CS 260 – Homework #1

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Problem 1 (1.10 from text)

For each distinct pair (i, j) of functions determine if f_i is dominated by f_j, and if f_j is dominated by f_i . Show your reasoning. Assume f and g are positive functions on [0, ∞)

$$f_1(n) = n^2$$

$$f_2(n) = n^2 + 1000n$$

$$f_3(n) = \begin{cases} n, & if\ n\ is\ odd \\ n^3, & if\ n\ is\ even \end{cases}$$

$$f_4(n) = \begin{cases} n, & if\ n \leq 100 \\ n^3, & if\ n > 100 \end{cases}$$

Solution

$f_1(n) \in O(f_2(n))$? $c(n^2 + 1000n) \geq n^2, \forall\, n > n_0$ $c = 2\,; n_0 = 1$ $2n^2 + 2000n \geq n^2$ $n^2 + 2000n \geq 0$ $n(n + 2000) \geq 0$ $\therefore f_1(n) \in O(f_2(n))\ \forall\, n \geq 1$	$f_2(n) \in O(f_1(n))$? $cn^2 \geq n^2 + 1000n, \forall\, n > n_0$ $c = 3\,; n_0 = 1$ $3n^2 \geq n^2 + 1000n$ $2n^2 + 1000n \geq 0$ $2n(n - 500) \geq 0$ $\therefore f_2(n) \in O(f_1(n))\ \forall\, n \geq 500$

$f_1(n) \in O\big(f_3(n)\big)$?

$\underline{n\ is\ odd}$

$cn \geq \mathrm{n}^2, \forall n > n_0$

$c \geq \mathrm{n}$

$Doesn't\ work\ as\ n \to \infty$

$\underline{n\ is\ even}$

$cn^3 \geq \mathrm{n}^2, \forall n > n_0$

$c = 1\ ; n_o = 1$

$n^3 \geq n^2$

$n^2(n-1) \geq 0$

$\therefore\ f_1(n) \cancel{\in} O\big(f_3(n)\big)\ \forall\ odd\ n$

$\therefore\ f_1(n) \in O\big(f_3(n)\big)\ \forall\ even\ n > 0$

$f_3(n) \in O\big(f_1(n)\big)$?

$\underline{n\ is\ odd}$

$cn^2 \geq \mathrm{n}, \forall n > n_0$

$c = 1\ ; n_0 = 1$

$n^2 \geq n$

$n^2 - n \geq 0$

$n(n-1) \geq 0$

$\underline{n\ is\ even}$

$cn^2 \geq \mathrm{n}^3, \forall n > n_0$

$c \geq n$

$Doesn't\ work\ as\ n \to \infty$

$\therefore\ f_3(n) \in O\big(f_1(n)\big)\ \forall\ odd\ n > 0$

$\therefore\ f_3(n) \cancel{\in} O\big(f_1(n)\big)\ \forall\ even\ n$

$f_1(n) \in O\big(f_4(n)\big)$?

$\underline{n \leq 100}$

$cn \geq \mathrm{n}^2, \forall n > n_0$

$c \geq \mathrm{n}$

$Doesn't\ work\ as\ n \to \infty$

$\underline{n > 100}$

$cn^3 \geq \mathrm{n}^2, \forall n > n_0$

$c = 1\ ; n_o = 100$

$n^3 \geq n^2$

$n^2(n-1) \geq 0$

$\therefore\ f_1(n) \cancel{\in} O\big(f_4(n)\big)\ \forall\ 0 \leq n \leq 100$

$\therefore\ f_1(n) \in O\big(f_4(n)\big)\ \forall\ n > 100$

$f_4(n) \in O\big(f_1(n)\big)$?

$\underline{n \leq\ 100}$

$cn^2 \geq \mathrm{n}, \forall n > n_0$

$c = 1\ ; n_0 = 1$

$n^2 \geq n$

$n^2 - n \geq 0$

$n(n-1) \geq 0$

$\underline{n > 100}$

$cn^2 \geq \mathrm{n}^3, \forall n > n_0$

$c \geq n$

$Doesn't\ work\ as\ n \to \infty$

$\therefore\ f_4(n) \in O\big(f_1(n)\big)\ \forall\ 0 \leq n \leq 100$

$\therefore\ f_4(n) \cancel{\in} O\big(f_1(n)\big)\ \forall\ n > 100$

$f_2(n) \in O\big(f_3(n)\big)$?

<u>n *is odd*</u>

$cn \geq \mathrm{n}^2 + 1000n, \forall n > n_0$

$c \geq \mathrm{n} + 1000$

$Doesn't\ work\ as\ n \rightarrow \infty$

<u>n *is even*</u>

$cn^3 \geq \mathrm{n}^2 + 1000n, \forall n > n_0$

$c = 1\ ; n_o = 1$

$n^3 - n^2 - 1000n \geq 0$

$n(n^2 - n - 1000) \geq 0$

$roots\ in\ above\ quadratic\ \approx 33, -31$

$\therefore f_2(n) \cancel{\in} O\big(f_3(n)\big)\ \forall\ odd\ n$

$\therefore f_2(n) \in O\big(f_3(n)\big)\ \forall\ even\ n > 33$

$f_3(n) \in O\big(f_2(n)\big)$?

<u>n *is odd*</u>

$cn^2 \geq \mathrm{n}, \forall n > n_0$

$c = 1\ ; n_0 = 1$

$n^2 \geq n$

$n^2 - n \geq 0$

$n(n-1) \geq 0$

<u>n *is even*</u>

$c(n^2 + 1000n) \geq \mathrm{n}^3, \forall n > n_0$

$cn + 1000c \geq n^2$

$Doesn't\ work\ as\ n \rightarrow \infty$

$\therefore f_3(n) \in O\big(f_2(n)\big)\ \forall\ odd\ n > 0$

$\therefore f_3(n) \cancel{\in} O\big(f_2(n)\big)\ \forall\ even\ n$

$f_2(n) \in O\big(f_4(n)\big)$?

<u>$n \leq 100$</u>

$cn \geq \mathrm{n}^2 + 1000n, \forall n > n_0$

$c \geq \mathrm{n} + 1000$

$Doesn't\ work\ as\ n \rightarrow \infty$

<u>$n > 100$</u>

$cn^3 \geq \mathrm{n}^2 + 1000n, \forall n > n_0$

$c = 1\ ; n_o = 100$

$n^3 - n^2 - 1000n \geq 0$

$n(n^2 - n - 1000n) \geq 0$

$roots\ in\ above\ quadratic\ \approx 33, -31$

$\therefore f_2(n) \cancel{\in} O\big(f_4(n)\big)\ \forall\ 0 < n \leq 100$

$\therefore f_2(n) \in O\big(f_4(n)\big)\ \forall\ n > 100$

$f_4(n) \in O\big(f_2(n)\big)$?

<u>$n \leq 100$</u>

$c(n^2 + 1000n) \geq \mathrm{n}, \forall n > n_0$

$c = 1\ ; n_0 = 0$

$n^2 + 999n \geq 0$

$n(n + 999) \geq 0$

<u>$n > 100$</u>

$c(n^2 + 1000n) \geq \mathrm{n}^3, \forall n > n_0$

$cn + 1000c \geq n^2$

$Doesn't\ work\ as\ n \rightarrow \infty$

$\therefore f_4(n) \in O\big(f_2(n)\big)\ \forall\ 0 \leq n \leq 100$

$\therefore f_4(n) \cancel{\in} O\big(f_2(n)\big)\ \forall\ n > 100$

$f_3(n) \in O\big(f_4(n)\big)$?

$\underline{n\ is\ odd\ and\ \leq 100}$

$cn \geq n, \forall n > n_0$

$c = 2\ ; n_0 = 1$

$2n \geq n$

$n \geq 0$

$\underline{n\ is\ odd\ and > 100}$

$cn \geq \mathrm{n}^3, \forall n > n_0$

$c \geq n^2$

$Doesn't\ work\ as\ n \to \infty$

$\underline{n\ is\ even\ and\ \leq 100}$

$cn^3 \geq \mathrm{n}, \forall n > n_0$

$c = 1; n_0 = 1$

$n^3 - n \geq 0$

$n(n^2 - 1) \geq 0$

$\underline{n\ is\ even\ and > 100}$

$cn^3 \geq \mathrm{n}^3, \forall n > n_0$

$c = 2\ ; n_o = 100$

$2n^3 \geq n^3$

$n^3 \geq 0$

$\therefore f_3(n) \in O\big(f_4(n)\big)\ \forall\ odd\ 0 \leq n \leq 100$

$\therefore f_3(n) \cancel{\in} O\big(f_4(n)\big)\ \forall\ odd\ n > 100$

$\therefore f_3(n) \in O\big(f_4(n)\big)\ \forall\ even\ n \geq 0$

$f_4(n) \in O\big(f_3(n)\big)$?

$\underline{n\ is\ odd\ and\ \leq 100}$

$cn \geq n, \forall n > n_0$

$c = 2\ ; n_0 = 1$

$2n \geq n$

$n \geq 0$

$\underline{n\ is\ odd\ and > 100}$

$cn \geq \mathrm{n}^3, \forall n > n_0$

$c \geq n^2$

$Doesn't\ work\ as\ n \to \infty$

$\underline{n\ is\ even\ and\ \leq 100}$

$cn^3 \geq \mathrm{n}, \forall n > n_0$

$c = 1; n_0 = 1$

$n^3 - n \geq 0$

$n(n^2 - 1) \geq 0$

$\underline{n\ is\ even\ and > 100}$

$cn^3 \geq \mathrm{n}^3, \forall n > n_0$

$c = 2\ ; n_o = 100$

$2n^3 \geq n^3$

$n^3 \geq 0$

$\therefore f_4(n) \in O\big(f_3(n)\big)\ \forall\ odd\ 0 \leq n \leq 100$

$\therefore f_4(n) \cancel{\in} O\big(f_3(n)\big)\ \forall\ odd\ n > 100$

$\therefore f_4(n) \in O\big(f_3(n)\big)\ \forall\ even\ n \geq 0$

Problem 2 (1.12 from text)

Let N be the set of positive integers, and let R be the set of real numbers. For each of the procedures in Exercise 1.12, provide a function
t: N -> R such that the computing time function of the procedure is dominated by *t*. Make sure your function *t* is reasonably tight.

Solution

a. procedure *matmpy* (*n*: integer);

```
        var
          i, j, k: integer; O(1)
        begin
          for i := 1 to n do   O(n)
            for j := 1 to n do begin   O(n)
                C[i, j] := 0;   O(1)
                for k := 1 to n do    O(n)
                  C[i, j] := C[i, j,] + A[i, k] * B[k, j]   O(1)
              end
          end
```

$$f(t) = \max\{\, O(1), O(n) * O(n) * O(1) * O(n) * O(1)\,\}$$
$$f(t) = O(n^3)$$

b. procedure *mystery* (*n*: integer);

```
      var
        i, j, k: integer; O(1)
      begin
        for i:= 1 to n-1 do  O(n)
        for j:= i + 1 to n do    O(n)
          for k := 1 to j do   O(n)
            { some statement requiring O(1) time }    O(1)
      end
```

$$f(t) = \max\{\, O(1), O(n) + O(n) * O(n) * \ O(1)\} = \ O(n^2) + O(n)$$
$$f(t) = O(n^2)$$

c. procedure *veryodd* (*n*: integer);

```
    var
        i, j, x, y: integer;  O(1)
    begin
      for i := 1 to n do   O(n)
        if odd(i) then begin O(1)
           for j := i to n do  O(n)
              x := x + 1;  O(1)
           for j := 1 to i do O(n)
              y := y + 1  O(1)
        end
    end
```

$$f(t) = \max\{\, O(1), O(n) * O(1) * \left(O(n) * O(1) + O(n) * O(1)\right)\}$$
$$= O(n) * 2O(n)$$
$$f(t) = O(n^2)$$

d. function *recursive* (*n*: integer) : integer;

```
        begin
          if n <= 1 then O(1)
             return (1)  O(1)
          else
            return (recursive(n-1) + recursive(n-1)) ?
        end
```

recursive(2) requires 2 additional recursive calls
recursive(3) requires 6 additional recursive calls
recursive(4) requires 14 additional recursive calls
recursive(5) requires 30 additional recursive calls

mapping this as a function gives:

$$f(t) = 2^n - 2 = O(2^n)$$