1. [By Zhong Wang]

Question 1

The asympototically sorted order is:

$$(lgn)^n > (1+0.001)^n > n^{lgn} > n^{lglgn} = (lgn)^{lgn} > n^2 > nlgn > n > lg^{1000}n$$
$$> lgn^{1000} = lgn > lg_{1000}n > n^{1/lgn} = 1$$

2. [By Rupert Freeman]

2. The master method applies in each instance. I use a, b, f and ε as they are used in the notes.

   (a) $a = 2, b = 3, f(n) = 1$. Compare $f(n)$ with $n^{\log_3 2} = n^\alpha$ for some $0 < \alpha < 1$. If we choose $\varepsilon = \frac{\alpha}{2}$ then $f(n) = O(n^{\alpha-\varepsilon})$. This is case 1. So $T(n) = \Theta(n^{\log_3 2})$.

   (b) $a = 5, b = 4, f(n) = n$. Compare $f(n)$ with $n^{\log_4 5} = n^\alpha$ for some $1 < \alpha < 2$. If we choose $\varepsilon = \frac{\alpha-1}{2}$ then $f(n) = O(n^{\alpha-\varepsilon})$. This is case 1. So $T(n) = \Theta(n^{\log_4 5})$.

   (c) $a = 8, b = 2, f(n) = n^3$. Compare $f(n)$ with $n^{\log_2 8} = n^3$. $f(n) = \Theta(n^3)$. This is case 2. So $T(n) = \Theta(n^3 \lg n)$.

   (d) Let $m = \lg n \Leftrightarrow n = 2^m$. Then $T(n) = T(2^m) = T((2^m)^{\frac{1}{2}}) + 1 = T(2^{\frac{m}{2}}) + 1$. Writing $S(m) = T(2^m)$ gives us $S(m) = S(\frac{m}{2}) + 1$ to which we may apply the master method. $a = 1, b = 2, f(m) = 1$. Compare $f(m)$ with $n^{\log_2 1} = n^0 = 1$. $f(m) = \Theta(1)$ therefore this is case 2 and $S(m) = \Theta(\lg m)$. Hence $S(m) = T(2^m) = T(n) = \Theta(\lg m) = \Theta(\lg \lg n)$.

3. You can use recursion tree, but should be rigorous.
[By Rupert Freeman] Not perfect, but good enough to tell you the strategy.

3. For any $n_0 \in \mathbb{N}$, we have that $T(n) = \Theta(1)$ for all $n < n_0$. We are then able to pick c large enough that $T(n) = \Theta(1) \leq cn \log^2 n$ for all $1 \leq n < n_0$. Now suppose that $T(k) \leq cn \log^2 n$ for $k < n$. We have

$$\begin{aligned}
T(n) &= 2T(\frac{n}{2}) + O(n \log n) \\
&\leq 2c\frac{n}{2} \log^2 \frac{n}{2} + O(n \log n) \\
&= cn(\log^2 n - 2 \log n \log 2 + \log^2 2) + O(n \log n) \\
&\leq cn \log^2 n - 2cn \log n \log 2 + cn \log^2 2 + kn \log n \\
&= cn \log^2 n - n(2c \log 2 \log n - c \log^2 2 - k \log n) \\
&\leq cn \log^2 n
\end{aligned}$$

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whenever $2c \log 2 \log n - c \log^2 2 - k \log n \geq 0$ which we will always be able to achieve by choosing c to be much larger than k.

4. Everyone got full credit.