LECTURE 14

LOCAL VOLATILITY MODELS:
CALIBRATION
DUPIRE EQUATION
HEDGE RATIOS
EXOTIC OPTIONS

Looking Ahead

Local vol and Calibration

Dupire Equation and Justification of Rules of Thumb

Hedge Ratios of Vanillas

Values of Exotics

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Hedging Rules

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Stochastic Volatility Models

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Jump Diffusion Models

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Guest Speakers
Michael Kamal - April 15
Jackie Rosner - April 20

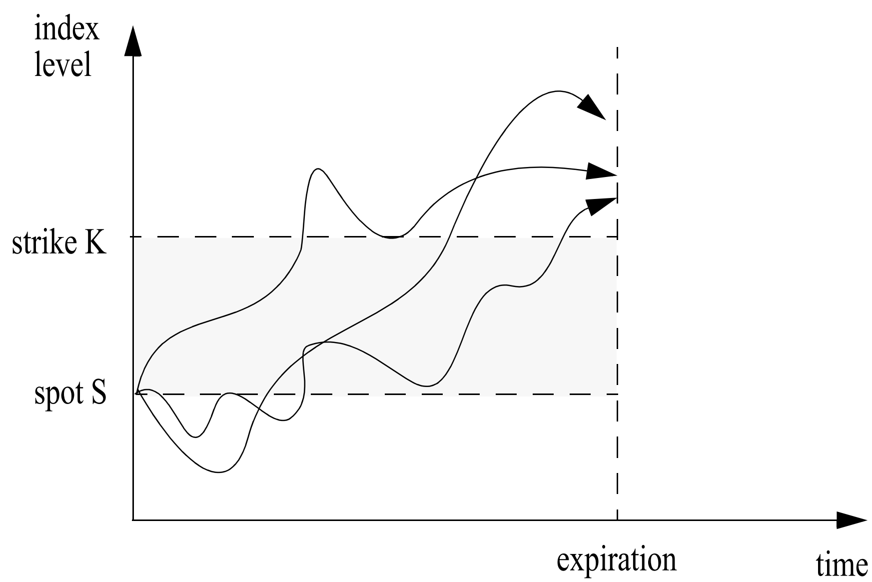
If you have questions come to my office hours or see me some other time by appointment.

Where We Were: The Rule of 2: Understanding The Relation Between Local and Implied Volatilities

The implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option on the path between spot and strike.

Therefore local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

You can understand this intuitively. Consider the implied volatility $\Sigma(S,K)$ of a slightly out-of-the-money call option with strike K when the index is at S. Any paths that contribute to the option value must pass through the region between S and K, shown shaded in the figure below. The volatility of these paths during most of their evolution is determined by the local volatility in the shaded region.



Implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot and strike.

Local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

This could apply to squares of volatilities too, approximately. We'll see more rigorous results shortly.

Moving Forward: Implied Trees and Calibration

$\frac{dS}{S} = \mu dt + \sigma(S, t)dZ$. We went from $\sigma(S, t)$ to $\Sigma(S, t, K, T)$.

In reality, one observes discretely spaced implied volatilities $\Sigma(S, t, K_i, T_i)$ for discrete strikes K_i and expirations T_i, and one wants to calibrate a local volatility surface $\sigma(S, t)$.

"Implied trees" are a generalization of implied volatility. Implied volatility is a single variable defining a Black-Scholes tree; the implied tree is a representation of local volatility defining an evolution of a stock price consistent with options prices.

Binomial framework in the paper *The Volatility Smile and Its Implied Tree* Derman, E. & I. Kani. "Riding on a Smile." RISK, 7(2) Feb.1994, pp. 139-145. Also *The Local Volatility Surface* Continuous-time derivation of the same results: Dupire, *Pricing With A Smile*, RISK, January 1994, pages 18-20.

The key point of these papers is that there is a unique stochastic stock evolution process with a variable continuous volatility $\sigma(S, t)$ that can fit all options prices and their *continuous* implied volatilities $\Sigma(S, t, K, T)$. The tree representation of this stochastic process is called the *implied tree*.

Implied variables are widely used in finance and proprietary trading.

- *The Volatility Smile and Its Implied Tree* shows that the local volatility at each node is determined numerically and non-parametrically from options prices.
- Inverse problem: going backwards from output to input.
- Analogous to finding a potential in physics from viewing the way particles move under its influence.
- Theoretically straightforward, it turns out, but is rather difficult in practice. It's an "ill-posed" problem. Beginning with sparse implied volatilities and interpolating them into a surface, small changes in the interpolated input can cause dramatic changes in the output.
- In practice, it may be better to assume some parametric form for the local volatility function and then find the parameters that make the tree's option prices match as closely as possible the market's option prices.
- In a later class we'll discuss some methods of calibrating local volatilities to implied volatilities.

Dupire's Equation for Local Volatility

This equation describes the mathematical relationship between implied and local volatility.

Assuming continuity, you can derive the local volatility from the options prices (or their corresponding BS implied volatilities) in a simple way. Once you find the local volatilities from the implied volatilities surface, you can use them to build a tree or MC, and then use it to value options.

Breeden-Litzenberger formula: $p(S, t, K, T) = e^{r(T-t)} \frac{\partial^2}{\partial K^2}(C_{mkt}(S, t, K, T))$

p is the density or risk-neutral probability function that tells you the no-arbitrage price $p(S, t, K, T)dK$ of earning \$1 if the future stock price at time T lies between K and $K + dK$, determined by the second derivative of the market option price.

Even better: $\sigma(S, t)$ itself can be found from market prices of options and their derivatives. For zero interest rates and dividends, the local volatility at the stock price K is given by:

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T}C(S, t, K, T)}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

This is the continuous version of the procedure we used to construct a local volatility binomial tree.

If interest rates are non-zero,

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T} C(S, t, K, T) + rK\frac{\partial}{\partial K} C(S, t, K, T)}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

This is the mathematically correct generalization of the notion of forward stock volatilities consistent with current implied volatilities.

Understanding the Equation

We can interpret the equation in economic terms.

$$\frac{\partial C}{\partial T} = \frac{C(S, t, K, T + dT) - C(S, t, K, T)}{dT}$$

is proportional to an infinitesimal calendar spread for standard calls with strike K,

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{(dK)^2}$$

is proportional to an infinitesimal butterfly spread for standard calls with strike κ.

Therefore the local variance $\sigma^2(K, T)$ at stock price K and time T is proportional to the ratio of the price of a calendar spread to a butterfly spread.

A calendar spread and a butterfly spread are combinations of tradeable options, and so the local volatility can be extracted from traded options prices (if they are available)!

Intuitively

A calendar spread

$$C(S, t, K, T + dT) - C(S, t, K, T)$$

measure the risk-neutral probability $p(S,t,K,T)$ of the stock moving from S at time t to K at time T, times the variance $\sigma^2(K, T)$ at K and T that is responsible for the adding option value.

$$\text{calendar spread} \sim p(S, t, K, T)\sigma^2(K, T).$$

But, according to the Breeden-Litzenberger, the probability

$$p(S, t, K, T) \sim \frac{\partial^2}{\partial K^2}C(S, t, K, T) \sim \text{butterfly spread}$$

So, roughly speaking, combing the two equations above, we have

$$\text{calendar spread} \approx \text{butterfly spread} \times \sigma^2(K, T)$$

or

$$\sigma^2(K, T) \approx \frac{\text{calendar spread}}{\text{butterfly spread}}$$

Using The Equation

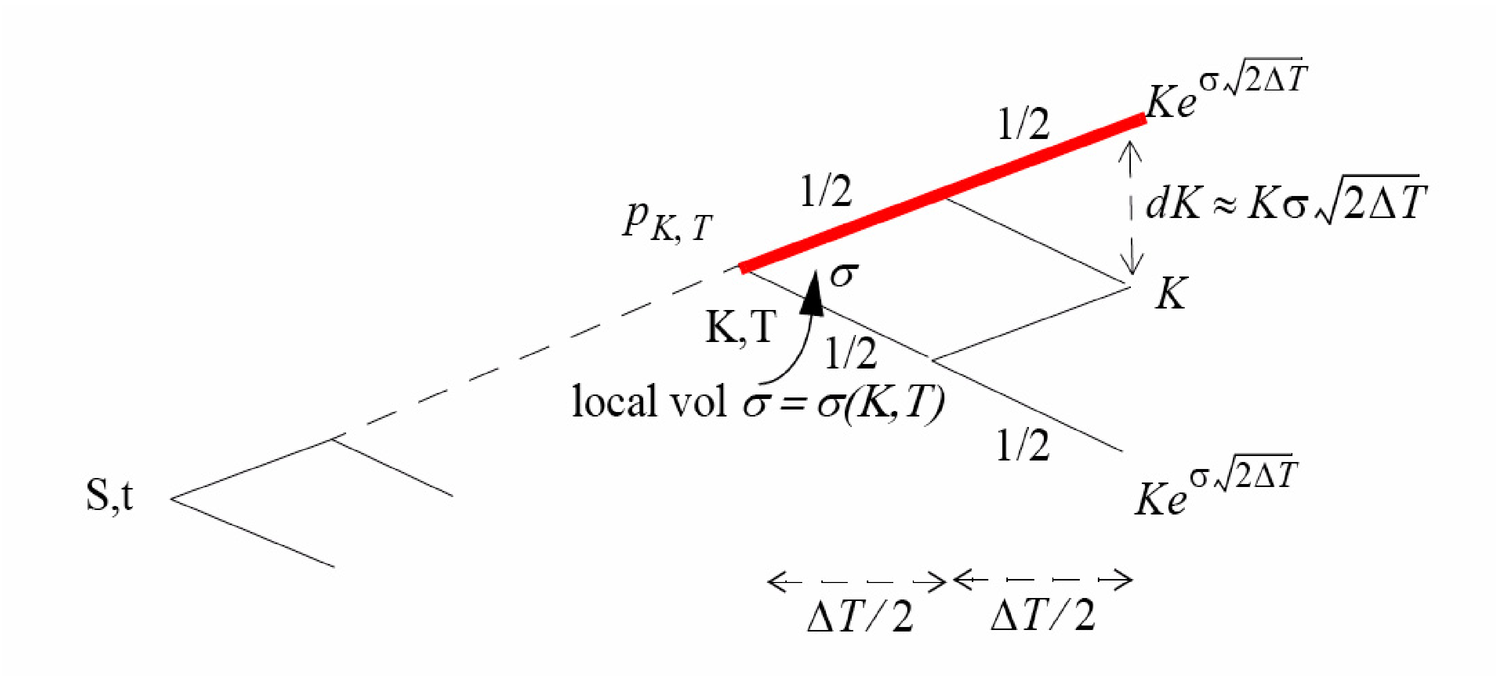
$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}} \quad \text{means} \quad \frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} - \frac{\sigma^2}{2}K^2\frac{\partial^2 C}{\partial K^2} = 0$$

resembles Black-Scholes equation with t replaced by T and S replaced by K.

- Black-Scholes equation holds for any contingent claim on S, relating the value of *any option* at S, t to the value of that option at $S + dS, T + dT$.
- This holds only for standard calls (or puts) and relates the value of a *standard* option with strike and expiration at K, T to the same option with strike and expiration $K + dK, T + dT$ when S, t is kept fixed.
- Find $\sigma(K, T)$ and hence build an implied local volatility tree from options prices and their derivatives.
- One consistent model that values all standard options correctly rather than having to use several different inconsistent Black-Scholes models with different underlying volatilities.
- Volatility arbitrage trading. You can calculate the future local volatilities implied by options prices and then see if they seem reasonable. If some of them look too low or too high in the future, you can think about buying or selling future butterfly and calendar spreads to l make a bet on future volatility.

A (Clever) Poor Man's Derivation of the Dupire Equation in a Binomial Framework.

Let's use a Jarrow-Rudd tree that goes from (S,t) to (K,T) through **two half-periods of time** $\Delta T/2$,keeping interest rates zero for pedagogical simplicity.



The calendar spread obtains all its optionality from moving up the heavy red line:

$$C(S,t,K,T+dT) - C(S,t,K,T) \equiv \frac{\partial C}{\partial T}\Delta T = \overset{\text{probability}}{p_{K,T}\frac{1}{4}} \times \overset{\text{payoff}}{dK}$$

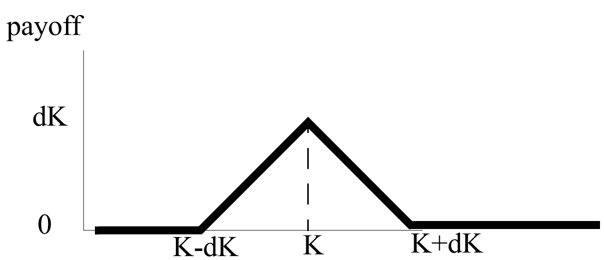
Value of the calendar spread per unit time is $\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,\,T} \times \frac{dK}{dT}$

where $p_{K,\,T}$ is the risk-neutral probability of getting to (K, T) and $dK \approx \Delta K = K\sigma\sqrt{2\Delta T}$.

We can get $p_{K,\,T}$ from a **butterfly spread portfolio**

$p_{K,\,T}$ is the value of a portfolio that pays \$1 if the stock price is at node K, and zero for all other nodes at time T.

The butterfly spread portfolio $C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)$ pays



Dividing by dK produces a payoff which is \$1 at the node K and zero at adjacent nodes.

$$p_{K,\,T} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{dK} = \frac{\partial^2 C}{\partial K^2} dK$$

Combining the expression for $p_{K,T}$ with $\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,T} \times \frac{dK}{dT}$

$$\frac{\partial C}{\partial T} = \frac{1}{4} p_{K,T} \frac{\Delta K}{\Delta T} = \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{(\Delta K)^2}{\Delta T}$$

$$= \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{[K\sigma\sqrt{2\Delta T}]^2}{\Delta T}$$

$$= \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

so that the local volatility $\sigma(K, T)$ is given by

$$\sigma^2(K, T) = \frac{\partial C}{\partial T} \Big/ \left(\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}\right)$$ **You can regard this as a *definition* of the effective local volatility from options prices and has meaning beyond the model.**

A More Rigorous Formal Proof of Dupire's Equation

The stochastic PDE for the risk-neutral stock price with stochastic σ: $\frac{dS_t}{S_t} = rdt + \sigma(S_t, t, .)dZ_t$

Call value at time t:

$$C_t(K, T) = e^{-r(T-t)}E\left\{[S_T - K]_+\right\}$$

where E denotes the q-measure risk-neutral expectation over S_T and **all** other stochastic variables.

Examine the derivatives of the call value that enter the Dupire equation.

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)}E\{\theta(S_T - K)\}$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)}E\{\delta(S_T - K)\}$$

To find $\left.\frac{\partial C}{\partial T}\right|_K$ we need to take account of both the change in T and the corresponding change in S_T through Ito's Lemma.

$$C_t(K, T) = e^{-r(T-t)} E\left\{[S_T - K]_+\right\}$$

$$d_T C\Big|_K = E\left\{\frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S_T} dS_T + \frac{1}{2}\frac{\partial^2 C}{\partial S_T^2}(dS_T)^2\right\}$$ expectation in risk-neutral measure over S_T

$$= E\left\{-rC\,dT + e^{-r\tau}\theta(S_T - K)\,dS_T + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)S_T^2\,dT\right\}$$

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$$= E\left\{-rC\,dT + e^{-r\tau}\theta(S_T - K)(rS_T\,dT) + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)K^2\,dT\right\}$$

$$= E\left\{-re^{-r\tau}\theta(S_T - K)(S_T - K)\,dT + e^{-r\tau}\theta(S_T - K)rS_T\,dT + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(K, T, .)K^2\,dT\right\}$$

$$= e^{-r\tau}E\left\{rK\theta(S_T - K)\,dT + \frac{1}{2}\delta(S_T - K)\sigma^2(K, T, .)K^2\,dT\right\}$$

$$= -rK\frac{\partial C}{\partial K}dT + \frac{1}{2}E\left\{\sigma^2(K, T, .)\right\}\frac{\partial^2 C}{\partial K^2}K^2\,dT$$

Then the change in the value of C when S_T and T change is given by

$$\left.\frac{\partial C}{\partial T}\right|_K = -rK\frac{\partial C}{\partial K} + \frac{1}{2}E\left\{\sigma^2(K, T, .)\right\}\frac{\partial^2 C}{\partial K^2}K^2$$

$$E\left\{\sigma^2(K, T, .)\right\} = \frac{\left(\left.\frac{\partial C}{\partial T}\right|_K + rK\left.\frac{\partial C}{\partial K}\right|_T\right)}{\left.\frac{1}{2}\frac{\partial^2 C}{\partial K^2}\right|_T K^2}$$

Define the local volatility as: $\sigma^2(K, T) \equiv E\left\{\sigma^2(K, T, .)\right\}$ and this is the Dupire equation.

Denominator is positive, and one can use dominance arguments to show that the numerator is too.

For another more classic proof using the Fokker-Planck equation, see the Appendix of Derman, E. & I. Kani. “Riding on a Smile.” RISK, 7(2) Feb.1994, pp. 139-145.

An Exact Relationship Between Local and Implied Volatilities and Its Consequences (Homework Problem)

For zero interest rates and dividend yields, we derived $\sigma^2(K, T) = \left(2\frac{\partial C}{\partial T}\bigg|_K\right) \Big/ \left(K^2\frac{\partial^2 C}{\partial K^2}\bigg|_T\right)$

Quoting in terms of BS implied vols: $C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T))$

By carefully using the chain rule for differentiation and the formulas for the Black-Scholes Greeks:

$$\sigma^2(K, T) = \frac{2\frac{\partial \Sigma}{\partial T} + \frac{\Sigma}{T-t}}{K^2\left(\frac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{T-t}} + d_1\frac{\partial \Sigma}{\partial K}\right\}^2\right)}$$

where $d_1 = \frac{\ln(S/K)}{\Sigma\sqrt{T-t}} + \frac{\Sigma\sqrt{T-t}}{2}$, and $\Sigma = \Sigma(S, t, K, T)$ is a function of S, t, K, T.

This formula is the generalization of the notion of forward volatilities in a no-skew world to local volatilities in a skewed world.

We can now prove rigorously the previous relations we intuited between implied local volatility.

Implied variance is average of local variance if there no skew.

$\Sigma(S, t, K, T)$ is independent of strike K, $\frac{\partial\Sigma}{\partial K} = 0$ with no skew at all. Then, writing $\tau = T - t$

$$\frac{1}{2}\sigma^2(K, T) = \frac{\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{2\tau}}{K^2\frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}}\right\}^2} = \tau\Sigma\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma^2}{2}$$

$$\sigma(\tau)^2 = \frac{\partial}{\partial\tau}(\Sigma^2\tau)$$

$$\tau\Sigma^2(\tau) = \int_0^\tau \sigma^2(u)du$$

the standard result that expresses the total variance as an average of forward variances.

Near the money, for no term structure, the slope of the skew w.r.t strike is 1/2 the slope of the local volatility w.r.t. spot

$\Sigma = \Sigma(K)$ alone, independent of expiration, and $\dfrac{\partial\Sigma}{\partial\tau} = 0$.

Assume a *weak linear* dependence of the skew on K, so that we keep only terms of order $\dfrac{\partial\Sigma}{\partial K}$,

assuming $\left(\dfrac{\partial\Sigma}{\partial K}\right)^2$ and $\dfrac{\partial^2\Sigma}{\partial K^2}$ are negligible.

Then

$$\sigma^2(K, T) = \frac{2\dfrac{\partial\Sigma}{\partial T} + \dfrac{\Sigma}{T-t}}{K^2\left(\dfrac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\dfrac{\partial\Sigma}{\partial K}\right)^2 + \dfrac{1}{\Sigma}\left\{\dfrac{1}{K\sqrt{T-t}} + d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)} \approx \frac{\dfrac{\Sigma}{\tau}}{\dfrac{K^2}{\Sigma}\left(\left\{\dfrac{1}{K\sqrt{\tau}} + d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + d_1 K\sqrt{\tau}\dfrac{\partial\Sigma}{\partial K}\right\}^2}$$

and so

$$\sigma(K) = \frac{\Sigma(K)}{1 + d_1 K\sqrt{\tau}\dfrac{\partial\Sigma}{\partial K}}$$

Close to at-the-money, $K = S + \Delta K$. Then

$$d_1 \approx \frac{\ln S/K}{\Sigma\sqrt{\tau}} \approx -\frac{(\Delta K)}{S(\Sigma\sqrt{\tau})} \approx -\frac{(\Delta K)}{K(\Sigma\sqrt{\tau})}$$

so that

$$\sigma(K) \approx \frac{\Sigma(K)}{1 - \frac{(\Delta K)}{\Sigma}\frac{\partial\Sigma}{\partial K}} \approx \Sigma(K)\left(1 + \frac{(\Delta K)}{\Sigma}\frac{\partial\Sigma}{\partial K}\right) \approx \Sigma(K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$$

Therefore since, $K = S + \Delta K$ $\qquad \sigma(S+\Delta K) \approx \Sigma(S+\Delta K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$

and so, since $\sigma(S+\Delta K) \approx \sigma(S) + (\Delta K)\frac{\partial\sigma}{\partial S} = \Sigma(S) + (\Delta K)\frac{\partial\sigma}{\partial S}$ and $\Sigma(S+\Delta K) \approx \Sigma(S) + (\Delta K)\frac{\partial\Sigma}{\partial K}$

we obtain $\qquad \frac{\partial}{\partial S}\sigma(S) = 2\left(\frac{\partial\Sigma}{\partial K}\right)\Big|_{K=S}$

The local volatility $\sigma(S)$ grows twice as fast with stock price *S* as the implied volatility $\Sigma(K)$ grows with strike!

Implied volatility is an harmonic average over local volatility at short expirations.

For zero rates and dividends: $\sigma^2(K,T) = \dfrac{2\dfrac{\partial\Sigma}{\partial\tau} + \dfrac{\Sigma}{\tau}}{K^2\left(\dfrac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\dfrac{\partial\Sigma}{\partial K}\right)^2 + \dfrac{1}{\Sigma}\left\{\dfrac{1}{K\sqrt{\tau}} + d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)}$

Multiplying top and bottom by τ: $\sigma^2(K,T) = \dfrac{2\tau\dfrac{\partial\Sigma}{\partial\tau} + \Sigma}{K^2\left(\tau\dfrac{\partial^2}{\partial K^2}(\Sigma) - d_1\tau\sqrt{\tau}\left(\dfrac{\partial\Sigma}{\partial K}\right)^2 + \dfrac{1}{\Sigma}\left\{\dfrac{1}{K} + \sqrt{\tau}d_1\dfrac{\partial\Sigma}{\partial K}\right\}^2\right)}$

As $\tau \to 0$, this becomes the o.d.e. $\sigma^2(K,T) = \dfrac{\Sigma}{K^2\left(\dfrac{1}{\Sigma}\left\{\dfrac{1}{K} + \sqrt{\tau}d_1\dfrac{d\Sigma}{dK}\right\}^2\right)} = \dfrac{\Sigma^2}{\left\{1 + \sqrt{\tau}Kd_1\dfrac{d\Sigma}{dK}\right\}^2}$

Now $\sqrt{\tau}Kd_1 \to \dfrac{K\ln(S/K)}{\Sigma}$ as $\tau \to 0$, and we obtain $\sigma = \dfrac{\Sigma}{1 + \dfrac{K}{\Sigma}\dfrac{d\Sigma}{dK}\ln(S/K)}$

Transforming from K into the new variable $x = \ln(S/K)$ we can rewrite this as the o.d.e.

$$\frac{\Sigma}{1-\frac{x}{\Sigma}\frac{d\Sigma}{dx}} = \sigma$$

In the homework you are asked to show that the solution is $\frac{1}{\Sigma(x)} = \frac{1}{x}\int_{o}^{x}\frac{1}{\sigma(y)}dy$

In other words, at very short times to expiration, the implied volatility is the harmonic mean of the local volatility as a function of $\ln S/K$ between spot and strike.

This is intuitively reasonable, more sensible than an arithmetic mean.

Suppose that $\sigma(y)$ falls to zero above a certain level S, so that the stock price can never diffuse higher. Then the implied volatility of any option with a strike above that level should be zero.

If $\Sigma(x) = \frac{1}{x}\int_{o}^{x}\sigma(y)dy$, an ordinary arithmetic mean, then its value would be non-zero, which is impossible if the stock can never reach the strike. In contrast, for the harmonic mean in Equation , if $\sigma(y)$ becomes zero anywhere in the range between spot and strike, then the implied volatility for that strike, $\Sigma(x)$, becomes zero too, which is as to be expected.

There is an intuitive way to understand the harmonic mean.

If the stock's volatility were infinite, it would be transparent to diffusion. Think of it as a medium.

$1/\sigma^2(\ln S)$ is the time taken to diffuse through the medium for $\ln S$.

$\int_{\ln S}^{\ln K} 1/\sigma^2$ is the total diffusion time. Or $\int_{\ln S}^{\ln K} 1/\sigma$ is the square root of the total diffusion time.

$\int_o^x \frac{1}{\sigma(y)} dy$ is roughly the total $\sqrt{\text{diffusion time}}$ computed from the sum of local $\sqrt{\text{diffusion times}}$.

But the Total $\sqrt{\text{time}}$ = total distance / Average Volatility = $\dfrac{\ln S/K}{\Sigma}$

$\frac{x}{\Sigma(x)} = \int_o^x \frac{1}{\sigma(y)} dy$ The average volatility is found from the total time and total distance.

This is similar to the statement that, for a car with a velocity that varies with position, the **total time** for the trip is the sum of the local times. The average velocity is *not* the average of the local velocities. The **average velocity is the total distance divided by the total time** and is therefore the harmonic average of the local velocities.

$$T = \int_0^D \frac{dx}{v(x)} \equiv \frac{D}{\bar{\bar{V}}}$$

$$\frac{1}{\bar{\bar{V}}} = \frac{1}{D}\int_0^D \frac{dx}{v(x)}$$