LECTURE 17

Stochastic Volatility Models: A Variety of Approaches

Looking Ahead

Stochastic Volatility Models

Jump Diffusion Models

Guest Speakers

Michael Kamal - April 15

Jackie Rosner - April 20

If you have questions come to my office hours or see me some other time by appointment.

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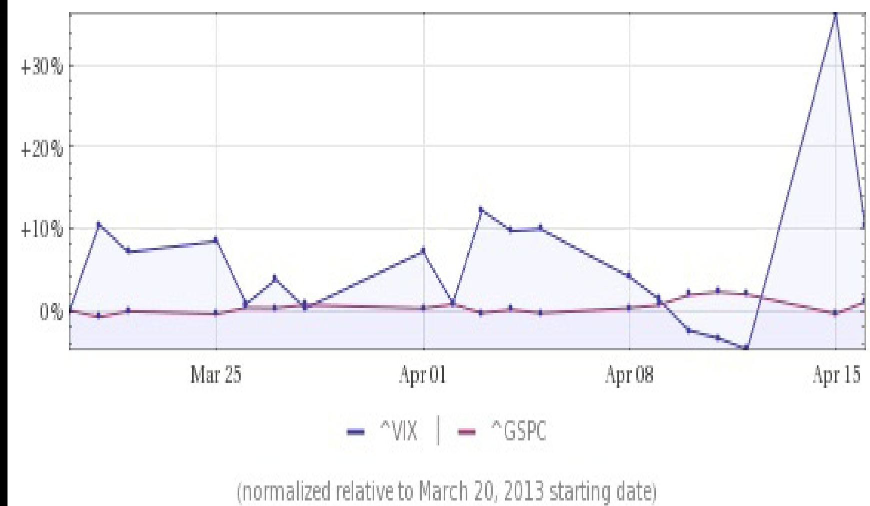
April 15 2013 Market Move
Compare Static and Dynamic Ratios

S&P down in mini-crash; Vix up; Skew steepens a little; think about hedge ratio.

VIX



VIX and S&P anticorrelated

April 12: S&P 1588; VIX 12%
April 15: $&P 1552; VIX 17%

Static: K = 1500

1 mo $K\dfrac{\partial\Sigma}{\partial K}$ between 95 and 100 went from $\dfrac{4\%}{5\%}$ to $\dfrac{4.4\%}{5\%}$; $\dfrac{\partial\Sigma}{\partial K}$ went from 4/(1500 x 0.05)=**0.050** to 0.057

GRAB

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95) Templates | 96) Actions | 97) Hide Settings | Volatility Skew

1) Skew Analysis | 2) Term Structure | 3) Vol Surface

	Und	Src	AO	Date	Mkt		Und	Src	AO	Date	Mkt
1.	SPX	Market	C	04/12/13	M	3.	SPX	Market	TD	04/16/13	L
2.	SPX	Market	C	04/15/13	M	4.	SPX	Market	1M	03/16/13	L

View: Table | Term: Fixed | Strike: % Money | Value: Imp Vol

Term	Expiry	85%	90%	95%	97.5%	100%	102.5%	105%	110%	115%	120%
1) SPX Index Market 4/12/2013 1588.85 Mid Call											
1 MO	05/12/2013	18.446	18.446	14.497	12.373	10.476	9.344	9.493	11.981	12.075	12.075
2 MO	06/11/2013	17.174	17.509	14.416	12.763	11.293	10.184	9.627	10.133	10.134	10.131
3 MO	07/11/2013	19.444	17.398	14.619	13.228	11.954	10.889	10.145	9.974	11.020	11.088
6 MO	10/09/2013	19.615	17.446	15.380	14.409	13.502	12.663	11.923	10.927	10.824	11.553
1 YR	04/07/2014	19.806	18.295	16.841	16.150	15.489	14.866	14.298	13.338	12.673	12.360
1.5 YR	10/04/2014	19.964	18.764	17.642	17.115	16.621	16.155	15.721	14.963	14.368	13.940
2 YR	04/02/2015	20.335	19.315	18.361	17.913	17.483	17.072	16.686	15.983	15.378	14.880
2) SPX Index Market 4/15/2013 1552.36 Mid Call											
1 MO	05/15/2013	22.020	22.020	17.992	15.745	13.616	12.013	11.398	13.183	13.314	13.314
2 MO	06/14/2013	19.632	19.921	16.864	15.199	13.664	12.371	11.436	11.061	11.028	11.021
3 MO	07/14/2013	21.581	19.273	16.518	15.140	13.846	12.696	11.771	11.014	11.889	12.105
6 MO	10/12/2013	20.703	18.611	16.599	15.647	14.751	13.912	13.148	11.990	11.607	12.252
1 YR	04/10/2014	20.485	19.056	17.664	16.994	16.344	15.724	15.146	14.127	13.344	12.854
1.5 YR	10/07/2014	20.598	19.436	18.325	17.793	17.287	16.806	16.354	15.547	14.899	14.417
2 YR	04/05/2015	20.869	19.859	18.907	18.460	18.030	17.618	17.229	16.519	15.905	15.395

Dynamic:

April 12: S&P 1588; VIX 12%

April 15: $&P 1552; VIX 17%

$$\frac{\partial \Sigma_{atm}}{\partial S} = \frac{5\%}{36} = 0.14$$

and we saw

$$\frac{\partial \Sigma}{\partial K} = 0.05$$

$$\frac{\left(\frac{\partial \Sigma_{atm}}{\partial S}\right)}{\left(\frac{\partial \Sigma}{\partial K}\right)} = 2.8 \text{ -- larger than local vol.}$$

Which Model to Use?

Use local vol if the asset has a volatility that seems to depend on market level.

Use local vol as a way of hedging exotics from vanillas, heuristically.

But must use stochastic vol if the option has a value strongly dependent on the level of volatility, e.g. forward start options.

More later.

Introduction to Stochastic Volatility

Approaches to Stochastic Volatility Modeling

The local volatility model is a special case of a stochastic volatility in which.

$\sigma(S, t)$ is 100% correlated with the stochastic stock price.

But volatility is independently stochastic too.

Several approaches to stochastic volatility:

- Allow the instantaneous stock volatility σ itself to be truly stochastic:

  (i) σ is stochastic and independent of S, *and then add correlation* to obtain the skew;

  (ii) $\sigma = \sigma(S)$ so we begin with a skew, *and then add volatility* to that skew.

- BGM-type models. Let the Black-Scholes implied volatilities $\Sigma(K, t)$ be stochastic. There are then strong constraints on the evolution of the B-S implied volatilities in order to avoid arbitrage.

- Stochastic implied tree models that begin with a local volatility model, which already projects the future no-arbitrage implied tree from a snapshot of current market options prices.
  Then allow these trees themselves to vary stochastically.
  Here again there are strong no-arbitrage conditions on the evolution.

Comment: Modeling stochastic volatility is much more complex than modeling local volatility.
We will develop models and study the character of the solutions and their smile.

References:

- Wilmott, “Derivatives” (Several chapters on stochastic volatility).
- Chapter 2 of Fouque, Papanicolaou and Sircar, “Derivatives in Financial Markets with Stochastic Volatility,” Cambridge University Press.
- Hull and White: Journal of Finance XLII, No 2, June 87, pp 281-300.
- Gatheral: The Implied Volatility Surface

Wilmott is perhaps the easiest place to start.
Gatheral has lots of math details on the analytic solutions to these models and their properties.

Our elliptical path:

1. The SDEs for stochastic volatility
2. Mean reversion
3. Intuitive qualitative look at skew from adding stochastic volatility to Black-Scholes.
4. Another approach: Local Stochastic Volatility. The SABR model that begins with a local volatility $\sigma = \sigma(S)$ and then adds stochasticity to the evolution of the local volatility
5. Risk-neutral valuation: The riskless hedge and the resultant PDE for the value of the option
6. The Hull-White solution when the correlation is zero
7. Monte Carlo solutions to the PDE more generally
8. Semi-analytic solutions and the asymptotic properties of the smile
9. Hedging in a stochastic volatility model

The Stochastic Differential Equation for Stochastic Volatility Models

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t$$

$$d\sigma = \text{several possibilities discussed below}$$

$$V = \sigma^2$$

The Hull-White stochastic volatility model with GBM:

$$\frac{dV_t}{V_t} = \alpha_t dt + \xi dW_t \qquad \text{Hull-White}$$

ξ is the volatility of volatility; typical fluctuations of volatility can be very large. How large?

Realized and implied volatilities, like interest rates and credit spreads, are parameters rather than prices and are likely both mean-reverting variables.

Stochastic Mean Reversion and its Qualities

Ornstein-Uhlenbeck models:

$$dY = \alpha(m - Y)dt + \beta dW \quad \text{Ornstein Uhlenbeck}$$

First let's solve it for $\beta = 0$ with zero volatility and no stochastic variability.

$$dY = \alpha(m - Y)dt \qquad Y(t) = m + (Y_0 - m)e^{-\alpha t}$$

As *t* gets large, the initial position Y_0 becomes irrelevant.

Alan Lewis estimates of half-life of volatility is between a few weeks and more than a year.
Not completely accurate, since volatility tends to jump up, and then stay high for a long time. There is a stickiness or persistence to high and low volatilities.

For non-zero volatility

$$Y(t) = m + (Y_0 - m)e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dW_s \qquad \text{Eq.1.1}$$

The contribution of random previous moves to the long-term value of Y(t) damps out exponentially.

This solution satisfies the stochastic differential equation, as shown below.

$$\begin{aligned} dY(t) &= -\alpha(Y_0 - m)e^{-\alpha t} + \beta dW_t - \beta\alpha\int_0^t e^{-\alpha(t-s)}dW_s \\ &= -\alpha\left[Y(t) - m - \cancel{\beta\int_0^t e^{-\alpha(t-s)}dW_s}\right] + \beta dW_t \cancel{-\beta\alpha\int_0^t e^{-\alpha(t-s)}dW_s} \\ &= \alpha[m - Y(t)] + \beta dW_t \end{aligned}$$

The cross-sectional mean $\overline{Y(t)}$ of Y(t) at time *t*, averaged over all increments dW_s.

$$\overline{Y(t)} = m + (Y_0 - m)e^{-\alpha t}$$

so that the average displacement at time *t* is just the deterministic one.

The variance of the displacements at time *t* by making use of the fact that the dW_s are independent

$$E[dW_s dW_u] = du ds \delta(u - s)$$

Sketch of proof of the line above:

Consider the discrete case where successive increments of the stochastic variable W(t) are given by the index *i* along the path as dW_i at each time t_i, before we go to the continuous limit. Then

$$E(dW_i)^2 = dt$$

$$E[dW_i dW_j] = 0 \text{ independent increments}$$

$$\text{Thus } E[dW_i dW_j] = \delta_{i,j} dt$$

In the continuous limit as i and j approach each other

$$E[dW(t)dW(u)] = \delta(t-u)dudt$$

because $\delta(t-u)du$ is zero everywhere except at u = t, and integrates to a value of 1, which is the continuous analog of $\delta_{i,j}$ which is 1 when i = j, zero elsewhere, and sums to 1 over all j.

So

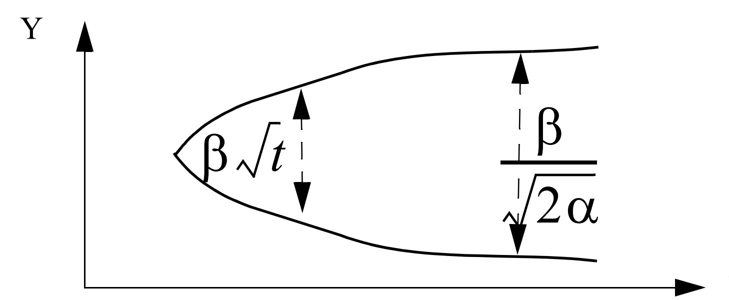
$$E[Y(t)-\overline{Y(t)}]^2 = \beta^2 \int_0^t \int_0^t e^{-\alpha(t-s)} e^{-\alpha(t-u)} E[dW_s dW_u]$$

$$= \beta^2 \int_0^t \int_0^t e^{-2\alpha t} e^{\alpha(s+u)} du\, ds\, \delta(u-s)$$

$$= \beta^2 \int_0^t ds\, e^{-2\alpha t} e^{2\alpha s}$$

$$= \frac{\beta^2}{2\alpha}(1-e^{-2\alpha t})$$

For small times *t*, variance behaves like $\beta^2 t$, which is like standard Brownian motion.



As $t \to \infty$ variance is $\frac{\beta^2}{2\alpha}$.

As α gets larger, the variance gets smaller. Here is a rough sketch of the distribution of the process over time.

At time $t \approx 1/(2\alpha)$ the variance grows no larger. In contrast, for regular Brownian motion, the linear dependence of the variance on *t* continues for all time.

Some stochastic volatility models

Most start from traditional GBM with no smile:

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

Then make σ stochastic too.

The simplest mean-reverting stochastic volatility one can write is **normal**

$$d\sigma = \alpha(m - \sigma)dt + \beta dW$$

Snag: volatility can become negative.

Lognormal models of variance

$$dV = \alpha(m - V)dt + \beta V dW$$

Arithmetic variance βV vanishes as the variance becomes zero. Therefore, the variance can never become negative.

Heston Square Root, popular because analytic solution: $dV = \alpha(m - V)dt + \xi\sqrt{V}dW$
Cf. Cox, Ingersoll and Ross interest rates. Non negative. Analytic solutions and their derivation are available in Heston's original paper, as well as in the books of Lewis and Gatheral.

Two stochastic variables, S and σ, and a correlation

$$dZdW = \rho dt$$

The Black-Scholes formula is NOT the solution. Even **standard options prices are different**.
Model must be calibrated.

Pros: Stochastic volatility is more realistic.

> Three parameters -- volatility, volatility of volatility and its correlation – give a rich structure and a generally sensible dynamics.

Cons: Volatility evolution is even less well understood than stock price movement.

> Models are unlikely to be accurate.
> Correlation is at least as stochastic as volatility itself.
> Shape of skew for small expirations is difficult to match.

An Intuitive Look at Stochastic Volatility Models Starting from Black Scholes Point of View

Assume rates and dividends are zero. Add stochastic volatility to BS:

$$
\begin{aligned}
dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma
\end{aligned}
$$

Now suppose that we constructed a riskless hedge that is long the call C and short just enough stock and enough volatility σ so that the hedged portfolio is instantaneously riskless.

Then

$$
dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma
$$

We don't know the value of the partial derivatives in the above equation, since we haven't applied the methods of risk-neutral valuation to determine the partial differential equation for the value of the option with both stochastic volatility and stock price. In order to proceed further we will replace the unknown partial derivatives by their values in the Black-Scholes model, hoping that these capture the approximate contribution to the P&L from the stochastic volatility.

Then

$$\frac{\partial C_{BS}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial S^2}\sigma^2 S^2 = 0$$

Then expected change in the value of the hedged portfolio from stochastic volatility is approximately

$$\frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma$$

volga,
butterfly spread

vanna
risk reversal

Cheat a little more by using the BS derivatives of $C_{BS}(S, t, K, T, r, \sigma)$ in the Ito expansion.

$$\frac{\partial C}{\partial \sigma} = \frac{Se^{-\frac{1}{2}d_1^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}}$$ vega is always positive

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}(d_1 d_2) = \frac{S\sqrt{\tau}N'(d_1)}{\sigma}\left(\frac{(\ln S/K)^2}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right)$$ volga is mostly positive except atm

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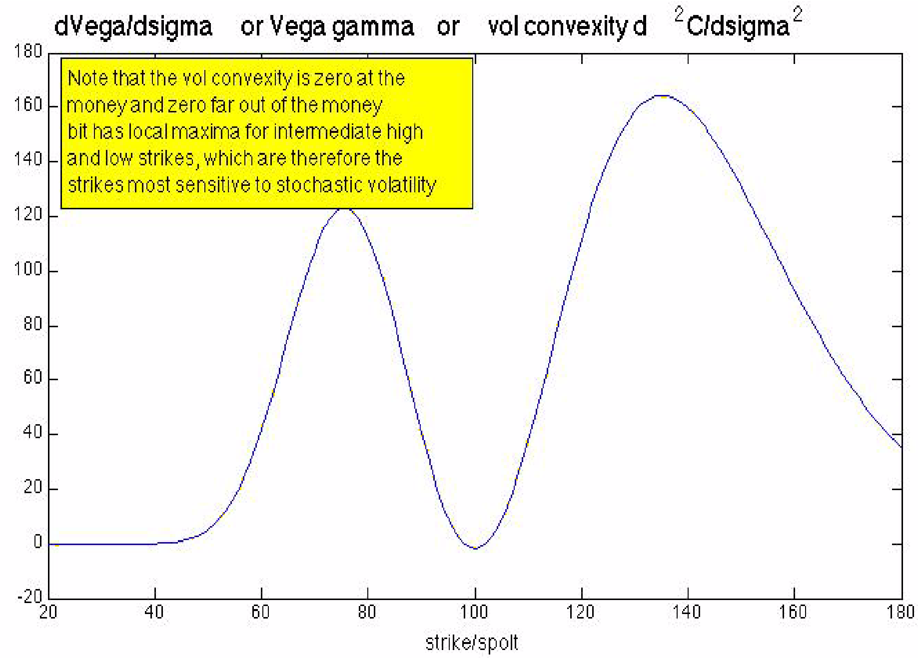


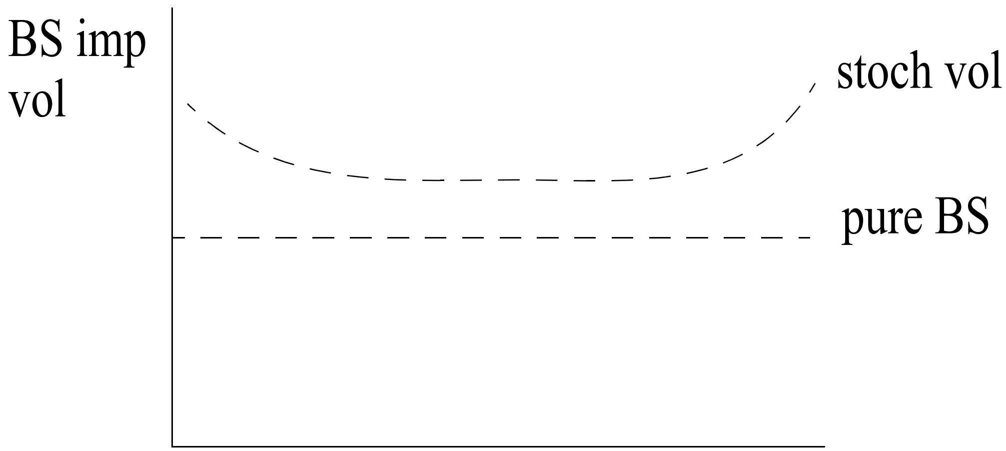
Mostly positive convexity, with peaks on either side.

$$\frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma$$

A hedged option is long gamma, long volatility and **long volatility of volatility**, esp out of money or deep in the money.

The Vanna Volga model/method for exotics involves approximately replicating an exotic option with vanillas that have the same vega, volga and vanna in a Black-Scholes no-smile world, and then turning on the smile to see the effect of moving away from Black-Scholes, *assuming* Black Scholes derivatives hold. (Later)

If volatility is volatile, then the convexity in volatility adds value to the option away from at-the-money, and adds value to out-of-the-money options relative to at-the-money options in a U-shaped smile.

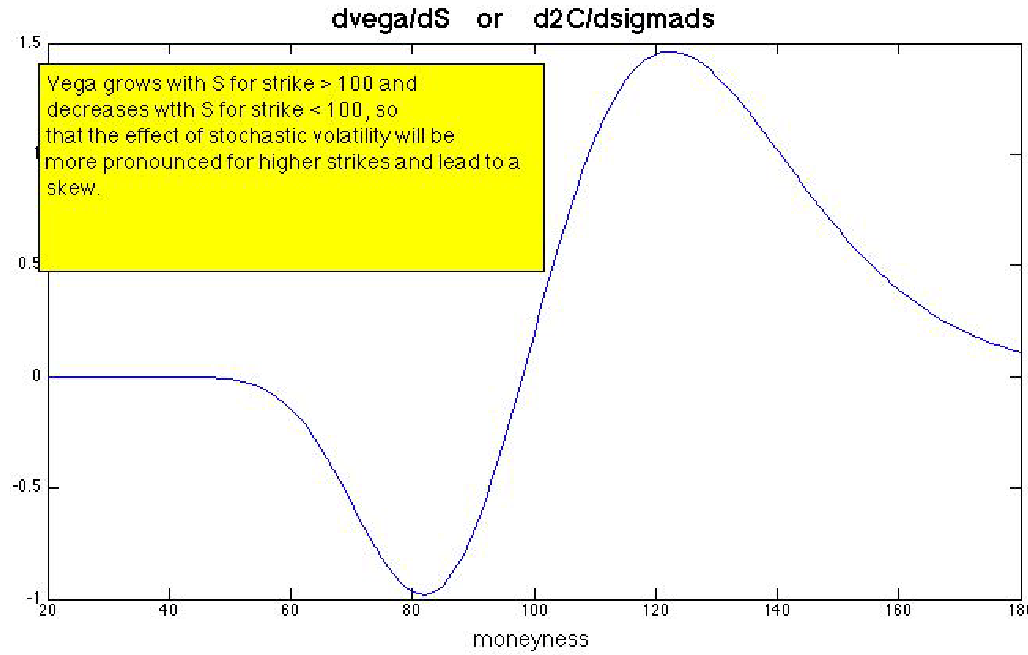


Similarly, one can plot the Black-Scholes vanna $\frac{\partial^2 C}{\partial\sigma\partial S}$ the rate of change of vega with spot S

$$\frac{\partial C}{\partial\sigma} = \frac{Se^{-\frac{1}{2}\left(\frac{\ln S/K}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}\sqrt{\tau}}{\sqrt{2\pi}} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right)$$

$$\frac{\partial^2 C}{\partial\sigma\partial S} = \left[1 - Sd_1\frac{\partial d_1}{\partial S}\right]\frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right) = \left[\frac{1}{2} - \frac{\ln(S/K)}{\sigma^2\tau}\right]\frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right)$$

$$\underbrace{\frac{1}{2}\frac{\partial^2 C}{\partial\sigma^2}d\sigma^2}_{\text{volga}} + \underbrace{\frac{\partial^2 C}{\partial S\partial\sigma}dSd\sigma}_{\text{vanna}}$$



For typical values of σ and τ, vanna will be positive when the call option is out of the money ($K > S$) and negative when the call option is in the money ($K < S$). If E[dSd σ] is positive (if the stock price and its volatility are positively correlated), the vanna term will enhance the P&L and hence value of a Black-Scholes option at high strikes and reduce it at low strikes. The opposite is the case if the correlation is negative. Since the equity index skew is typically negative, with low strikes carrying greater implied volatility than high ones, we can guess that in a stochastic volatility model we will require a negative correlation between the index and its volatility in order to reflect the skew.

Crude usefully intuitive ways to understand the effect of stochastic volatility on the smile.

A Preliminary Look: Start From Local Volatility with a Skew and Perturbatively Add Stochasticity: Stoch Local Vol

Add a stochastic element to a local volatility model.

$$\frac{dS}{S} = \alpha S^{\beta - 1} dW$$

$$d\alpha = \xi \alpha dZ \qquad \text{SABR model}$$

$$dZ dW = \rho dt$$

For $\beta = 1$ this is geometric Brownian motion with no smile, else it's CEV with skew.

α is the stochastic part of the smile-inducing local volatility, and ξ is the volatility of volatility.

Assume $\rho = 0$ and β close to 1 (small skew): estimate the skew using our knowledge of local vol.

For $\xi = 0$ we know the implied volatility is roughly average of the local volatilities S to K:

$$\Sigma_{LV}(S, t, K, T, \alpha) = \frac{\alpha}{2}[S^{\beta - 1} + K^{\beta - 1}]$$

Taylor expansion in K for β close to 1: $\Sigma_{LV}(S, t, K, T, \alpha) \approx \frac{\alpha}{S^{1-\beta}}\left[1 + \frac{(\beta - 1)}{2}\ln\frac{K}{S}\right]$

a linear skew with negative slope, $\frac{\partial \Sigma}{\partial K} \approx \frac{\partial \Sigma}{\partial S}$ atm.

Now switch on the stochastic volatility $\xi \neq 0$ There is a range of possible α values. Estimate C_{SLV} in this Stochasticized Local Vol model as average of the BS prices over the range of α:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \alpha))\phi(\alpha)d\alpha$$

Taylor expand this about the mean $\bar{\alpha}$ for small volatility of volatility:

$$C_{SLV} = \int C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha} + (\alpha - \bar{\alpha})))\phi(\alpha)d\alpha$$

$$\approx \int \left\{ C_{BS}(\Sigma_{LV}(S, t, K, T, \bar{\alpha})) + \frac{\partial C_{BS}}{\partial \alpha}(\alpha)\Big|_{\alpha = \bar{\alpha}}\cancel{(\alpha - \bar{\alpha})} + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\alpha)\Big|_{\alpha = \bar{\alpha}}(\alpha - \bar{\alpha})^2) \right\}\phi(\alpha)d\alpha$$

$$\approx C_{BS}(\bar{\alpha}) + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\bar{\alpha}))var(\alpha)$$

Look at implied Black-Scholes volatility Σ_{SLV} deviation away from $\xi = 0$:

$$C_{SLV} \equiv C_{BS}(\Sigma_{SLV}) \approx C_{BS}(\Sigma_{LV}\overline{(\alpha)} + \{\Sigma_{SLV} - \Sigma_{LV}\})$$

$$\approx C_{BS}(\Sigma_{LV} + \{\Sigma_{SLV} - \Sigma_{LV}\})$$

$$\approx C_{BS}(\bar{\alpha}) + \frac{\partial C_{BS}}{\partial \Sigma_{LV}}(\Sigma_{SLV} - \Sigma_{LV})$$

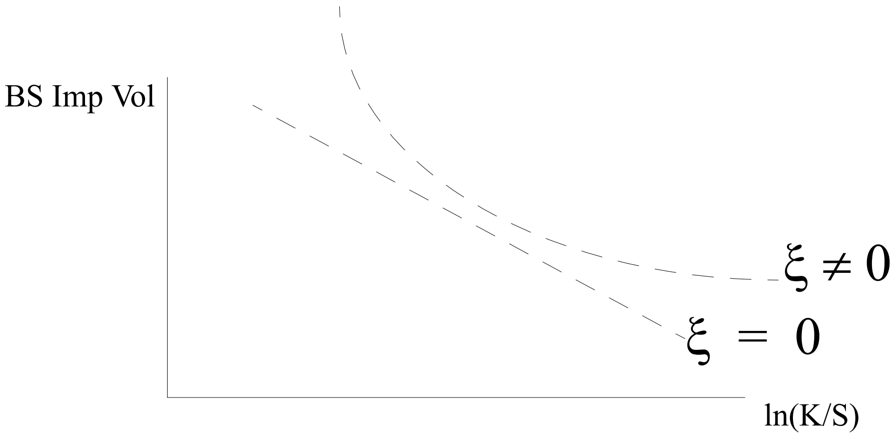
Comparing the above two equations, we obtain

$$\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha}) + \frac{\frac{1}{2}\frac{\partial^2}{\partial \alpha^2}(C_{BS}(\alpha))\Big|_{\alpha = \bar{\alpha}} \quad var(\alpha)}{\frac{\partial C_{BS}}{\partial \Sigma_{LV}}}$$

Evaluate the BS derivatives above **for small times to expiration and close to at-the-money**:

$$\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha})\left\{1 + \frac{1}{2}\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2\left[\ln\frac{S}{K}\right]^2\right\} \qquad \text{Eq.1.2}$$

The local volatility smile is altered by the addition of a quadratic term in $\ln\frac{S}{K}$



No need for correlation between volatility and stock price in order to obtain a smile if we start from local volatility.

Risk-neutral Valuation And Stochastic Volatility Models

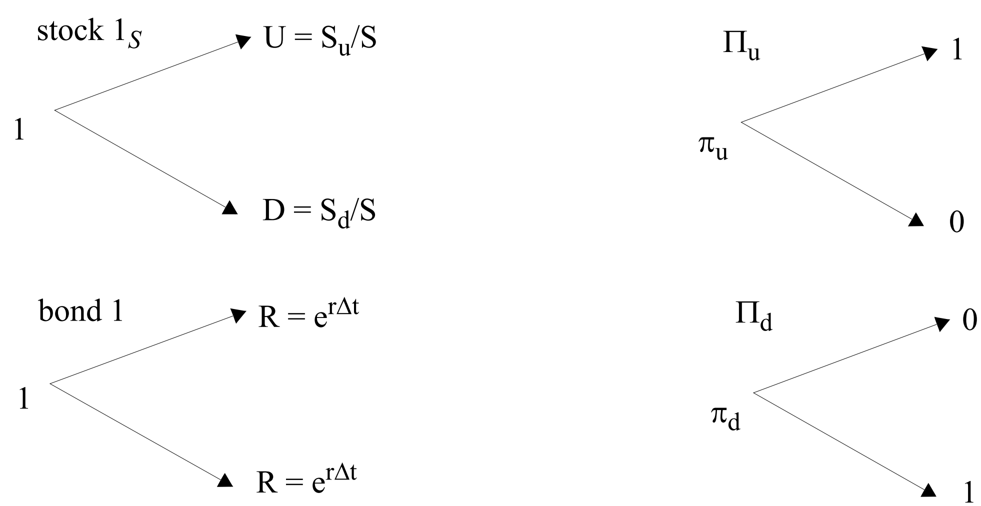
Arbitrage-free options valuation:

Hedge away all the risk instantaneously.
Need enough securities to span all the possible states of the world dynamically.
Then the riskless hedged portfolio must earn the risk-free rate.

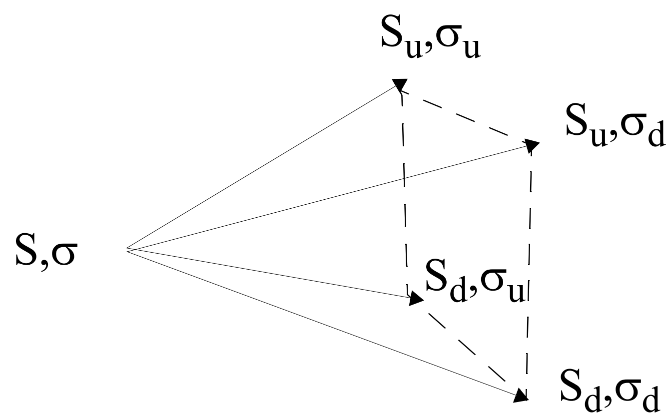
Recall how we derived Black-Scholes with a stochastic stock price:

Arrow-Debreu securities, Π_u and Π_d span the space of payoff states.

Two securities: two final states. Hence you can value any instrument irrespective of outcome or its probability.



Stochastic volatility: S **and** σ can vary: σ_u and σ_d differ; there is a correlation between S and σ



4 possible final states
Need 4 four Arrow-Debreu securities that pay $1.
But we know only two: S and B.
We would need to know the *price of the volatility* σ today in order to span the other states.

But volatility is not a security or a traded variable, it's a parameter.
Instead, we can only hedge options with other options to hedge the volatility sensitivity.

Cf: Vasiçek interest-rate model.
You cannot hedge the interest-rate exposure of a bond with "interest rates". You must hedge the interest-rate sensitivity of one bond with another.

If we hedge options only with shares of stock perfect replication is impossible. ***If*** you can also use other options, and if you *assume* you know the stochastic process for options as well as stock prices, then you can derive an arbitrage-free formula for options values. But do we? Nevertheless …

Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dWdZ &= \rho dt \end{aligned} \qquad \text{Eq.1.3}$$

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

$$d\Pi = dt\begin{bmatrix} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{bmatrix}$$

$$+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right)$$

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma}$$

$$\text{Then } d\Pi = dt\left[\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right]$$

No riskless arbitrage: $$d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$$

$$\frac{\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\dfrac{\partial V}{\partial S} - rV}{\dfrac{\partial V}{\partial \sigma}}$$

$$= \frac{\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\dfrac{\partial U}{\partial S} - rU}{\dfrac{\partial U}{\partial \sigma}}$$

$$= \phi(S, \sigma, t) \qquad \text{separation of variables}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don't know the value of the function φ!