Lecture 3: Transactions Costs; The Smile: Constraints &Problems

Recapitulation of Lecture 2

- The Black-Scholes PDE is equivalent to setting the Sharpe ratios of the option and the stock equal to each other, instantaneously.

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S}$$

- Hedging Options Means Betting On Volatility
When you hedge at implied volatility, the net P&L of the hedged curved position during time $\Delta t = \frac{1}{2}\Gamma(\sigma^2 - \Sigma^2)S^2\Delta t$
This P&L is path-dependent and random,and so although it is deterministic at each instant, the final P&L is unknown.

  Hedging at (suppposedly known) realized volatility σ_r:The final P&L of a hedged derivative V purchased at implied volatility and hedged at realized volatility is $V_r - V_i$, and known in advance if we know σ_r, but the fluctuations in the P&L along the way are random.

- The P&L of Hedged Trading Strategies

$$\underbrace{(C_0 - \Delta_0 S_0)e^{r(T-t)}}_{\text{initial hedge}} = \underbrace{(C_T - \Delta_T S_T)}_{\text{final hedge}} + \underbrace{\int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b}_{\text{rebalancing}}$$

- Hedging Errors from Discrete Hedging at the Correct Volatility
At the money:

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial \sigma}\frac{\sigma}{\sqrt{n}}$$

  The standard deviation of a constant volatility σ measured discretely with n samples is $\frac{\sigma}{\sqrt{2n}}$. This is not negligible; traders can reduce it by putting together portfolios of options whose volatility sensitivities tend to cancel each other, so that the net vega of the portfolio is much smaller than the sum of the magnitudes of the individual vegas.

This Lecture:

The smile in various markets
The difficulties the smile presents for trading desks and for theorists
Pricing and hedging
How volatility varies; what people mean by volatility
How to graph the smile?
Delta as a plotting parameter
Parametrizing options prices: delta, strike and their relationship
Estimating the effects of the smile on delta and on exotic options
Reasons for a smile
No-riskless-arbitrage bounds on the size of the smile
Fitting the smile
Black-Scholes is wrong: what can replace it?
Behavioral reasons for the smile

3.1 No Matter How You Hedge …

From Eq 2.24 of Lecture 2 gave another expression for the value of an option in terms of rehedging:

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[dS_\tau - S_\tau r d\tau] e^{-r(T-\tau)}$$

Here the LHS is the value of the option based on the hedging and the value of the final payoff. And this is the formula for *any* delta one uses, not necessarily the Black-Scholes delta.

Now, assume that $\mu = r$ and hence with GBM $dS - Srdt = \sigma S dZ$

Then

$$C_0 = C_T e^{-r(T-t)} - \int_t^T \Delta(S_\tau, \tau)[\sigma S] e^{-r(T-\tau)} dZ$$

Now let's take expected values over all stochastic movements on the stock.

Then

$$E[C_0] = E[C_T] e^{-r(T-t)}$$

where the expected value of the last term with the dZ is zero of course.

The equation above is simply the Black-Scholes formula when you take the expected value over the lognormal distribution of the stock price at expiration.

> Thus irrespective of the hedge ratio, the expected value of the call is the discounted expected value of the payoff, no matter how you hedge, no matter what hedging formula you use for delta, even if you don't hedge at all, provided that you use $\mu = r$

3.2 The Effect of Transactions Costs

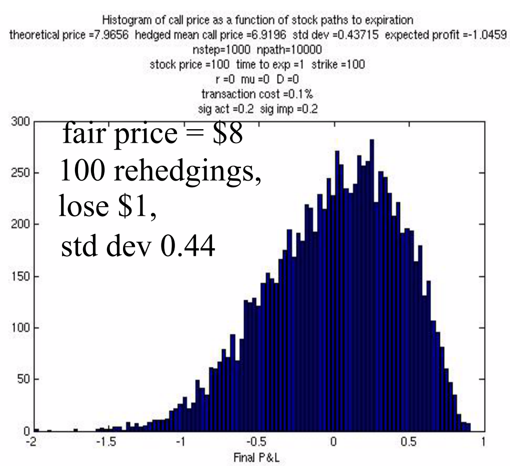
3.2.1 Simulation

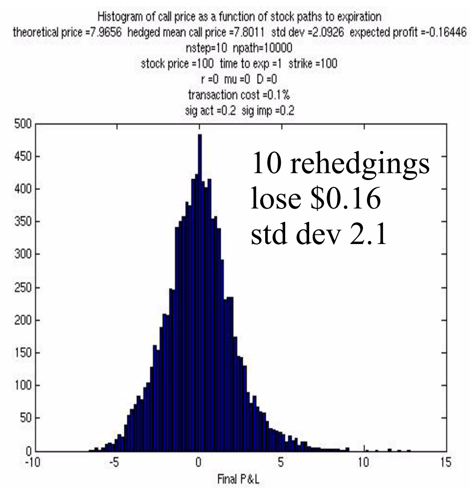
Suppose it costs money to buy and sell the stock each time you rehedge. Then, not only is the P&L uncertain because of the discrete hedging schedule and the consequent inaccuracy of the hedge ratio, but in addition the cost of hedging also lowers the fair value of the option if you buy it, and raises the cost to you if you sell it.

In the examples that follow, we assume a simple transactions cost proportional to the cost of the shares traded, and hedge at the realized volatility.

Rehedging at regular intervals

You can rehedge at every step, no matter how little or how much stock you need to trade to rebalance.





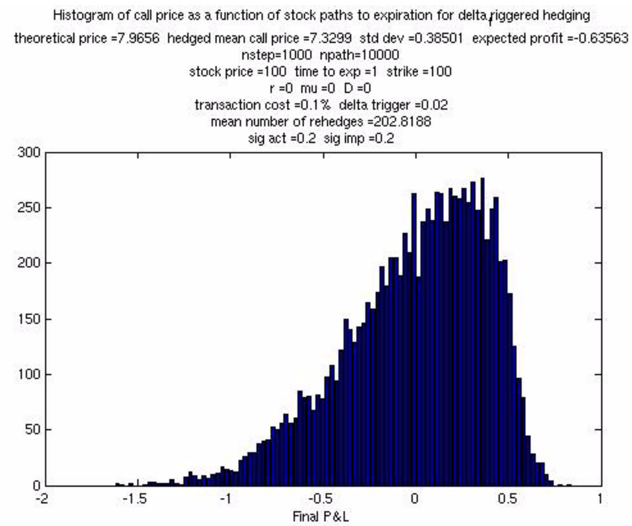
Notice that logically, the more frequently you rehedge, the more accurately you replicate the option; however, the more you rehedge the more of your profit you give away to transactions costs. Correspondingly the less you rehedge, the less profit you relinquish; but, correspondingly, the less certain that profit is. When you hedge in practise, you might want to figure out the optimal hedge ratio.

Rehedging triggered by changes in the hedge ratio

Another way to rehedge more efficiently is to trigger the rehedging on a substantial change in delta; you only rehedge when there is a big change in the hedge ratio. This is a more sensible means of hedging, but the computation converges more slowly. Here is an example of hedging an at-the-money call with a delta trigger of 0.02 or 2% and a transactions cost of 0.1%.

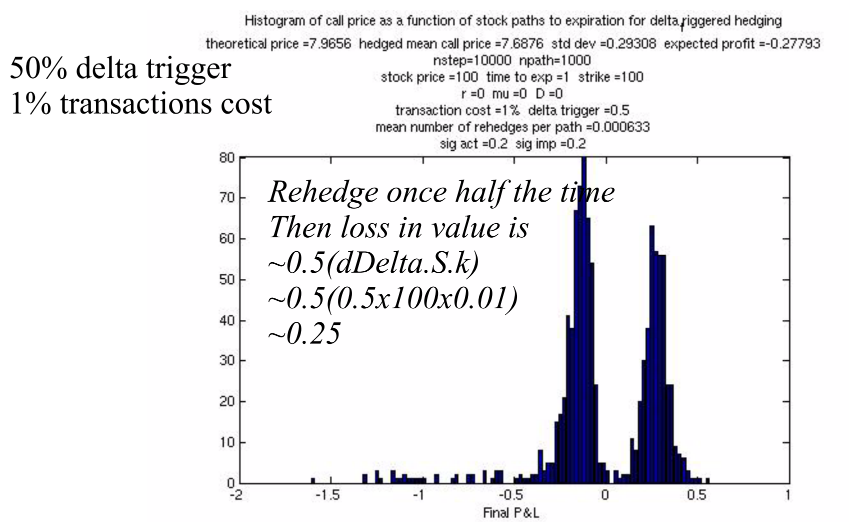
2% delta trigger
0.1% trans cost
202 hedges
-0.63 loss
std dev 0.38

better proft
less variation



Comparing this to the similar case where you rehedge at every step over 1000 steps, we see that the loss owing to the the transactions cost is smaller, and the standard deviation of the P&L is smaller too.

Here is the option being rehedged only when the delta changes by 50 percentage points and with a transactions cost of 1%.



The distribution is bimodal. The reason is that if you rehedge only when the delta of the option changes by 50 points, then rehedges only occur when the stock makes a substantial move up or down in order to achieve such a large change in the delta. Hence one set of final call prices involve no transactions costs (over the paths where delta changed by less than 50 points) and hence lie above the mean; the other set of call final call prices involve one rehedging and its cost (over the paths where delta did change by 50bp or more) and hence lie below the mean.

3.2.2 Analytical Approximations to Transactions Cost

Read this section to educate yourself, but we probably won't cover it in class.

In Lecture 2 we showed that the hedging error when the hedge volatility and the realized volatility are identical is given by

$$HE = \sum_{i=1}^{n} \frac{1}{2}\Gamma_i S_i^2 \sigma_i^2 (\varepsilon_i^2 - 1)\Delta t \qquad \text{Eq.3.1}$$

To leading order in Δt, the mean of the hedging error is zero, and the variance is $o([\Delta t]^2)$. If the option has time T to expiration, then the total number of rehedgings is $T/(\Delta t)$, so that the variance in the hedging error is $o\left(\frac{T[\Delta t]^2}{\Delta t}\right) = o(T\Delta t)$ which vanishes as $\Delta t \to 0$. Hedging continuously captures exactly the value of the option.

Now let's see what happens when you include transactions costs. To make things simple, let's consider the case where every time you trade the stock (buying *or* selling), you pay a fraction *k* of the cost of the shares traded.

Assume that you rehedge an option C with value *C* every time Δt passes. Then, every time you rehedge, you have to trade a number of shares equal to

$$\Delta(S + \delta S, t + \Delta t) - \Delta(S, t) \sim \frac{\partial^2 C}{\partial S^2}\delta S$$

Then the cost of this rebalancing is the value of number of shares traded times the fraction *k*, that is

$$\left|\frac{\partial^2 C}{\partial S^2}\delta S \times (kS)\right|$$

where the absolute value reflects the fact that you pay a positive transaction cost irrespective of whether you buy or sell shares.

If $\delta S = \mu S\Delta t + \sigma S\sqrt{\Delta t}Z$, then to order $(\Delta t)^{1/2}$ the expected transactions cost in time Δt is

$$E\left[\left|\frac{\partial^2 C}{\partial S^2}\sigma S^2 Z\sqrt{\Delta t}k\right|\right]$$

Since the expected value of $|Z|$ is not zero, the expected hedging cost in time Δt is non-zero too. For an option with time to expiration *T*, there are $T/(\Delta t)$ rehedgings, so that the total cost of rehedging is of order $\frac{T}{\Delta t}\sqrt{\Delta t} \sim \frac{1}{\sqrt{\Delta t}} \to \infty$ as the time between rehedgings goes to zero.

3.2.3 A PDE Model of Transactions Costs

One can approach transactions costs even more analytically in the framework of Hoggard, Whaley & Wilmott (see Wilmott's book *Derivatives*.)

Let

$$dS = \mu S dt + \sigma S Z\sqrt{dt}$$

From Lecture 2, the P&L of a hedged position when one includes transactions costs is given by

$$\begin{aligned} d\text{P\&L} &= dV - \Delta dS - \text{cash spent on transactions costs} \\ &= \left(\frac{\partial V}{\partial S} - \Delta\right)\sigma S Z\sqrt{dt} + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - \mu\Delta S\right)dt - \kappa S|N| \end{aligned}$$

where we have set the dividend yield D and the riskless rate *r* to zero, *N* is the number of shares traded to rehedge the initially riskless portfolio at the next interval, and the modulus sign reflects the fact that transactions costs are paid for both buying and selling shares.

Now we hedge the initial portfolio by choosing as usual $\Delta = \frac{\partial}{\partial S}V(S, t)$. After time δt we have to rehedge, so that

$$\begin{aligned} N(S, t) &= \frac{\partial}{\partial S}V(S + \delta S, t + \delta t) - \frac{\partial}{\partial S}V(S, t) \\ &\approx \frac{\partial^2 V}{\partial S^2}\Delta S \\ &\approx \frac{\partial^2 V}{\partial S^2}\sigma S Z\sqrt{\Delta t} \end{aligned}$$

to leading order in δt, and notice that N itself is stochastic.

Approximately, therefore, the average number of shares traded is

$$E[N] = \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S |Z| \sqrt{\delta t} = \sqrt{\frac{2}{\pi}} \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S \sqrt{\delta t}$$

with an average transactions cost

$$\sqrt{\frac{2}{\pi}} \frac{\partial^2 V}{\partial S^2} \kappa \sigma S^2 \sqrt{\delta t}$$

The expected value of the change in the P&L is therefore given by

$$\begin{aligned} dE[\text{P\&L}] &= E\left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - \mu \Delta S - \sqrt{\frac{2}{\pi \delta t}} \frac{\partial^2 V}{\partial S^2} \kappa \sigma S^2\right) dt\right] \\ &\approx \left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi \delta t}} \frac{\partial^2 V}{\partial S^2} \kappa \sigma S^2\right) dt\right] \end{aligned}$$

A reasonable expectation is that the holder of this on-average hedged portfolio would expect to earn the riskless rate. (The "on-average" means that we have averaged the transactions costs rather than keep them stochastic as they should be.) In that case, since the value of the hedged portfolio is $V - S\frac{\partial V}{\partial S}$, the expected value of the portfolio a time dt later should be $r\left(V - S\frac{\partial V}{\partial S}\right) dt$.

Inserting this expression into the LHS of the equation above leads to the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 - \sqrt{\frac{2}{\pi \delta t}} \left|\frac{\partial^2 V}{\partial S^2}\right| \kappa \sigma S^2 + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.3.2}$$

This is a modification of the Black-Scholes partial differential equation with a nonlinear additional term proportional to the absolute value of $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

Because of the nonlinearity, the sum of two solutions to the equation is not necessarily a solution too; you cannot assume that the transactions costs for a portfolio of options is the sum of the transactions costs for hedging each option in isolation.

For a single long position in a call or a put, $\frac{\partial^2 V}{\partial S^2} \ge 0$, so we can drop the modulus sign. Equation 3.2 then becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.3.3}$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}}$$

This is the Black-Scholes equation with a modified reduced volatility, first derived by Leland, and the option is worth less. For a short position, the effective volatility is enhanced, given by

$$\hat{\sigma}^2 = \sigma^2 + 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}}$$

and the option is worth more.

The effective volatility is

$$\hat{\sigma} \approx \sigma \pm \kappa\sqrt{\frac{2}{\pi\delta t}} \qquad \text{Eq.3.4}$$

For very small δt this expression diverges and the approximation becomes invalid.

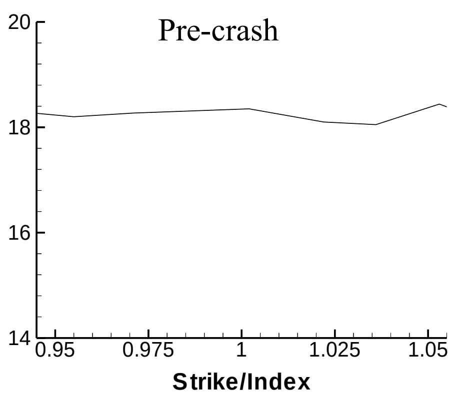
3.3 More About The Smile

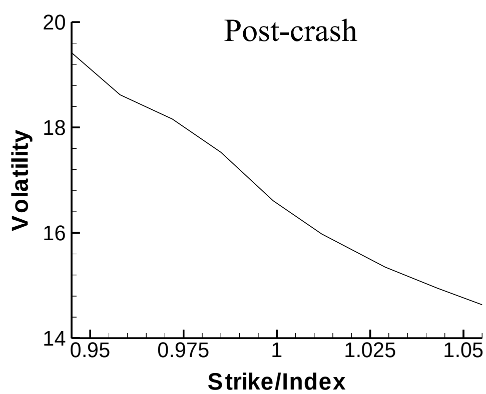
The Columbia Smile Generated by a Truck with Stochastic Volatility in 2004

3.3.1 Equity index smiles: a reminder

Since the '87 crash there has been a persistent skewed structure in Black-Scholes implied volatilities in most world equity option markets.

Representative implied volatility skews of S&P 500 options. (a) Pre-crash. (b) Post-crash. Data taken from M. Rubinstein, "Implied Binomial Trees" *J. of Finance*, 69 *(*1994) pp. 771-818.

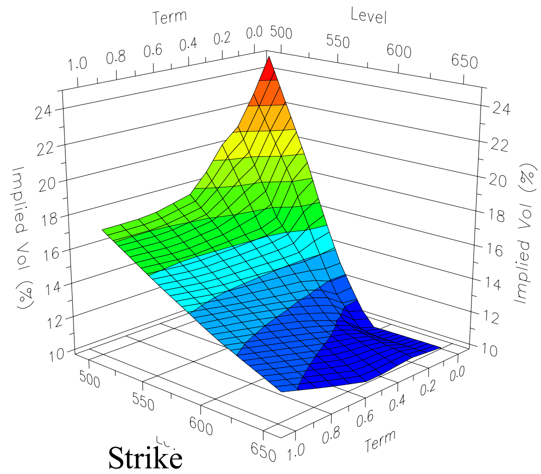




The Black-Scholes model assumes that volatility is independent of strike and time to expiration. But the Black-Scholes model has no simple way of allowing the implied volatility of the *stock* to depend upon the *option* strike or time. The stock's volatility cannot be influenced by the option whose price you quote.

Here is an old but typical S&P smiles plotted against strike K.

S&P
September 27, 1995.



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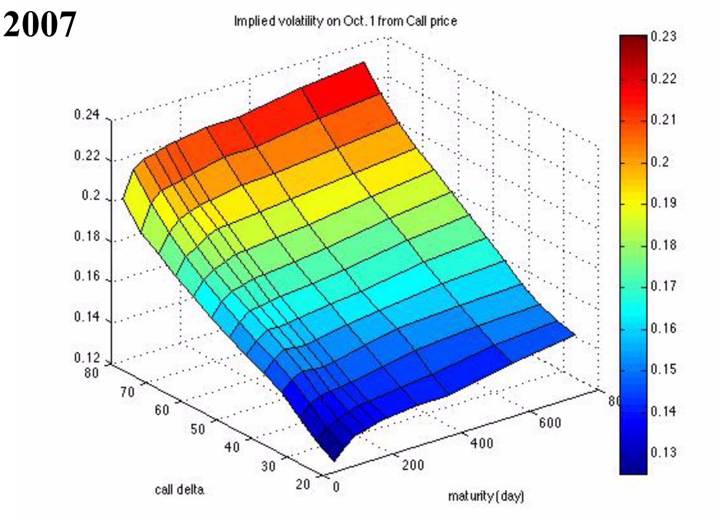

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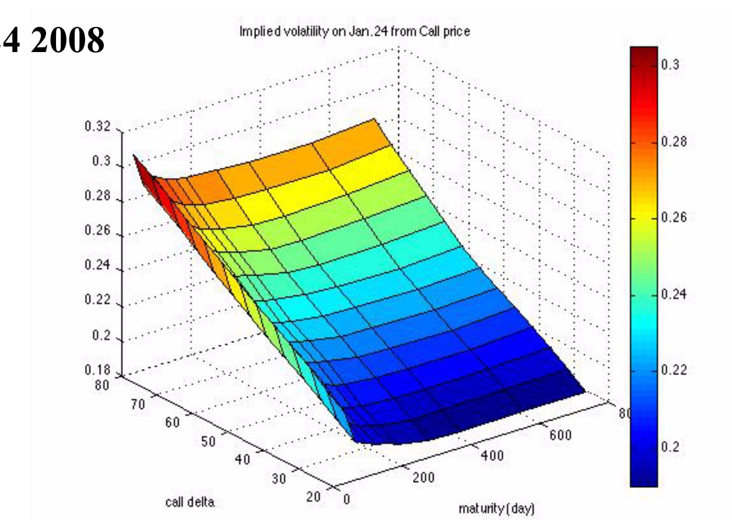


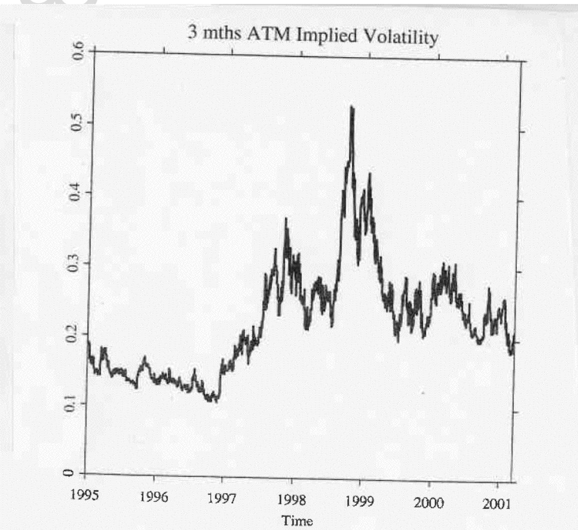


Fig. 2.13. The three-months ATM IV levels of DAX index options

short-term implieds
move more
than long-term

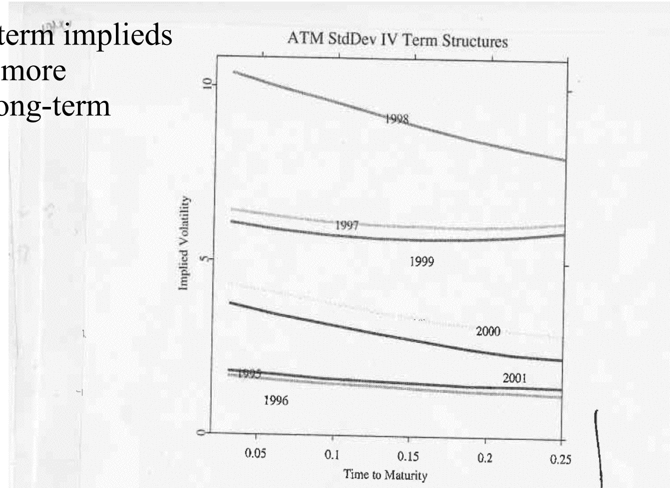


Fig. 2.12. ATM standard deviation of IV term structures between 1995 and 2001, computed from smoothed surfaces

negative correlation
during crisis

From Fengler's book

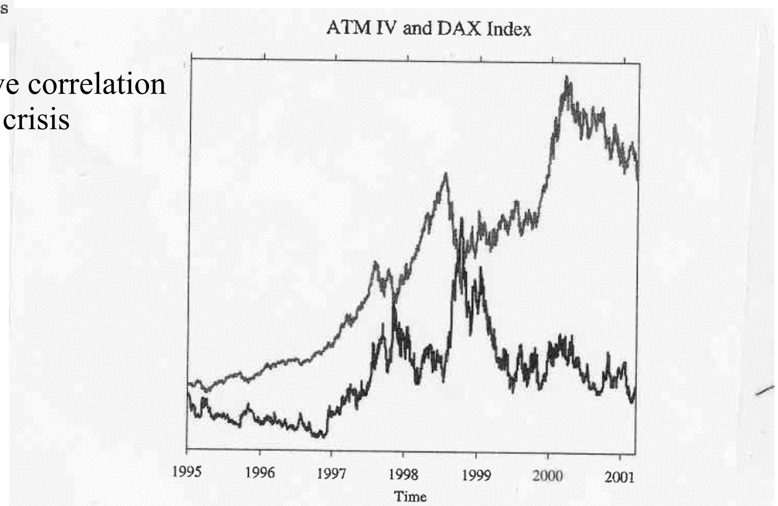


Fig. 2.14. German DAX $\times 10^{-4}$ (*upper line*) and three-months ATM IV levels (*lower line*), also given in Fig. 2.12

FIGURE 3.1. Implied Volatility as a Function of Strike/Spot for Different Expirations. (Crash-o-phobia: A Domestic Fear Or A Worldwide Concern? Foresi & Wu JOD Winter 05

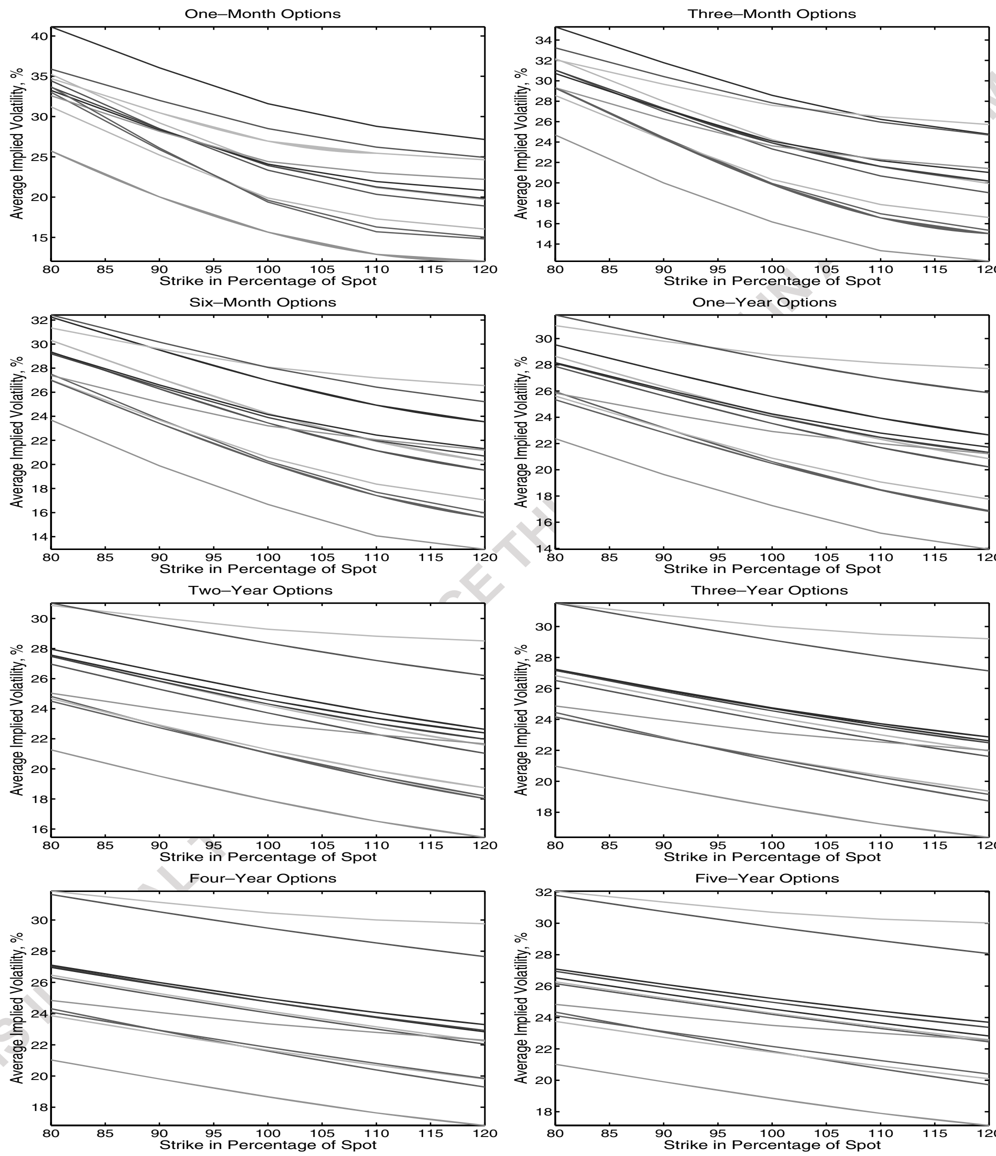
The quoting convention is the Black-Scholes implied volatility

Notice the patterns that persist across all indexes:

- out-of-the-money puts have higher implied Black-Scholes volatilities than out-of-the-money calls. (Why?)
- The slope of implied volatility against strike as a percentage of spot is negative, even for long maturities, though not as steep as for short maturities.

E X H I B I T 2

Implied Volatility Smirk on Major Equity Indexes



Lines represent the sample averages of the implied volatility quotes plotted against the fixed moneyness levels defined as strike prices as percentages of the spot level. Different panels are for options at different maturities. Data are daily from May 31, 1995, to May 31, 2005, spanning 2,520 business days for each series. The 12 lines in each panel represent the 12 equity indexes listed in Exhibit 1.

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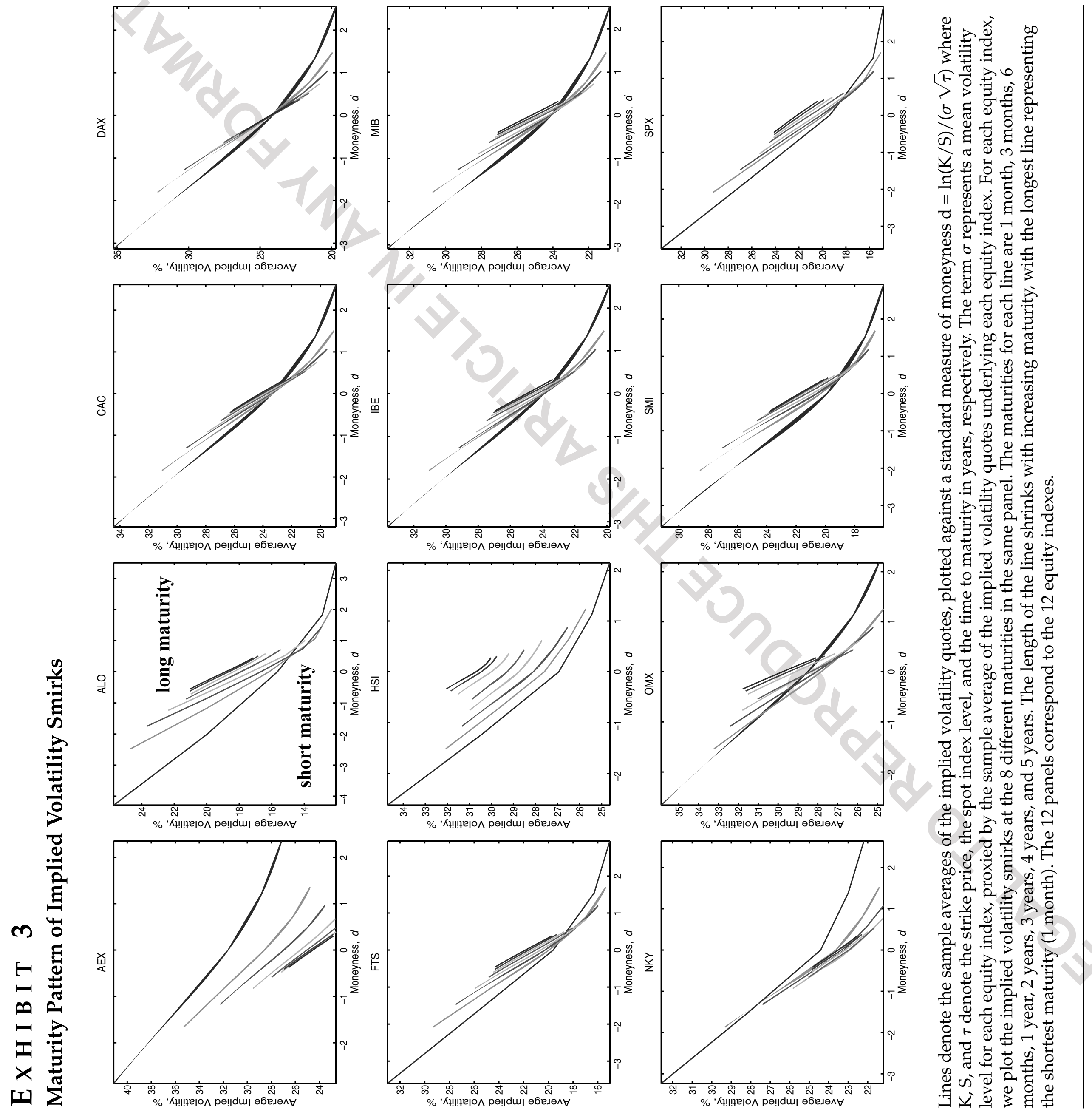
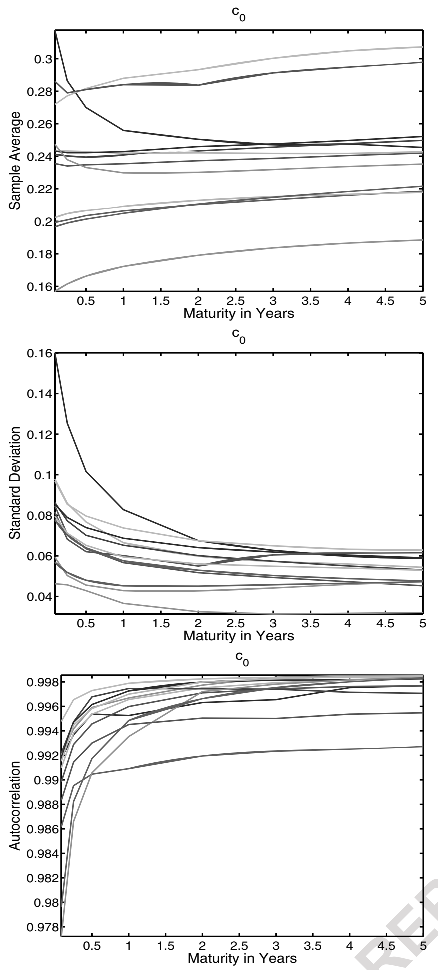
FIGURE 3.2. Implied Volatility as a Function of $\left(\log\dfrac{\text{Strike}}{\text{Spot}}\right)/(\sigma\sqrt{\tau})$

related to d_1



When plotted against the number of standard deviations between the log of the strike and the log of the spot price for a lognormal process, the slope of the skew actually increases with expiration. Whatever is happening to cause this doesn't fade away with future time.

FIGURE 3.3. Behavior of implied volatility level c_0 as a function of option expiration.



term structure of implied volatility is roughly flat

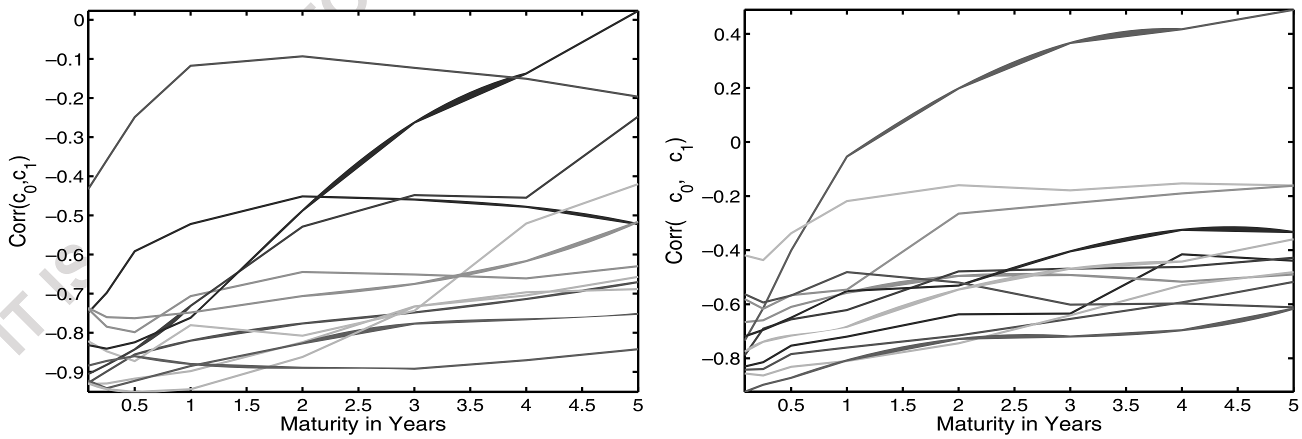
volatility of volatility decreases with expiration, suggesting mean reversion or stationarity for the instantaneous volatility evolution

daily autocorrelation of implied volatility is large, and larger for longer maturities
[excitement or depression tends to continue]

FIGURE 3.4. The cross-correlation between volatility level and slope of the skew is large.

EXHIBIT 5

Cross Correlations between Volatility Level and Smirk Slope



Lines denote the cross-correlation estimates between the volatility level proxy (c_0) and the volatility smirk slope proxy (c_1). The left panel measures the correlation based on daily estimates, the right panel measures the correlation based on daily changes of the estimates.

short-term slope tends to get more negative as volatility increases

3.3.2 Some characteristics of the equity implied volatility smile

- Volatilities are steepest for small expirations as a function of strike, shallower for longer expirations.

- The minimum volatility as a function of strike occurs near atm strikes o strikes corresponding to slightly otm call options.

- Low strike volatilities are usually higher than high-strike volatilities, but high strike volatilities can also

- The term structure is usually increasing but can change depending on views of the future. After large sudden market declines, the implied volatility out-of- the-money calls may be greater than for atm calls, reflecting an expectation that the market may rebound.

- The volatility of implied volatility is greatest for short maturities, as with Treasury rates.

- There is a negative correlation between changes in implied atm volatility and changes in the underlying asset itself. [Fengler: $\rho = -0.32$ for the DAX in the late 90s, for three-month expirations.]

- Implied volatility appears to be mean reverting with a life of about 60 days.

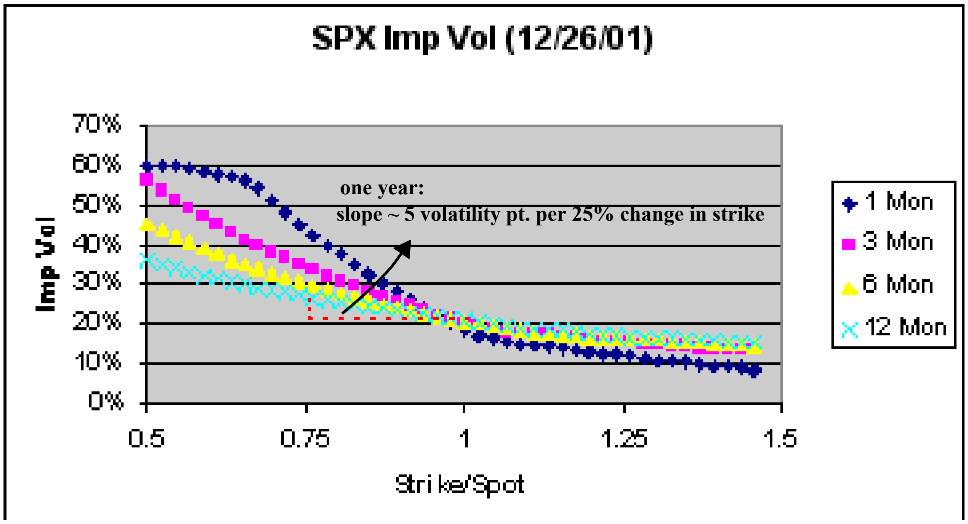
- Implied volatility tends to rise fast and decline slowly.

- Shocks across the surface are highly correlated. There are a small number of principal components or driving factors. We'll study these effects more closely later in the course.

- Implied volatility is usually greater than recent historical volatility..

3.3.3 Different Smiles in Different Markets

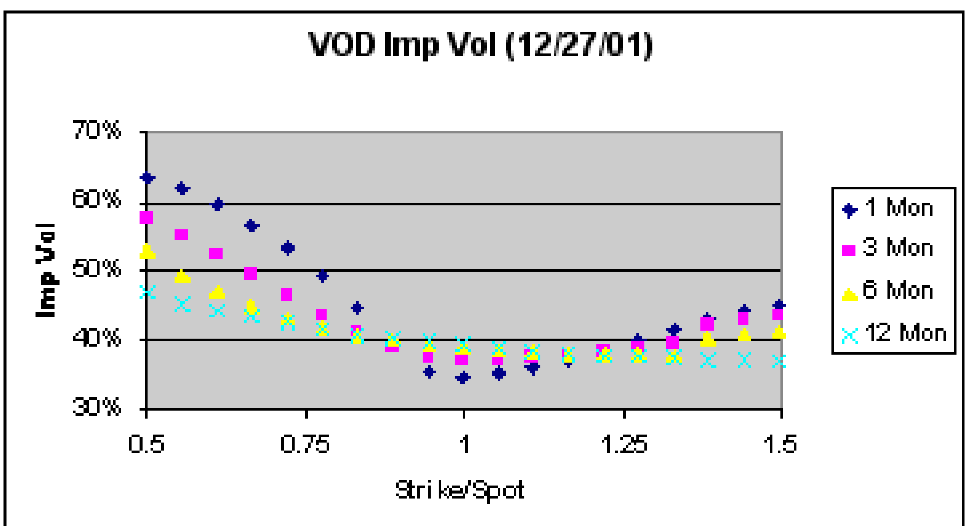
Here are some smiles for the S&P 500, plotted a little differently:



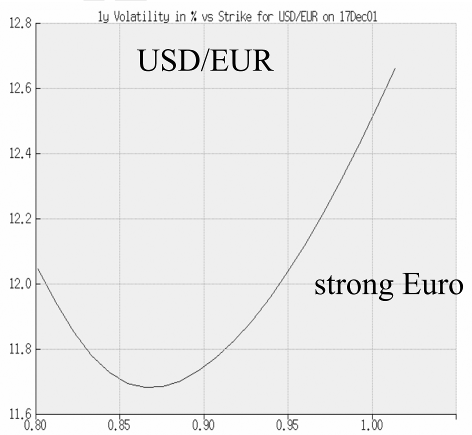
Indexes generally have a negative skew. The slope here for a one-year option is of order 5 volatility points per 250 S&P points, or about $\frac{0.05}{250} = 0.0002$. Note that the slope for a 3-month option is about twice as much, which roughly confirms the idea that the smile depends on $\frac{(\ln K/S)}{(\sigma\sqrt{\tau})}$, because a four-fold decrease in time to expiration then implied a doubling of the slope of the smile. The magnitude of the slope of the one-month option volatility is about 23 volatility points per 250 S&P points, or about 0.001.

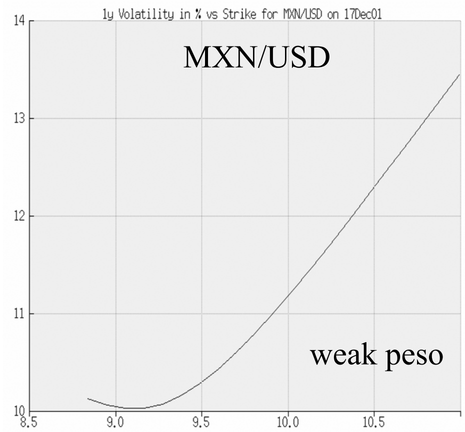
3.3.4 Single stock smiles

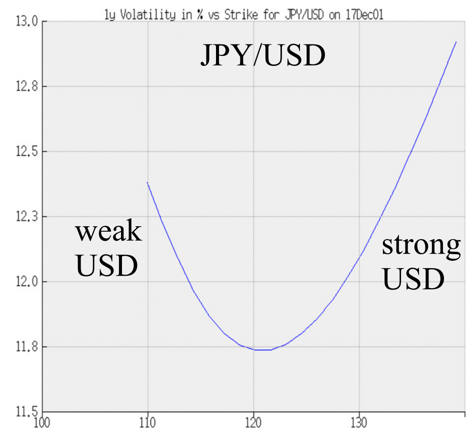
A single stock smile is more of an actual smile with both sides turning up..



3.3.5 Some currency smiles....







ATM Strike = 0.90 9.85 123.67

The smiles are more symmetric for “equally powerful” currencies, less so for “unequal” ones. Equally powerful currencies are likely to move up or down. There are investors for whom a move down in the dollar is painful, but there are investors for whom a move down in the yen, i.e. up in the dollar, is equally painful. Hence, there is a motive for symmetry. FX smiles tend to be more symmetric and resemble a real smile.

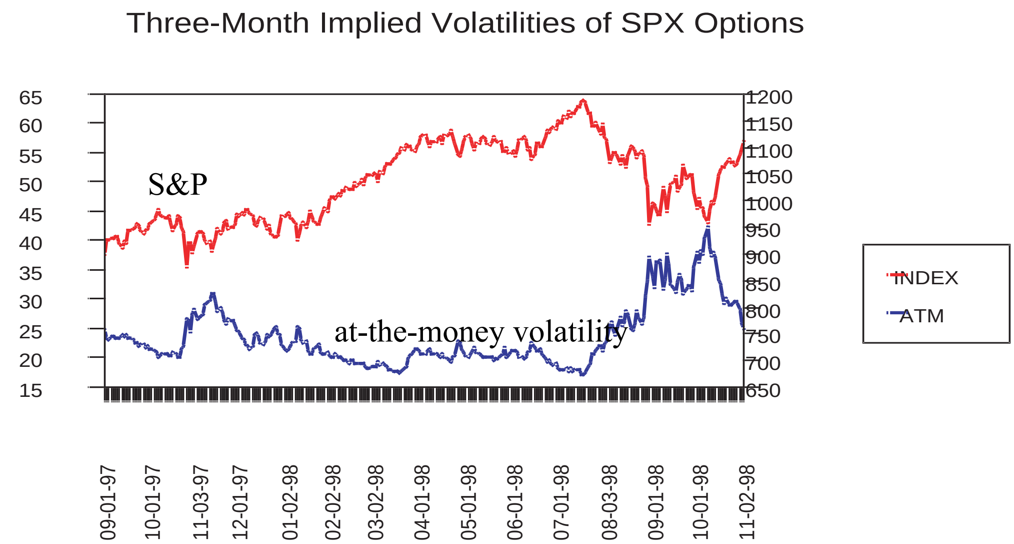
Equity index smiles tend to be skewed to the downside. The big painful move for an index is a downward move, and needs the most protection. Upward moves hurt almost no-one. An option on index vs. cash is very different and much more asymmetric than an option on JPY vs. USD.

Single-stock smiles tend to be more symmetric than index smiles. Single stock prices can move dramatically up or down. Indexes like the S&P when they move dramatically, move down.

Interest-rate or swaption volatility, which we will not consider much in this course, tend to be more skewed and less symmetric, with higher implied volatilities corresponding to lower interest rate strikes. This can be partially understood by the tendency of interest rates to move normally rather than lognormally as rates get low.

3.3.6 Variation of implied volatility and the smile over time

Example: here is the behavior of “volatility” itself as time passes.



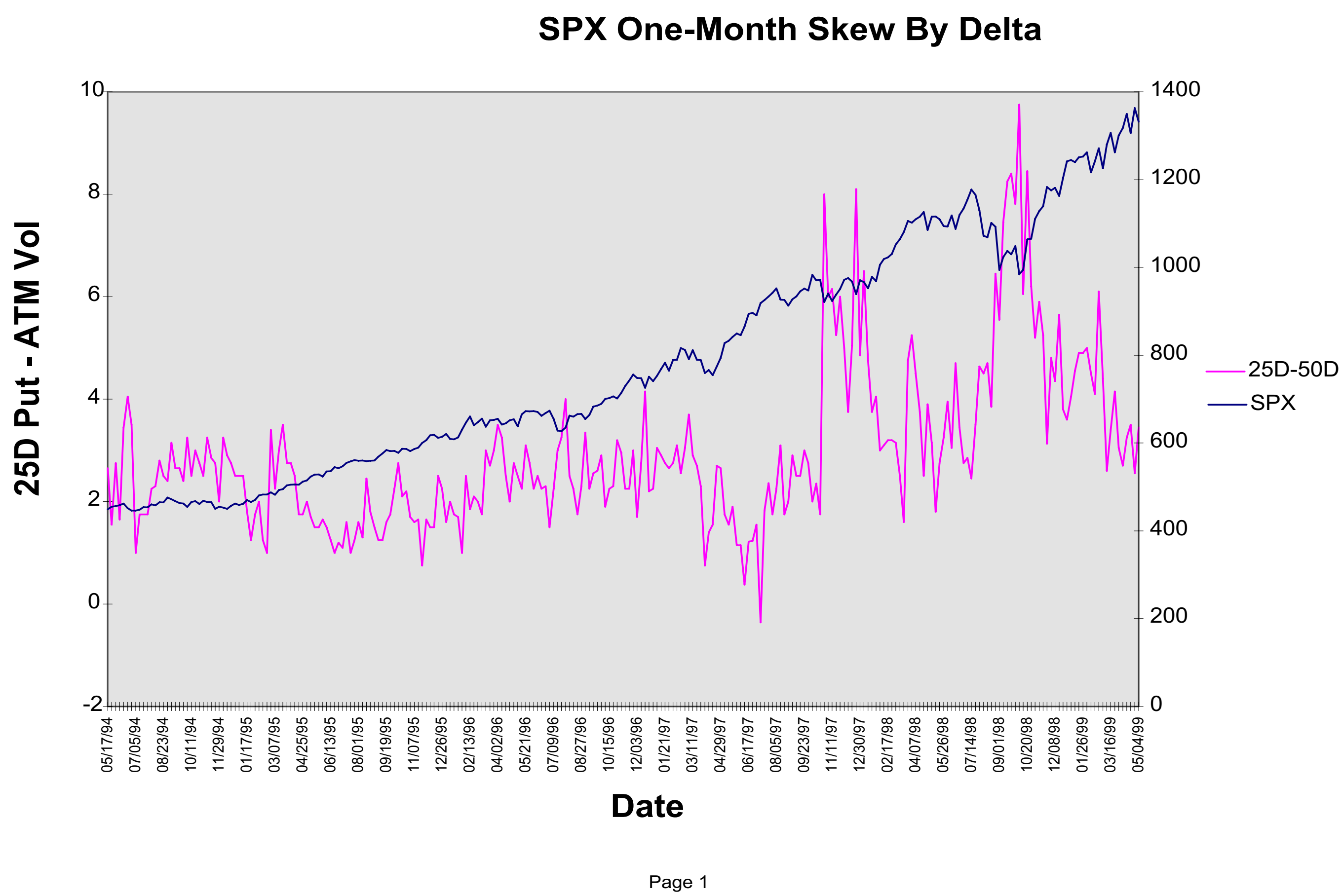
Here “volatility” goes up as the index goes down, and vice versa, but the volatility plotted is the *at-the-money* volatility $\Sigma(S, t, S, T)$ which is the implied volatility of a different option each day, because as the index level S changes the atm strike level changes. ATM volatility is therefore not the volatility of a particular option you own.

If the index’s negative smile doesn’t move as time passes and the index level changes, then at-the-money volatility will go up when the index goes down simply because the atm strike moves down with index level, and lower strikes have higher implied volatilities. Thus, some of the apparent correlation in the figure above would occur even if $\Sigma(S, t, K, T)$ didn’t change with S at all. How much of the correlation is true co-movement and not incidental?

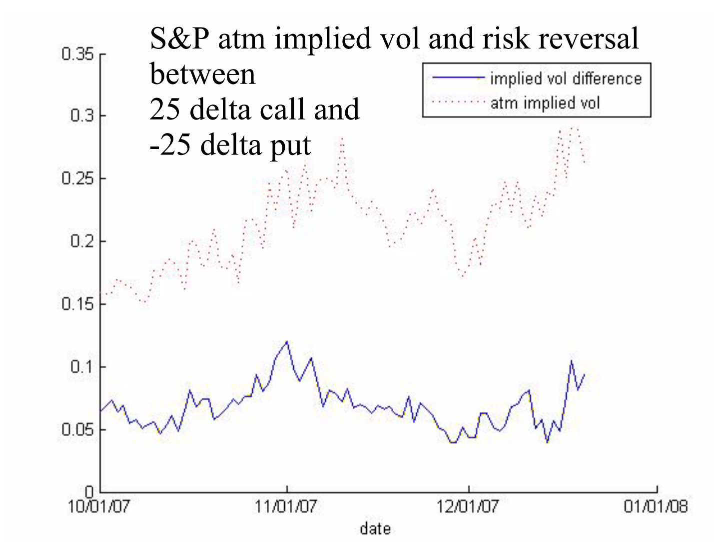
A NOTE ABOUT FIGURES OF SPEECH: People in the market often talk about how “volatility changed.” One must be very careful in speaking about volatility because there are so many different kinds of volatility. There is realized volatility σ, at-the-money volatility, and implied volatility for a *definite strike*, $\Sigma = \Sigma(S, t;K, T)$ which can vary with S,t and K,T. When you talk about the change in Σ, what are you keeping fixed?

For example, at-the-money volatility is $\Sigma_{atm} = \Sigma(S, t;S, T)$ which constrains strike to equal spot. When you talk about how this moves, it’s a very different quantity from volatility of an option with a fixed strike. It’s a little like the difference between talking about the yield of the 2016 bond and the yield of the ten-year constant maturity bond over time. Those are different things: one ages and the other doesn’t.

The index skew's variation with time and with market level



More recently:notice how skew varies with atm implied volatility.



3.4 Consequences of the Smile for Trading

What are the consequences of the smile for people concerned with trading and hedging?

Obviously, the assumed dynamics of the underlyer in the Black-Scholes model is inconsistent with the smile. However, liquid standard call and put options prices are obtained from the market and simply *quoted* via the Black-Scholes formula, no matter how they were generated, and so the model doesn't really matter that much for pricing in a market-making or manufacturing business. (The model does matter if you wanted to generate your own idea of fair options values and then arbitrage them against market prices, but that is a very risky long-term business.)

However, if you have a position in standard options that you want to hedge, then, even if the price is known, the hedge ratios are model-dependent, and if you don't get it right you cannot replicate the option accurately. Furthermore, if you want to take positions in illiquid OTC exotic options, their values must be estimated from a model. The question in both of these cases is of course: which model?

3.5 How to Graph the Smile?

We observe implied volatility as a function of strike at a given time t_0 when the stock or index is at S_0; that is we are given only $\Sigma(S_0, t_0, K, T)$ as a sort of snapshot. Our problem is similar to that of yield curve modeling: you see the yield curve at one instant and wonder what happens to it later. Similarly, if we are interested in volatility, what we want to know is its dynamic behavior as a function of S and t, namely $\Sigma(S,t;K,T)$, assuming implicitly that the Black-Scholes Σ is the appropriate way to indicate value.

We can plot $\Sigma(\ )$ against strike K, moneyness K/S, forward moneyness K/S_F, $(\ln K/S)/(\sigma\sqrt{\tau})$ or even more generally $\Delta = N(d_1)$, which depends on stock price, strike, time to expiration and volatility.

Traders usually like to plot the smile against Δ because they believe that the shape of the smile changes less with time and stock price level when it is plotted against Δ. There are some other good practical reasons for preference too:

- Plotting implied volatilities against Δ immediately indicates the hedge to put on for an option at that strike.
- Since every option of any strike and maturity has a Δ, you can compare the implied volatilities of different expirations and strikes on one scale.
- Finally, Δ is approximately equal to the risk-neutral probability $N(d_2)$ that the option will expire with $S > K$ – that is in the money – and therefore seems like a sensible behavioral variable that traders might care about. Plotting it against Δ embodies the notion that what matters for an option's price is how likely it is to move into the money from wherever it is now.

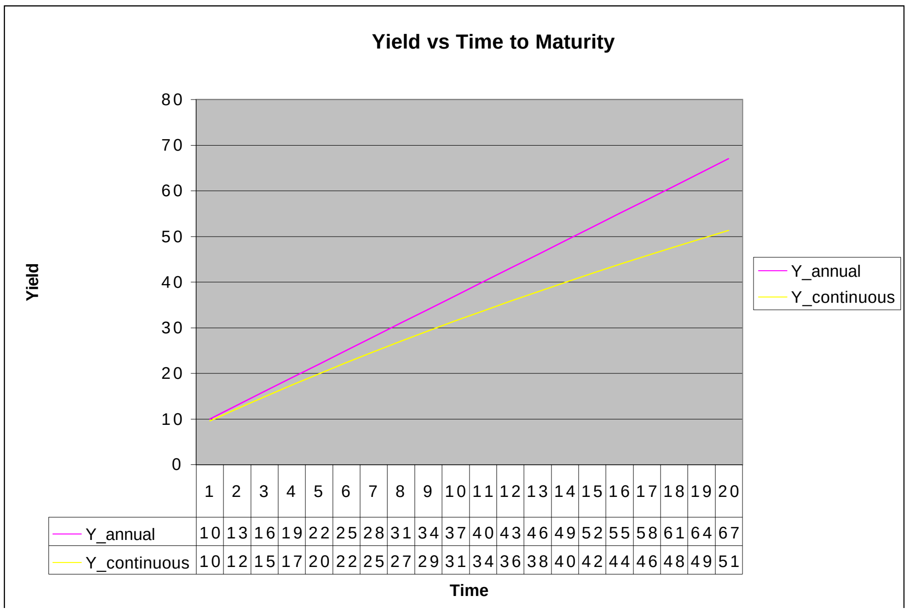
3.5.1 Plotting the Smile - variable choice can matter

The right scale really depends on the process that determines the evolution of the stock price and its volatility.

If volatility follows one process and you plot it using a different quoting convention, you can see spurious dynamics that results from your bad choice of variables. Let's look at a simple example in the world of interest rates. Suppose that people think about the yield to maturity on a bond, and always use the annual compounding convention, so that the present value of a \$100 payment delivered at a time t in the future is $100/(1+y_a)^t$. If we choose to think about yields as continuously compounded, then we would write the same present value as $100 \times e^{-y_c t}$, and since the values are equal, $y_c = \ln(1+y_a)$.

Now suppose that the risk premium for longer-term bonds is such that people require higher yield for longer maturity, with annual yield proportional to maturity, so that the graph of yield vs. maturity is a straight line.

Suppose that we measured yields using continuously compounded yield to maturity. Then we would observe different-looking yield curves because of the change in convention. A linearly sloped yield curve for y_a would look non-linear for y_c and would seem to be a puzzle.



Another example is the case where stock evolution is arithmetic rather than geometric Brownian motion. The lognormal volatility of an arithmetic Brownian motion with constant volatility is not itself constant, but varies inversely with the level of the underlyer. Here again, plotting lognormal volatility against underlyer level would lead to a mysterious dependence on underlyer level which we could understand as being equivalent to a constant volatility with arithmetic Brownian motion.

Using the wrong quoting convention can distort the simplicity of the underlying dynamics. Perhaps the Black-Scholes model uses the wrong dynamics for stocks and therefore the smile looks peculiar in that quoting convention. That's the underlying hope behind advanced models of the smile.