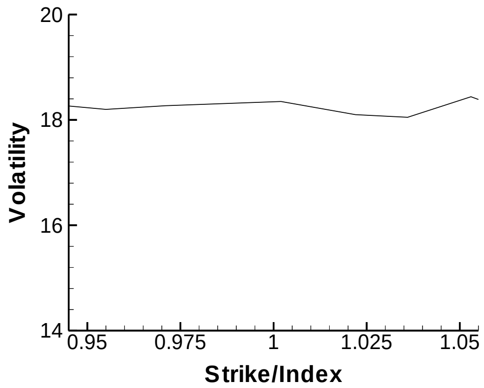
LECTURE 2

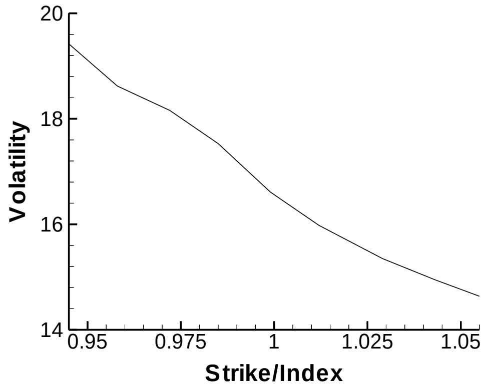
PRINCIPLES OF VALUATION

Recap: The Smile

- Before 1987



- After 1987.



- The volatility of a *stock* itself cannot depend upon the option with which you choose to view it.
- So description of underlier is incorrect.

Development of the Smile

- Volatility smile has spread to most other options markets.
- Traders and quants in every product area have had to model the smile.
- No area where model risk/uncertainty is more of an issue than in the modeling of the volatility smile.
- DAX Implied Volatility Surface 2008: Typical

2 Matthias R. Fengler

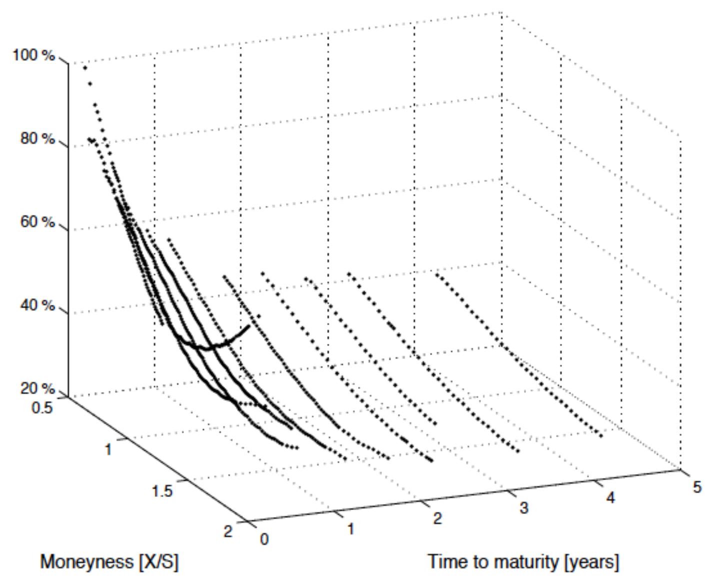


Fig. 1. IV surface of DAX index options from 28 Oct. 2008, traded at the EUREX. IV given in percent across a spot moneyness metric, time to expiry in years.

SVI Parameterization

- A commonly used parameterization of the implied volatility smile for a fixed expiration T is given by

$$\sigma_I^2(m, T) = a + b[\rho(m - c) + \sqrt{(m - c)^2 + \theta^2}]$$

where $m = \ln\frac{K}{F}$ is the forward moneyness, and $F = Se^{rT}$ is the forward price of the underlying stock, and T is the time to expiration of the option.

- But we must worry about arbitrage violations, since this is a parameter, not a price. (Cf. interest rates)

Stylized Facts about the Smile for Equity Indexes

- Steep for short expiration, flatter for longer ones.
- Negatively skewed almost always
- Term structure is upward sloping except in crises, when skew also steepens.

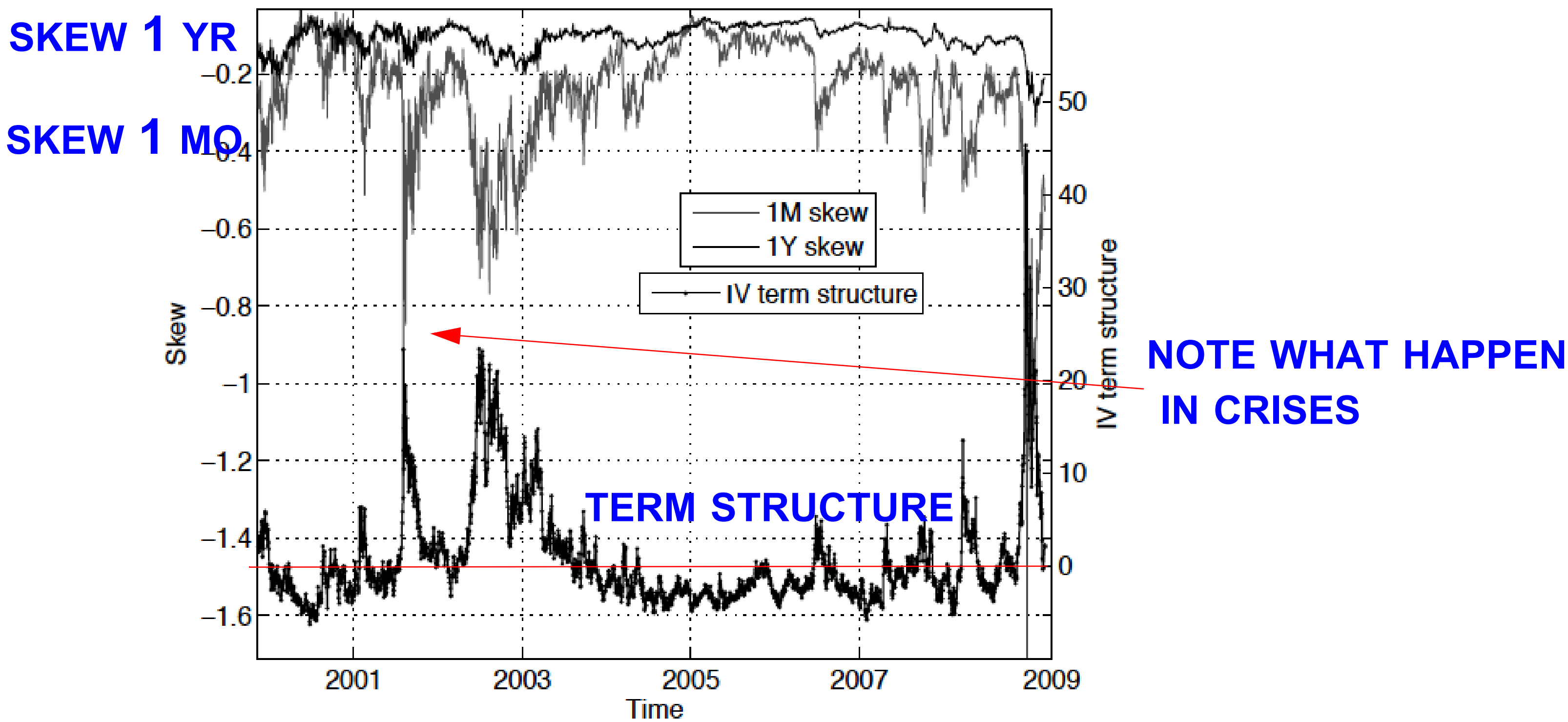


Fig. 3. Time series of 1M and 1Y IV skew (left axis, gray line and black line respectively) and time series of the IV term structure (right axis, black dotted line). Skew is defined as $\left.\frac{\partial \hat{\sigma}^2}{\partial m}\right|_{m=0}$, where m is log-forward moneyness. The derivative is approximated by a finite difference quotient. IV term structure is the difference between 1M ATM and 1Y ATM in terms of percentage points. Negative values indicate an upward sloping term structure.

- Changes in the at-the-money implied volatility and the index are negatively correlated.

corr = -0.69

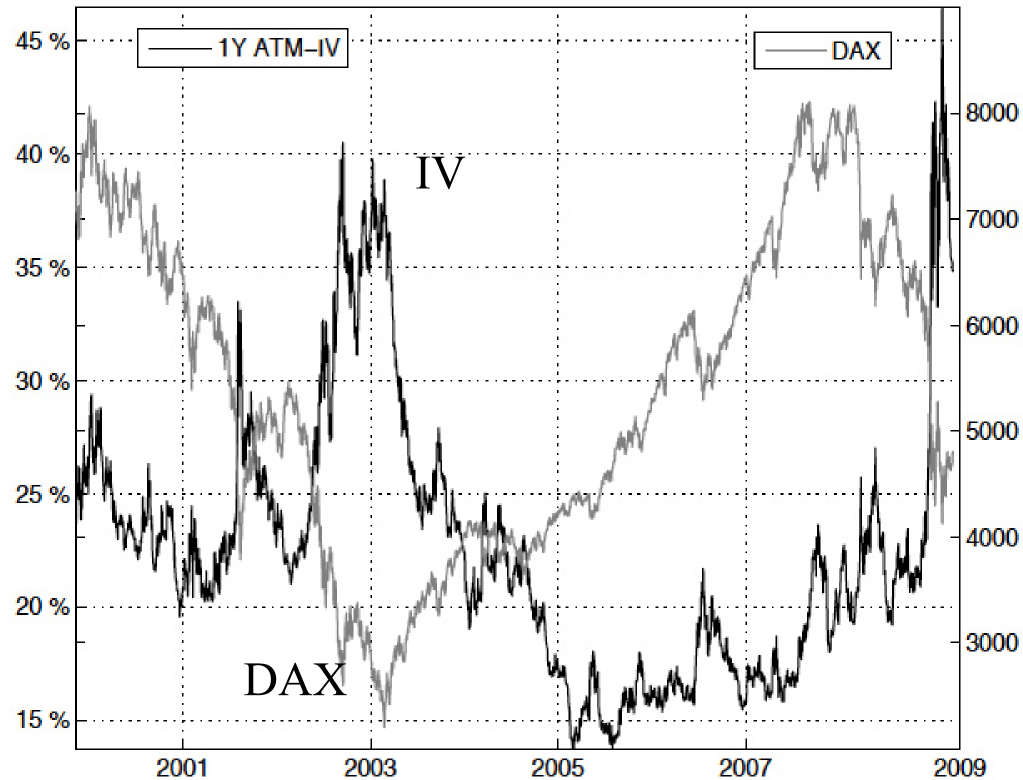
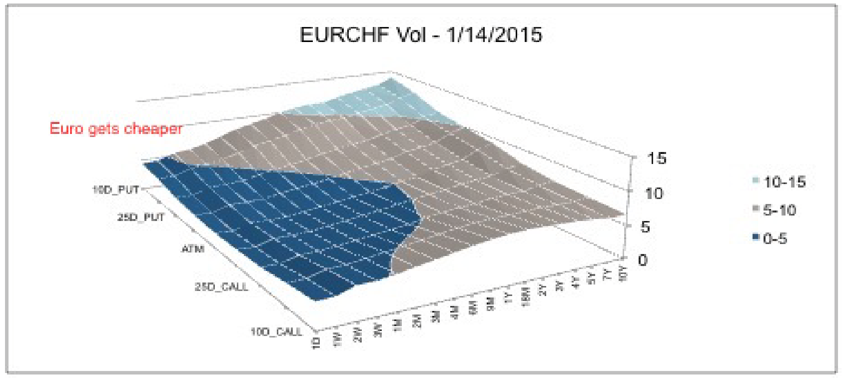


Fig. 2. Time series of 1Y ATM IV (left axis, black line) and DAX index closing prices (right axis, gray line) from 2000 to 2008.

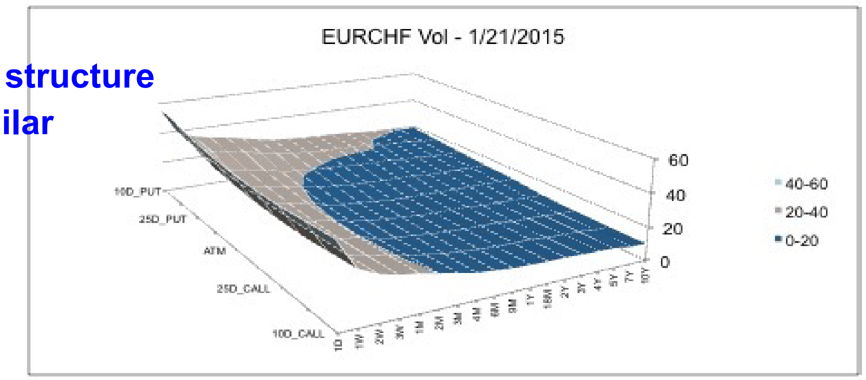
- Implied volatility seems to be mean-reverting.

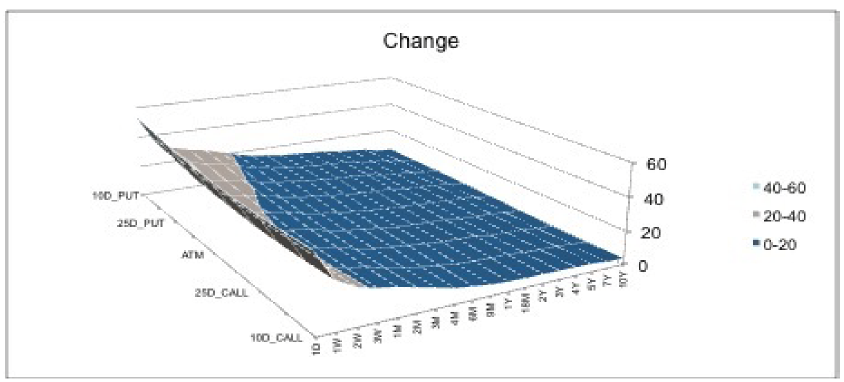
The EURCHF Skew at the End of the Swiss Franc Floor[1]



Vol of options on 1 Euro with strike in CHF

Change in level
Change in term structure
Skew stays similar





1. Michael C. Eichenberger

PRINCIPLES OF VALUATION

The Principles of Financial Valuation

Fundamental Theorem of Finance. *Security prices exclude arbitrage if and only if there exists a strictly positive value functional, under the technical restrictions that the space of portfolios and the space of contingent claims are locally convex topological vector spaces and the positive cone of the space of contingent claims is compactly generated, that is, there exists a compact set K of X (not containing the null element of X) such that*

$$C = \{x \in X : x \geq 0\} = \bigcup_{\lambda \geq 0} \lambda K.$$

What other fields have theorems? Why can you value options?

So formal and axiomatic. But we are studying financial "**engineering**".

Don't get misled by mathematics, theorems, lemmas. The world may not respect them. Understand them, but we're dealing with the real world.

- What is financial engineering?
  Cf. Mechanical engineering, Electrical engineering, Bio-engineering.

  Science seeks to discover the fundamental principles that describe the world, and is usually reductive.
  Engineering is about using those principles, constructively, for a purpose.

  Financial engineering, layered above financial science, would be the study of how to create functional financial devices – convertible bonds, warrants, default swaps, etc. – that perform in desired ways.
- What is financial science?
  Our theories don't describe the behavior of assets very well.
  Stock evolution isn't Brownian.

Price & Value

- Price = what you have to pay to acquire a security.
  Value is what it is worth. The price is fair when it is equal to the value.
- Judging value, in even the simplest way, involves the construction of a model or theory.
- Black: markets are efficient when prices are 1/2 to 2 times value.

The Purpose of Models

- Example: Valuing a Park Ave apartment.
- Models are used to rank securities by value.
- Models are used to interpolate or extrapolate from liquid prices to illiquid prices.
- Models transform linear quantities you can have intuition about into nonlinear dollar values.

Styles of Modeling

- Absolute vs. Relative Value.
- Absolute Value: Quantum mechanics is a theory of the world, absolute.
  Geometric Brownian motion is a *model* of absolute valuation, but not correct.
- Relative Value: illiquid --> liquid
  Derivatives are like molecules made out of simpler atoms.
  Relative valuation is less ambitious
  Black-Scholes tells you the price of an option in terms of the price of a stock and a bond.
- We are taking the view point of an options trading desk, as manufacturers or arbitrageurs
  Derivatives can be constructed or deconstructed
  Stocks to Options, Exotics to Vanilla
  Fruit salad: What is *the implied price* of pears

The One Commandment of Quantitative Finance

If you want to know the value of a security, use the price of another security that's as similar to it as possible.

The *law of one price*, or *the principle of no riskless arbitrage:*

Any two securities with identical future payoffs, no matter how the future turns out, should have identical current prices.
Almost everything in finance follows from this.

Valuation by Replication

Target security
Replicating portfolio, a collection of more *liquid* securities that, collectively, has the same future payoffs as the target *no matter how the future turns out.*

The target's value is then simply that value of the replicating portfolio.

No matter how the future turns out: **the science of markets: a model**

Replication: **engineering**

Styles of Replication

Static: rarely possible: put-call parity and variance swaps

Dynamic: BSM

Dynamic replication is very elegant but makes many assumptions.

Neftci's *Principles of Financial Engineering*
"Real life complications make dynamic replication a much more fragile exercise than static replication. The problems that are encountered in static replication are well known. There are operational problems, counterparty risk, and so the theoretically exact synthetics may not be identical to the original asset. There are liquidity problems and other transactions costs. But all these are relatively minor and in the end, static replicating portfolios used in practice generally provide good synthetics.

With dynamic replication, these problems are magnified, because the underlying positions needs to be readjusted many times. For example, the effect of transaction costs is much more serious if dynamic adjustments are required frequently. Similarly, the implications of liquidity problems will also be more serious. But more importantly, the real-life use of dynamic replication methods brings forth *new* problems that would not exist with static synthetics."

We have to worry not just about current liquidity and bid-ask spreads, but about how they vary in the future.

Dynamic replication is imperfect; it depends upon models, which imply assumptions and the approximations involved in working in discrete time steps.

Even if the theory is easy, “the strategy needs to be implemented using appropriate position-keeping and risk-management tools. The necessary software and human skills required for these tasks may lead to significant new costs, but also to many jobs producing and taking care of these tools.

Finally, dynamic replication is often used to replicate securities with nonlinear payoffs. This leads to exposure to the level of volatility, and who knows what the future level of volatility will be. Managing exposure to volatility can be much more difficult than managing exposure to interest rates or currencies, because there are (almost) no underliers to trade.”

- So, in this course, *first try use static replication* for valuing new securities. If we cannot, *then we will use dynamic replication*.
- Models are unreliable guides to the world of finance, and because you don’t know which is the right one, *it’s best to use as little modeling as possible. And, if you have to use a model, it’s always good to use more than one so you understand the model-dependence of your result.*

Implied Variables and Realized Variables

Physics models start from today and **predict the future**.

Financial models think about the future and **predict values today**.

What matters is not only what will happen, but what people *think* will happen.

What people think will happen affects what happens today.

Realized variables describe what actually happens.

Financial models calibrate the future to current known prices to produce implied variables about the future that match known prices today. One then has to compare these implied values to the future values that are actually realized as time passes.

Implied variables describe what people think will happen filtered through a model.

Should you hedge with implied volatility or realized volatility? Later lectures.

Testing Models

- Here are some counter-intuitive and interesting remarks about quantitative finance by Fischer Black.
- “It's better to 'estimate' a model than to test it. I take 'calibration' to be a form of estimation, so I'm sympathetic with it, so long as we don't take seriously the structure of a model we calibrate. Best of all, though, is to 'explore' a model.”
- “My job, I believe, is to persuade others that my conclusions are sound. I will use an array of devices to do this: theory, stylized facts, time-series data, surveys, appeals to introspection and so on.”
- “In the real world of research, conventional tests of [statistical] significance seem almost worthless.”
- No-one could make these remarks about models or theories in physics, chemistry or engineering. Think about the differences between these fields and quantitative finance, and why that should be the case.

MODELING MARKETS:

The Efficient Market Hypothesis/Model

Experience shows that it is difficult or impossible to successfully and consistently predict what's going to happen to the stock market tomorrow based on all the information you have today.

The EMH formalizes this experience by stating that it is impossible to beat the market, because current prices reflect all current economic and market information.

Jiu-jitsu approach: *I can't figure out how things work, so I'll make the inability to do that a principle.*

Uncertainty, Risk & Return

Quantifiable Uncertainty or Risk

What do you mean when you say there's a 1/8 chance of throwing 3 heads in succession?

Frequentist Probabilities

Tossing a coin: history doesn't matter

Unquantifiable Uncertainty:

What do you mean when you say that there's a small probability of a revolution overthrowing the United States government?

The likelihood of a revolution in some country or the probability of a terrorist attack. The chance that an earthquake of magnitude 6.7 or greater will occur before the year 2030 in the San Francisco Bay Area.

No way of honestly estimating probabilities.

In human affairs frequentist probabilities are not known and history matters.

Most financial models assume that unquantifiable uncertainties are actually frequentist. We are mostly going to do this too, but need to remember the assumptions we make.

Agent-based models perhaps go one step closer to reality, but are still models.

First, Modeling A Share of Stock

- A company is a tremendously complex and structured endeavor. Consider Apple.

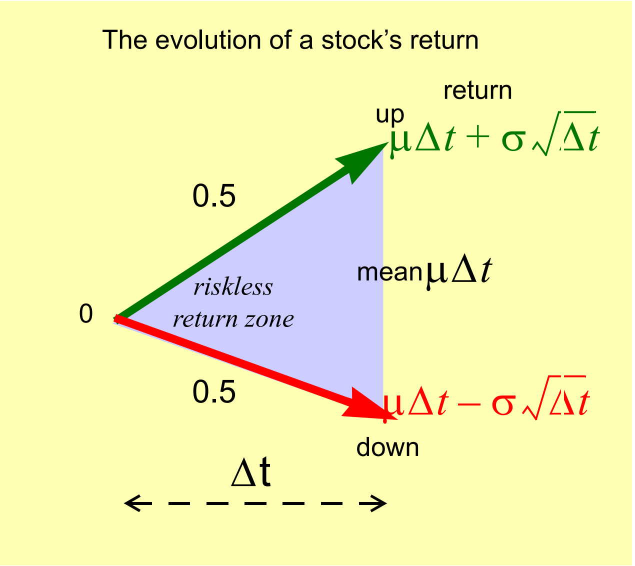
  The value of the organization is reflected in the price to buy or sell just *one* incremental share of the company. Financial modeling aims to tell you what you should pay today for a share of its future performance.

  The EMH model treats a stock as a simple kind of atom that undergoes a simple kind of quantifiable uncertainty.

  A stock's most important feature is the uncertainty of its return.

  No Arbitrage means the riskless return is a convex combination of up and down.

  This naive either-up-or-down model captures much of the inherent risk of owning a stock and many other securities. But not all.



  The Efficient Market Models often uses Geometric Brownian Motion to describe return distributions. This means that all we care about is volatility and return; these parameters specify the entire distribution of returns. Obviously not strictly true, but let's see where it takes us.

  What return should we expect for a given volatility?

  The law of one price will give us the answer.

The Law of One Price Relates Risk to Return

We can extend the law of one price (identical payoffs have identical prices*)* to demand that *identical expected risks have identical expected returns*.

But some risks can be avoided. Therefore the principle: *identical* ***unavoidable*** *expected risks have identical expected returns*.

How can you avoid risk?

There are three ways to reduce or avoid risk:

(i) dilution with a riskless bond

(ii) diversification

(iii) hedging

We'll combine these with the law of one price to derive everything we can use.

[Finance For Future Generations]

- Feynman: One sentence about physics to guide future civilizations

- Feynman: One sentence about biology to guide future civilizations:

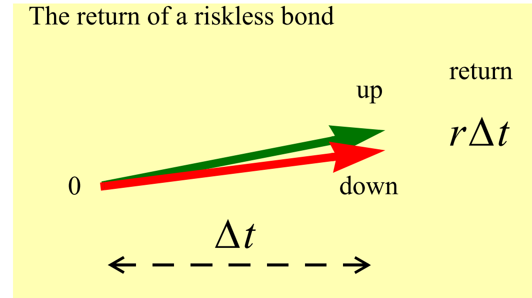
- One sentence about finance to guide future civilizations:
  If you can hedge away all correlated risk, and then diversify over all uncorrelated risk, you should expect to earn the riskless return.

- This is a sensible principle.
  The trouble is that correlation and diversification can’t really be carried out because risk is not really purely statistical. It can’t be specified for all time by a stochastic pde or Monte Carlo.

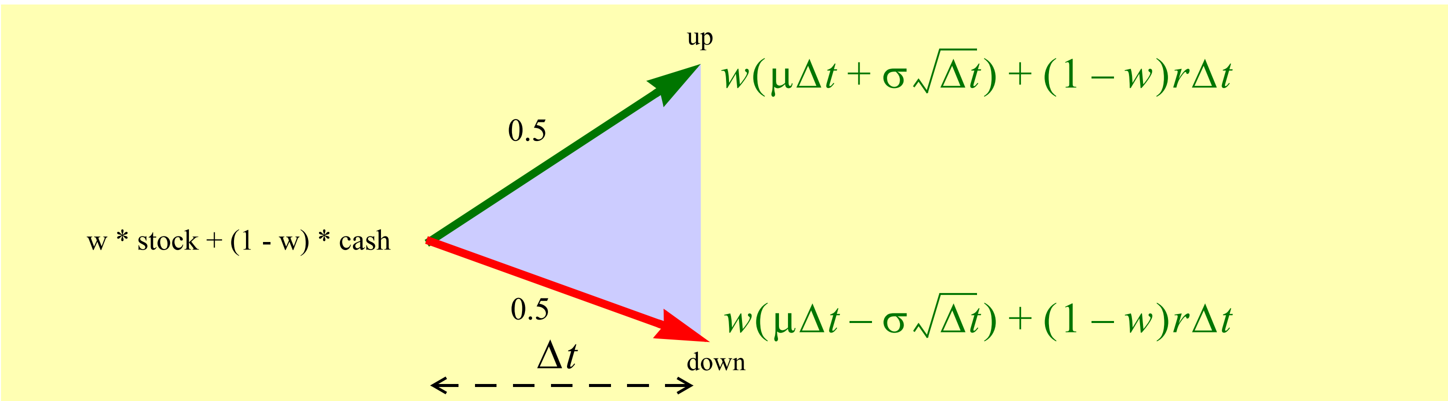
- Bedazzled

(i) Risk Reduction by Dilution Means Risk and Return are Proportional

By adding a riskless bond with zero volatility to the stock of volatility σ and expected return μ, you reduce both the risk and return of your investment.



Consider a mixture of w% risky stock with volatility σ and (1 - w)% riskless bond.



The expected return for this mixture is $w\mu\Delta t + (1-w)r\Delta t = r\Delta t + w(\mu - r)\Delta t$.
The volatility of returns is $w\sigma$.

Thus extra risk of magnitude $w\sigma$ must generate extra return $w(\mu - r)\Delta t$

$$\frac{\mu - r}{\sigma} = \lambda \qquad \mu - r = \lambda\sigma$$

Excess return is proportional to risk. By law of one price, must be true for all securities. A similar result holds for options values, and is equivalent to the Black-Scholes equation.

(ii) Risk Reduction by Diversification Means λ is Zero

If you can accumulate a portfolio of so many uncorrelated unavoidable risks that they cancel in the limit as the number of stocks become large, the portfolio's net volatility σ approaches zero.

Then, by the law of one price, it must produce an excess return of zero for the entire portfolio.

But the excess return of the entire portfolio is the weighted sum of the excess returns of each individual member of the portfolio, each of which is proportional to their individual non-zero volatility via the Sharpe ratio.

Hence the Sharpe ratio in the equations above for this portfolio must be zero. But the Sharpe ratio is the same for all portfolios of stocks, so that $\lambda = 0$ in general. Thus,

$$\mu = r$$

All stocks must be expected to earn the riskless rate if you can diversify.

(iii) Risk Reduction by Hedging

You can't always diversify because stocks are sensitive to the entire market M.

Let ρ be the correlation of the returns between stock S with volatility σ and the market M with volatility σ_M

$$\frac{dS}{S} = \mu dt + \sigma\left(\sqrt{1-\rho^2}\,dZ + \rho dZ_M\right)$$

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

You can *hedge away* the M-related risk of any stock to create an M-neutral portfolio:

$dS_M = dS - \Delta dM$ has no market risk if $\Delta = \rho(\sigma/\sigma_M)\frac{S}{M} \equiv \beta\frac{S}{M}$.

This M-neutral stock has expected return $\dfrac{\mu - \beta\mu_M}{1-\beta}$ sensitive only to the volatility of the stock.

By diluting it, we can show that its excess return of the M-neutral stock must then be proportional to its residual volatility. Furthermore, by diversifying over many M-neutral stocks we can show that the M-neutral stock can expect only the riskless rate r, so that[1]

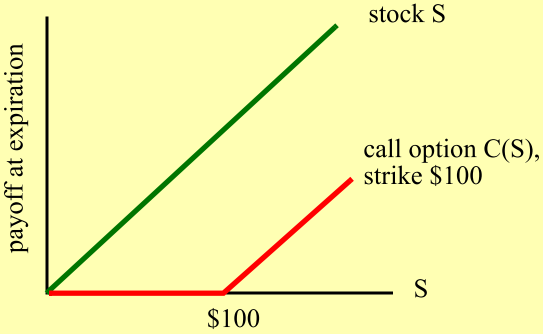
$$(\mu - r) = \beta(\mu_M - r) \qquad \text{CAPM in "Efficient Markets"}$$

Do you believe this?

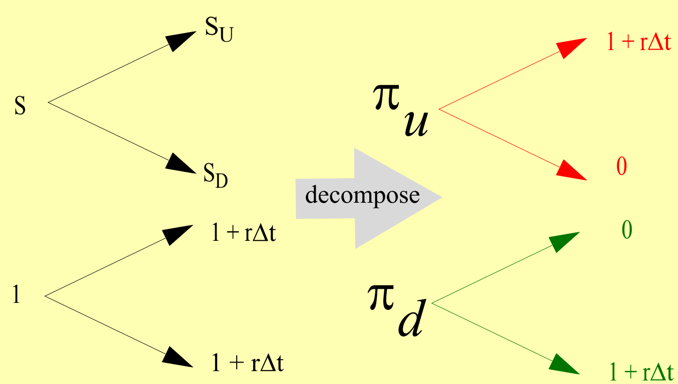
1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://www.ederman.com/new/docs/qf-market_bubbles.pdf

Derivative Valuation by Replication

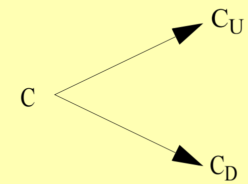
A stock is straight, an option curved.



Binary price trees for a stock S and a $1 investment in a riskless bond. The stock and bond can be decomposed into a security $\pi_{\mathbf{u}}$ that pays off only in the up state and a security $\pi_{\mathbf{d}}$ that pays off only in the down state.



You can replicate an option's nonlinear payoff over each instant by suitable investments in the elemental securities τ_u and τ_d.



$(1+r\Delta t)C = (C_U).\pi_{\mathbf{u}} + (C_D).\,\pi_{\mathbf{d}}$

A derivative is a contract whose payoff depends on the price of a "simpler" *underlier*. The most relevant characteristic is the *curvature* of its payoff *C(S)*, as illustrated for a simple call option. **What is the value of curvature?**

You can use linear algebra to decompose the stock and bond into a basis of two more elemental securities π_u and π_d, each respectively paying $\$(1+r\Delta t)$ in only one of the final states.

Then you can replicate the payoff of any non-linear function *C(S)* over the next instant of time Δt, no matter into which state the stock evolves. Note that the portfolio consisting of both π_u and π_d is riskless and is therefore worth $1.

The value of the option is the price of the mixture of stock and bond that replicates it. The coefficients depend on the difference between the up-return and the down-return at each node, that is, on the stock's volatility σ.

The choice-of-currency/numeraire trick

You can use any currency to value a security if markets are efficient.

A convenient choice of currency can greatly simplify thinking about a problem, and sometimes reduce its dimensionality.

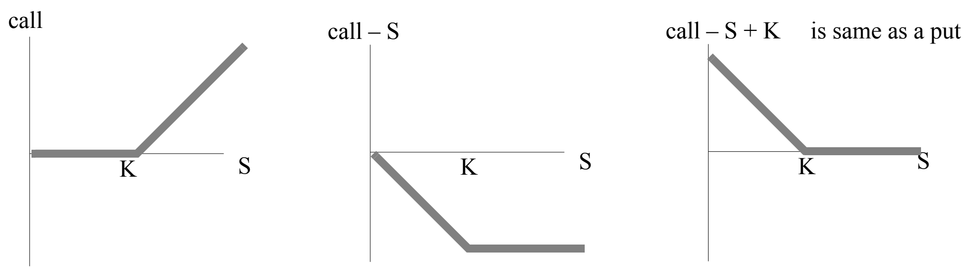
Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modeled by choosing a bond itself as the natural valuation currency.

METHODS OF REPLICATION

Static Replication

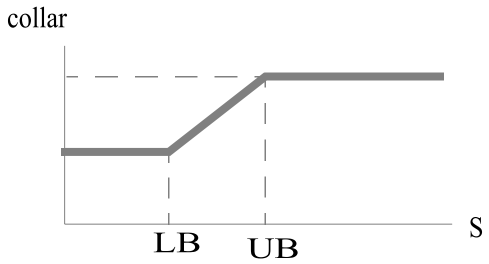
If you can create a static replicating portfolio for your payoff, you have very little model risk.

European put from a call: Put-Call Parity



Thus price of put = price of call - price of stock + PV(K).

A collar is a very popular instrument for portfolio managers who have made some gains during the year and now want to make sure they keep some upside but don't lose too much downside.



You can write the payoff as

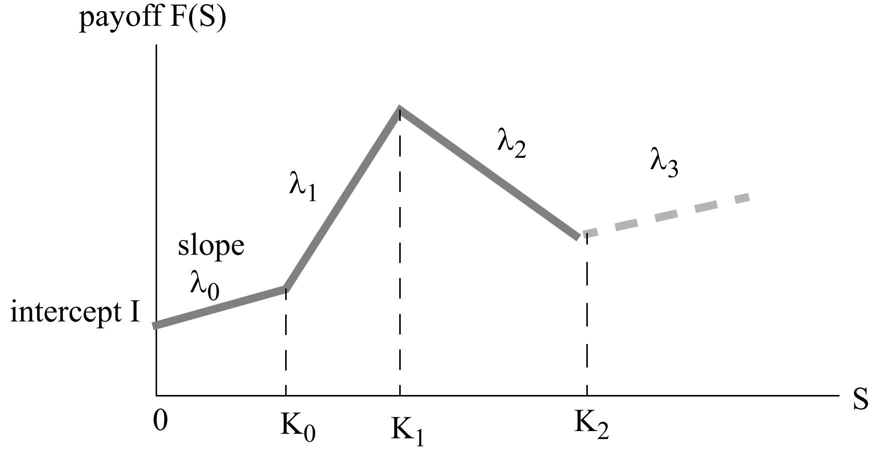
$$LB + call(S, LB) - call(S, UB)$$

Using put-call parity: $S + put(S, LB) - call(S, UB)$.

Its popularity forces dealers to be short puts and long calls.

Generalized European payoffs:

Piecewise-linear function of the terminal stock price S



Replicating portfolio consisting of a zero-coupon bond ZCB(I) plus a series of calls $C(K_i)$:

$$ZCB(I) + \lambda_0 S + (\lambda_1 - \lambda_0)C(K_0) + (\lambda_2 - \lambda_1)C(K_1) + \ldots$$

whose value can be determined from market prices.

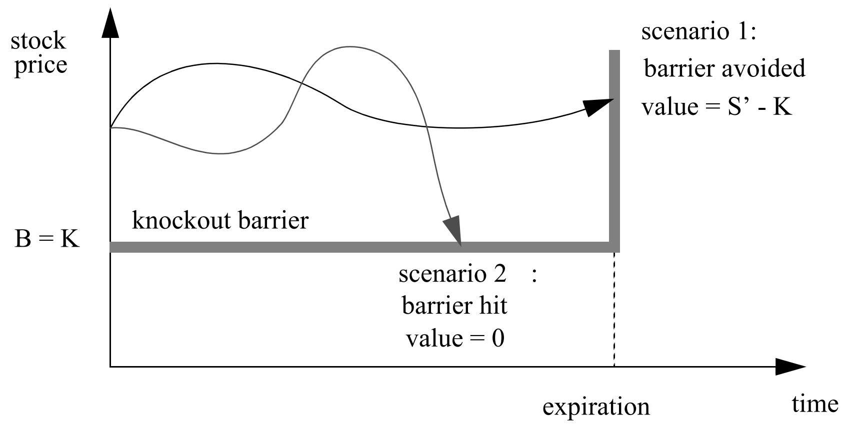
Static Hedge for a Down-and-Out Call with Strike = Barrier

This option has a time-dependent boundary.

Stock price S and dividend yield d, strike K and out-barrier $B = K$.

Scenario 1 in which the barrier is avoided and the option finishes in-the-money.

Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



In scenario 1 the call pays out $S' - K$, the payoff of a forward contract with delivery price K worth

$$F = Se^{-dt} - Ke^{-rt}$$

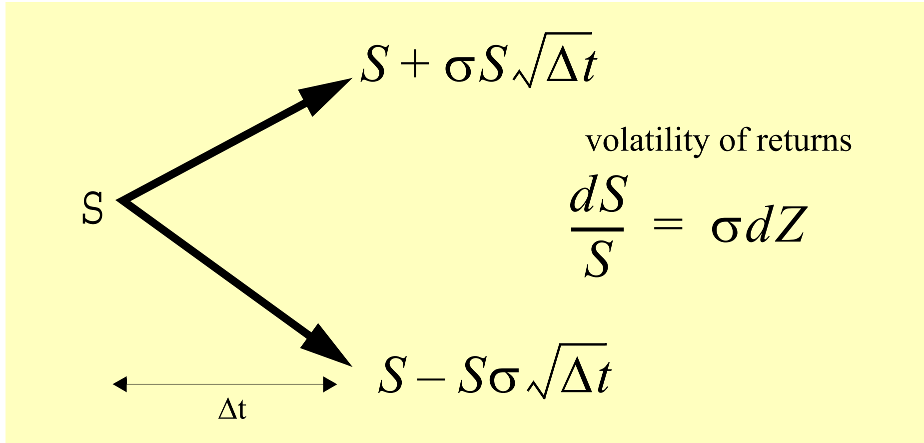
For paths in scenario 2, the down-and-out call immediately expires with zero value. In that case, the above forward F that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$. This is close to zero.

When the stock hits the barrier you must sell the forward to end the trade.

DYNAMIC REPLICATION

Quick Derivation of the Black-Scholes PDE

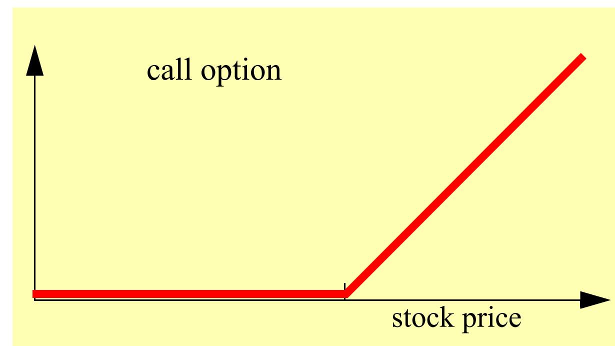
Assume GBM with zero rates for simplicity.



In time Δt, $\Delta S \approx \sigma S\sqrt{\Delta t}$.

The stock S is a primitive, linear underlying security that provides a linear position in ΔS.

If you are long an option, you profit whether the stock goes up or down! The call has curvature, or convexity. $\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$



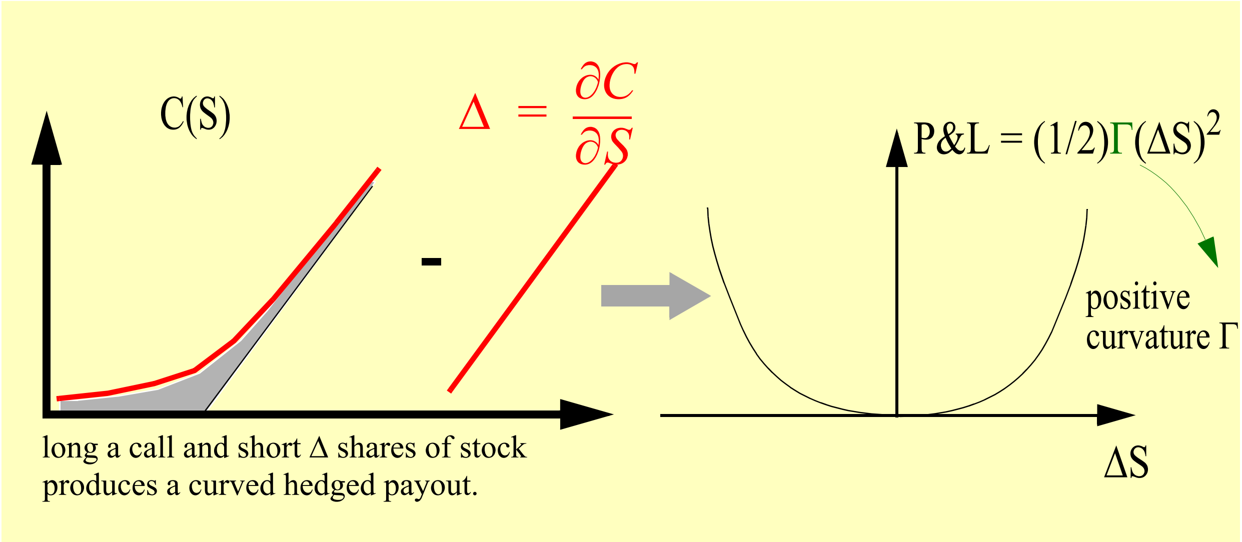
What is the fair price for *C(S,K,t,T)*?

We can do a Taylor series expansion on the unknown price *C( )* and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S+\Delta S, t+\Delta t) = C(S,t) + \left.\frac{\partial C}{\partial t}\right|_{S,t} \Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,t} \Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,t} \frac{(\Delta S)^2}{2} + \ldots$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

If you hedge away the linear term in ΔS by shorting $\Delta = \frac{\partial C}{\partial S}$ shares the profit and loss of the hedged option position looks like this:



Positive convexity generates a profit or loss that is quadratic in (ΔS).

What Should You Pay for Convexity?

Suppose we think we know the future volatility of the stock, Σ.

Over time Δt, the stock should move an amount $\Delta S = \pm\Sigma S Z(0,1)\sqrt{\Delta t}$.

Binomially, this corresponds to $\Delta S = \pm\Sigma S\sqrt{\Delta t}$ with $(\Delta S)^2 = \Sigma^2 S^2 \Delta t$.

Change in value from the movement in stock price $= \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t)$

Change in value from passage of time $= \Theta(\Delta t)$ where $\Theta = \frac{\partial C}{\partial t}$

Total change in value of the hedged position is $d\text{P\&L} = d(C - \Delta S) = \frac{1}{2}\Gamma(\Sigma^2 S^2 \Delta t) + \Theta(\Delta t)$

If we know Σ, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it behaves like a riskless bond and must earn zero interest: $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

The Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

time decay and curvature are linked

$$C_{BS}(S, K, \Sigma, t, T) = SN(d_1) - KN(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

The option's Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate r is non-zero, we will show in a subsequent chapter that

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

...