ 1 - \frac{1}{2}(1-\beta-\rho\lambda) \log K/F + \frac{1}{12}\left[(1-\beta)^2 + (2-3\rho^2)\lambda^2\right] \log^2 K/F \right\}$$
$$\lambda = \frac{\nu}{\alpha} F^{1-\beta}$$

Skew depends on both local vol and stochastic vol components,
λ measures the relative strength of stoch vol vs local vol.

A Subtlety with SABR: Condition on dF!

- What does d(atmVol)/dF even mean? (Stoch vol?)
- Rewrite SABR dynamics:

$$
\begin{aligned}
dF &= \alpha F^{\beta-1} F dZ \\
d\alpha &= d\alpha_F + d\alpha_\perp \\
d\alpha_F &= \nu\alpha\rho dZ = \nu\alpha\rho\left[\frac{dF}{\alpha F^{\beta-1}F}\right] \\
d\alpha_\perp &= \nu\alpha\sqrt{1-\rho^2}\,dZ_\perp \\
E[dZ dZ_\perp] &= 0
\end{aligned}
$$

Recognize that: $E\left[\frac{d\alpha}{dF}\middle| dF\right] = \alpha\frac{\rho\lambda}{F}$

To derive β=2 as an expected value !

Highly Recommended Homework 2

- Suppose the current vol surface is given by:

  $\sigma(K)=30-0.5(K-100)$ when the forward is 100. Also suppose that the surface only moves by parallel shifts (i.e., skew stays constant, but atmvol changes)

  For each value of beta=0,1,2, complete this table (3 tables in total)

Forward	**ATMVol**	**vol(K=90)**	**vol(K=100)**	**vol(K=110)**
100	30	35	30	25
90				
110				

- Finish the argument that shows that the skew multiplier is 2.0 for the simplified SABR model

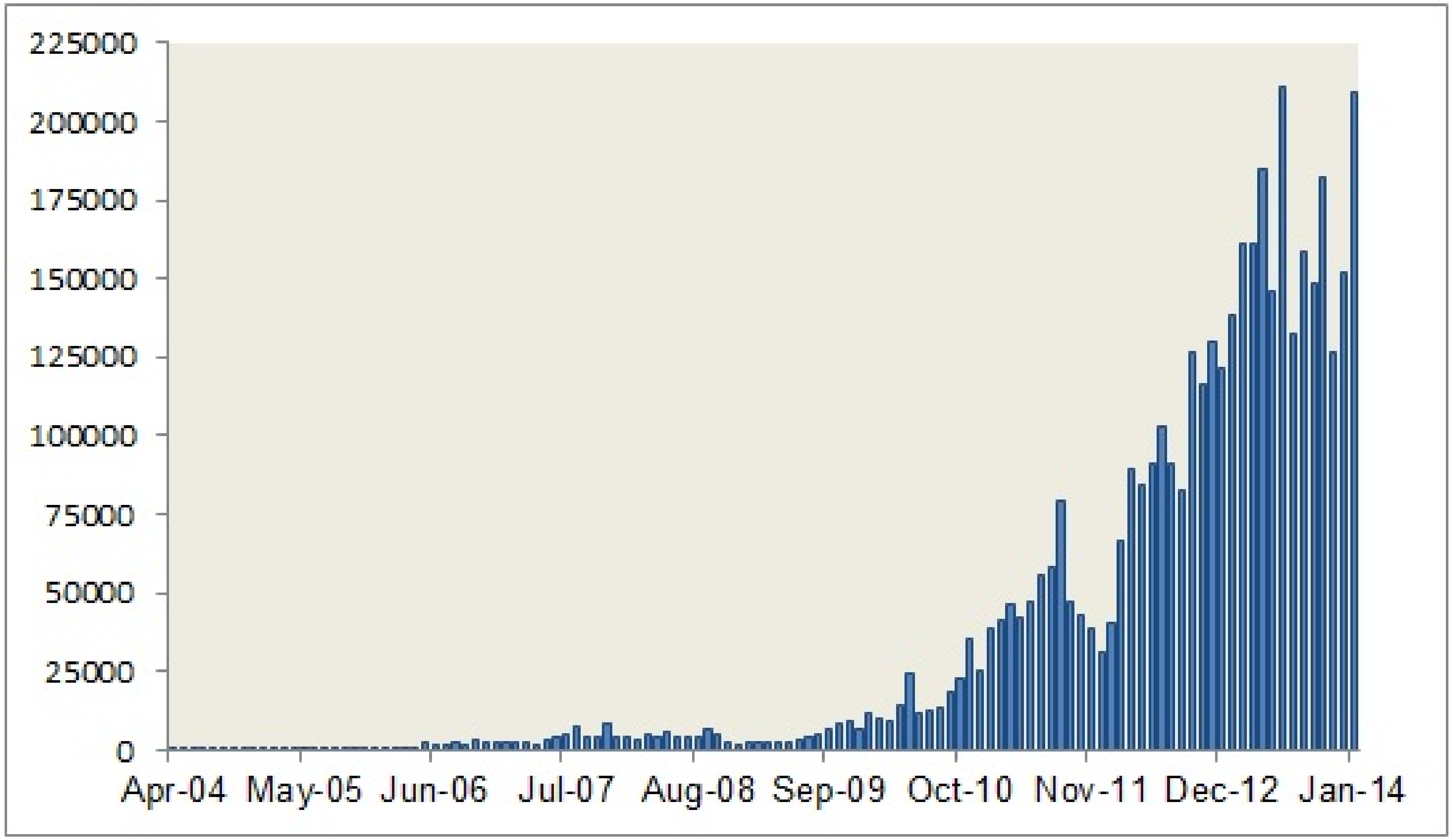
4. Volatility Derivatives: Forward Variance, VIX Futures and VIX Options

A Brief, Partial and Biased History of *Listed* Vol. Derivatives and Indices.

- 1993: VIX index introduced, defined as essentially the ATM vol. of 3 month options.
- ~1998: Variance swaps gain critical mass (OTC), in part because replication argument (through options) is practical.
- 2003: VIX index redefined to correspond to a 30-day fair variance, quoted as a volatility (i.e. take the square root)
- 2004: VIX futures start trading
- 2006: VIX options start trading.
- 2009: VXX ETN issued: Rolling strategy that targets a 30-day weighed maturity by rebalancing the front two VIX futures.
- 2010: Options on VXX

VIX Futures (2014)

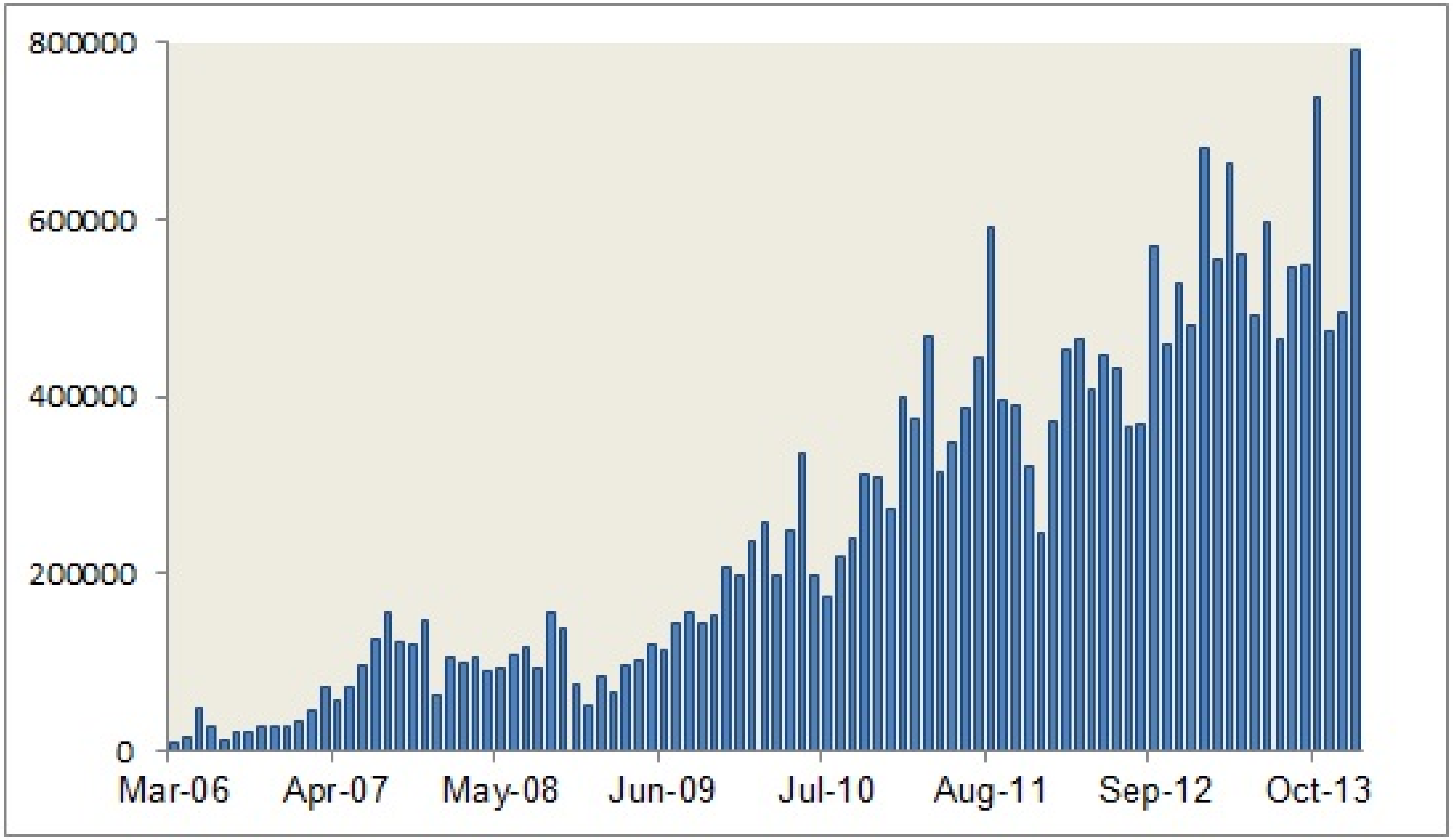
Average daily volume for VIX Futures trading appears to be 209,611 which is just over 1,000 contracts short of the record set i June 2013 of 210,674. Average daily volume since 2004 by month appears in the chart below.



From CBOE website

VIX Options (2014)

Preliminary numbers show average daily VIX Option volume at 792,668 contracts for January. This is more than 50,000 contracts per day higher than the previous record of 738,295 which was set in October of last year. The chart below shows the average daily volume by month for VIX Option trading since 2006.

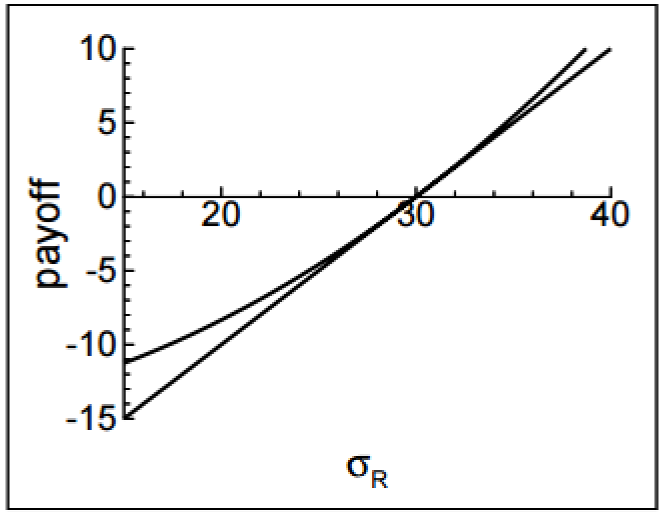


From CBOE website

Volatility vs Variance Swaps: Convexity is Key

Old Goldman paper on variance swaps recognized the convexity bias inherent in volswaps: Can't make a line out of a parabola!

FIGURE 11. Payoff of a volatility swap (straight line) and variance swap (curved line) as a function of realized volatility, for $K_{vol} = 30\%$.



$$\sigma_R - K_{vol} \approx \frac{1}{2K_{vol}}(\sigma_R^2 - K_{vol}^2)$$

$$\text{convexity bias} = \frac{1}{2K_{vol}}(\sigma_R - K_{vol})^2$$

$$K_{vol} < \sqrt{K_{var}}.$$

Dynamic Replication of a Volatility Swap

In principle, some of the risks inherent in the static approximation of a volatility swap by a variance swap could be reduced by dynamically trading new variance contracts throughout the life of the volatility swap. This dynamic replication of a volatility swap by means of vari-

(1998) and Ledoit (1998)]. When there is a liquid market in variance swaps, these models may be useful in hedging volatility swaps and other variance derivatives.

VIX Index, Futures and Options

- VIX index: current 30-day fair variance quoted as a volatility: $VIX = \sqrt{FV(t, t+30d)}$
- VIX future: Converges to the VIX spot at settlement (T) Today's price for what VIX will be in the future:

$$VIXFut(t,T) = E_t\left[\sqrt{FV(T, T+30d)}\right]$$

Same convexity bias that applied to variance vs volswaps applies here:

VIX future <= fwd. variance quoted as a vol.

- VIX Option: Standard European options, expiration dates coincide with VIX Future settlement dates.

Estimating the convexity correction in 2005?

- The convexity bias of VIX future vs fwd SPX variance depends on vol. of vol. How would you estimate? Hedge?
- Not easy back in 2005 (before VIX options). Could try to use Heston, SABR to calibrate vol. of vol. form SPX vol surface.
- Clever argument by Dupire: Simple dynamic strategy based on Ito's lemma: $V_T^2 = V_0^2 + \int_0^T dV_t(2V_t) + \int_0^T (dV_t)^2$

Suggests you can almost synthesize fwd. variance by dynamically trading VIX futures: the last term accumulates the daily hedging errors. Could estimate from historical data:

$$V_0^2 = E_0\left[V_T^2 - \int_0^T (dV_t)^2\right]$$

VIX Options to the Rescue

- VIX options ought to provide a better estimate of vol. of vol. (Sounds circular: using options to price the underlyer?)
- How do we estimate? Which expiry? Strike? Is it the raw implied vol? Or some transformation.
- Surprisingly simple and powerful link.

Current fwd. var and VIX futures prices are related:

(fwd var as vol)^2 = (VIX future)^2 + (Price of a Strip of VIX options)

This strip of VIX options is a *constant weight* portfolio (not 1/K^2) comprised of call options with strikes above the current VIX futures price, and puts struck below.

This pricing relation leads defines (is defined by?) a *static* hedging strategy

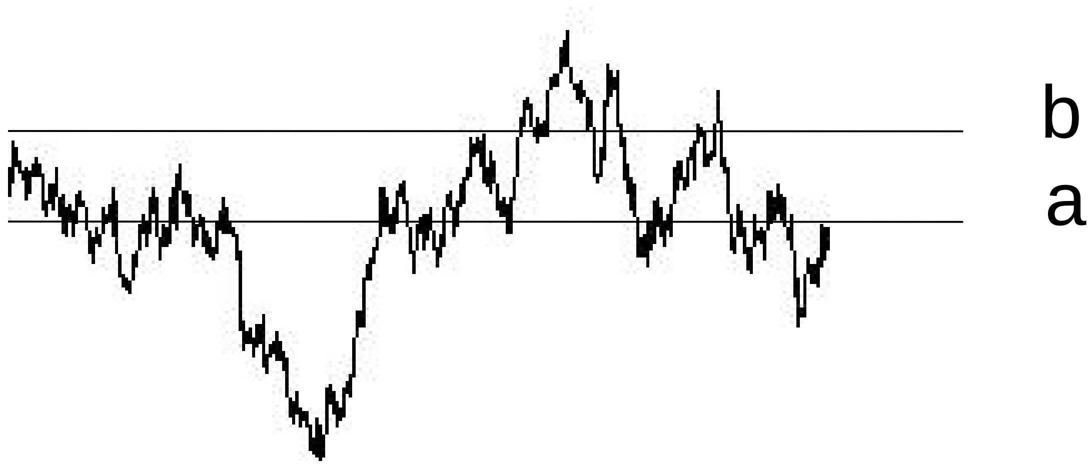
4. Volatility Derivatives: Corridor Variance Swaps

Corridor Variance Swap

- Accrue realized variance only when index is within a predetermined band. (The indicator function is 1 when S is within the band, 0 outside)

$$(\text{Norm. factor}) \sum_i \log\left(\frac{S_{i+1}}{S_i}\right)^2 1_{a<S<b} \; - K$$

Amazingly, almost same
hedging strategy as variance swap



Except: Only replicate with options with strikes a<K<b, hold 1/S shares when within the range.

Why does this work?

Revisit Derivation of Variance Swap Replication

$$\text{realized var} = \sum_i \left(\frac{\Delta S_i}{S_i}\right)^2$$

$$\log(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

$$x^2 \approx 2\left[x - \log(1+x)\right]$$

$$\text{realized var} \approx \sum_i 2\left[\left(\frac{\Delta S_i}{S_i}\right) - \log\left(\frac{S_{i+1}}{S_i}\right)\right]$$

Special property of log: Log(A/B)= log(A)-log(B) leads to terminal log payoff

$$-log\left(\frac{S_1}{S_0}\right) - log\left(\frac{S_2}{S_1}\right) - log\left(\frac{S_3}{S_2}\right) \cdots - log\left(\frac{S_N}{S_{N+1}}\right)$$

$$= -log\left(\frac{S_N}{S_0}\right)$$

Extend to Corridor VarSwap

$$\text{realized var} = \sum_i \left(\frac{\Delta S_i}{S_i}\right)^2 1_{a<=S_i<=b}$$

$$\text{realized var} \approx \sum_i 2\left[\left(\frac{\Delta S_i}{S_i}\right) - \log\left(\frac{S_{i+1}}{S_i}\right)\right] 1_{a<=S_i<=b}$$

Indicator function is 1 when S is in the accrual range [a,b], 0 otherwise

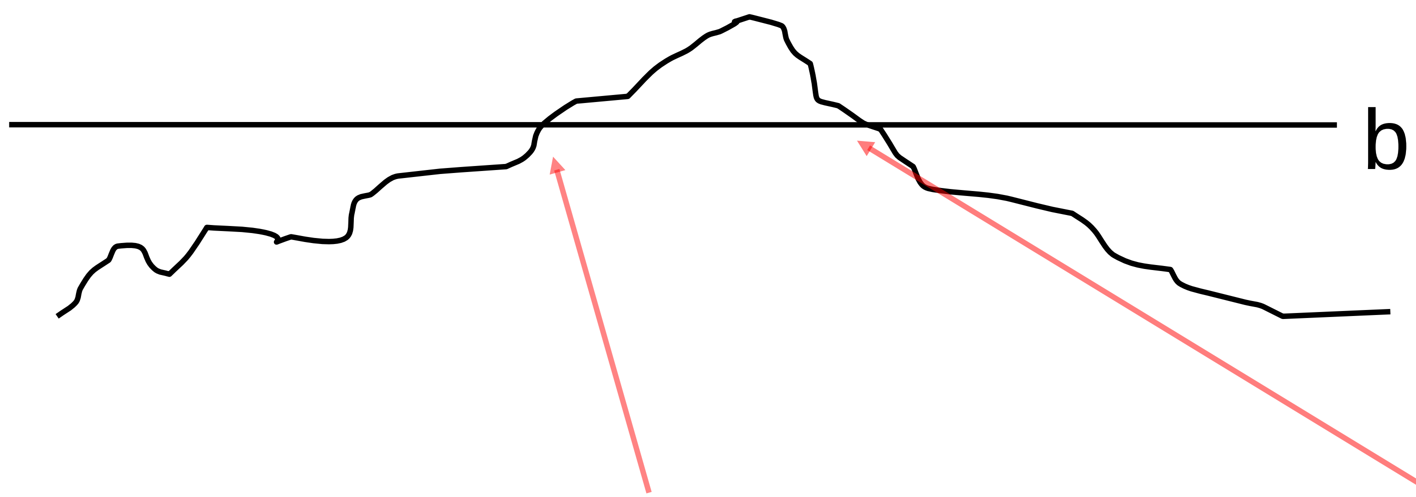
There is still a lot of cancellation of log terms

$$\text{Sum of Log Terms} = \log\left(\frac{\max\left(\min\left(b, S_T\right), a\right)}{S_0}\right)$$

Cap terminal stock price at b, floor at a

Crossing Above and Returning

- Stock path crosses above b at time i and returns at time j



$$-log\left(\frac{S_1}{S_0}\right) - log\left(\frac{S_2}{S_1}\right) \cdots - log\left(\frac{b}{S_i}\right) + 0 + 0 + \cdots + 0 + 0 + -log\left(\frac{S_j}{b}\right) - log\left(\frac{S_{j+1}}{S_j}\right) + \cdots$$

$$\approx -log\left(\frac{S_N}{S_0}\right)$$

Recommended Homework 3

- Work out the details of the relation between VIX futures, VIX options and forward variance:

  - Justify the pricing relation
  - Work out the static hedging strategy: What is the weight of the options? Do you need to hold VIX futures? How many?

- Work out details of corridor variance swap replication.
- What does a forward starting *corridor* variance swap capture?