LECTURE 16

Wrap-Up on Local Volatility Models

Relation between Skew Statics and Dynamics.

Stochastic Volatility Models

Looking Ahead

Values and Hedge Ratios of Exotics

Wrap up on Local Volatility Models

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Hedging Rules and their Efficacy/ Statics and Dynamics

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Stochastic Volatility Models

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Jump Diffusion Models

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Guest Speakers
Michael Kamal - April 15
Jackie Rosner - April 20

If you have questions come to my office hours or see me some other time by appointment.

Lookback Call Deltas With A Smile are Different from Black-Scholes

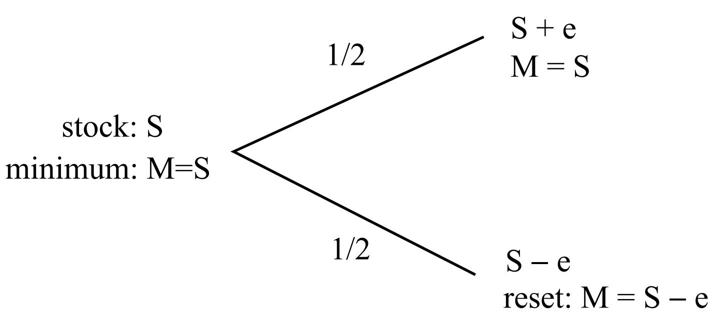
Claim: In the Black-Scholes model at inception, a lookback call has a delta of approximately zero:

Indicative Proof: Let $V(S, M, \tau)$ be the value of the lookback call when the stock is S and the minimum value of the index so far is M, and τ is the time to expiration.

First: intuitively, when $M = S$, a little increase in S is the same as a correspondingly small decrease in M, so that

$$\left.\frac{\partial V}{\partial S}\right|_{M = S} = -\left.\frac{\partial V}{\partial M}\right|_{M = S}$$

Second: consider the next infinitesimal move $e = \sigma\sqrt{\tau}$ in the stock price:



$$\begin{aligned} V(S, S, \tau) &= \frac{1}{2}V(S+e, S, \tau - d\tau) + \frac{1}{2}V(S-e, S-e, \tau - d\tau) \\ &\approx V(S, S, \tau) - \frac{\partial V}{\partial M}\frac{e}{2} + O(\tau) \end{aligned}$$

Then by backwards induction in the risk neutral world with zero interest rates, to leading order.

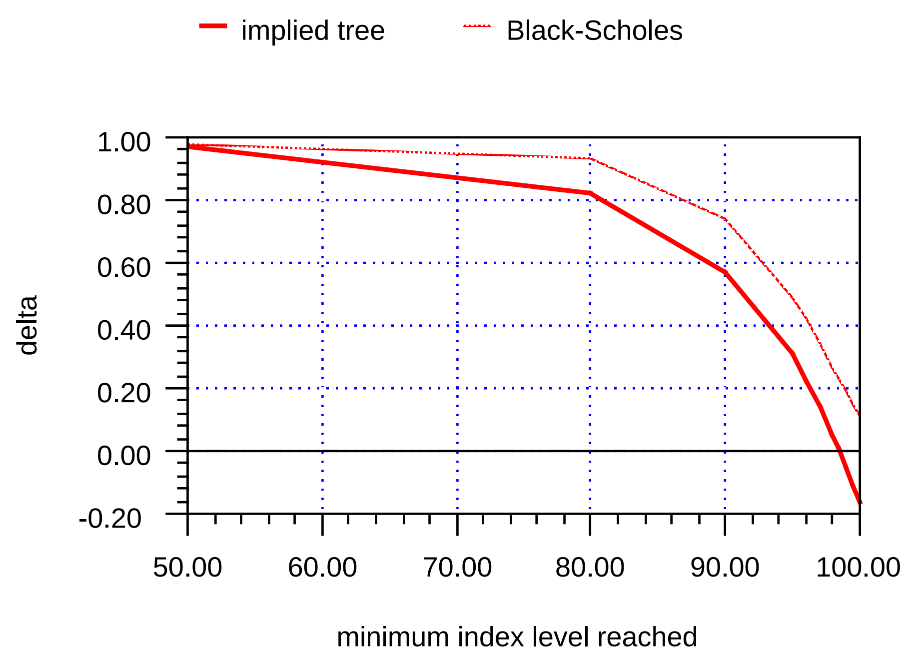
Thus for M = S, $\frac{\partial V}{\partial S} \equiv \frac{\partial V}{\partial M} = 0$ and so the delta of the lookback is approximately zero under these conditions.

Local volatility model deltas compared to Black-Scholes deltas for the one-year lookback call described above, for a range of minima M.

- Black-Scholes deltas are calculated at the Black-Scholes implied volatility of 15.6%.
- The local-volatility delta of an at-the-money lookback call is negative – to hedge a long call position you must actually go long the index! Increasing the stock price decreases the value of the option! It's advantageous for the stock price to first drop.
- The delta of the lookback call is always lower than in Black-Scholes.
- The mismatch is greatest where volatility sensitivity is largest, where M = S.

  The mismatch is smallest when the lowest level previously reached is much lower than the current index level, since M is unlikely to change and the lookback is effectively a forward contract with zero volatility sensitivity.

FIGURE 1.1. The delta of a one-year call with a three-month lookback period that has identical prices in the implied tree model and the Black-Scholes world with no skew. The current market level is assumed to be 100.



Practical Calibration of Local Volatility Models

In practice we are given implied volatilities $\Sigma(K_i, T_i)$ and must calibrate a smooth local volatility function. To use Dupire's equation, we need a smooth implied volatility surface that is at least twice differentiable in the strike direction and once differentiable in the time direction, and that doesn't violate arbitrage.

But all we have is discrete points with noise. They have to be smoothed.

.The most straightforward way to do this is to write down a smooth parametric *non-arbitrage-violating* form for the implied volatilities, and then compute the parameters that minimize the distance between computed and observed standard options prices.

One can then calculate the local volatilities by taking the appropriate derivatives of the implieds. One difficulty with this method is how to determine a realistic form of the parametrization, particularly on the wings where prices are hard to obtain.

Another way is to write down a plausible formula for local vols, since as long as they are positive, they will guarantee no arbitrage violation. But who knows what the tails should look like?

Other methods involve splines (nonparametric) or semiparametric interpolations.

There are many papers on this.

SVI (Stochastic Volatility Inspired) Parametrization

In its original formulation (Gatheral, 2004), SVI model is defined at each maturity T in terms of the 5 parameters a, b, ρ, m, σ such that the square of the implied volatility $\theta(K,T)$ is

$$\begin{aligned} \theta^2(K,T) &= v(x,T) = a + b\Big(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2}\Big) \\ x &= \ln(K/F(T)) \end{aligned} \tag{1}$$

where $F(T)$ is the forward and the parameters lie in the following definition domain

$$b > 0 \tag{2}$$

$$\sigma \geq 0 \tag{3}$$

$$\rho \in [-1,1] \tag{4}$$

$$a \geq -b\sigma\sqrt{1-\rho^2}. \tag{5}$$

(Rogers & Tehranchi, 2008) derived the necessary condition for no-strike arbitrage

$$\forall x,\, \forall T, \quad \Big|\frac{\partial v(x,T)}{\partial x}\Big| \leq 4, \tag{6}$$

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

Problems and Benefits of Local Volatility Models

Inadequacy of the Short-Term Skew

For equity indexes, future short-term local volatilities have less skew than current short-term implied volatilities.Therefore the short-term future skew in a local volatility model is too flat. One needs the threat of jumps in the near future to produce a short-term skew.

A good model would look more or less time-invariant.

On the other hand, all financial models need recalibration; even in Black-Scholes, the implied volatility changes from day to day. Local volatility models, like Black-Scholes, must be recalibrated regularly; they allow the valuation of exotic options consistent with the volatility surface for vanilla options, and are widely used as a means of valuing exotics that will be hedged with vanillas.

The question is: to what extent do they mirror the behavior of realized volatility?

They are not good for options on volatility. Why?

Local Vol May Provide Better Hedge Ratios During Volatility Regimes

The best hedge is the one that minimizes the variance of the P&L of the hedged portfolio. If the replication were exact, the variance of the P&L would vanish.

Compare local volatility hedge ratios and P&L vs. Black-Scholes hedge ratios and P&L.

Call $C(S, t, K, T, \Sigma(S, t, K, T))$

Hedged portfolio $\Pi = C - \Delta S$

$$\Pi_{BS} = C - \Delta_{BS} S$$

$$\Pi_{loc} = C - \Delta_{loc} S$$

The difference between the BS-hedged P&L and the local-volatility-hedged P&L for a move δS:

$$\delta\Pi_{loc} - \delta\Pi_{BS} = (-\Delta_{loc} + \Delta_{BS})\delta S \equiv \varepsilon \times \delta S$$

In a negatively skewed market,

$$\Delta_{loc} = \Delta_{BS} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \approx \Delta_{BS} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \le \Delta_{BS}$$

and so $\varepsilon = (\Delta_{BS} - \Delta_{loc}) \ge 0$.

From the theory of hedging in Lecture 2, the individual P&L's are:

$$\delta\Pi_{BS} = \frac{1}{2}\Gamma_{BS}S^2\left[\sigma_R^2 - \Sigma^2\right]\delta t$$

$$\delta\Pi_{loc} = \frac{1}{2}\Gamma_{loc}S^2\left[\sigma_R^2 - \sigma(S, \delta t)^2\right]\delta t$$

since the p.d.e. for local volatility is the Black-Scholes equation with Σ replaced by $\sigma(S, \delta t)$, and if the hedging is perfect, $\delta\Pi_{loc}$ should vanish.

$\delta\Pi_{BS}$ is positive or negative depending on whether σ_R is greater or less than Σ.
$\delta\Pi_{loc}$ is positive or negative depending on whether σ_R is greater or less than $\sigma(S, \delta t)$.

What happens to $\delta\Pi_{BS}$ in actual markets?

Combining the equation, we see that the BS P&L in a world with negative skew is

$$\delta\Pi_{BS} = \delta\Pi_{loc} - \varepsilon \times \delta S = \frac{1}{2}\Gamma_{loc}S^2\left[\sigma_R^2 - \sigma(S, \delta t)^2\right]\delta t - \varepsilon \times \delta S$$

The gamma term is quadratic and non-directional, and depends on the volatility mismatch, and the δS hedging mismatch is linear and directional.

Crepey: Four different market regimes: indexes can move up or down, with high or low realized volatility compared to instantaneous local volatility.

TABLE 1. Types of Markets: Equity index markets have the characteristics of the yellow cells.

Volatility / Direction	volatile	non-volatile
up	$\sigma_R > \sigma(S, \delta t), \delta S > 0$	$\sigma_R < \sigma(S, \delta t), \delta S > 0$
down	$\sigma_R > \sigma(S, \delta t), \delta S < 0$	$\sigma_R < \sigma(S, \delta t), \delta S < 0$

For volatile down markets (a fast sell-off), both terms for $\delta\Pi_{BS}$ are positive, and the errors to $\delta\Pi_{BS}$ are additive. The Black-Scholes P&L differs from zero (the perfect hedge value) due to two additive contributions.

For non-volatile up markets (slow rise), both terms are negative and the same is true.

For slow sell-offs or fast rises, the two terms have opposite signs, and the hedging errors tend to cancel. Therefore, the Black-Scholes hedging strategy will perform worst in fast sell-offs or slow rises, which is just what characterizes negatively skewed equity index markets.

Crepey has also backtested the hedging of actual options to show that the P&L of a hedged portfolio has less variance under the local volatility hedging strategy.

Regimes of Volatility

How do implied volatilities of options with definite strikes actually behave as stock price changes and time passes? What happens to at-the-money volatility?

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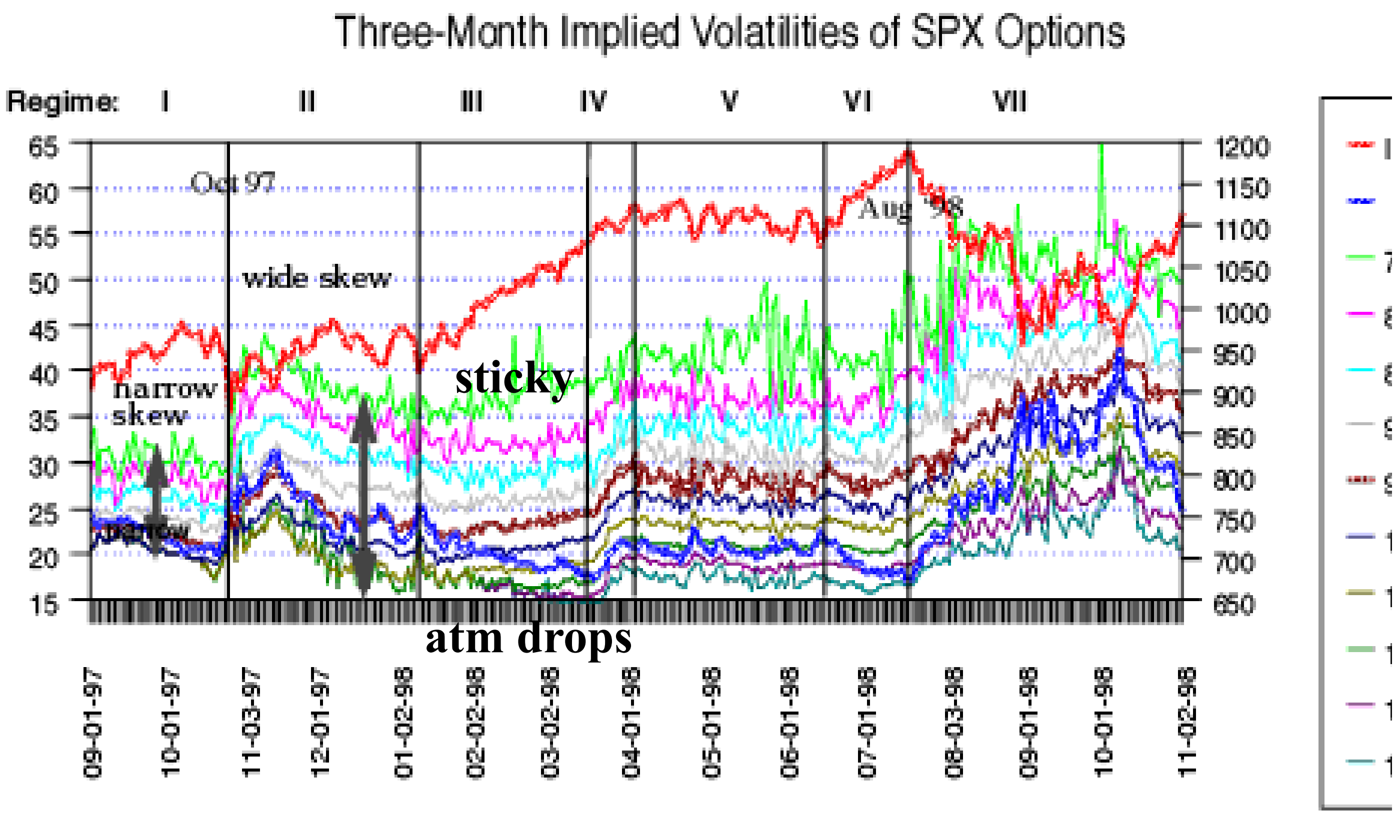
FIGURE 3. The implied volatilities of S&P 500 options from September 1997 through October 1998.



Heuristic Rules & Models for Variation of Implied Volatility Σ: Skew Relates Statics to Dynamics

Traders like heuristic rules for the B-S *quoting parameter* Σ rather than models of stochastic evolution.

Specify what **doesn't change** rather than what changes. Invariance principles or symmetries.

The Sticky Strike Rule.
The Sticky Delta Rule.
The Sticky Local Volatility Rule.

The Sticky Strike Rule

Each option of a definite strike maintains its initial implied volatility – hence the "sticky strike" appellation. This is the simplest "model" of implied volatility:

$$\Sigma(S, K, t) = \Sigma_0 - b(K - S_0) \qquad \text{Sticky Strike Rule, independent of } S \text{ for all } t$$

(We have assumed $b(t) \equiv b$, independent of t. b can change, especially during crisis periods.)

Intuitively, "sticky strike" is a poor man's inconsistent attempt to preserve the BS model.
As S moves, each option keeps the exactly the same constant future instantaneous volatility in its evolution, inconsistently different for different options.

Implied volatility for an option of strike K is independent of S, and therefore $\Delta = \Delta_{BS}$.

TABLE 2. Volatility behavior using the sticky-strike rule.l

Quantity	Behavior
Fixed-strike volatility:	is independent of index level
At-the-money volatility $\Sigma_{atm}(S)$:	$\Sigma_{atm}(S, t) = \Sigma_0 - b(S - S_0)$ which decreases as index level increases
Exposure Δ:	$= \Delta_{BS}$

You can think of this model as representing Irrational Exuberance. Σ_{atm} decreases as S increases.

The Sticky Delta/Moneyness Rule

It's easier to start by explaining the related concept of sticky moneyness.

Sticky moneyness means that an option's volatility depends only on its moneyness K/S or, roughly, $K - S$

$$\Sigma(S, K, t) = \Sigma_0 - b(K - S) \qquad \text{Sticky Moneyness Rule}$$

Intuition: the volatility of the most liquid option, should stay constant as the index moves. Similarly, a 10% out-of-the-money should always have same volatility.

It's a scale-invariant model of common sense and moderation.

For a roughly linear skew $\Sigma \approx \Sigma(S - K)$

Therefore implied volatility must rise when S rises

In the Black-Scholes model, Δ_{BS} depends on K and S through the moneyness K/S, so that "sticky moneyness" is equivalent to "sticky delta," with an at-the-money being $\Delta_{BS} \approx 0.5$.

Sticky delta means that the implied volatility must be purely a function of Δ_{BS}, i.e. $\dfrac{\ln S/K}{\Sigma\sqrt{\tau}}$.

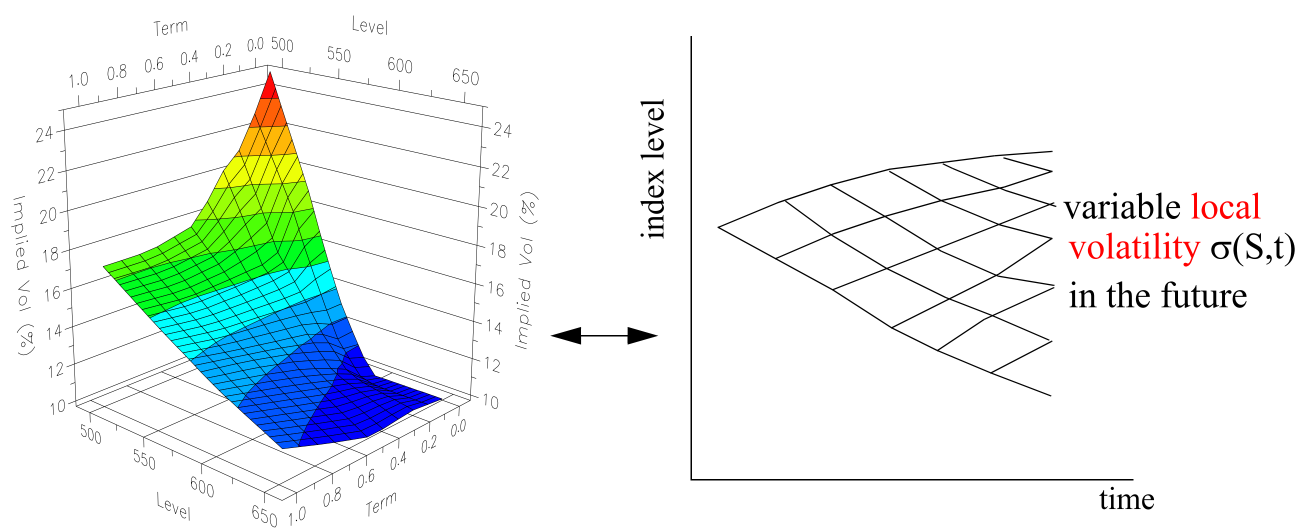
TABLE 3. Index Volatility behavior using the sticky-delta/moneyness rule.

Quantity	Behavior
Fixed-strike volatility:	increases as index level increases
At-the-money volatility:	is independent of index level
Exposure Δ:	$> \Delta_{BS}$

The (Sticky) Local Volatility Model

All current index options prices determine a single **consistent** unique set of local volatilities.

The implied tree corresponding to a given implied volatility surface.



The implied tree/local volatility model attributes the implied volatility skew to the market's expectation of higher realized volatilities and higher implied volatilities if the index moves down.

As the index level within the tree rises, you can see that the local volatilities decline, monotonically and (roughly) linearly, in order to match the linear strike dependence of the negative skew.

$$\Sigma(S, K, t) = \Sigma_0 - b(K + S - 2S_0) \qquad \text{Local Volatility Model, symmetric in K,S}$$

At-the-money volatility is given by

$$\Sigma_{atm}(S, t) = \Sigma_0 - 2b(S - S_0)$$

Implied volatilities decrease as K or S increases.
At-the-money implied volatility decreases twice as fast.

TABLE 4. Equity index volatility behavior in the sticky implied tree model.

Quantity	Behavior
Fixed-strike volatility:	decreases as index level increases
At-the-money volatility:	decreases twice as rapidly as index level increases
Exposure Δ:	$< \Delta_{BS}$

In this regime the options market experiences fear. The implied tree model implicitly assumes the skew arises from a fear of higher market volatility in the event of a fall, and assumes that after the fall, atm market volatility will rise twice as fast.

Summary of the Rules

Assume the current skew linear $\Sigma(S_0, K) = \Sigma_0 - b(K - S_0)$.

Sticky	General functional form for future implied volatility	Linear approximation: Future skew when stock price is S	Model with this property
Strike	$\Sigma(S, K) = f(K)$	$\Sigma(S, K) = \Sigma_0 - b(K - S_0)$	Black-Scholes[a]
Moneyness	$\Sigma(S, K) = f(K/S)$	$\Sigma(S, K) \approx \Sigma_0 - b(K - S)$	Stochastic volatility[b], jump diffusion
Implied tree/local volatility	$\Sigma(S, K) = f(K, S)$	$\Sigma(S, K) \approx \Sigma_0 - b(K + S - 2S_0)$ because Σ is approximately the average of the local volatilities between spot and strike.	Local volatility[c]
Delta	$\Sigma(S, K) = f(\Delta)$	$\Sigma(S, K) \approx \Sigma_0 - b[0.5 - \Delta_{\text{call}}(S, K, t, T)]$ or $\Sigma(S, K) \approx \Sigma_0 - b'\left[\frac{\ln K/S}{\Sigma\sqrt{\tau}}\right]$ Note that Δ is itself a function of Σ!	?

a. The Black-Scholes model corresponds roughly to the sticky strike rule of thumb, but it cannot honestly accommodate a skew, because all implied volatilities are the same irrespective of strike in the Black-Scholes model. So, although people use it, it's not really consistent from a theoretical point of view

b. In stochastic volatility models, there is another stochastic variable, the volatility itself, and so $\Sigma(S, K) = f(K/S)$ only if the other stochastic variable doesn't change.

c. **Crepey, Quantitative Finance 4 (Oct. 2004) 559-579,** argues that the local volatility hedging is the best for equities markets, in that it gets things right when the market moves a lot and isn't very wrong otherwise. See next page.

Volatility Surface Dynamics: What does skew tell us about change in atm vol with spot? (Kamal, Gatheral)

Parametrize implied volatility as mixture of sticky moneyness S-dependence:

$$\Sigma(S, K) \approx \Sigma_0 - b(K-S) - d(S-S_0)$$

$$\Sigma_{atm} \approx \Sigma_0 - d(S-S_0)$$

$$\frac{d\Sigma_{atm}}{dS} = -d \qquad \frac{d\Sigma}{dK} = -b \qquad \beta = \frac{d\Sigma_{atm}}{dS} \Big/ \frac{d\Sigma}{dK} = \frac{d}{b}$$ both easy to observe

Sticky moneyness: d = 0, $\beta = 0$

Sticky strike: d = b, $\beta = 1$

Sticky local volatility: d = 2b, $\beta = 2$

Actually, $\beta \approx 1.5$, between sticky strike and local volatility on average, though there may be regimes.

If $\beta = 1.5$ then **d = 1.5b** and so

$$\begin{aligned}\Sigma(S, K) &\approx \Sigma_0 - b(K-S) - d(S-S_0) \\ &\approx \Sigma_0 - b(K-S) - \frac{3}{2}b(S-S_0) \\ &\approx \Sigma_0 - b(K-S_0) - \frac{1}{2}b(S-S_0)\end{aligned}$$

This is somewhere between sticky moneyness and local volatility models.

With negative skew, when the market goes down, vol goes up a little, less than implied tree.

In contrast, Sticky Strike says $\Sigma(S, K) = \Sigma_0 - b(K-S_0) = \Sigma_0 - b(K-S) - b(S-S_0)$

The smile moves intraday 1.5 times more than sticky strike predicts! Moving up when S decreases.

Introduction to Stochastic Volatility

Approaches to Stochastic Volatility Modeling

The local volatility model is a special case of a stochastic volatility in which.

$\sigma(S, t)$ is 100% correlated with the stochastic stock price.

But volatility is independently stochastic too.

Several approaches to stochastic volatility:

- Allow the instantaneous stock volatility σ itself to be truly stochastic:

  (i) σ is stochastic and independent of S, *and then add correlation* to obtain the skew;

  (ii) $\sigma = \sigma(S)$ so we begin with a skew, *and then add volatility* to that skew.

- BGM-type models. Let the Black-Scholes implied volatilities $\Sigma(K, t)$ be stochastic. There are then strong constraints on the evolution of the B-S implied volatilities in order to avoid arbitrage.

- Stochastic implied tree models that begin with a local volatility model, which already projects the future no-arbitrage implied tree from a snapshot of current market options prices.
  Then allow these trees themselves to vary stochastically.
  Here again there are strong no-arbitrage conditions on the evolution.

Comment: Modeling stochastic volatility is much more complex than modeling local volatility.
We will develop models and study the character of the solutions and their smile.

References:

- Wilmott, “Derivatives” (Several chapters on stochastic volatility).
- Chapter 2 of Fouque, Papanicolaou and Sircar, “Derivatives in Financial Markets with Stochastic Volatility,” Cambridge University Press.
- Hull and White: Journal of Finance XLII, No 2, June 87, pp 281-300.
- Gatheral: The Implied Volatility Surface

Wilmott is perhaps the easiest place to start.
Gatheral has lots of math details on the analytic solutions to these models and their properties.

Our path:

1. The SDEs for stochastic volatility models
   The no-riskless-arbitrage hedge and the resultant PDE for the value of the option
   The Hull-White solution when the correlation is zero
   Monte Carlo solutions to the PDE more generally
   Semi-analytic solutions and the asymptotic properties of the smile;
2. The SABR model that begins with a local volatility $\sigma = \sigma(S)$ and then adds stochasticity to the evolution of the local volatility
3. Hedging in a stochastic volatility model

The Stochastic Differential Equation for Stochastic Volatility Models

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t$$

$$d\sigma = \text{several possibilities discussed below}$$

$$V = \sigma^2$$

The Hull-White stochastic volatility model with GBM:

$$\frac{dV_t}{V_t} = \alpha_t dt + \xi dW_t \qquad \text{Hull-White}$$

ξ is the volatility of volatility; typical fluctuations of volatility can be very large.

Realized and implied volatilities, like interest rates and credit spreads, are parameters rather than prices and are likely both mean-reverting variables.

Stochastic Mean Reversion and its Qualities

Ornstein-Uhlenbeck models:

$$dY = \alpha(m - Y)dt + \beta dW \qquad \text{Ornstein Uhlenbeck}$$

First let's solve it for $\beta = 0$ with zero volatility and no stochastic variability.

$$dY = \alpha(m - Y)dt \qquad Y(t) = m + (Y_0 - m)e^{-\alpha t}$$

As t gets large, the initial position Y_0 becomes irrelevant.

Alan Lewis estimates of half-life of volatility is between a few weeks and more than a year.
Not completely accurate, since volatility tends to jump up, and then stay high for a long time. There is a stickiness or persistence to high and low volatilities.

For non-zero volatility

$$Y(t) = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dW_s \qquad \text{Eq.1.1}$$

The contribution of random previous moves to the long-term value of Y(t) damps out exponentially.

This solution satisfies the stochastic differential equation, as shown below.

$$
\begin{aligned}
dY(t) &= -\alpha(Y_0 - m)e^{-\alpha t} + \beta dW_t - \beta\alpha\int_0^t e^{-\alpha(t-s)}dW_s \\
&= -\alpha\left[Y(t) - m - \beta\int_0^t e^{\cancel{-\alpha(t-s)}}dW_s\right] + \beta dW_t \cancel{-\beta\alpha\int_0^t} e^{-\alpha(t-s)}dW_s \\
&= \alpha[m - Y(t)] + \beta dW_t
\end{aligned}
$$

The cross-sectional mean $\overline{Y(t)}$ of Y(t) at time *t*, averaged over all increments dW_s.

$$\overline{Y(t)} = m + (Y_0 - m)e^{-\alpha t}$$

so that the average displacement at time *t* is just the deterministic one.

The variance of the displacements at time *t* by making use of the fact that the dW_s are independent

$$dW_s dW_u = du ds \delta(u - s)$$

$$[Y(t) - \overline{Y(t)}]^2 = \beta^2 \int_0^t \int_0^t e^{-\alpha(t-s)} e^{-\alpha(t-u)} dW_s dW_u$$

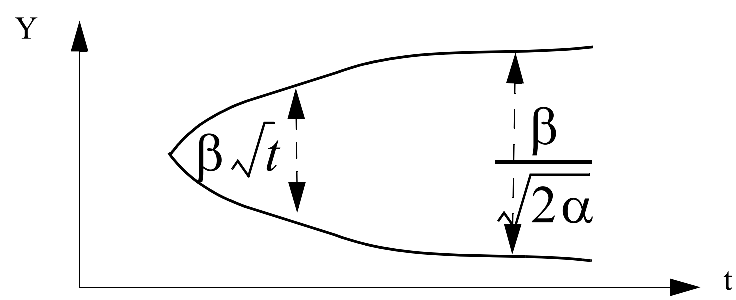
$$= \beta^2 \int_0^t \int_0^t e^{-2\alpha t} e^{\alpha(s+u)} du\, ds\, \delta(u-s)$$

$$= \beta^2 \int_0^t ds\, e^{-2\alpha t} e^{2\alpha s}$$

$$= \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$$

For small times *t*, variance behaves like $\beta^2 t$, which is like standard Brownian motion.

As $t \to \infty$ variance is $\frac{\beta^2}{2\alpha}$.



As α gets larger, the variance gets smaller. Here is a rough sketch of the distribution of the process over time.

At time $t \approx 1/(2\alpha)$ the variance grows no larger. In contrast, for regular Brownian motion, the linear dependence of the variance on *t* continues for all time.

Some stochastic volatility models

Most start from traditional GBM with no smile:

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

Then make σ stochastic too.

The simplest mean-reverting stochastic volatility one can write is **normal**

$$d\sigma = \alpha(m - \sigma)dt + \beta dW$$

Snag: volatility can become negative.

Lognormal models of variance

$$dV = \alpha(m - V)dt + \beta V dW$$

Arithmetic variance βV vanishes as the variance becomes zero. Therefore, the variance can never become negative.

Heston Square Root, popular because analytic solution: $dV = \alpha(m - V)dt + \xi\sqrt{V}dW$
Cf. Cox, Ingersoll and Ross interest rates. Non negative. Analytic solutions and their derivation are available in Heston's original paper, as well as in the books of Lewis and Gatheral.

Two stochastic variables, S and σ, and a correlation

$$dZdW = \rho dt$$

The Black-Scholes formula is NOT the solution. Even **standard options prices are different**. Model must be calibrated.

Pros: Stochastic volatility is more realistic.

> Three parameters -- volatility, volatility of volatility and its correlation – give a rich structure and a generally sensible dynamics.

Cons: Volatility evolution is even less well understood than stock price movement.

> Models are unlikely to be accurate.
> Correlation is at least as stochastic as volatility itself.
> Shape of skew for small expirations is difficult to match.

An Intuitive Look at Stochastic Volatility Models Starting from Black Scholes Point of View

Add stochastic volatility to BS: $dC \sim \text{usual stuff} + \frac{\partial^2 C}{\partial\sigma^2}(d\sigma)^2 + \frac{\partial^2 C}{\partial S \partial\sigma} dS d\sigma$

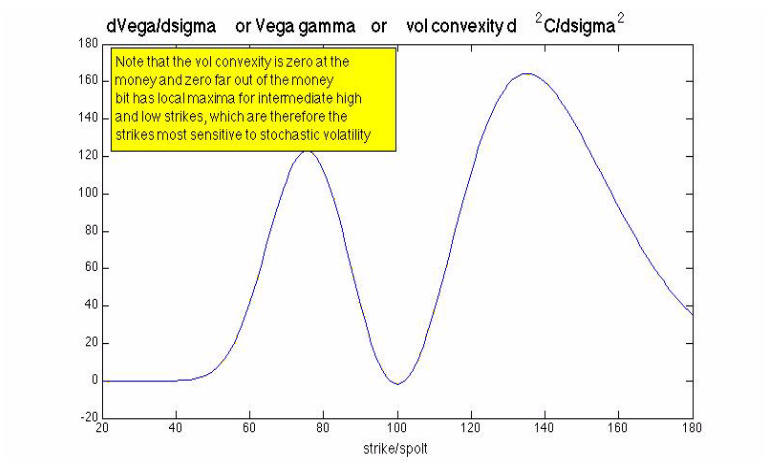
Cheat a little by using the BS derivatives of $C_{BS}(S, t, K, T, r, \sigma)$ in the Ito expansion.

$$\frac{\partial C}{\partial \sigma} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}} \text{ is always positive}$$

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right) \text{ which is positive}$$

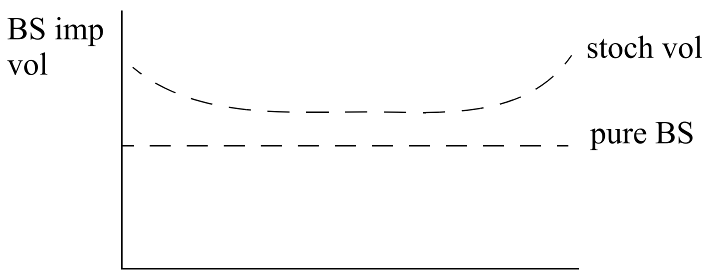
except when S/K is close to unity.

Mostly positive convexity, with peaks on either side.

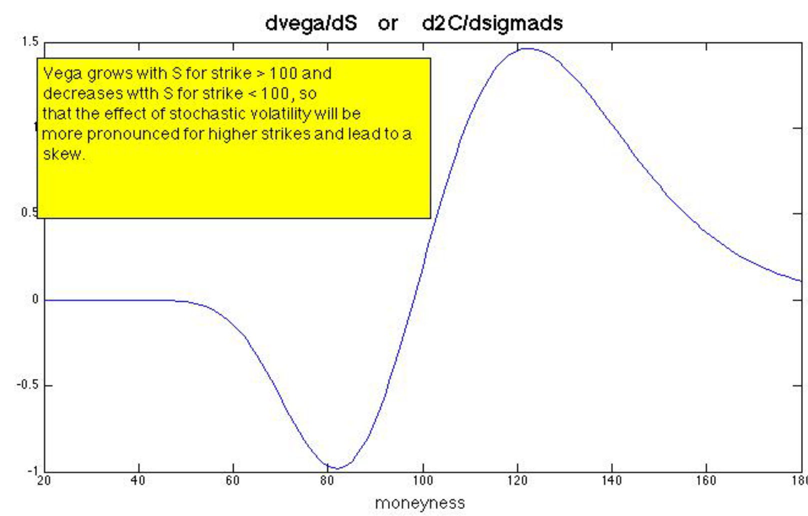


A hedged option is long gamma, long volatility and long volatility of volatility.

If volatility is volatile, then the convexity in volatility adds value to the option away from at-the-money, and adds value to out-of-the-money options relative to at-the-money options in a U-shaped smile.



Similarly, one can plot the Black-Scholes $\frac{\partial^2 C}{\partial\sigma\partial S}$, the rate of change of vega with spot S:



The volatility sensitivity is asymmetric about moneyness, and thus when volatility is volatile and correlated with the stock price, adds asymmetric value to the options and hence produces a skew.

Crude usefully intuitive ways to understand the effect of stochastic volatility on the smile.

A Preliminary Look at Stochastic Volatility: Start From Local Volatility with a Skew and Add Stochasticity

Add a stochastic element to a local volatility model.

$$\frac{dS}{S} = \alpha S^{\beta - 1} dW$$

$$d\alpha = \xi \alpha dZ \qquad \text{SABR model}$$

$$dZdW = \rho dt$$

For $\beta = 1$ this is geometric Brownian motion with no smile, else it's CEV with skew.
α is the stochastic part of the smile-inducing local volatility, and ξ is the volatility of volatility.

Assume $\rho = 0$ and β close to 1 (small skew): estimate the skew using our knowledge of local vol.

For $\xi = 0$ the implied volatility is roughly the average of the local volatilities S to K:

$$\Sigma_{LV}(S, t, K, T, \alpha) = \frac{\alpha}{2}[S^{\beta - 1} + K^{\beta - 1}]$$

Taylor expansion in K for β close to 1: $\Sigma_{LV}(S, t, K, T, \alpha) \approx \frac{\alpha}{S^{1-\beta}}\left[1 + \frac{(\beta - 1)}{2}\ln\frac{K}{S}\right]$

a linear skew with negative slope, $\frac{\partial\Sigma}{\partial K} \approx \frac{\partial\Sigma}{\partial S}$ atm.

Now switch on the stochastic volatility $\xi \neq 0$ There is a range of possible α values. Estimate C_{SLV} in this Stochasticized Local Vol model as average of the BS prices over the range of α:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \alpha))\phi(\alpha)d\alpha$$

Taylor expand this about the mean $\bar{\alpha}$ for small volatility of volatility:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha} + (\alpha - \bar{\alpha})))\phi(\alpha)d\alpha$$

$$\approx \int \left\{ C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha})) + \frac{\partial C_{BS}}{\partial \alpha}\cancel{(\bar{\alpha})(\alpha - \bar{\alpha})} + \frac{1}{2}\frac{\partial^2}{\partial \bar{\alpha}^2}(C_{BS}(\bar{\alpha})(\alpha - \bar{\alpha})^2) \right\} \phi(\alpha)d\alpha$$

$$\approx C_{BS}(\bar{\alpha}) + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\bar{\alpha}))var(\alpha)$$

Look at implied Black-Scholes volatility Σ_{SLV} deviation away from $\xi = 0$:

$$
\begin{aligned}
C_{SLV} \equiv C_{BS}(\Sigma_{SLV}) &\approx C_{BS}(\Sigma_{LV}\overline{(\alpha)} + \{\Sigma_{SLV} - \Sigma_{LV}\}) \\
&\approx C_{BS}(\Sigma_{LV} + \{\Sigma_{SLV} - \Sigma_{LV}\}) \\
&\approx C_{BS}(\bar{\alpha}) + \frac{\partial C_{BS}}{\partial \Sigma_{LV}}(\Sigma_{SLV} - \Sigma_{LV})
\end{aligned}
$$

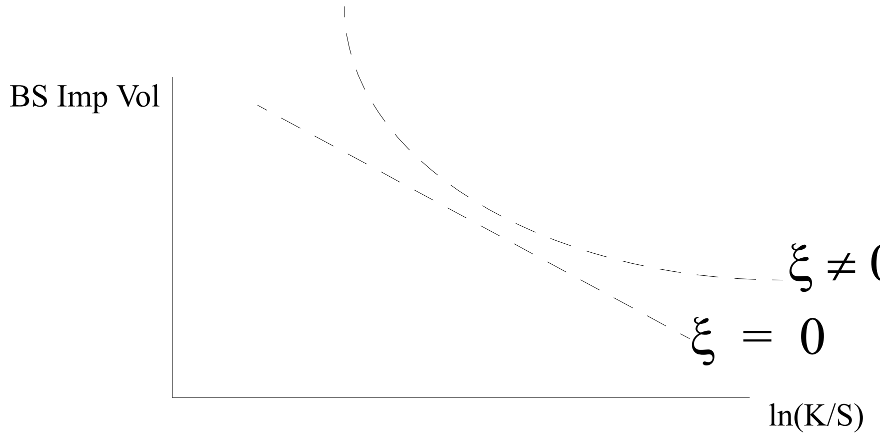
Comparing the above two equations, we obtain

$$
\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha}) + \frac{\frac{1}{2}\frac{\partial^2}{\partial \bar{\alpha}^2}(C_{BS}(\bar{\alpha}))var(\alpha)}{\frac{\partial C_{BS}}{\partial \Sigma_{LV}}}
$$

Evaluate the BS derivatives above **for small times to expiration and close to at-the-money**:

$$
\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha})\left\{1 + \frac{1}{2}\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2\left[\ln\frac{S}{K}\right]^2\right\} \qquad \text{Eq.1.2}
$$

The local volatility smile is altered by the addition of a quadratic term in $\ln\frac{S}{K}$



No need for correlation between volatility and stock price in order to obtain a smile if we start from local volatility.

Risk-neutral Valuation And Stochastic Volatility Models

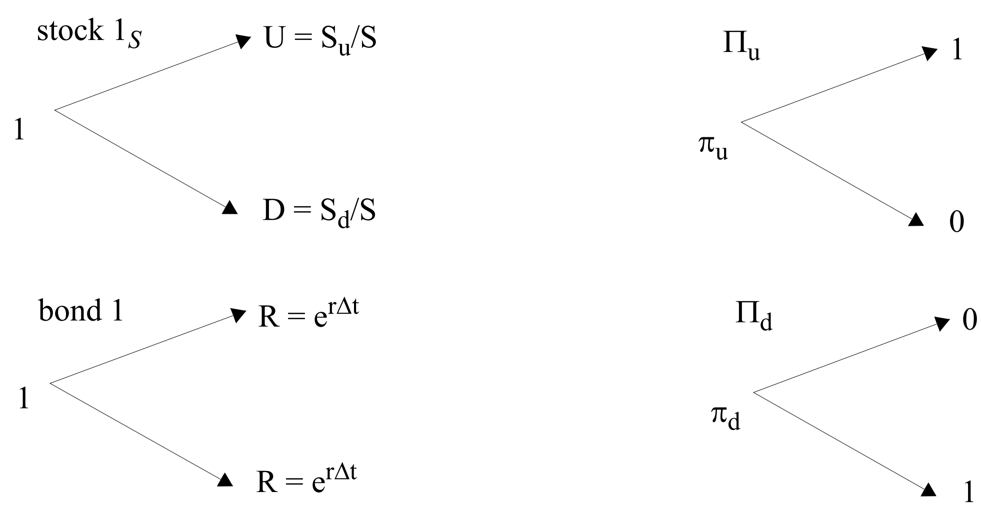
Arbitrage-free options valuation:

Hedge away all the risk instantaneously.
Need enough securities to span all the possible states of the world dynamically.
Then the riskless hedged portfolio must earn the risk-free rate.

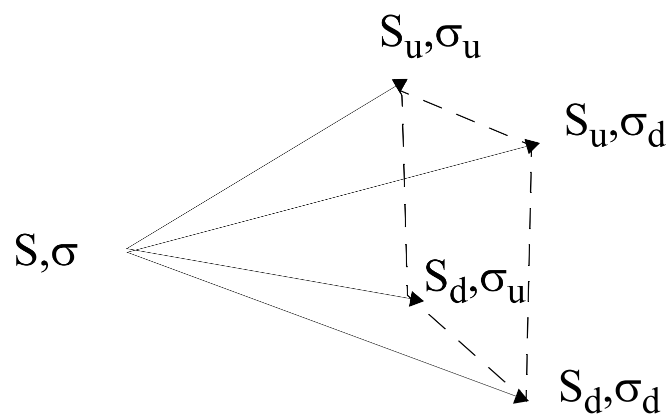
Recall how we derived Black-Scholes with a stochastic stock price:

Arrow-Debreu securities, Π_u and Π_d span the space of payoff states.

Two securities: two final states. Hence you can value any instrument irrespective of outcome or its probability.



Stochastic volatility: S **and** σ can vary: σ_u and σ_d differ; there is a correlation between S and σ



4 possible final states
Need 4 four Arrow-Debreu securities that pay $1.
But we know only two: S and B.
We would need to know the *price of the volatility* σ today in order to span the other states.

But volatility is not a security or a traded variable, it's a parameter.
Instead, we can only hedge options with other options to hedge the volatility sensitivity.

Cf: Vasiçek interest-rate model.
You cannot hedge the interest-rate exposure of a bond with "interest rates". You must hedge the interest-rate sensitivity of one bond with another.

If we hedge options only with shares of stock perfect replication is impossible. ***If*** you can also use other options, and if you *assume* you know the stochastic process for options as well as stock prices, then you can derive an arbitrage-free formula for options values. But do we? Nevertheless …

Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \qquad \text{Eq.1.3} \\
dW dZ &= \rho dt
\end{aligned}
$$

The coefficients $p(S, \sigma, t)$ and $q(S, \sigma, t)$ are general functions.

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$:

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

$$
\begin{aligned}
d\Pi = {} & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\
& -\Delta dS \\
& -\delta\left(\frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 dt + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho dt\right)
\end{aligned}
$$

Collecting the *dt*, *dS* and $d\sigma$ terms

$$d\Pi = dt\left[\begin{array}{c}\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right)\end{array}\right]$$

$$+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right)$$

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma}\Big/\frac{\partial U}{\partial \sigma}$$

$$\text{Then } d\Pi = dt\left[\begin{array}{c}\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}\mathrm{q}^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right)\end{array}\right]$$

No riskless arbitrage: $$d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$$

Comparing the last two equations

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho - rV$$

$$-\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho - rU\right)$$

$$+ r\Delta S = 0$$

But $\Delta = \dfrac{\partial V}{\partial S} - \delta\dfrac{\partial U}{\partial S}$ and so

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV$$

$$= \delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU\right)$$

But $\delta = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma}$ so

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial \sigma}}$$

$$= \frac{\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial \sigma}}$$

$$= \phi(S, \sigma, t) \qquad \text{separation of variables}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{q}^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don't know the value of the function ϕ!

The Sharpe-ratio meaning of $\phi(S, \sigma, t)$ in terms of Sharpe ratios

See what PDE says about expected risk and return of the option V using Ito's lemma:

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\
dWdZ &= \rho dt
\end{aligned}
$$

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + \frac{\partial V}{\partial \sigma}p dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\
&\quad + \frac{\partial V}{\partial S}\sigma S dZ + \frac{\partial V}{\partial \sigma}q dW \\
&\equiv \mu_V V dt + V\sigma_{V_S} dZ + V\sigma_{V_\sigma} dW
\end{aligned}
$$

$$
\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right]
$$

$$
\sigma_{V_S} = \frac{S}{V}\frac{\partial V}{\partial S}\sigma \qquad \sigma_{V_\sigma} = \frac{1}{V}\frac{\partial V}{\partial \sigma}q \qquad \sigma_V \equiv \sqrt{\sigma_{V_S}^2 + \sigma_{V_\sigma}^2 + 2\rho\sigma_{V_S}\sigma_{V_\sigma}}
$$

where σ_{V_S} and σ_{V_σ} are the partial volatilities of option V with total volatility σ_V

We can rewrite the LHS of the stochastic volatility PDE for the value of the option as

$$\frac{1}{V}\left[\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2+\frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2+\frac{\partial^2 V}{\partial S\partial\sigma}\sigma q S\rho\right]\equiv\mu_V-\mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right)-p\left(\frac{\partial V}{\partial\sigma}\frac{1}{V}\right)$$

Substituting this into the stochastic vol PDE

$$\mu_V-\mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right)-p\left(\frac{\partial V}{\partial\sigma}\frac{1}{V}\right)+r\frac{S}{V}\frac{\partial V}{\partial S}+\phi(S,\sigma,t)\frac{1}{V}\frac{\partial V}{\partial\sigma}-r\;=\;0$$

or

$$(\mu_V-r)\;=\;\frac{S}{V}\frac{\partial V}{\partial S}(\mu-r)+\frac{1}{V}\frac{\partial V}{\partial\sigma}(p-\phi)$$

or

$$(\mu_V-r)\;=\;\sigma_{V_S}\frac{(\mu-r)}{\sigma}+\sigma_{V_\sigma}\frac{(p-\phi)}{q}$$

or

$$\frac{(\mu_V-r)}{\sigma_V}\;=\;\frac{\sigma_{V_S}}{\sigma_V}\frac{(\mu-r)}{\sigma}+\frac{\sigma_{V_\sigma}}{\sigma_V}\frac{(p-\phi)}{q}$$

Excess return per unit of risk for the option the excess return per unit of risk for the stock and the excess return per unit of risk for volatility.

ϕ plays the role for stochastic volatility that the riskless rate *r* plays for a stochastic stock price.

In the Black-Scholes case, r is the rate at which the stock price must grow in order that option payoffs can be discounted at the riskless rate.

Similarly, ϕ is the drift that volatility must undergo in order that option prices with stochastic volatility be discounted at the riskless rate.

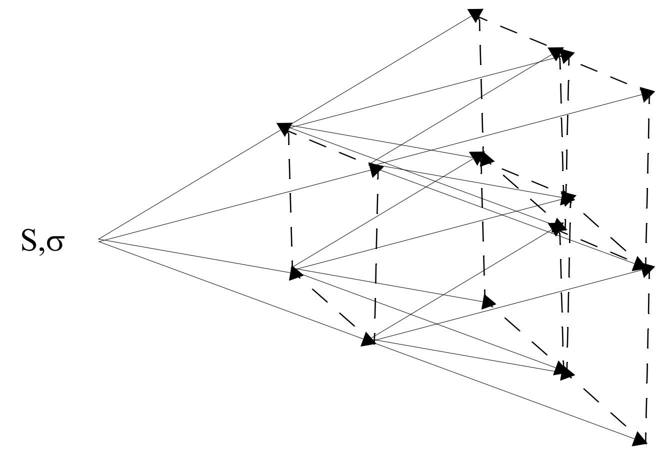
φ is not equal to r because σ is not traded. ϕ is the rate at which volatility σ must grow in order that the price of the option V grows at the rate r when you can hedge away all risk.

From a calibration point of view, ϕ must be chosen to make option prices grow at the riskless rate.

If we know the market price of just one option U, and we assume an evolution process for volatility, $d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the value of U obtained from PDE matches its market price.

Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.

Of course, it may be naive to assume that just one option can calibrate the entire volatility evolution process.)

Note that even though the payoffs of the option are the same as in the Black-Scholes world, the evolution process of the stock is different, and so the option price will be different too.

The Characteristic Solution to the Stochastic Volatility Model.

$$V = \exp(-r\tau) \sum_{\text{all paths}} p(path) \times \text{payoff}\big|_{path}$$

the usual discounted risk-neutral expected present value of the payoffs, where the expectation is taken over all future evolution paths of the stock, and $p(path)$ is the risk-neutral probability for that path.

Characterize each path by its terminal stock price S_T and the (average) **path variance** σ_T along that path.

$$\sigma_T^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

Rewrite sum as $$V = \exp(-r\tau) \sum_{\text{all } \sigma_T} \sum_{\substack{\text{paths of all } S_T \\ \text{for fixed } \sigma_T}} p(\sigma_T, S_T)\text{payoff}\big|_{path}$$

where $p(\sigma_T, S_T)$ is the probability of a particular terminal stock price and average path volatility.

If stock movements and volatility are uncorrelated ($\rho = 0$), then $p(\sigma_T, S_T) = p(\sigma_T)p(S_T|\sigma_T)$, so

$$V = \sum_{\text{all } \sigma_T} p(\sigma_T)\left[\exp(-r\tau) \sum_{\substack{\text{paths of all } S_T \\ \text{for fixed } \sigma_T}} p(S_T|\sigma_T)\text{payoff}\Big|_{path}\right]$$

Now for lognormal evolution [ ] is equal to the Black-Scholes formula, so that

$$V = \sum_{\sigma_T} p(\sigma_T) \times BS(S, K, r, \sigma_T, T)$$

"Mixing Theorem": The stochastic volatility solution for zero correlation is the weighted sum over the Black-Scholes solutions for different volatilities. (Hull and White)

For non-zero correlation there are similar formulas $V = E[BS(S'(\sigma_T, \rho), K, r, \overline{\sigma_T}'(\rho), T)]$

where the stock price in the Black-Scholes formula is shifted to $S'(\sigma_T, \rho)$ and the volatility $\overline{\sigma_T}$ is shifted to $\overline{\sigma_T}'(\rho)$ so it's not quite as useful or intuitive.

[Refs: Fouque, Papanicolaou and Sircar book, and Roger Lee, *Implied and Local Volatilities under Stochastic Volatility*, International Journal of Theoretical and Applied Finance, 4(1), 45-89 (2001).]

The Smile That Results From Stochastic Volatility

The zero-correlation smile depends on moneyness

Mixing: average BS solutions over the volatility distribution to get the stochastic volatility solution.
Example: path volatility can be one of two values, either high or low, with equal probability:

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)]$$

Homogeneity: $$C_{SV} = \frac{1}{2}\left[SC_{BS}\left(1, \frac{K}{S}, \sigma_H\right) + SC_{BS}\left(1, \frac{K}{S}, \sigma_L\right)\right] = Sf\left(\frac{K}{S}\right)$$

Now, defining BS Σ by $$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}\left(1, \frac{K}{S}, \Sigma\right)$$

and so $$\Sigma = g\left(\frac{K}{S}\right)$$

Implied volatility is a function of moneyness in stochastic volatility models with zero correlation (conditional on the state of volatility itself not changing).

Deriving Euler's equation: $$\frac{\partial\Sigma}{\partial S} = \left(-\frac{K}{S^2}\right)g' \qquad \frac{\partial\Sigma}{\partial K} = \frac{1}{S}g' \qquad S\frac{\partial\Sigma}{\partial S} + K\frac{\partial\Sigma}{\partial K} = 0$$

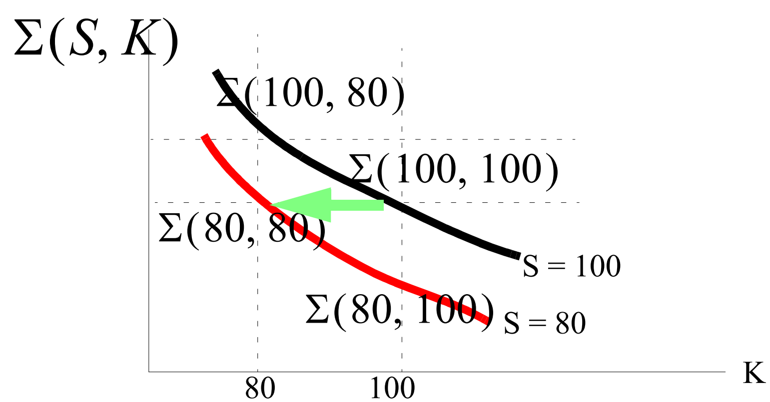
Close to at-the-money, $$\frac{\partial \Sigma}{\partial S} \approx -\frac{\partial \Sigma}{\partial K}$$

just the opposite of what we got with local volatility models.

Close to at-the-money

$$\Sigma \approx \Sigma(S - K)$$

In stochastic volatility models, *conditioned on the current volatility remaining the same,*



Note that the volatility of all options drops when the stock price drops.

Of course if the volatility itself changes, then the whole curve can move.