

# Causal discovery: constraint-based methods

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## Preliminaries

Markov equivalence class for DAGs

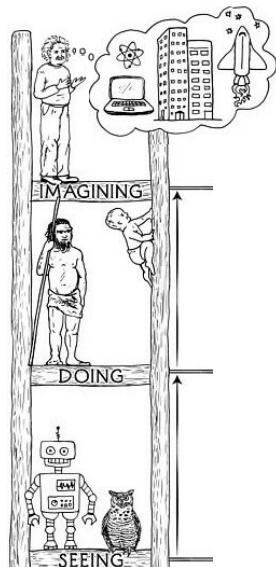
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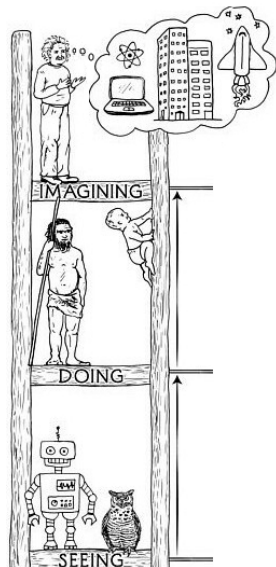
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# Where does a causal graph come from?



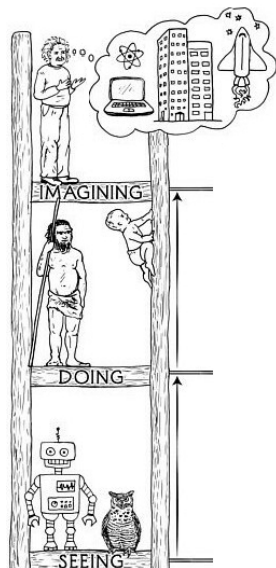
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- ▶ Theory
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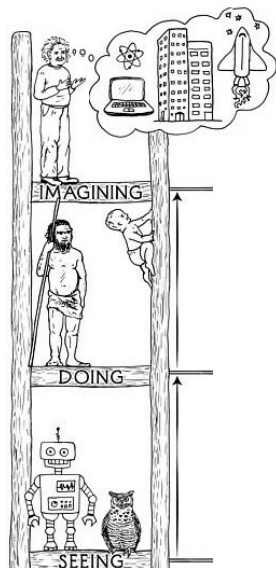
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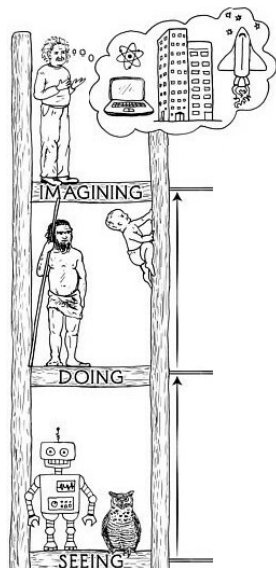
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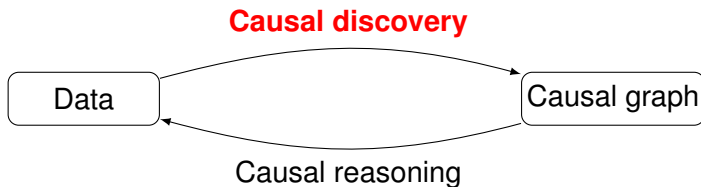
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- ▶ Observations
  - ▶ Correlation does not imply causation!

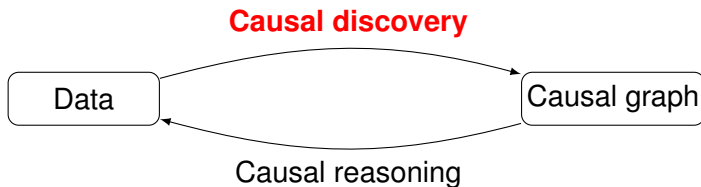




# Causal discovery (1/2)

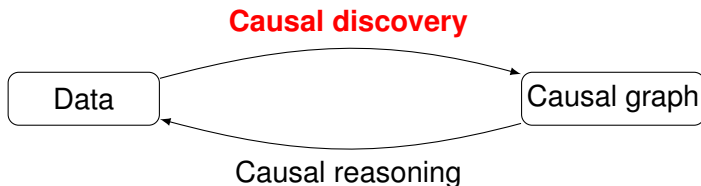


# Causal discovery (1/2)



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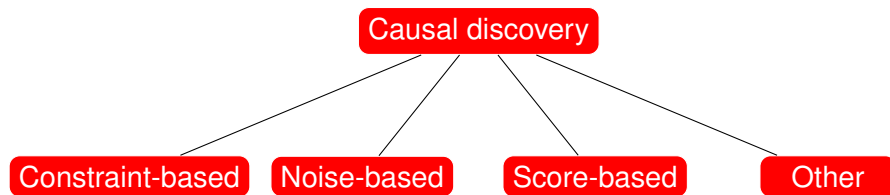
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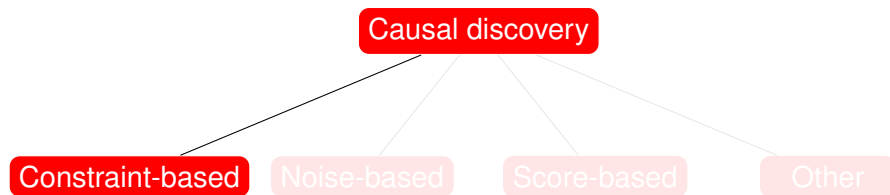
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But it is possible under **additional assumptions**.

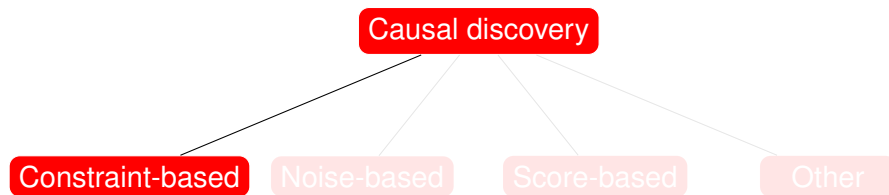
## Causal discovery (2/2)



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Constraint-based: run local tests of independence to create constraints on space of possible graphs.

# Recap about causal graphical models (1/2)

**Parental Markov Condition** Given  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ ,

$$\forall X \in \mathcal{V}, X \perp\!\!\!\perp_P \mathcal{V} \setminus \{Parents(X), Descendants(X)\} \mid Parents(X).$$

**Collider**  $X \rightarrow Z \leftarrow Y$ .

**V-structure (or unshielded colliders, or immorality)** If the two parent vertices are not adjacent, the collider is a v-structure.

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## Recap about causal graphical models (2/2)

**Theorem (probabilistic implications of d-separation)** Given a DAG  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , a distribution  $P(\mathcal{V})$  compatible with  $\mathcal{G}$  and disjoint sets  $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subset \mathcal{V}$ :

- (i)  $\mathcal{X} \perp\!\!\!\perp_{\mathcal{G}} \mathcal{Y} \mid \mathcal{Z} \Rightarrow \mathcal{X} \perp\!\!\!\perp_P \mathcal{Y} \mid \mathcal{Z}$  in every distribution  $P$  compatible with  $\mathcal{G}$  (Also known as the global Markov property);
- (ii) If  $\mathcal{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathcal{Y} \mid \mathcal{Z}$ , then there exists a distribution  $P$  compatible with  $\mathcal{G}$  such that  $\mathcal{X} \not\perp\!\!\!\perp_P \mathcal{Y} \mid \mathcal{Z}$ .

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# Markov equivalence for DAGs

## Causal sufficiency

$\forall X \leftarrow Z \rightarrow Y, \text{ if } X, Y \in \mathcal{V} \text{ then } Z \in \mathcal{V}.$

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**Theorem (Markov equivalence for DAGs)** Two DAGs  $\mathcal{G}_1$  and  $\mathcal{G}_2$  Markov equivalent (have the same d-separations) *iff* they have the same skeleton and the same v-structures.

Proof in (Verma and Pearl, 1990)

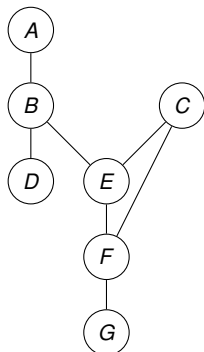
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- Skeleton is the undirected graph with same adjacencies

# A characterization of Markov equivalence classes for DAGs (1/2)

**Completed partially directed acyclic graph (CPDAG)** Let  $[\mathcal{G}]$  be the Markov equivalence class of a DAG  $\mathcal{G}$ . The CPDAG  $\mathcal{G}^*$  of  $\mathcal{G}$  is the graph:

- ▶ With the same skeleton as  $\mathcal{G}$ ;
- ▶ Where an edge is directed in  $\mathcal{G}^*$  iff it occurs as a directed edge with the same orientation in every graph in  $[\mathcal{G}]$ ;
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**Lemma** Let  $\mathcal{G}_1^*$  and  $\mathcal{G}_2^*$  denote two CPDAGs then  $\mathcal{G}_1^* = \mathcal{G}_2^*$  iff  $\mathcal{G}_1$  and  $\mathcal{G}_2$  belong to the same Markov equivalent class.

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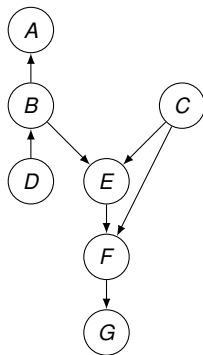
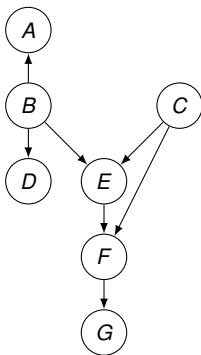
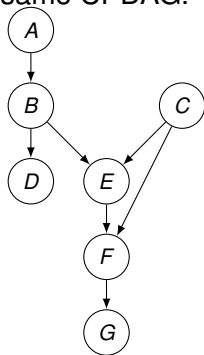
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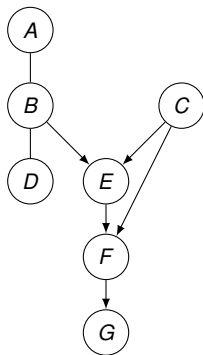


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# Constraint based question

Main question: Given  $P(\mathcal{V})$  a compatible probability distribution of  $\mathcal{G}$ , can we discover  $\mathcal{G}^*$  the CPDAG of  $\mathcal{G}$ ?

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**Because**  $X \perp\!\!\!\perp_P Y \mid Z \not\Rightarrow X \perp\!\!\!\perp_G Y \mid Z$ .

**Faithfulness** We say that a graph  $\mathcal{G}$  and a compatible probability distribution  $P$  are faithful to one another if all and only the conditional independence relations true in  $P$  are entailed by the Markov condition applied to  $\mathcal{G}$ .

# faithfulness and d-sep

**Theorem (implication of faithfulness on d-sep)**  $P(\mathcal{V})$  is faithful to directed acyclic graph  $\mathcal{G}$  with vertex set  $\mathcal{V}$  iff for all disjoint sets of vertices  $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subset \mathcal{V}$ ,  $\mathcal{X} \perp\!\!\!\perp_P \mathcal{Y} \mid \mathcal{Z}$  iff  $\mathcal{X} \perp\!\!\!\perp_{\mathcal{G}} \mathcal{Y} \mid \mathcal{Z}$ .

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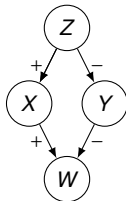
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**Proof:** Follows immediately by Theorem (probabilistic implication on d-separation) and by Definition of faithfulness.

# Violation of faithfulness (1/2)

## Example 1: Canceling out

Consider



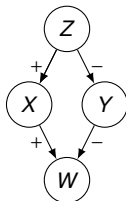
where

- ▶  $Z = \epsilon_Z$
- ▶  $X = a_{ZX} \times Z + \epsilon_X$
- ▶  $Y = a_{ZY} \times Z + \epsilon_Y$
- ▶  $W = a_{XW} \times X - \frac{a_{ZX}a_{XW}}{a_{ZY}} \times Y + \epsilon_W$

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- ▶  $Y = a_{zy} \times Z + \epsilon_y$
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By canceling out

- ▶  $Z \perp\!\!\!\perp_P W$

# Violation of faithfulness (2/2)

## Example 2: Determinism

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where

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By determinism

- ▶  $X \perp\!\!\!\perp_P Z \mid Y$



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# Finding skeleton and v-structures

**Theorem (faithfulness, adjacencies and v-structures)** If  $P(\mathcal{V})$  is faithful to some directed acyclic graph, then  $P(\mathcal{V})$  is faithful to directed acyclic graph  $\mathcal{G}$  with vertex  $\mathcal{V}$  iff:

- ▶ For  $X, Y \in \mathcal{V}$ ,  $X$  and  $Y$  are adjacent iff  $\forall \mathcal{S} \subseteq \mathcal{V} \setminus \{X, Y\}$ ,  $X \not\perp_P Y \mid \mathcal{S}$ ;
- ▶ For  $X, Y, Z \in \mathcal{V}$  such that  $X$  is adjacent to  $Z$  and  $Z$  is adjacent to  $Y$  and  $X$  and  $Y$  are not adjacent,  $X \rightarrow Z \leftarrow Y$  in  $\mathcal{G}$  iff  $\forall \mathcal{S} \subseteq \mathcal{V} \setminus \{X, Y\}$  such that  $Z \in \mathcal{S}$ ,  $X \not\perp_P Y \mid \mathcal{S}$ .

(proof on board)

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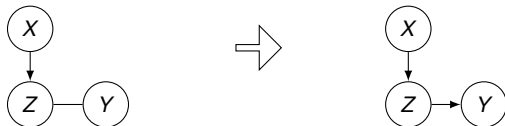
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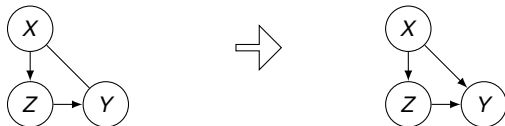
- ▶ Point 1 can be used to discover the skeleton of  $\mathcal{G}$  from  $P(\mathcal{V})$ ;
- ▶ Given the skeleton of  $\mathcal{G}$ , point 2 can be used to find all v-structures.

# Orientation rules

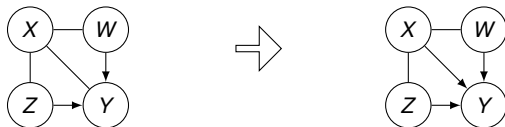
R1:



R2:



R3:



# Orientation rules correctness

**Pattern** A pattern of a DAG  $\mathcal{G}$  is a graph with the same skeleton as  $\mathcal{G}$  but where only v-structures are oriented.

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**Theorem (orientation completeness)** The result of recursively applying rules  $R1$ ,  $R2$ ,  $R3$  to a pattern of some DAG is a CPDAG.  
(proof in (Meek, 1995))

# The SGS algorithm

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## Algorithm 1 SGS

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**Input:**  $P(\mathcal{V})$

**Output:** CPDAG  $\mathcal{G}^*$

- 1: Form the complete undirected graph  $\mathcal{G}^*$  on vertex set  $\mathcal{V}$
  - 2: **for** all  $X - Y$  in  $\mathcal{G}^*$   
    and subsets  $S \subseteq \mathcal{V} \setminus \{X, Y\}$  **do**
  - 3:     **if**  $\exists S \subseteq \mathcal{V} \setminus \{X, Y\}$  such that  $X \perp\!\!\!\perp_P Y \mid S$  **then**
  - 4:         Delete edge  $X - Y$  from  $\mathcal{G}^*$
  - 5:     **end if**
  - 6: **end for**
  - 7: **for** all  $X - Z - Y$  in  $\mathcal{G}^*$  such that  $X \notin \text{Adj}(Y, \mathcal{G}^*)$  **do**
  - 8:     **if**  $\nexists S \subseteq \mathcal{V} \setminus \{X, Y\}$  such that  $Z \in S$  and  $X \perp\!\!\!\perp_P Y \mid S$  **then**
  - 9:         Orient  $X \rightarrow Z \leftarrow Y$  in  $\mathcal{G}^*$
  - 10:     **end if**
  - 11: **end for**
  - 12: Recursively apply rules R1-R3 until no more edges can be oriented
  - 13: **Return**  $\mathcal{G}^*$
- 

$\text{Adj}(Y, \mathcal{G})$ : Adjacencies of  $Y$  in  $\mathcal{G}$



# Correctness of SGS

**Theorem (correctness)** Assume the distribution  $P(\mathcal{V})$  is Markov and faithful to some DAG  $\mathcal{G}$  and assume that we are given perfect conditional independence information about all pairs of variables. Let  $\mathcal{G}^*$  be the CPDAG of  $\mathcal{G}$ . The SGS algorithm returns  $\mathcal{G}^*$ .

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**Proof:** By Theorem (faithfulness, adjacencies and v-structures), Theorem (orientation soundness) and Theorem (orientation completeness).

# Computational complexity of SGS

Running time of SGS depends *exponentially* on the *number of vertices* in the graph:

- ▶ For all pairs check all subsets;
- ▶ For all triples check all subsets.

# A better approach?

**Optimizing the procedure for skeleton construction**

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## Optimizing the procedure for skeleton construction

By the Parental Markov condition:

$X \notin \text{Adj}(Y, \mathcal{G})$  iff  $X \perp\!\!\!\perp_P Y \mid \text{Parents}(X, \mathcal{G})$  or  $X \perp\!\!\!\perp_P Y \mid \text{Parents}(Y, \mathcal{G})$

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Since the graph  $\mathcal{G}$  is unknown:

- ▶ The parent set is unknown ahead of time;
- ▶ We look at  $\mathcal{S} \subseteq Adj(X, \mathcal{G}')$  and  $\mathcal{S}' \subseteq Adj(Y, \mathcal{G}')$  for some  $\mathcal{G}'$  which is a supergraph of the true unknown skeleton;
- ▶ We can pursue an iterative strategy such that we increase the size of  $\mathcal{S}$  iteratively.

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## Optimizing the procedure for finding v-structures

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(proof on board)



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**Lemma (either d-sep or d-connect)** Given the distribution  $P(V)$  that is Markov and faithful to some DAG  $\mathcal{G}$ , if  $Z \in Adj(X, \mathcal{G})$ ,  $Z \in Adj(Y, \mathcal{G})$  and  $Y \notin Adj(X, \mathcal{G})$ , then either  $Z$  is in every set of variables that d-separates  $X$  and  $Y$  or it is in no set of variables that d-separates  $X$  and  $Y$ .

(proof on board)

$sepset(X, Y)$ : subset that permitted the separation of  $X$  and  $Y$  during the skeleton construction.

# A better approach?

## Optimizing the procedure for finding v-structures

**Lemma (either d-sep or d-connect)** Given the distribution  $P(V)$  that is Markov and faithful to some DAG  $\mathcal{G}$ , if  $Z \in Adj(X, \mathcal{G})$ ,  $Z \in Adj(Y, \mathcal{G})$  and  $Y \notin Adj(X, \mathcal{G})$ , then either  $Z$  is in every set of variables that d-separates  $X$  and  $Y$  or it is in no set of variables that d-separates  $X$  and  $Y$ .

(proof on board)

$sepset(X, Y)$ : subset that permitted the separation of  $X$  and  $Y$  during the skeleton construction.

R0: For all triples  $X - Z - Y \in \mathcal{G}^*$  such that  $Y \notin Adj(X, \mathcal{G}^*)$ , if  $Z \notin sepset(X, Y)$  then orient  $X \rightarrow Z \leftarrow Y$  in  $\mathcal{G}^*$ .

# The PC algorithm

---

## Algorithm 2 PC

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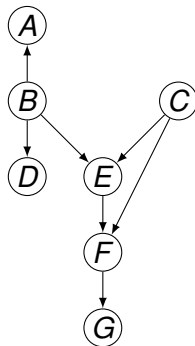
**Input:**  $P(\mathcal{V})$

**Output:** CPDAG  $\mathcal{G}^*$

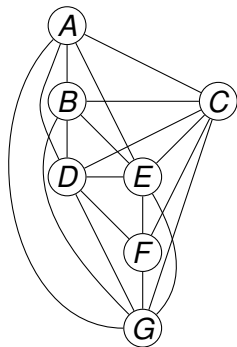
- 1: Form the complete undirected graph  $\mathcal{G}^*$  on vertex set  $\mathcal{V}$
  - 2: Let  $n = 0$
  - 3: **repeat**
  - 4:   **for** all  $X - Y$  in  $\mathcal{G}^*$  such that  $|Adj(X, \mathcal{G}^*) \setminus \{Y\}| \geq n$   
    and subsets  $\mathcal{S} \subseteq Adj(X, \mathcal{G}^*) \setminus \{Y\}$  such that  $|\mathcal{S}| = n$  **do**
  - 5:     **if**  $X \perp\!\!\!\perp_P Y \mid \mathcal{S}$  **then**
  - 6:       Delete edge  $X - Y$  from  $\mathcal{G}^*$
  - 7:       Let  $sepset(X, Y) = sepset(Y, X) = \mathcal{S}$
  - 8:     **end if**
  - 9:   **end for**
  - 10:   Let  $n = n + 1$
  - 11: **until** for each pair of adjacent vertices  $(X, Y)$ ,  $|Adj(X, \mathcal{G}^*) \setminus \{Y\}| < n$
  - 12: Apply R0
  - 13: Recursively apply rules R1-R3 until no more edges can be oriented
  - 14: **Return**  $\mathcal{G}^*$
-

## PC in action (1/3)

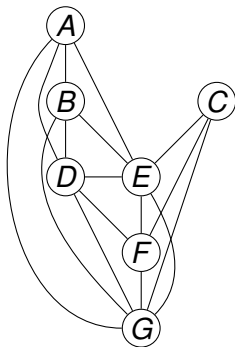
- ▶ Suppose the true graph on right;
- ▶ Assumptions: CMC, faithfulness, causal sufficiency.



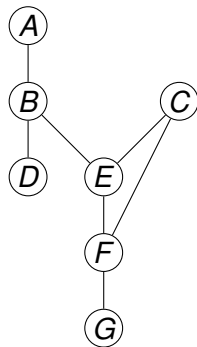
## Skeleton construction:



Initialization



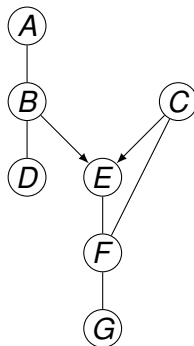
$|S| = 0$



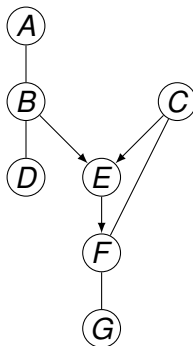
$|S| = 1$

## PC in action (3/3)

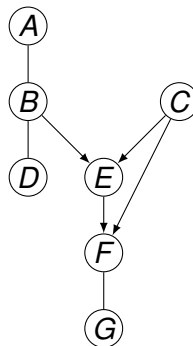
**Orientation:**



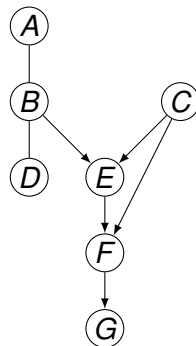
R0



R1



R2



R1

# Correctness of PC

**Theorem (correctness)** Assume the distribution  $P(\mathcal{V})$  is Markov and faithful to some DAG  $\mathcal{G}$  and assume that we are given perfect conditional independence information about all pairs of variables. Let  $\mathcal{G}^*$  be the CPDAG of  $\mathcal{G}$ . The PC algorithm returns  $\mathcal{G}^*$ .

(proof on board)

# Computational complexity of PC

Running time of PC depends *exponentially* on the *maximal degree* of the graph **but** for a fixed maximal degree running time over the *number of vertices* is *polynomial*.



# Exercise 1

Consider data that are generated from a chain  $X \rightarrow Y \rightarrow Z$ . Assuming that all assumptions are satisfied, which CPDAG would a constraint based causal discovery algorithm report?

If you could supply prior knowledge to the algorithm on only one arc that is required to be present, what arc (if any) would allow the entire structure to be learned? Explain briefly.

## Exercise 2

Consider data that truly come from a fork  $X \leftarrow Y \rightarrow Z$ . Assuming that all assumptions are satisfied, which CPDAG would a constraint based causal discovery algorithm report?

If you could supply prior knowledge to the algorithm on only one arc that is required to be present, what arc (if any) would allow the entire structure to be learned? Explain briefly.

## Exercise 3

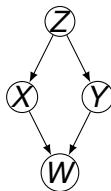
- ▶ Suppose the true graph on right;
- ▶ Assumptions: CMC, causal sufficiency, no deterministic relations;
- ▶ Generative process:

$$Z = \xi_z \quad \xi_z \sim N(0, 1);$$

$$X = a * Z + \xi_x \quad \xi_x \sim N(0, 1);$$

$$Y = b * Z + \xi_y \quad \xi_y \sim N(0, 1);$$

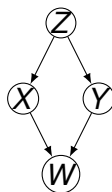
$$W = c * X - \frac{a * c}{b} * Y + \xi_w \quad \xi_w \sim N(0, 1).$$



- ▶ Given a compatible distribution what would be the output of the PC algorithm?

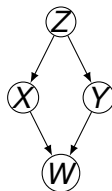
## Exercise 4

- ▶ Suppose the true graph on right;
- ▶ Assumptions: CMC, causal sufficiency, deterministic relations, no canceling out paths;
- ▶ Given a compatible distribution what would be the output of the PC algorithm?



## Exercise 5

- ▶ Suppose the true graph on right;
- ▶ Assumptions: CMC, faithfulness;
- ▶ Given a compatible distribution what would be the output of the PC algorithm if  $Z$  is unobserved?



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# Latent variables (1/2)

Consider  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with vertices  $\mathcal{V} = \mathcal{O} \cup \mathcal{L}$  such that

- ▶  $\mathcal{O}$  observable variables;
- ▶  $\mathcal{L}$  latent variables.

# Latent variables (1/2)

Consider  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with vertices  $\mathcal{V} = \mathcal{O} \cup \mathcal{L}$  such that

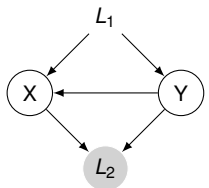
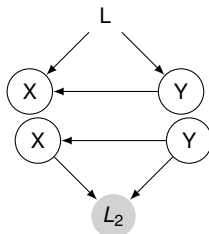
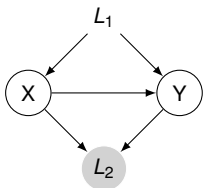
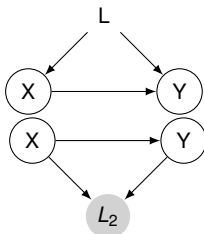
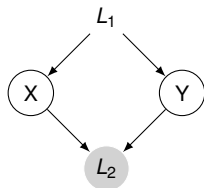
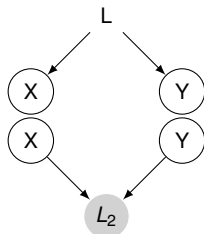
- ▶  $\mathcal{O}$  observable variables;
- ▶  $\mathcal{L}$  latent variables.

Latent variables are represented by a transparent border.



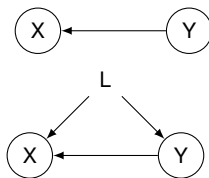
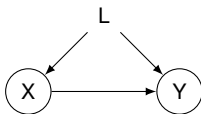
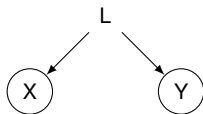
## Latent variables (2/2)

Assuming acyclicity, if two observed variables  $X$  and  $Y$  are statistically dependent:

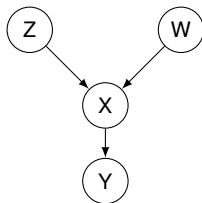


## Latent variables (2/2)

Assuming acyclicity, if two observed variables  $X$  and  $Y$  are statistically dependent:



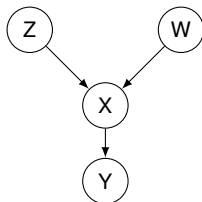
# Main structures (1/2)



Y-structure

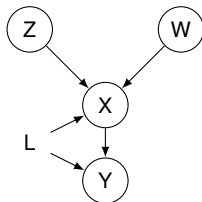
$$\begin{aligned} Z &\perp\!\!\!\perp_P W \\ Z &\not\perp\!\!\!\perp_P W \mid X \\ Y &\not\perp\!\!\!\perp_P Z \\ Y &\perp\!\!\!\perp_P Z \mid X \\ Y &\not\perp\!\!\!\perp_P W \\ Y &\perp\!\!\!\perp_P W \mid X \end{aligned}$$

# Main structures (1/2)



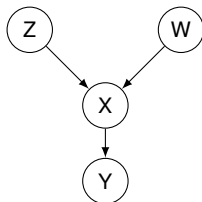
Y-structure

$$\begin{aligned}
 &Z \perp\!\!\!\perp_P W \\
 &Z \not\perp\!\!\!\perp_P W \mid X \\
 &Y \not\perp\!\!\!\perp_P Z \\
 &Y \perp\!\!\!\perp_P Z \mid X \\
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 \end{aligned}$$

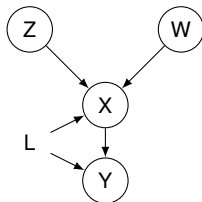


$$\begin{aligned}
 &Z \perp\!\!\!\perp_P W \\
 &Z \not\perp\!\!\!\perp_P W \mid X \\
 &Y \not\perp\!\!\!\perp_P Z \\
 &Y \not\perp\!\!\!\perp_P Z \mid X \\
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 &Y \not\perp\!\!\!\perp_P W \mid X
 \end{aligned}$$

# Main structures (1/2)



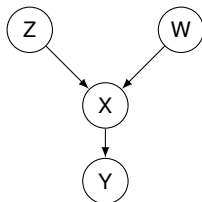
Y-structure



$$\begin{aligned} Z &\perp\!\!\!\perp_P W \\ Z &\not\perp\!\!\!\perp_P W \mid X \\ Y &\not\perp\!\!\!\perp_P Z \\ Y &\perp\!\!\!\perp_P Z \mid X \\ Y &\not\perp\!\!\!\perp_P W \\ Y &\perp\!\!\!\perp_P W \mid X \end{aligned}$$

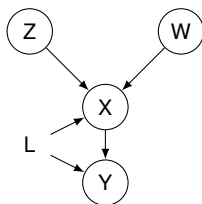
$$\begin{aligned} Z &\perp\!\!\!\perp_P W \\ Z &\not\perp\!\!\!\perp_P W \mid X \\ Y &\not\perp\!\!\!\perp_P Z \\ Y &\not\perp\!\!\!\perp_P Z \mid X \\ Y &\not\perp\!\!\!\perp_P W \\ Y &\not\perp\!\!\!\perp_P W \mid X \end{aligned}$$

# Main structures (1/2)



Y-structure

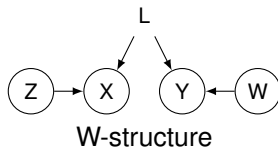
$$\begin{aligned} Z &\perp\!\!\!\perp_P W \\ Z &\not\perp\!\!\!\perp_P W \mid X \\ Y &\not\perp\!\!\!\perp_P Z \\ Y &\perp\!\!\!\perp_P Z \mid X \\ Y &\not\perp\!\!\!\perp_P W \\ Y &\perp\!\!\!\perp_P W \mid X \end{aligned}$$



$$\begin{aligned} Z &\perp\!\!\!\perp_P W \\ Z &\not\perp\!\!\!\perp_P W \mid X \\ Y &\not\perp\!\!\!\perp_P Z \\ Y &\not\perp\!\!\!\perp_P Z \mid X \\ Y &\not\perp\!\!\!\perp_P W \\ Y &\not\perp\!\!\!\perp_P W \mid X \end{aligned}$$

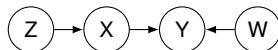
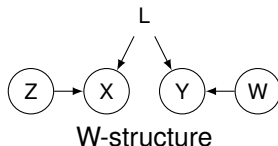
Pattern of independence can rule out latent confounding.

## Main structures (2/2)



$$\begin{array}{l} Z \not\perp_P X \\ X \not\perp_P Y \\ Y \not\perp_P W \\ Z \perp_P W \\ Z \perp_P Y \\ X \perp_P W \\ Z \not\perp_P Y \mid X \\ X \not\perp_P W \mid Y \end{array}$$

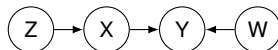
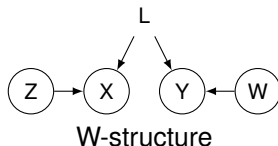
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 Z \not\perp_P X \\
 X \not\perp_P Y \\
 Y \not\perp_P W \\
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 \\
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 Y \not\perp_P W \\
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 Z \perp_P Y \mid X \\
 X \not\perp_P W \mid Y
 \end{array}$$



# Main structures (2/2)



$$Z \not\perp_P X$$

$$X \not\perp_P Y$$

$$Y \not\perp_P W$$

$$Z \perp_P W$$

$$Z \perp_P Y$$

$$X \perp_P W$$

$$Z \not\perp_P Y | X$$

$$X \not\perp_P W | Y$$

$$Z \not\perp_P X$$

$$X \not\perp_P Y$$

$$Y \not\perp_P W$$

$$Z \perp_P W$$

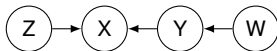
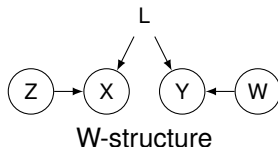
$$Z \not\perp_P Y$$

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# Main structures (2/2)



$$Z \not\perp_P X$$

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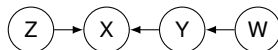
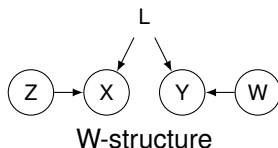
$$Z \perp_P Y$$

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# Main structures (2/2)



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$$Z \not\perp_P Y \mid X$$

$$X \not\perp_P W \mid Y$$

Pattern of independence can suggest latent confounding.

# The FCI algorithm

There exists an extension of the PC algorithm, called the FCI algorithm which relax the causal sufficiency assumption.

Procedure:

1. First step in PC to get skeleton
2. Additional step to prune more edges
3. Orient using similar rules of PC + additional rules

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# Conditional independence tests

With finite data, SGS, PC and FCI need a procedure for deciding whether  $X \perp\!\!\!\perp_P Y \mid \mathcal{S}$ .

In practice, test the null hypothesis:

$$H_0 : X \perp\!\!\!\perp_P Y \mid \mathcal{S}$$

and reject the null hypothesis if some test statistic  $T(x) < \alpha$ , where  $\alpha$  is a user-specified significance threshold. That is, if we reject the null hypothesis, we keep the edge, and if we fail to reject, we remove the edge.

# Examples of conditional independence tests

Tests	Assumptions
Fisher Z-transform	Linear, gaussian
$\chi^2$ test	Multinomial discrete
Kernel-based CI test	-
Local permutation test	-

**Theorem (consistency)** Assume the distribution  $P(\mathcal{V})$  is Markov and faithful to some DAG  $\mathcal{G}$ . Let  $\mathcal{G}^*$  be the CPDAG of  $\mathcal{G}$  and let  $\hat{\mathcal{G}}^*$  be the output of SGS, PC with some consistent conditional independence test and significance level  $\alpha$ . Then there is a sequence of  $\alpha_n \rightarrow 0 (n \rightarrow \infty)$  such that  $\lim_{n \rightarrow \infty} \Pr(\hat{\mathcal{G}}^* = \mathcal{G}^*) = 1$ .  
(proof in (Spirtes et al, 2000))



# Consistency

**Theorem (consistency)** Assume the distribution  $P(\mathcal{V})$  is Markov and faithful to some DAG  $\mathcal{G}$ . Let  $\mathcal{G}^*$  be the CPDAG of  $\mathcal{G}$  and let  $\hat{\mathcal{G}}^*$  be the output of SGS, PC with some consistent conditional independence test and significance level  $\alpha$ . Then there is a sequence of  $\alpha_n \rightarrow 0 (n \rightarrow \infty)$  such that  $\lim_{n \rightarrow \infty} \Pr(\hat{\mathcal{G}}^* = \mathcal{G}^*) = 1$ .  
(proof in (Spirtes et al, 2000))

Same result for FCI on MIPAG.

## Exercise 7

As the significance level is lowered to 0, what would you expect to happen to the graph skeleton learned by constraint based causal discovery algorithms? As the significance level is increased to 1? Explain.

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# Conclusion

- ▶ Under faithfulness and causal sufficiency constraint-based methods can discover a CPDAG (SGS, PC).
- ▶ Under faithfulness and causal sufficiency constraint-based methods can discover a MIPAG (FCI).
- ▶ Advantages:
  - ▶ Nonparametric (in principle);
  - ▶ PC and FCI are relatively scalable;
  - ▶ Lots of work on improvements.
- ▶ Drawbacks:
  - ▶ Cannot discover the entire true graph;
  - ▶ Faithfulness is not testable;
  - ▶ Cannot parallelize;
  - ▶ No confidence intervals;
  - ▶ Individual errors may propagate.

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# Some extensions

- ▶ Incorporating background knowledge;
- ▶ Order independent;
- ▶ Selection bias ( $R5'$ - $R7'$  in FCI);
- ▶ Really fast FCI;
- ▶ Time series.



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- ▶ Really fast FCI;
- ▶ Time series.

# Some extensions

- ▶ Incorporating background knowledge;
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## Direct inspirations for part 1

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## Additional readings

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