Causal Inference: Presenting Tools to Ascend the Ladder of Causation

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Sommaire

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Association

Bayesian networks
d-separation
Prediction
Do we need more than associations?

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1 Association

Bayesian networks

d-separation

Prediction

Do we need more than associations?

Notations

- Sets: $\mathbb{A} = \{X, Y, Z\}$
- Statistical independence: $\perp\!\!\!\perp_P$
- Statistical dependence: $\not\!\perp\!\!\!\perp_P$
- $P(Y = y \mid X = x) \equiv P(y \mid x)$

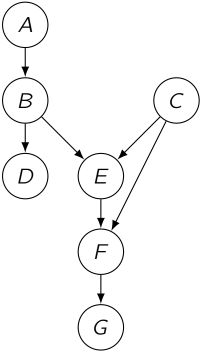
Directed graphs

A graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is said to be a directed graph iff

- $\mathbb{V}$ is the set of vertices (usually each vertex corresponds to a random variable),
- $\mathbb{E}$ is the set of edges,
- $\forall (X, Y) \in \mathbb{E},$ there is an arrow pointing from X to Y.

Directed graphs: basic concepts

Consider the following directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$:



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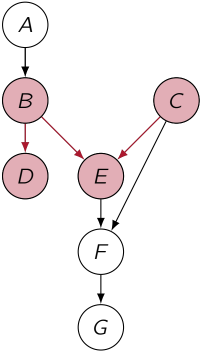

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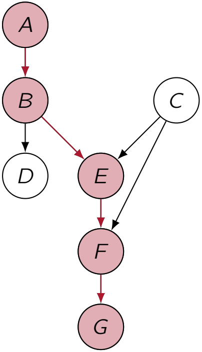

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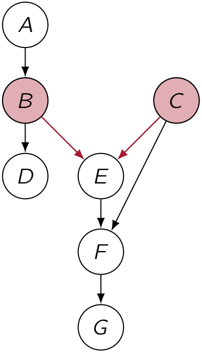
Parents: $Pa(E) = \{B, C\}$



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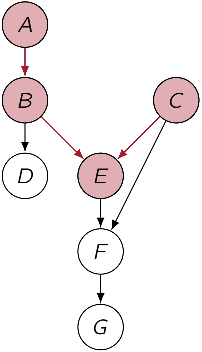

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Ancestors: $An(E) = \{A, B, C, E\}$

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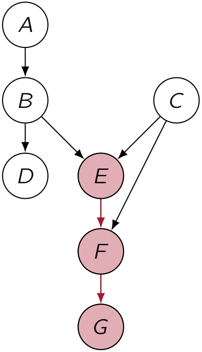

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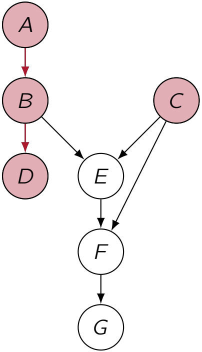

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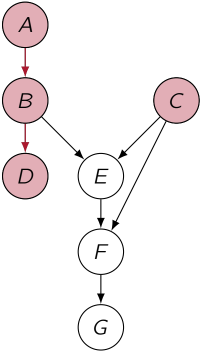

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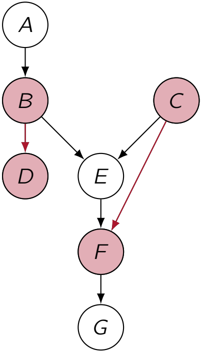

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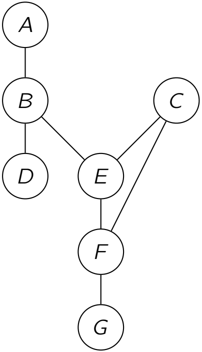

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Subgraph $\mathcal{G}[\mathbb{S}]$: $\mathcal{G}[\{B, C, D, F\}]$
Skeleton of $\mathcal{G}$

DAGs

A directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is said to be a directed acyclic graph (DAG) iff

$$\forall X \in \mathbb{V}, An(X) \cap Desc(X) = \{X\},$$

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From now on, we will only deal with DAGs.

Definition

A distribution $P(\mathbb{V})$ is compatible with a DAG $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ if

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A DAG $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is a Bayesian network iff there exists a joint distribution $P(\mathbb{V})$ that is compatible with $\mathcal{G}$.

Decomposing with respect to ancestral sets

Theorem

If P is compatible with $\mathcal{G}$ and $\mathbb{S} \subseteq \mathbb{V}$ is an ancestral set, then $P(\mathbb{S})$ is compatible with $\mathcal{G}[\mathbb{S}]$.

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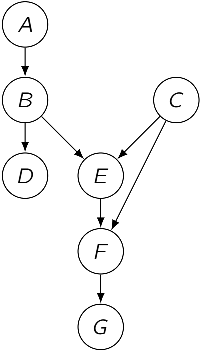

Bayesian network compatible with P

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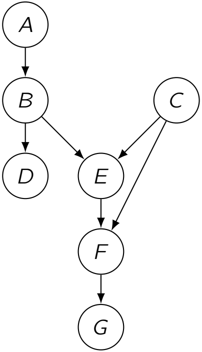

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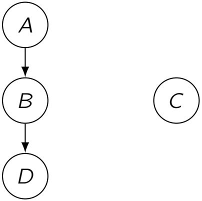

Bayesian network compatible with $P(A, B, C, D)$

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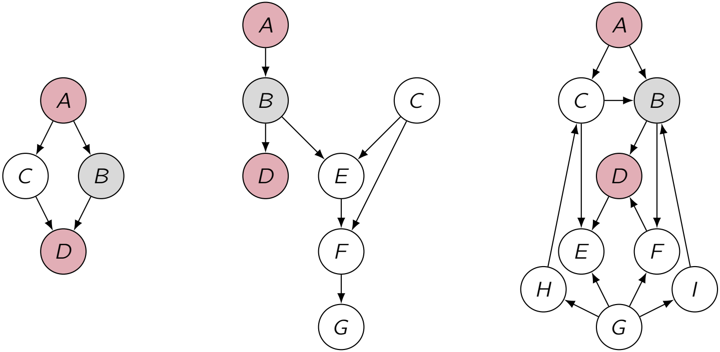
1

Association

Reading conditional independencies in graphs

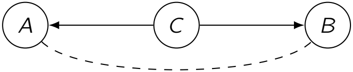
$$A \overset{?}{\perp\!\!\!\perp}_P D \mid B$$

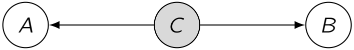


Basic structures

Basic structures

Fork (common cause principle)





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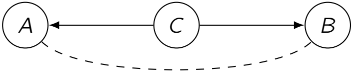
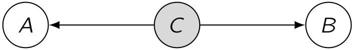
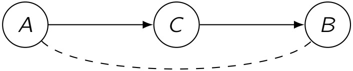
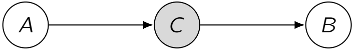




Chain





Basic structures

Fork (common cause principle)

Chain

V-structure

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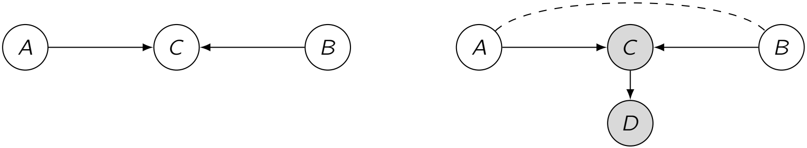
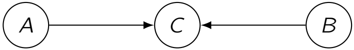
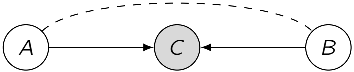


Artificial correlation in V-structures





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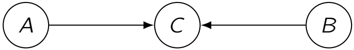

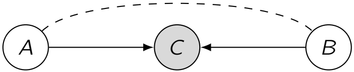

$$A = \begin{cases} \text{Mother carrier} \\ \text{Mother not carrier} \end{cases} \qquad B = \begin{cases} \text{Father carrier} \\ \text{Father not carrier} \end{cases}$$

$$C = (A \text{ or } B) = \begin{cases} \text{Child carrier} \\ \text{Child not carrier} \end{cases}$$

Artificial correlation in V-structures

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If $C = \text{Child carrier} \implies$

$$\begin{cases} \text{If } A = \text{Mother not carrier then } B = \text{Father carrier} \\ \text{If } B = \text{Father not carrier then } A = \text{Mother carrier} \end{cases}$$

Artificial correlation in V-structures

$$A = \begin{cases} \text{Asthmatic} \\ \text{Not asthmatic} \end{cases} \qquad B = \begin{cases} \text{Sleep apnea} \\ \text{No sleep apnea} \end{cases}$$

$$C = (A \text{ or } B) = \begin{cases} \text{Visit pulmonologist} \\ \text{No visit pulmonologist} \end{cases}$$

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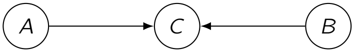

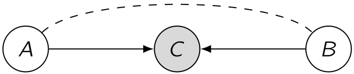

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If $C = \text{Visit pulmonologist} \implies$

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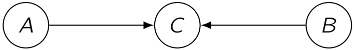
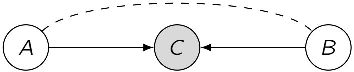
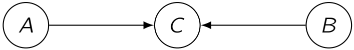
Artificial correlation in V-structures

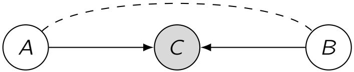




$A, B \sim U(-1, 1)$ $\qquad$ $\xi_c \sim N(0, \frac{1}{2})$ $\qquad$ $C = 2AB + \xi_c$

Artificial correlation in V-structures





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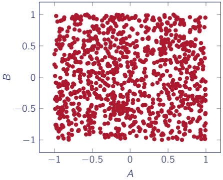
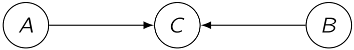
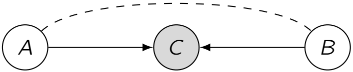


$Corr(A; B) = 0.002$

Artificial correlation in V-structures





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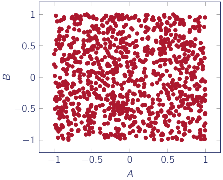

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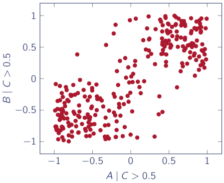

$Corr(A; B \mid C > 0.5) = 0.8$

Blocked paths

A triple such that $X \to Z \leftarrow Y$ is called a collider[1] . If the two parent vertices are not adjacent, the collider is a v-structure.

[1] We also refer to Z as the collider

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A path is said to be blocked by a set of vertices $\mathbb{Z} \in \mathbb{V}$ if:

- it contains a chain $A \rightarrow B \rightarrow C$ or a fork $A \leftarrow B \rightarrow C$ and $B \in \mathbb{Z}$; or
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A path that is not blocked is active.

[1] We also refer to Z as the collider

Given disjoint sets $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \subseteq \mathbb{V}$, we say that $\mathbb{X}$ and $\mathbb{Y}$ are d-separated by $\mathbb{Z}$ if every path between a vertex in $\mathbb{X}$ and a vertex in $\mathbb{Y}$ is blocked by $\mathbb{Z}$ and we write $\mathbb{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z}$.

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If one path is not blocked, we say that $\mathbb{X}$ and $\mathbb{Y}$ are d-connected given $\mathbb{Z}$ and we write $\mathbb{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z}$.

d-separation

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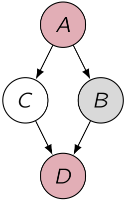
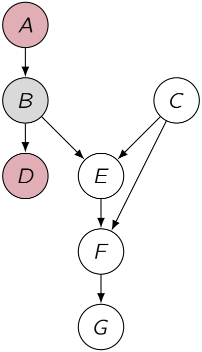
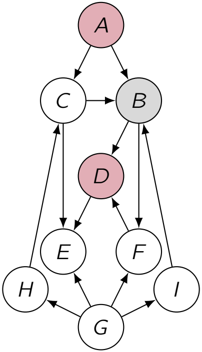
$\mathbb{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z} \Rightarrow \mathbb{X} \perp\!\!\!\perp_{P} \mathbb{Y} \mid \mathbb{Z}$
but
$\mathbb{X} \perp\!\!\!\perp_{P} \mathbb{Y} \mid \mathbb{Z} \not\Rightarrow \mathbb{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z}$
$\mathbb{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z} \not\Rightarrow \mathbb{X} \not\perp\!\!\!\perp_{P} \mathbb{Y} \mid \mathbb{Z}$
$\mathbb{X} \not\perp\!\!\!\perp_{P} \mathbb{Y} \mid \mathbb{Z} \not\Rightarrow \mathbb{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z}$

$$A \overset{?}{\perp\!\!\!\perp}_P D \mid B$$







Reading conditional independencies using d-separation

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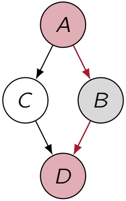

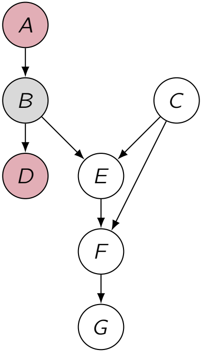

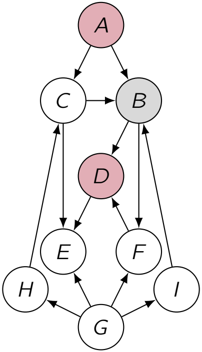

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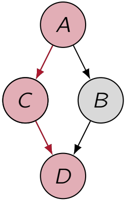

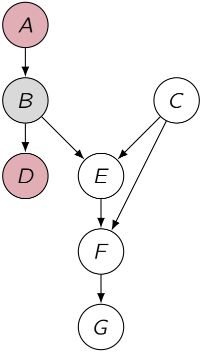

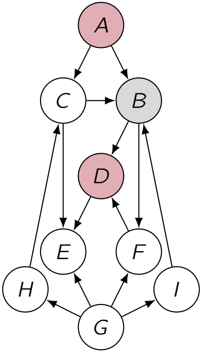

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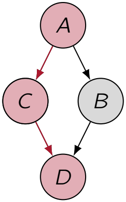

$< A, B, D >$ is not blocked

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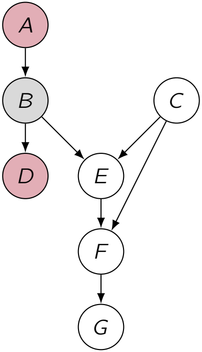

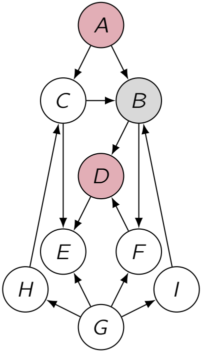

Reading conditional independencies using d-separation

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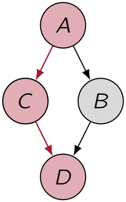

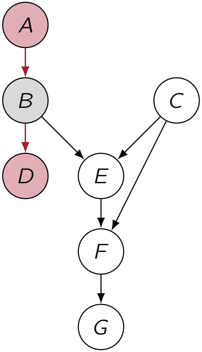

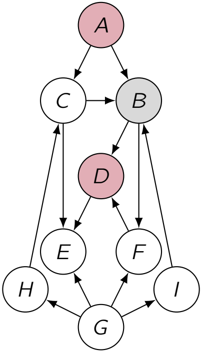

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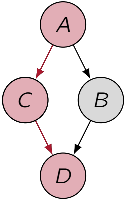

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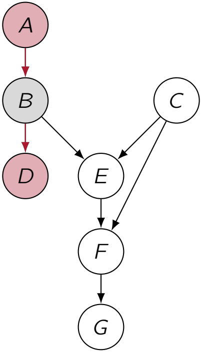

All paths are blocked

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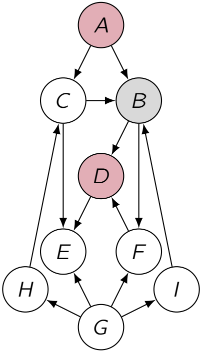

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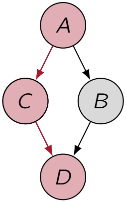

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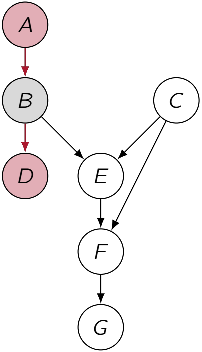

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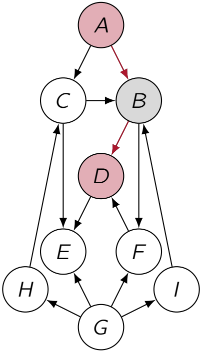

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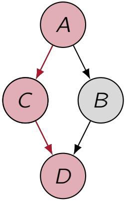

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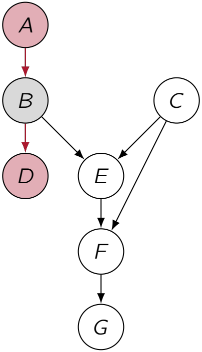

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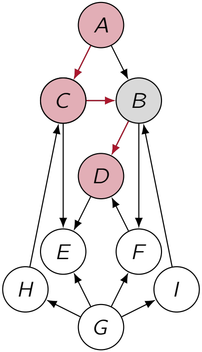

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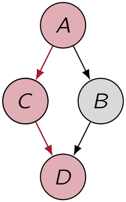

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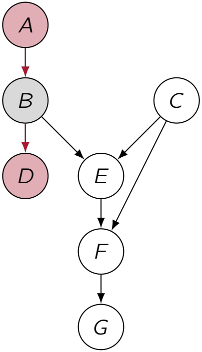

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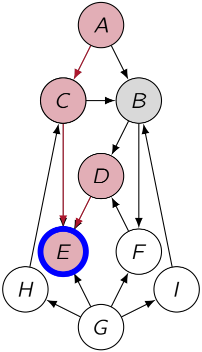

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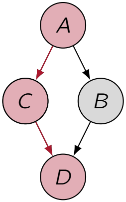

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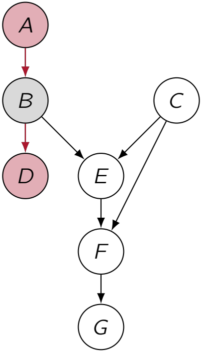

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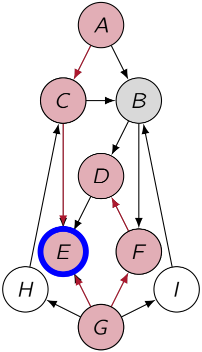

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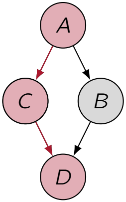

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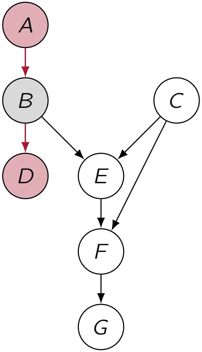

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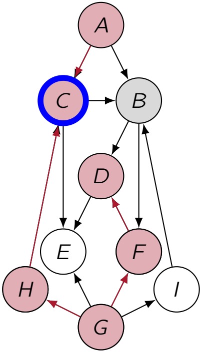

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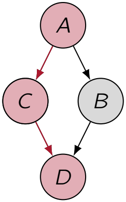

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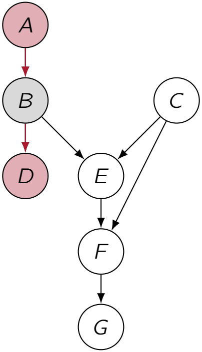

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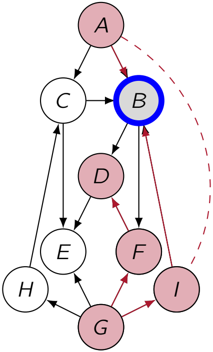

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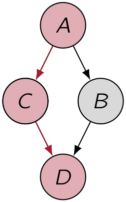

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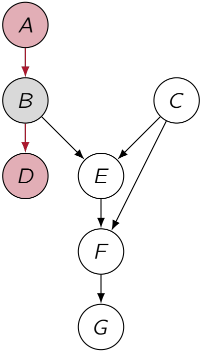

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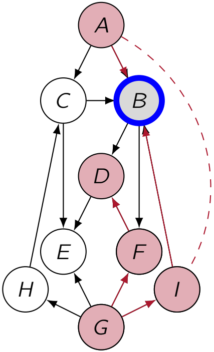

$< A, I, G, F, D >$ is not blocked

$$\implies A \overset{?}{\perp\!\!\!\perp}_P D \mid B$$

1

Association

Bayesian networks for prediction

Bayesian networks can be used to detect relevant variables that contains all the useful information for prediciton!

Bayesian networks can be used to detect relevant variables that contains all the useful information for prediciton! (Assuming the distribution does not change)

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Relevant variables for prediciting Y:

- Parents of Y;
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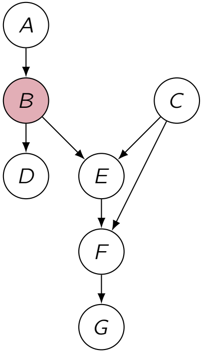

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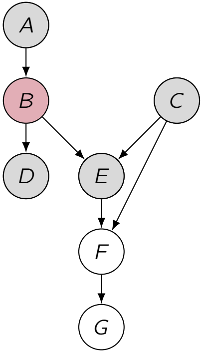
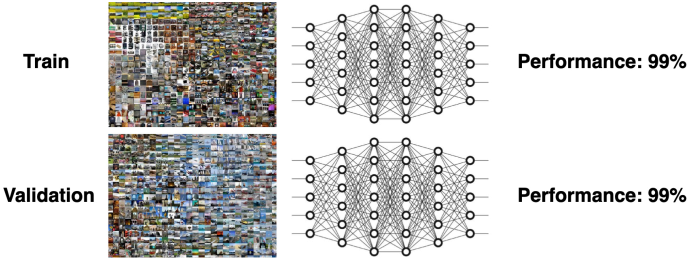


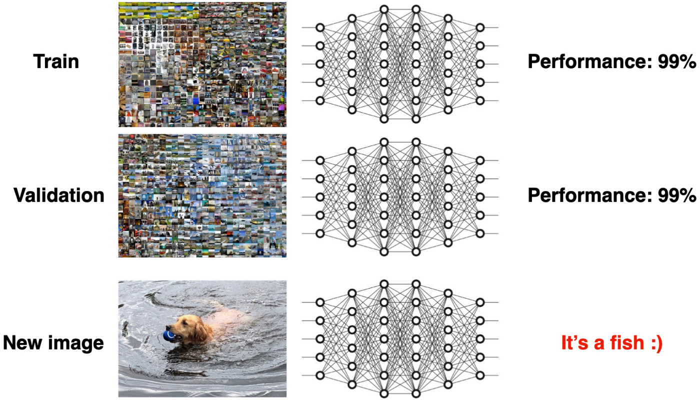
1
Association

Distribution shifts



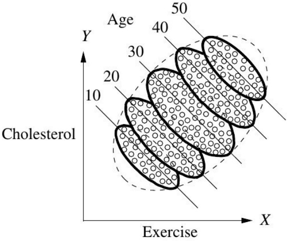
P. Cui and T. Zhang. Causal Inference and Stable Learning. Tutoriel at ICML, 2019.

Distribution shifts

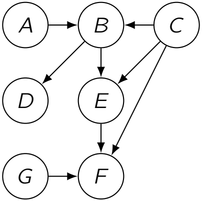


P. Cui and T. Zhang. Causal Inference and Stable Learning. Tutoriel at ICML, 2019.

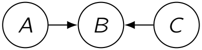
In a study, we measure weekly exercise and cholesterol levels for various age groups.



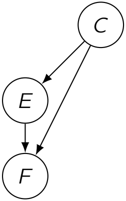
Suppose this DAG is a bayesian network compatible with P:



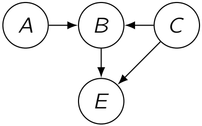
Do the following subgraphs represent Bayesian networks that are compatible with their respective distributions?



$P(A, B, C)$

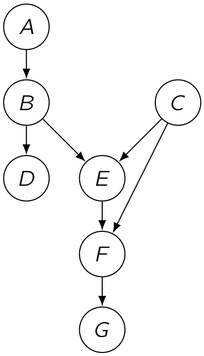


$P(C, E, F)$



$P(A, B, C, E)$

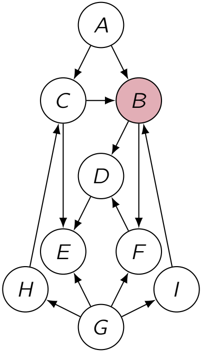
Exercise 2



- $B \perp\!\!\!\perp_P G \mid F$?
- $A \perp\!\!\!\perp_P F \mid E$?
- $B \perp\!\!\!\perp_P E \mid A, C, F$?

Exercise 3

What are the relevant variables for predicting B?



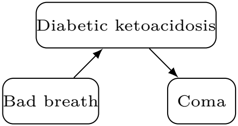
SANTÉ SORBONNE UNIVERSITÉ

2 Intervention

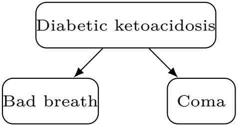
Causal graphs
Causal reasoning
Causal discovery

2 Intervention

Bayesian networks vs causal graph



Bad breath $\perp\!\!\!\perp_P$ Coma | Diabetic ketoacidosis
Bayesian network



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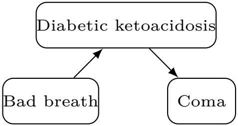
Bayesian networks vs causal graph



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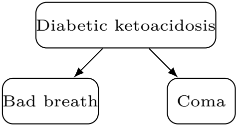
Bayesian network

Not a causal graph



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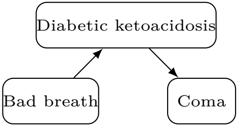

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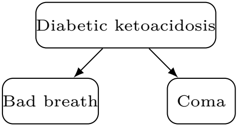
Bayesian network

Not a causal graph

Oracle for conditional independence



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Bayesian network

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Oracle for intervention

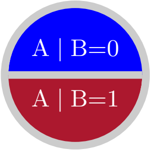
Conditioning vs Intervening (1/2)

Population

Conditioning vs Intervening (1/2)

Population

Sub-populations

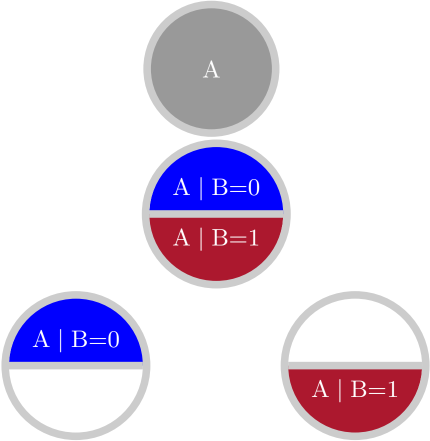


Conditioning vs Intervening (1/2)

Population

Sub-populations

Conditioning



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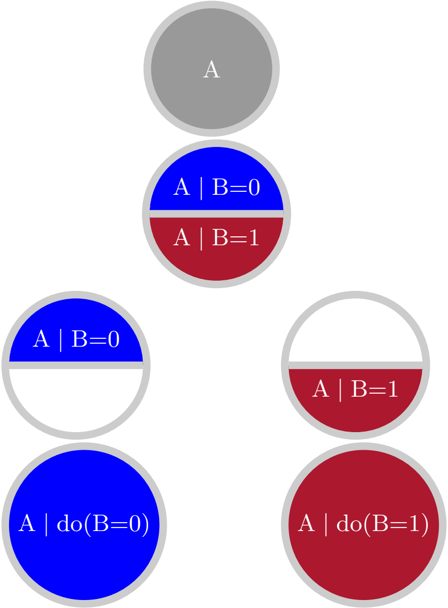
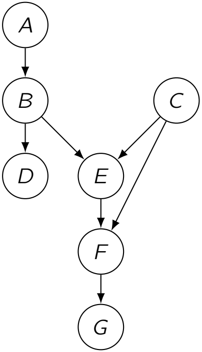
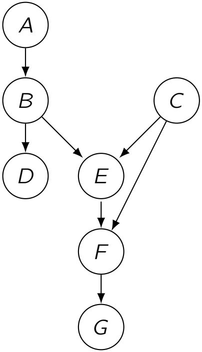


Conditioning vs Intervening (2/2)





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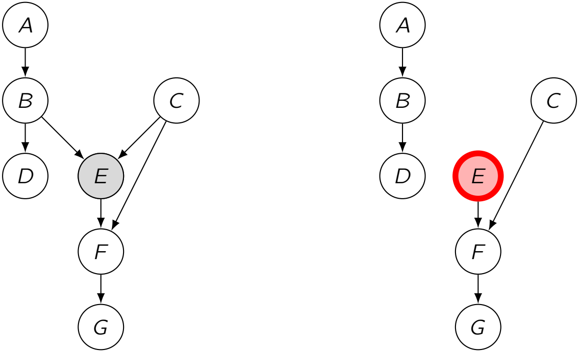

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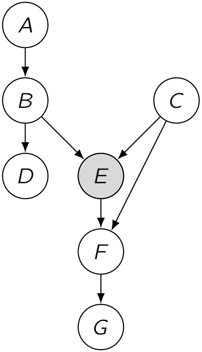

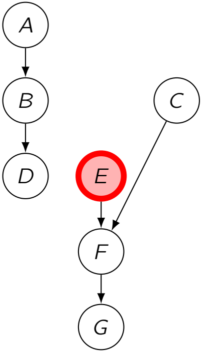

Note that there are two types of interventions:

- Structural (or hard) intervention
- Parametric (or soft) intervention

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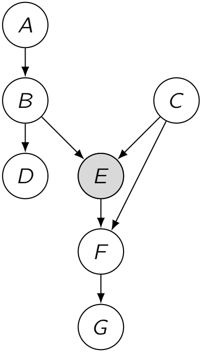

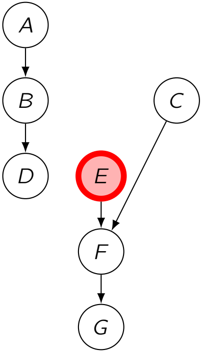

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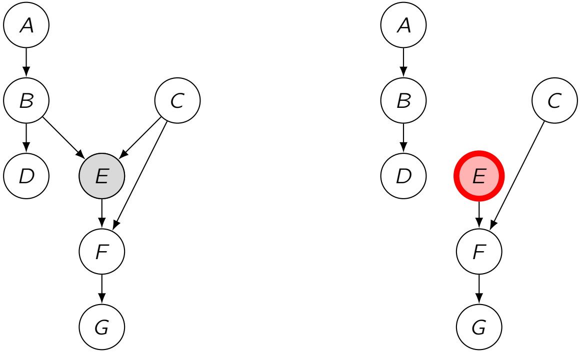

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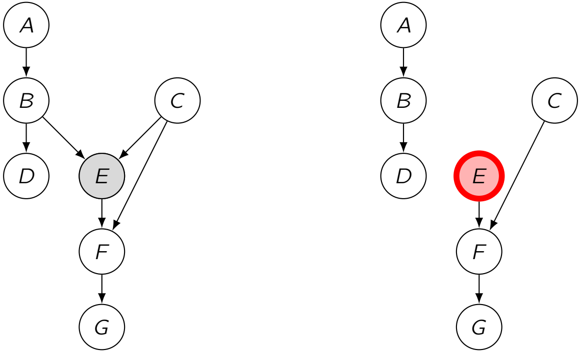

Note that there are two types of interventions:

- Structural (or hard) intervention (we will focus on this)
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The operator $do()$ is a way to denote (hard) interventions
For example $P(a, b, c, d, f, g \mid do(e))$ or $P_{E=e}(a, b, c, d, f, g)$

Truncated factorization [4]

If we intervene on a subset $\mathbb{S} \subset \mathbb{V}$, then

$$P(v_1, \cdots, v_d \mid do(s)) = \prod_{V_i \notin \mathbb{S}} P(v_i \mid Pa(v_i))$$

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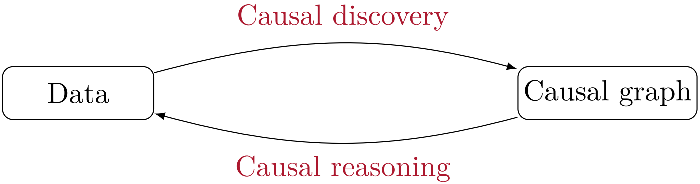
Also known as G-computation formula [2] or Manipulation theorem [10].

Let $P(\mathbb{V})$ be a probability distribution and let P_* denote the set of all interventional distributions $P(\mathbb{V} \mid do(s))$. A bayesian network $\mathcal{G}$ is said to be a causal graph compatible with P_* iff $\mathcal{G}$ and P_* satisfy the truncated factorization.

Working with causal graphs



2

Intervention

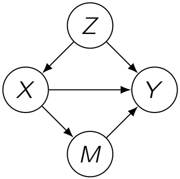
Causal graphs

Causal reasoning

Causal discovery

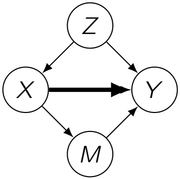
Different types of causal effects

There exists three main types of causal effects:



There exists three main types of causal effects:

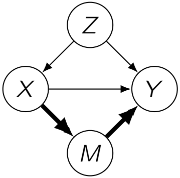
- Direct effect



Different types of causal effects

There exists three main types of causal effects:

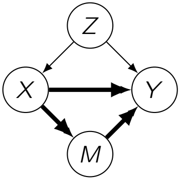
- Direct effect
- Indirect effect



Different types of causal effects

There exists three main types of causal effects:

- Direct effect
- Indirect effect
- Total effect



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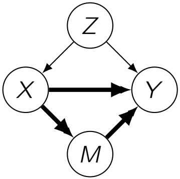

$$\text{Total effect of } X \text{ on } Y = P(y \mid do(x)) - P(y \mid do(x'))$$

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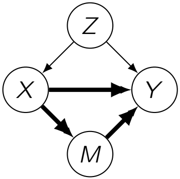

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where the $do()$ operator represents an intervention.

In the following, we will use the terms causal effect and total effect interchangeably.

Identifiability

The causal effect of an intervention $do(x)$ on a set of variables Y such that $Y \cap X = \emptyset$ is said to be identifiable if $P(y \mid do(x))$ is uniquely computable from $P(\mathbb{V})$.

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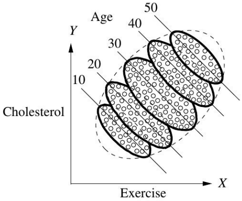
in other words, $P(Y \mid do(x))$ is identifiable if we can re-write it using a do-free expression.

Simpson paradox 1 and a simple solution

In a study, we measure weekly exercise and cholesterol levels for various age groups.



What is the effect of exercise on cholesterol $P(c \mid do(e))$?

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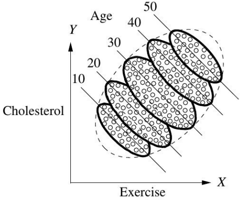

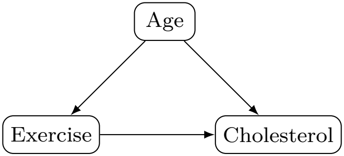

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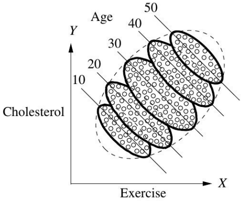

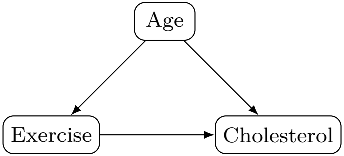

What is the effect of exercise on cholesterol $P(c \mid \color{red}{do(e)})$?

$$P(a, e, c) = P(a)P(e \mid a)P(c \mid a, e) \qquad \text{(Compatibility)}$$

$$P(a, c \mid \color{red}{do(e)}) = P(a)P(c \mid a, e) \qquad \text{(Truncated factorization)}$$

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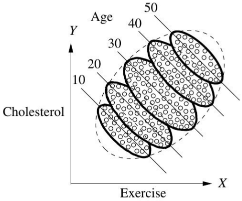

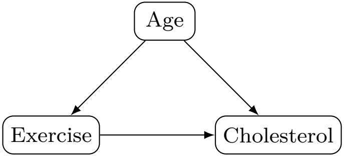

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$$P(c \mid do(e)) = \sum_a P(a)P(c \mid a, e) \qquad \text{(marginalizing)}$$

A simple solution: direct causes adjustment [4]

Theorem

Given a causal graph $\mathcal{G}$ in which a subset $\mathbb{V}$ of variables are measured, the causal effect $P(y \mid do(x))$ is identifiable whenever $\{X \cup Y \cup Parents(X)\} \subseteq \mathbb{V}$, and is given by:

$$P(y \mid do(x)) = \sum_{z \in Parents(x)} P(y \mid x, z) \Pr(z)$$

Limitations of the direct causes adjustment

- Sometimes the set of parents is too large. Is it possible to find a smaller set?

- Sometimes the set of observed parents is not sufficient for adjustment. Is it possible to find another set?

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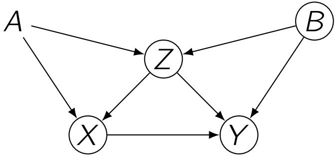


Back-door criterion

Consider a causal graph $\mathcal{G}$ and a causal effect $P(y \mid do(x))$. A set of variables $\mathbb{Z}$ satisfies the back-door criterion iff:

- no vertex in $\mathbb{Z}$ is a descendant of X;
- $\mathbb{Z}$ blocks every path between X and Y that contains an arrow into X.

Back-door criterion: using d-separation

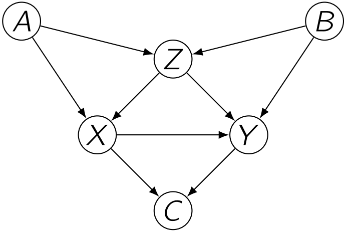
Consider a causal graph $\mathcal{G}$. The mutilated graph $\mathcal{G}_{\underline{X}}$ is a modified version of $\mathcal{G}$ that reflects the intervention on the variable X. This modification involves removing all edges going out of X.

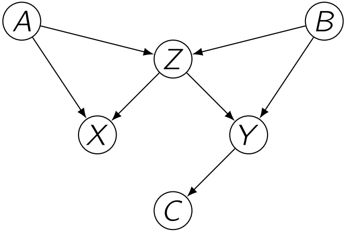
Consider a causal graph $\mathcal{G}$ and a causal effect $P(y \mid do(x))$. A set of variables $\mathbb{Z}$ satisfies the back-door criterion iff:

- $X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid \mathbb{Z}$

Back-door criterion: example

Causal graph $\mathcal{G}$ $P(y \mid do(x))$ Mutilated graph $\mathcal{G}_{\underline{X}}$



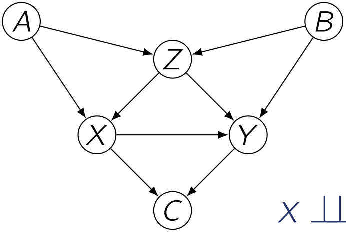


Back-door criterion: example

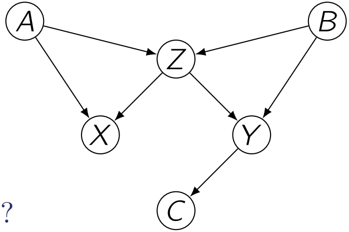
Causal graph $\mathcal{G}$

$P(y \mid do(x))$

Mutilated graph $\mathcal{G}_{\underline{X}}$

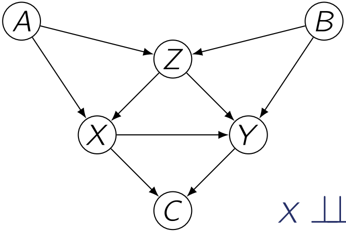


$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$?

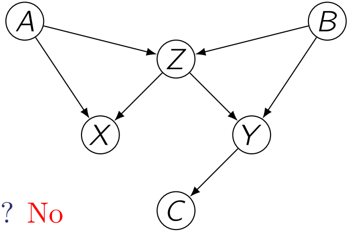


Back-door criterion: example

Causal graph $\mathcal{G}$ $\quad$ $P(y \mid do(x))$ $\quad$ Mutilated graph $\mathcal{G}_{\underline{X}}$



$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$? No



Back-door criterion: example

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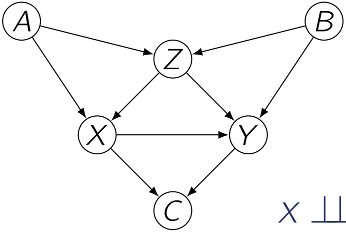

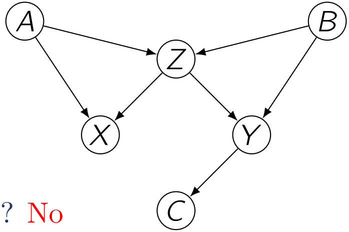

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid A$?

Back-door criterion: example

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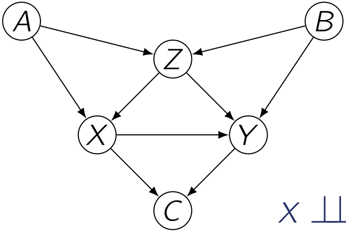

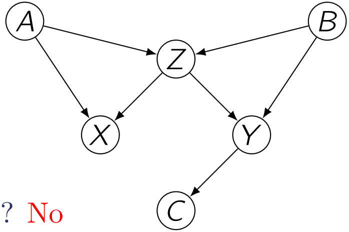

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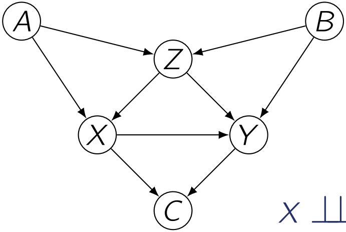
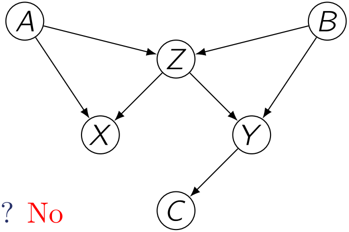
Causal graph $\mathcal{G}$



$P(y \mid do(x))$

Mutilated graph $\mathcal{G}_{\underline{X}}$



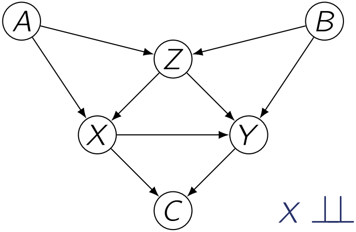
$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid A$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid B$?

Back-door criterion: example

Causal graph $\mathcal{G}$



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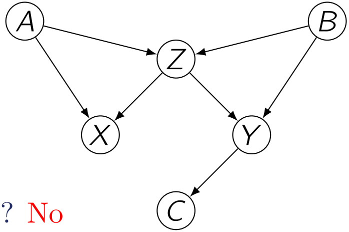

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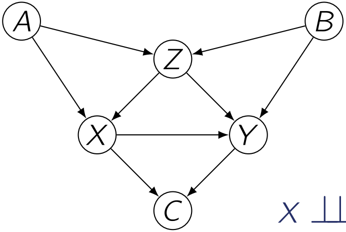
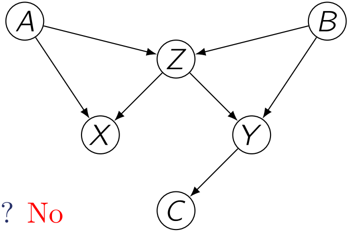
Back-door criterion: example

Causal graph $\mathcal{G}$



$P(y \mid do(x))$

Mutilated graph $\mathcal{G}_{\underline{X}}$



$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$? No

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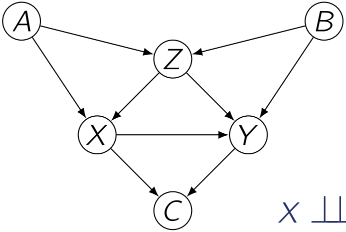
$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid B$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid C$?

Back-door criterion: example

Causal graph $\mathcal{G}$



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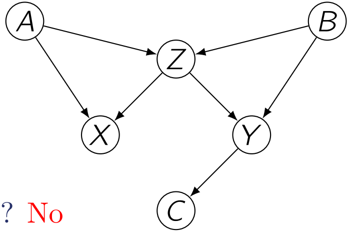
Mutilated graph $\mathcal{G}_{\underline{X}}$



$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid Z$? No

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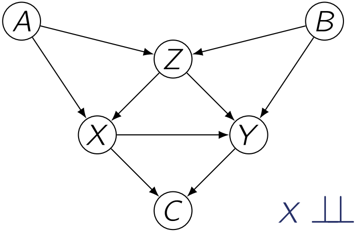
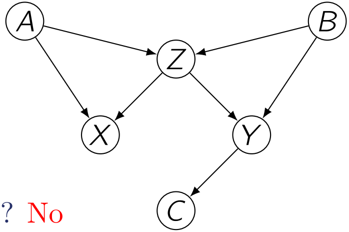
Back-door criterion: example

Causal graph $\mathcal{G}$ $\quad$ $P(y \mid do(x))$ $\quad$ Mutilated graph $\mathcal{G}_{\underline{X}}$





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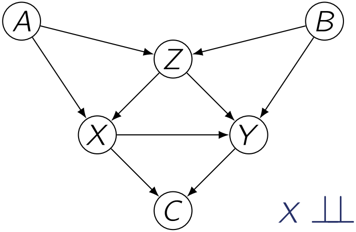
$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid B$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid C$? No

$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid A, B$?

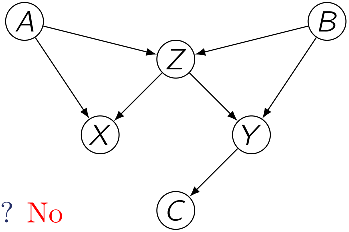
Back-door criterion: example

Causal graph $\mathcal{G}$



$P(y \mid do(x))$

Mutilated graph $\mathcal{G}_{\underline{X}}$



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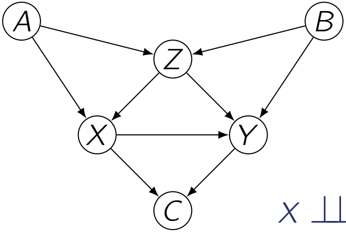
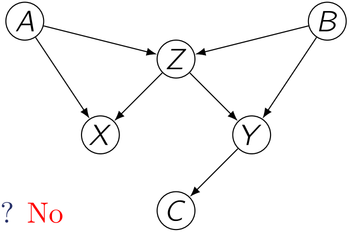
$X \perp\!\!\!\perp_{\mathcal{G}_{\underline{X}}} Y \mid A, B$? No

Back-door criterion: example

Causal graph $\mathcal{G}$ $\quad P(y \mid do(x))$ $\quad$ Mutilated graph $\mathcal{G}_{\underline{X}}$





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Back-door criterion: example

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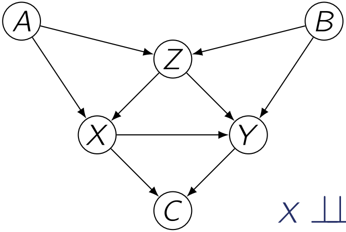

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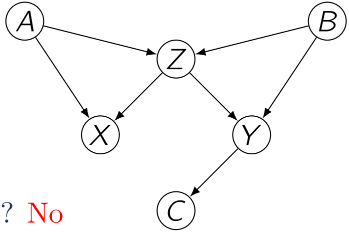

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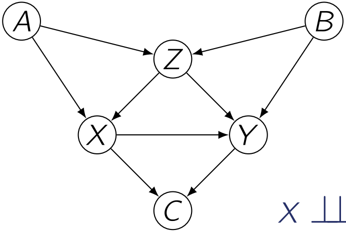

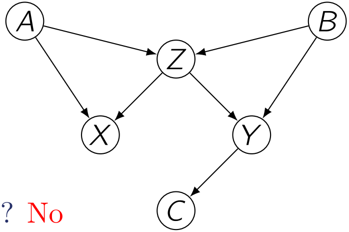

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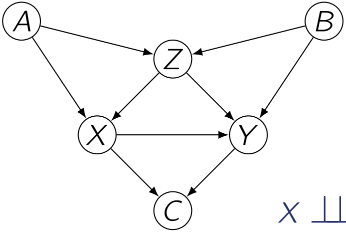

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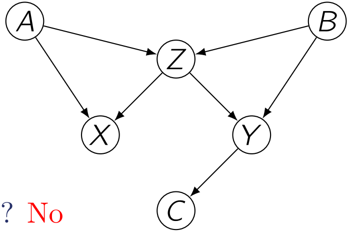

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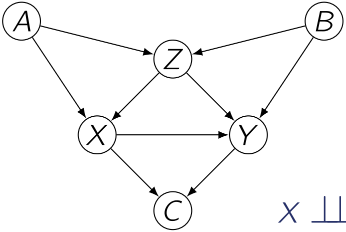

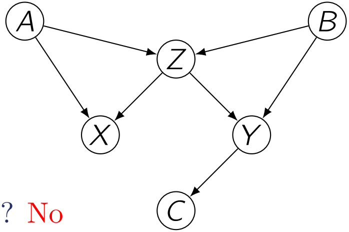

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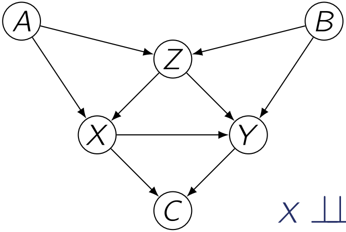

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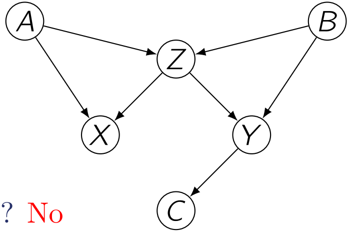

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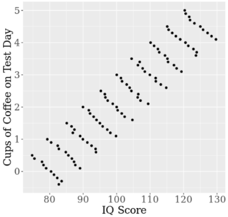

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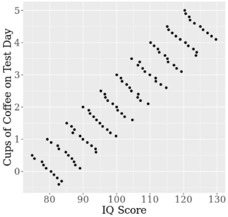

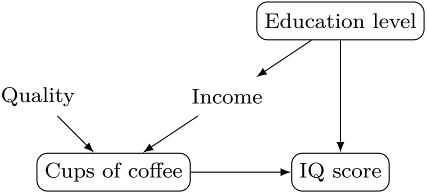

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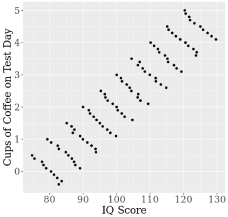

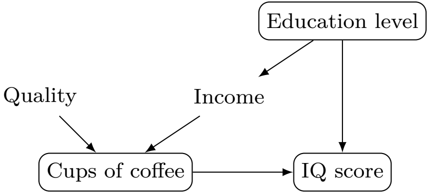

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Associational criterion for finding non-confounding

Let $\mathbb{Z}$ be the set of observed variables in a problem that are not affected by X. $\mathbb{Z}$ satisfies the associational criterion if each member Z of $\mathbb{Z}$ satisfied at least one of the following conditions:

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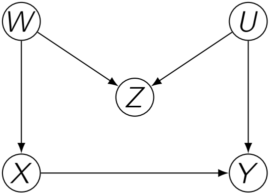

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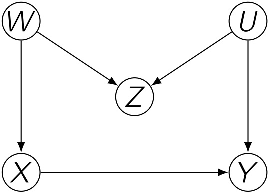

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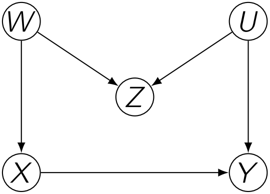

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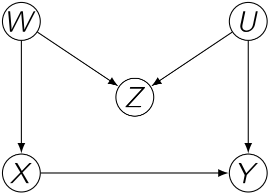

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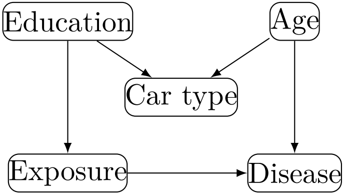

Car type is associated with Exposure and Car type is associated with Disease, conditional on Exposure

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Modified associational criterion for finding non-confounding

Let $\mathbb{Z}$ be the set of variables in a problem that are not affected by X. $\mathbb{Z}$ satisfies the associational criterion if it can be devided into two sets $\mathbb{Z}_1$ and $\mathbb{Z}_2$ that satisfy the following conditions:

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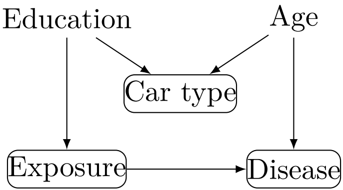

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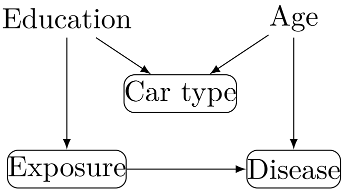

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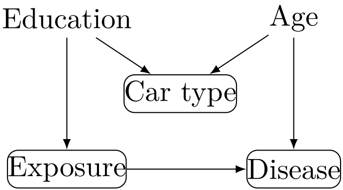

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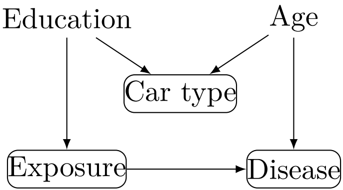

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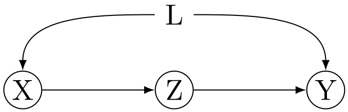

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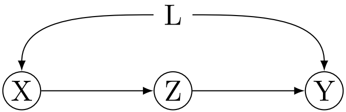

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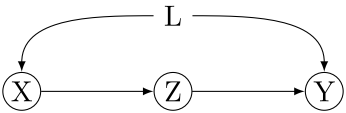

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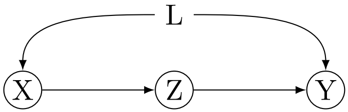

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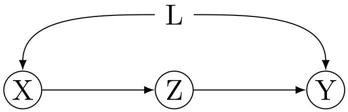

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Front-door criterion

Consider a causal graph $\mathcal{G}$ and a causal effect $P(y \mid do(x))$. A set of variables $\mathbb{Z}$ satisfies the front-door criterion iff:

- $\mathbb{Z}$ intercepts all directed paths from X to Y;
- There is no back-door path from X to $\mathbb{Z}$;
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Front-door adjustment

Theorem ([3])

If $\mathbb{Z}$ satisfies the front-door criterion relative to (X, Y) and if $P(x, z) > 0$, then the causal effect of X on Y is identifiable and is given by

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Smokers	Tar	% of cancer
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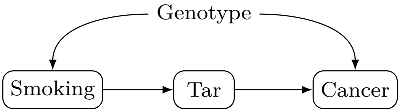

Simpson paradox 3 and the front-door in action

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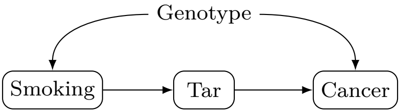

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The combination of the back-door and front-door criterions is also not complete.

A glimpse of the do-calculus [3]

The do calculus consists of three rules :

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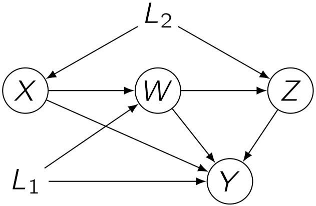
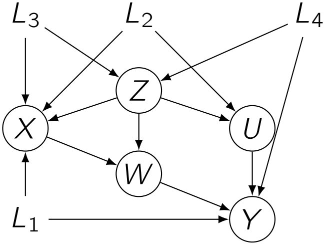




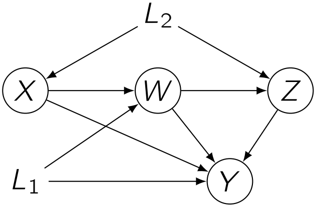
A glimpse of the do-calculus [3]

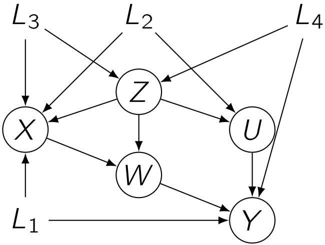
The do calculus consists of three rules :

- Rule 1: generalization of the d-seperation
- Rule 2: generalization of back-door criterion
- Rule 3: complicated

$P(y \mid do(x))$ is identifiable iff by repeatedly applying the rules of do-calculus, we obtain a do-free expression.

The do calculus is sound and complete!

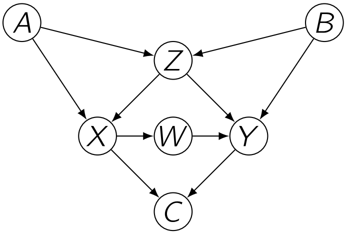




The ID algorithm [9] uses the do-calculus to automatically uncover the do-free expression.

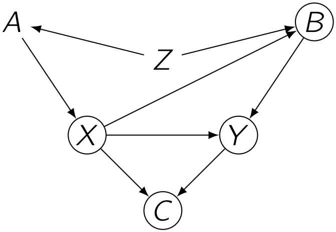
Exercise 4

Consider that in the following causal graph, only X and Y, and one additional variable can be measured. Which variable would allow the identification of $P(y \mid do(x))$?



Exercise 5

- Consider the following causal graph. List all sets of variables that satisfy the back-door criterion for $P(y \mid do(x))$;



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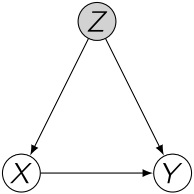

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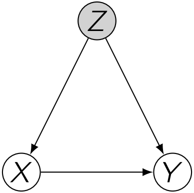

- Z blocks all back-door paths and it is not a descendant of X $\implies$ $\{Z\}$ is a good adjustment set.

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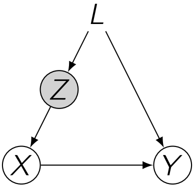

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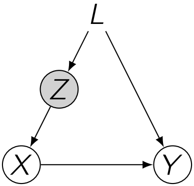

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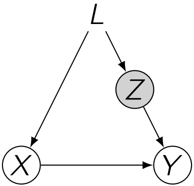

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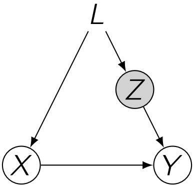

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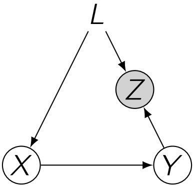

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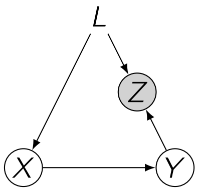

- Z activates a back-door path
  $\implies \{Z\}$ is a bad adjustment set.

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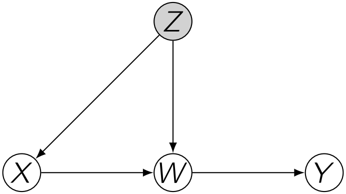

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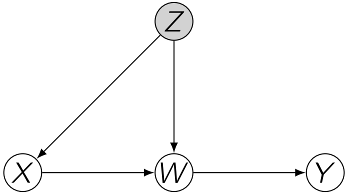

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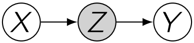

- Z d-separates X from Y
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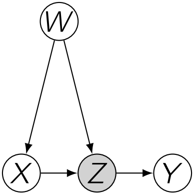

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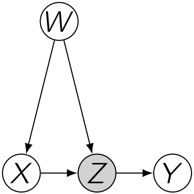

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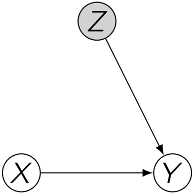

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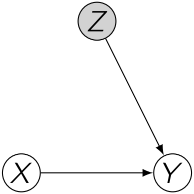

- Z does not open any back-door paths from X to Y $\implies \{Z\}$ is a neutral adjustment set;

- Adjusting on Z can reduces the variation of Y, and helps improve the precision of the estimate in finite samples.

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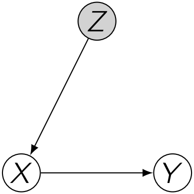

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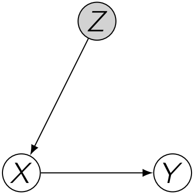

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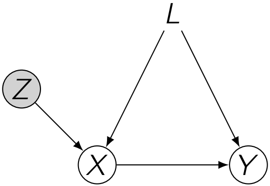

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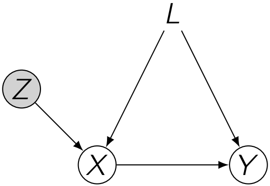

- Z does not block existing back-door path from X to Y $\implies$ $\{Z\}$ is a bad adjustment set;

- In linear models, adjusting on Z amplify any existing bias.

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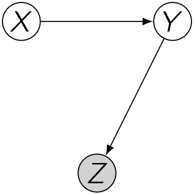

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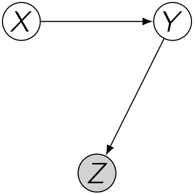

- Selection bias
  $\implies \{Z\}$ is a bad control.

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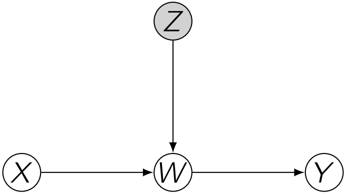

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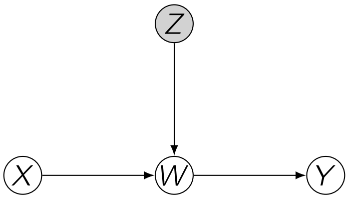

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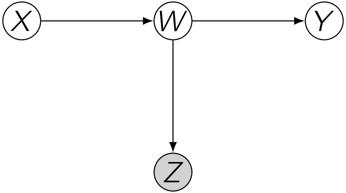

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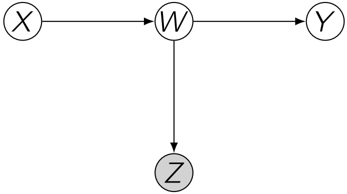

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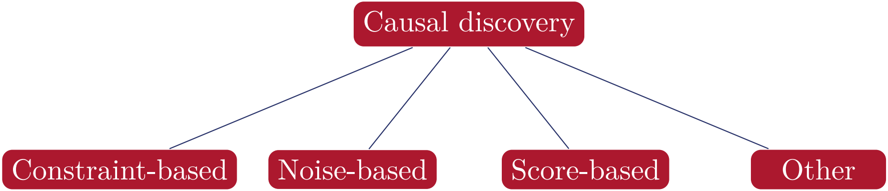
2

Intervention

Families of causal discovery methods



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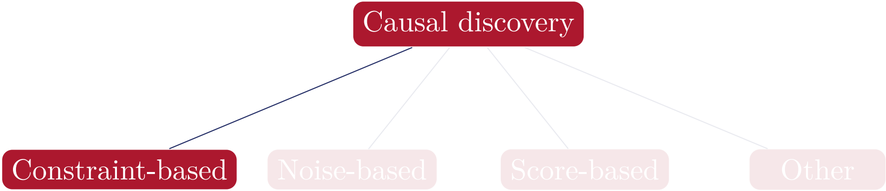

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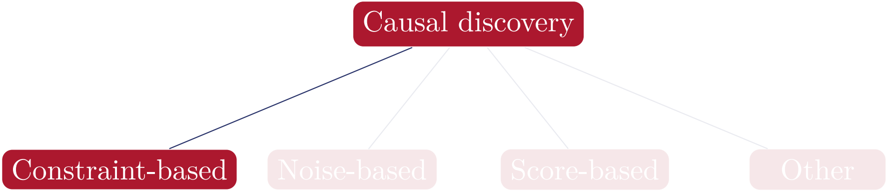

Constraint-based: run local tests of independence to create constraints on space of possible graphs.

Given observational data, is it possible to infer a causal graph using conditional independencies?

Causal discovery using conditional independencies

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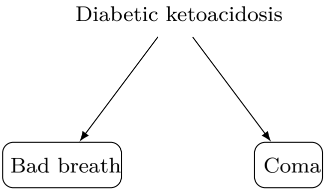

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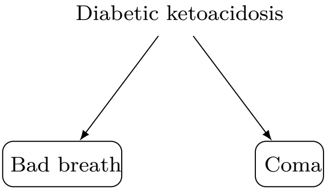

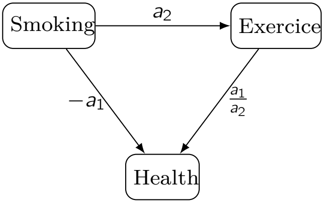

Additional assumptions

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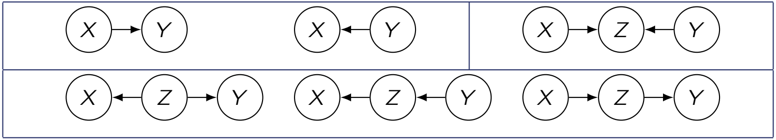
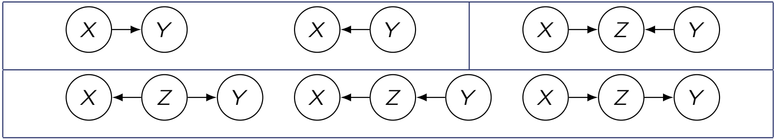


Markov equivalence class [12]

Equivalence in terms of conditional independence



Theorem

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Markov equivalence class [12]

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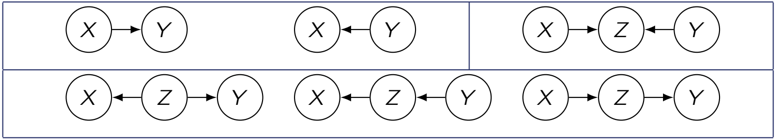

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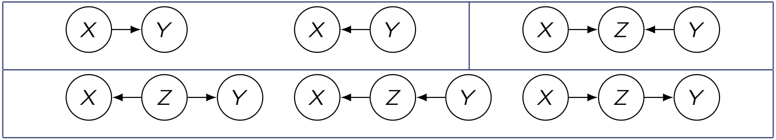

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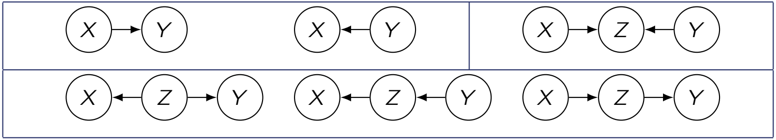

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Given observational data, is it possible to infer a CPDAG using conditional independencies under the assumptions of faithfulness and causal sufficiency? Yes!

Theorem

If $P(\mathbb{V})$ is faithful to some causal graph $\mathcal{G}$ with vertex $\mathbb{V}$ then:

- For $X, Y \in \mathbb{V}$, X and Y are adjacent iff $\forall \mathbb{S} \subseteq \mathbb{V} \backslash \{X, Y\}$, $X \not\perp\!\!\!\perp_P Y \mid \mathbb{S}$;

Finding skeleton and v-structures

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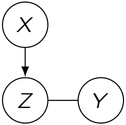
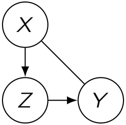
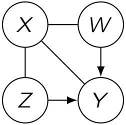
- Point 1 can be used to discover the skeleton of $\mathcal{G}$ from $P(\mathcal{V})$;
- Given the skeleton of $\mathcal{G}$, point 2 can be used to find all v-structures.

Orientation rules

Suppose we already found the skeleton and all v-structures:

PC-Rule 1: 

PC-Rule 2: 

PC-Rule 3: 

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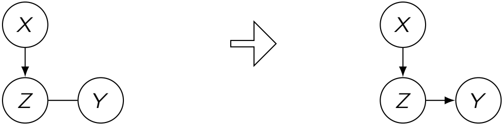

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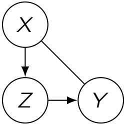
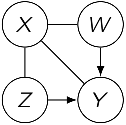


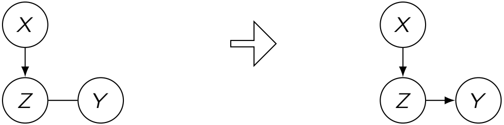
PC-Rule 3:



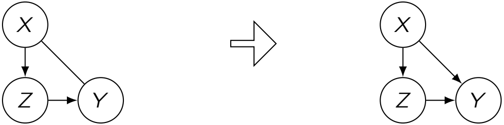
Orientation rules

Suppose we already found the skeleton and all v-structures:

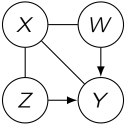
PC-Rule 1:



PC-Rule 2:

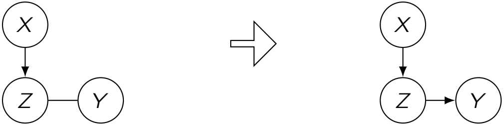


PC-Rule 3:

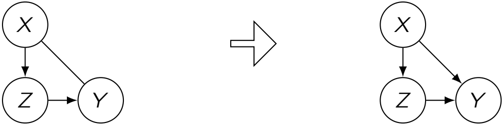


Suppose we already found the skeleton and all v-structures:

PC-Rule 1:



PC-Rule 2:



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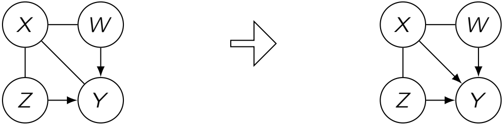

The PC algorithm [10]

- Step 1: skeleton construction:
  - Construct a complete non oriented graph
  - Prune unnecessary edges (optimal) from the skeleton using $\perp\!\!\!\perp_P$

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  - PC-Rule 1, 2, 3: deduce other orientations by contradiction to V-structures and acyclicity

The PC algorithm [10]

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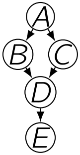
Theorem ([10])

Assume the distribution P is compatible and faithful to some causal graph $\mathcal{G}$ and assume that we are given perfect conditional independence information about all pairs of variables. The PC algorithm returns the CPDAG of $\mathcal{G}$.

PC in action

- Suppose the causal graph on the right
- Input: Observational data
- Output: CPDAG
- Assumptions: Markov condition, causal sufficiency, faithfulness



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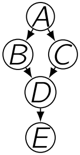
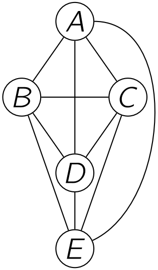


Skeleton construction:



$\boxed{card = 0}$

PC in action

- Suppose the causal graph on the right
- Input: Observational data
- Output: CPDAG
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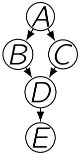
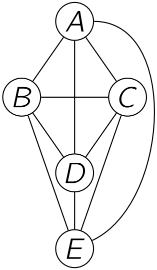
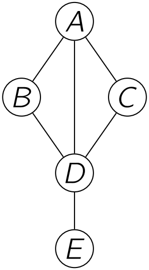


Skeleton construction:



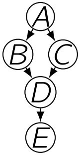
$\boxed{card = 0}$



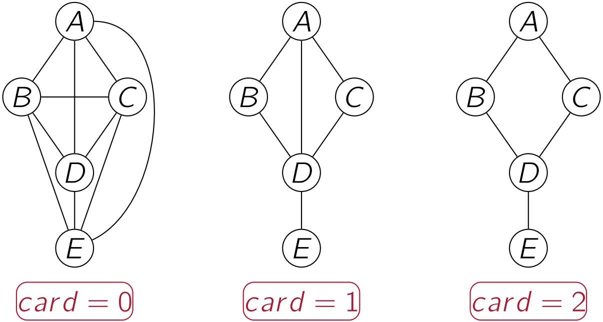
$\boxed{card = 1}$

PC in action

- Suppose the causal graph on the right
- Input: Observational data
- Output: CPDAG
- Assumptions: Markov condition, causal sufficiency, faithfulness



Skeleton construction:



PC in action

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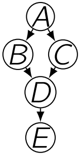
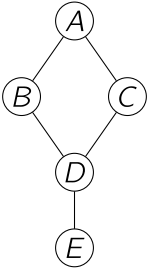


Orientation:



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- Suppose the causal graph on the right
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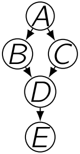

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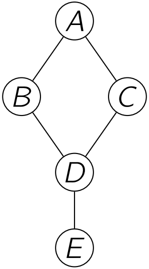

- PC-Rule 0

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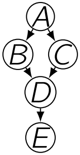

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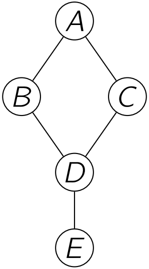
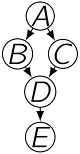


- PC-Rule 0
  - $B \perp\!\!\!\perp_P C \mid A \implies B \to D \leftarrow C$

PC in action



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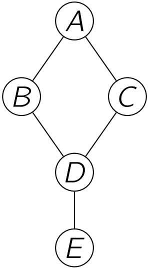

- PC-Rule 0
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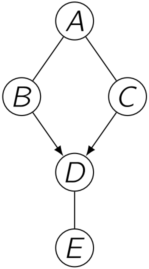

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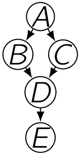

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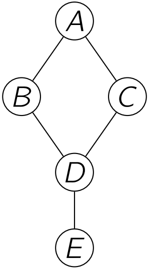
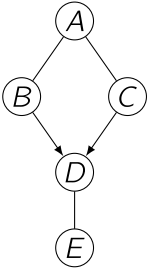


- PC-Rule 0
  - $B \perp\!\!\!\perp_P C \mid A \implies B \to D \leftarrow C$
- PC-Rule 1



PC in action

- Suppose the causal graph on the right
- Input: Observational data
- Output: CPDAG
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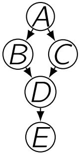
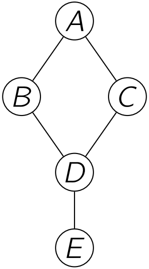
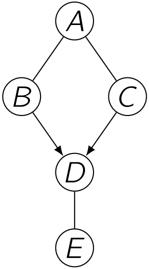


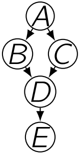
Orientation:



- PC-Rule 0
  - $B \perp\!\!\!\perp_P C \mid A \implies B \to D \leftarrow C$
- PC-Rule 1
  - $B \to D \;\&\; D - E \implies D \to E$

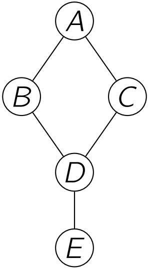


PC in action

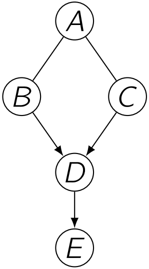


- Suppose the causal graph on the right
- Input: Observational data
- Output: CPDAG
- Assumptions: Markov condition, causal sufficiency, faithfulness

Orientation:



- PC-Rule 0
  - $B \perp\!\!\!\perp_P C \mid A \implies B \rightarrow D \leftarrow C$
- PC-Rule 1
  - $B \rightarrow D \;\&\; D - E \implies D \rightarrow E$



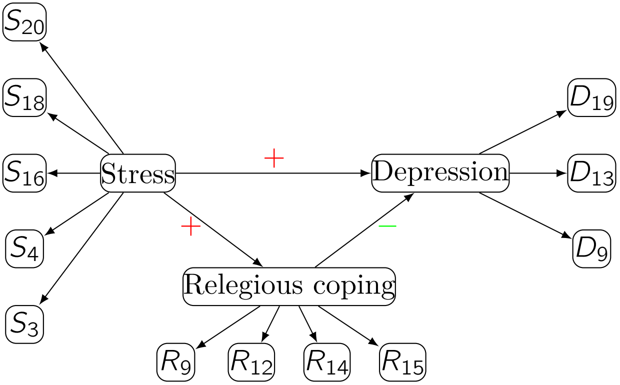
Incorporating background knowledge

The PC algorithm can effectively incorporate background knowledge in the form of:

- Forbidden edges
- Required edges
- Forbidden orientations
- Required orientations

MSW students ($N = 127$); 61 item survey (Likert Scale)

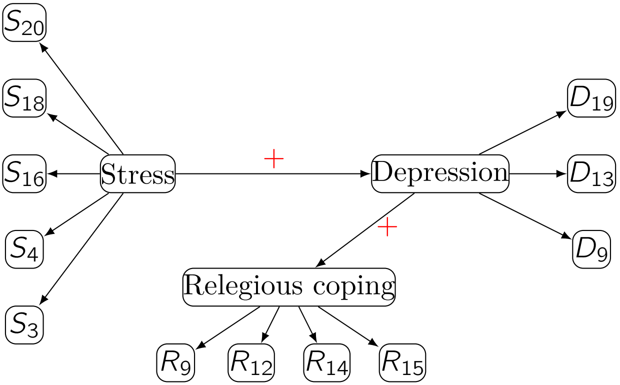
Specified graph



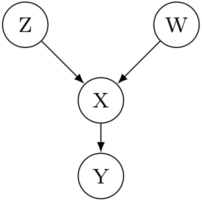
Real application of PC [7]

MSW students ($N = 127$); 61 item survey (Likert Scale)

Inferred graph (assuming stress is temporally prior)



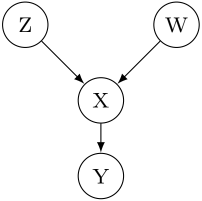
Beyond causal sufficiency (1/2)



Y-structure

$$Z \perp\!\!\!\perp_P W$$
$$Z \not\perp\!\!\!\perp_P W \mid X$$
$$Y \not\perp\!\!\!\perp_P Z$$
$$Y \perp\!\!\!\perp_P Z \mid X$$
$$Y \not\perp\!\!\!\perp_P W$$
$$Y \perp\!\!\!\perp_P W \mid X$$

Beyond causal sufficiency (1/2)



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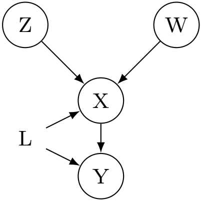

$$Z \perp\!\!\!\perp_P W$$
$$Z \not\!\perp\!\!\!\perp_P W \mid X$$
$$Y \not\!\perp\!\!\!\perp_P Z$$
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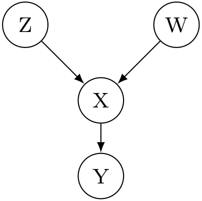
Beyond causal sufficiency (1/2)



Y-structure

$$Z \perp\!\!\!\perp_P W$$
$$Z \not\perp\!\!\!\perp_P W \mid X$$
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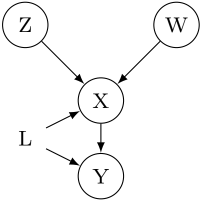

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$$Y \not\perp\!\!\!\perp_P Z$$
$$Y \not\perp\!\!\!\perp_P Z \mid X$$
$$Y \not\perp\!\!\!\perp_P W$$
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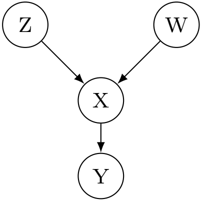
Beyond causal sufficiency (1/2)



Y-structure

$Z \perp\!\!\!\perp_P W$
$Z \not\!\perp\!\!\!\perp_P W \mid X$
$Y \not\!\perp\!\!\!\perp_P Z$
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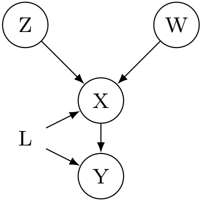

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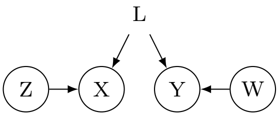
Pattern of independence can rule out hidden confounding.

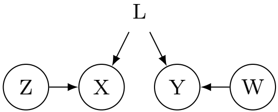
Beyond causal sufficiency (2/2)



W-structure

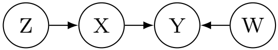
$$
\begin{gathered}
Z \not\perp\!\!\!\perp_P X \\
X \not\perp\!\!\!\perp_P Y \\
Y \not\perp\!\!\!\perp_P W \\
Z \perp\!\!\!\perp_P W \\
Z \perp\!\!\!\perp_P Y \\
X \perp\!\!\!\perp_P W \\
Z \not\perp\!\!\!\perp_P Y \mid X \\
X \not\perp\!\!\!\perp_P W \mid Y
\end{gathered}
$$

Beyond causal sufficiency (2/2)



W-structure

$$Z \not\perp\!\!\!\perp_P X$$
$$X \not\perp\!\!\!\perp_P Y$$
$$Y \not\perp\!\!\!\perp_P W$$
$$Z \perp\!\!\!\perp_P W$$
$$Z \perp\!\!\!\perp_P Y$$
$$X \perp\!\!\!\perp_P W$$
$$Z \not\perp\!\!\!\perp_P Y \mid X$$
$$X \not\perp\!\!\!\perp_P W \mid Y$$



$$Z \not\perp\!\!\!\perp_P X$$
$$X \not\perp\!\!\!\perp_P Y$$
$$Y \not\perp\!\!\!\perp_P W$$
$$Z \perp\!\!\!\perp_P W$$
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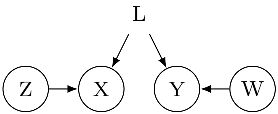

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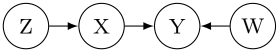

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$$X \perp\!\!\!\perp_P W$$
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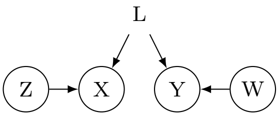
Beyond causal sufficiency (2/2)



W-structure

$Z \not\perp\!\!\!\perp_P X$
$X \not\perp\!\!\!\perp_P Y$
$Y \not\perp\!\!\!\perp_P W$
$Z \perp\!\!\!\perp_P W$
$Z \perp\!\!\!\perp_P Y$
$X \perp\!\!\!\perp_P W$
$Z \not\perp\!\!\!\perp_P Y \mid X$
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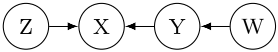

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$Z \not\perp\!\!\!\perp_P Y \mid X$
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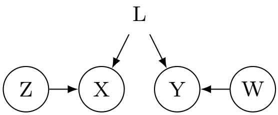

W-structure

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$$X \not\perp\!\!\!\perp_P Y$$
$$Y \not\perp\!\!\!\perp_P W$$
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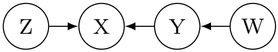

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Pattern of independence can suggest hidden confounding.

A glimpse of the FCI algorithm

The FCI algorithm extends the PC algorithm to accommodate hidden confounders:

A glimpse of the FCI algorithm

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- Skeleton construction is much more complicated;
- Orientation is done using 10 different rules.

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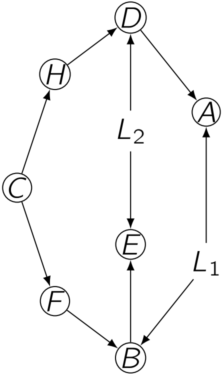

True graph

A glimpse of the FCI algorithm

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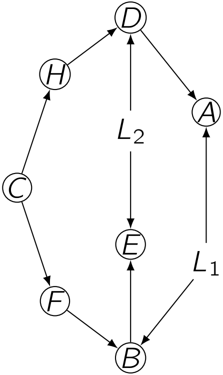
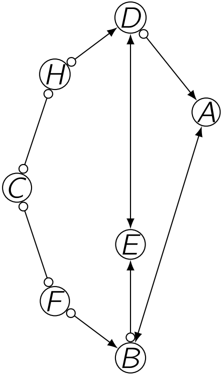


True graph



Inferred graph

Pros and cons of constraint-based methods

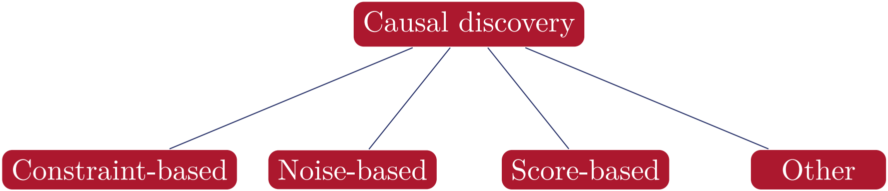
Pros:

- Non-parametric (in practice, it depends on the selected independence test)
- Intuitive

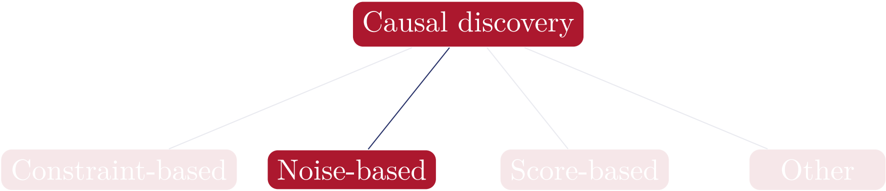
Cons:

- Recover a partially oriented graph
- The faithfulness assumption is not always accepted

Noise based


Causal discovery
Constraint-based
Noise-based
Score-based
Other

Noise based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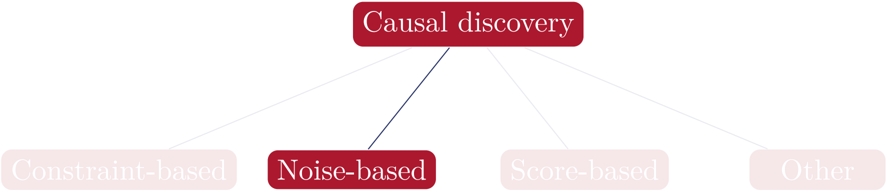
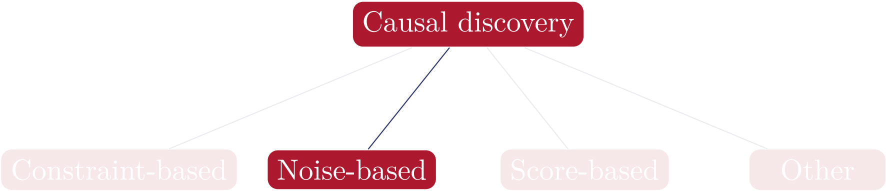


Noise-based: find footprints in the noise that imply causal asymmetry.

Noise based



Noise-based: find footprints in the noise that imply causal asymmetry.
Also known as semi-parametric-based or functional-based.

The intuition behind the noise (1/2)

$$Suppose \begin{cases} X := \xi_x \\ Y := 2X + \xi_y \end{cases}$$

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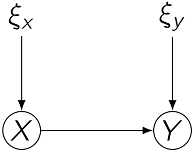
Assume that the noise follow a uniform distribution on $\{-1, 0, 1\}$

X	Y	$\xi_y = Y - 2X$	$\hat{\xi}_x = X - Y/2$
1	2	$0 \in \{-1, 0, 1\}$	$0 \in \{-1, 0, 1\}$
3	6	$0 \in \{-1, 0, 1\}$	$0 \in \{-1, 0, 1\}$
4	9	$1 \in \{-1, 0, 1\}$	$-0.5 \notin \{-1, 0, 1\}$

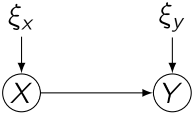
The intuition behind the noise (2/2)



$$M_1 : \begin{cases} X := f_x(\xi_x) \\ Y := f_y(X, \xi_y) \end{cases}$$

- $X \perp\!\!\!\perp_{\mathcal{G}} \xi_y$
- $Y \not\!\perp\!\!\!\perp_{\mathcal{G}} \xi_x$

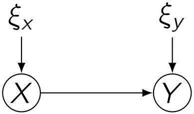
Suppose

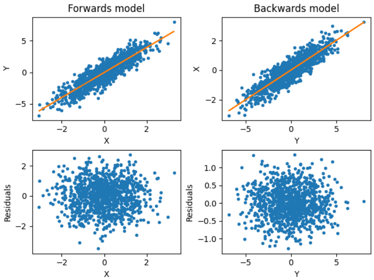


$$X \sim N(0,1)$$
$$\xi_y \sim N(0,1)$$
$$Y := 2X + \xi_y$$

The experiment that changed everything

Suppose





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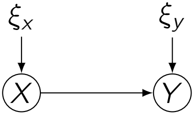

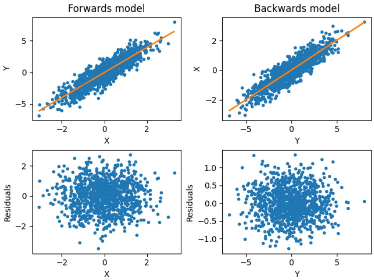

$$X \sim N(0,1)$$
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$$X \sim U(0,1)$$
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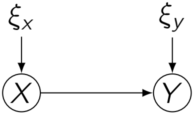
Suppose



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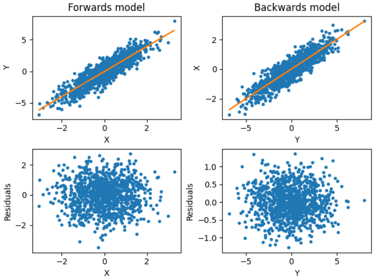

$$X \sim U(0,1)$$
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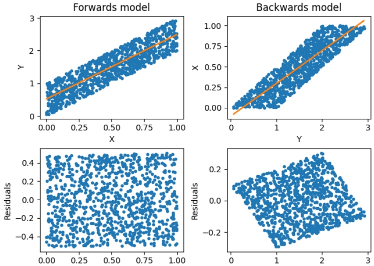
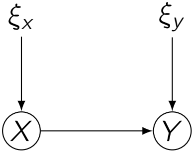


The linear case (1/2)



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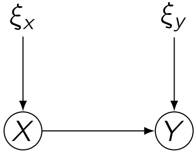
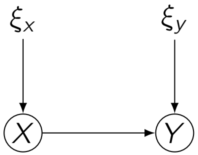


True model:

$$M_1 : \begin{cases} X := \xi_x \\ Y := aX + \xi_y \end{cases}$$

- $X \perp\!\!\!\perp_{\mathcal{G}} \xi_y$

The linear case (1/2)



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Backwards model:

$$M_2 : \begin{cases} Y := \hat{\xi}_y \\ X := bY + \hat{\xi}_x \end{cases}$$

- $Y \perp\!\!\!\perp_{\mathcal{G}} \hat{\xi}_x$?

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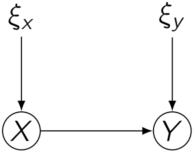

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- $Y \perp\!\!\!\perp_{\mathcal{G}} \hat{\xi}_x$?

$$\begin{aligned} \hat{\xi}_x &= X - bY \\ &= X - b(aX + \xi_y) \\ &= (1 - ba)X - b\xi_y \end{aligned}$$

The linear case (2/2)

$$Y = aX + \xi_y$$
$$\hat{\xi}_x = (1 - ba)X - b\xi_y$$

When $Y \perp\!\!\!\perp_P \hat{\xi}_x$?

The linear case (2/2)

$$Y = aX + \xi_y$$
$$\hat{\xi}_x = (1 - ba)X - b\xi_y$$

When $Y \perp\!\!\!\perp_P \hat{\xi}_x$?

Theorem (Darmois-Skitovich)

Let $X_1, \cdots, X_n$ be independent, non degenerate random variables. If for two linear combinations:

$$l_1 = a_1X_1 + \cdots + a_nX_n$$
$$l_2 = b_1X_1 + \cdots + b_nX_n$$

are independent, then each X_i is normally distributed.

Linear non-Gaussian models (LiNGAM) [8]

Theorem

Assume that $P(X, Y)$ admits the linear model

$$Y := aX + \xi_y, \qquad X \perp\!\!\!\perp_P \xi_y,$$

with continuous random variables X, ξ_y, and Y. Then there exists $b \in \mathbb{R}$ and a random variable $\hat{\xi}_x$ such that

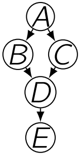
$$X := bY + \hat{\xi}_x, \qquad Y \perp\!\!\!\perp_P \hat{\xi}_x,$$

iff ξ_y and X are Gaussian.

Similar result for the multivariate case

A glimpse of LiNGAM in action

- Suppose the causal graph on the right
- Input: Observational data
- Output: Causal graph
- Assumptions: Markov condition, causal sufficiency, minimality, linearity, non-gaussianity



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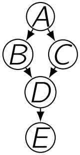
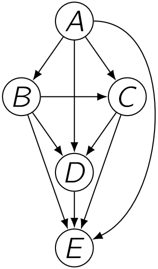
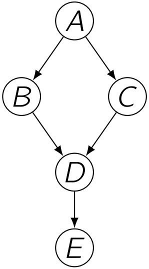


Causal order:



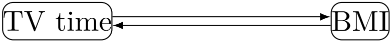
Pruning:

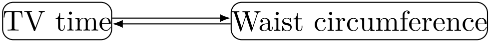


Real application of LiNGAM [1]

Young and middle-aged adults ($N = 2,060$); self-administered questionnaire for TV time

Specified graphs





Real application of LiNGAM [1]

Young and middle-aged adults ($N = 2,060$); self-administered questionnaire for TV time

Inferred graphs

$$\boxed{\text{TV time}} \longrightarrow \boxed{\text{BMI}}$$

$$\boxed{\text{TV time}} \longrightarrow \boxed{\text{Waist circumference}}$$

Extensions

- LiNGAM with hidden confounding
- Non-linear additive noise models
- Post non-linear additive noise models
- ...

Pros and cons of noise-based methods

Pros:

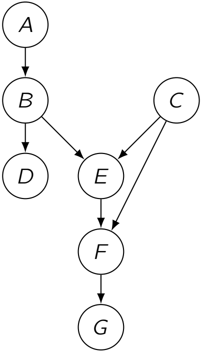
- Recover the complete causal graph

Cons:

- Semi-parametric

Exercise 7

What is the CPDAG of the following causal graph?



3

Counterfactuals

- Structural causal models
- Counterfactual reasoning
- Mediation analysis
- Difference with the potential outcome framework

3

Counterfactuals

https:
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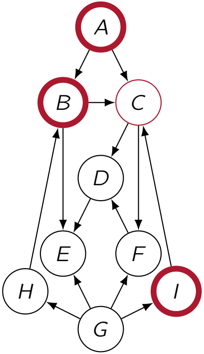
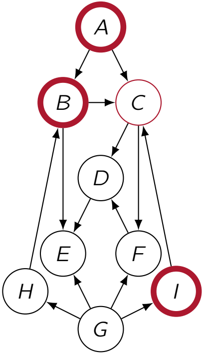
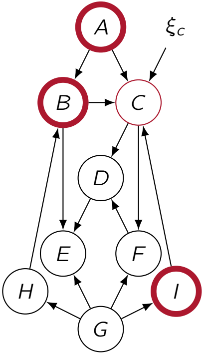
Causal mechanism





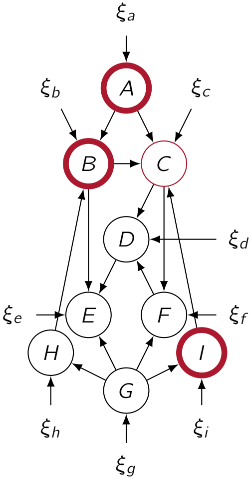
$$C = f(A, B, I)$$

Causal mechanism



$$C := f_c(A, B, I, \xi_c)$$

Causal mechanism



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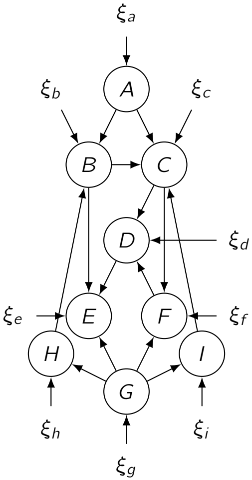
Structural causal model



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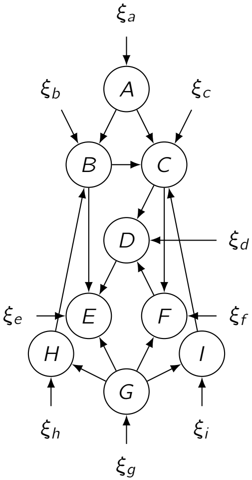
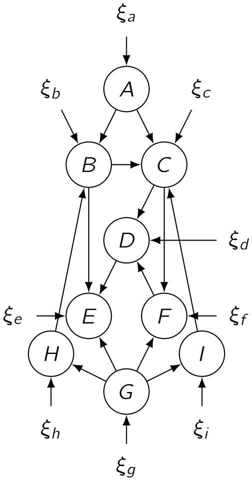


$$M : \begin{cases} A := f_a(\xi_a) \\ B := f_b(A, H, \xi_b) \\ C := f_c(A, B, I, \xi_c) \\ D := f_d(C, F, \xi_d) \\ E := f_e(B, G, \xi_e) \\ F := f_f(C, G, \xi_f) \\ G := f_g(\xi_g) \\ H := f_h(G, \xi_h) \\ I := f_i(G, \xi_i) \end{cases}$$



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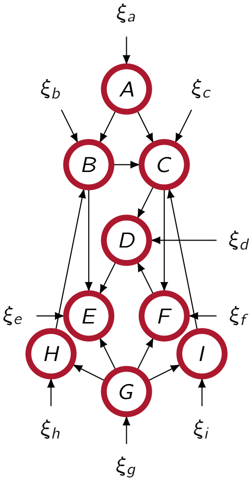
A structural causal model (SCM) is a tuple that contains:

Structural causal model



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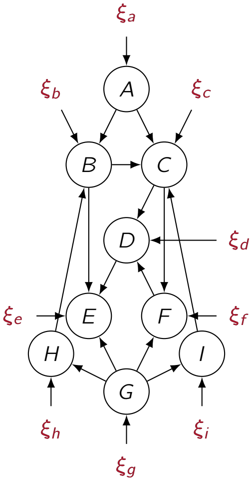
A structural causal model (SCM) is a tuple that contains:

- Endogenous variables



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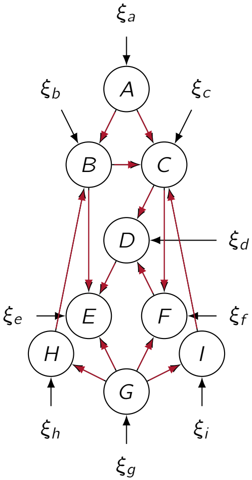

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A structural causal model (SCM) is a tuple that contains:

- Endogenous variables
- Exogenous variables
- Causal mechanisms for generating endogenous variables

Interventions in a structural causal model

$$
M : \begin{cases}
A := f_a(\xi_a) \\
B := f_b(A, H, \xi_b) \\
C := f_c(A, B, I, \xi_c) \\
D := f_d(C, F, \xi_d) \\
E := f_e(B, G, \xi_e) \\
F := f_f(C, G, \xi_f) \\
G := f_g(\xi_g) \\
H := f_h(G, \xi_h) \\
I := f_i(G, \xi_i)
\end{cases}
$$

Interventions in a structural causal model

$$M : \begin{cases} A := f_a(\xi_a) \\ B := f_b(A, H, \xi_b) \\ C := f_c(A, B, I, \xi_c) \\ D := f_d(C, F, \xi_d) \\ E := f_e(B, G, \xi_e) \\ F := f_f(C, G, \xi_f) \\ G := f_g(\xi_g) \\ H := f_h(G, \xi_h) \\ I := f_i(G, \xi_i) \end{cases} \qquad M_c : \begin{cases} A := f_a(\xi_a) \\ B := f_b(A, H, \xi_b) \\ C := c \\ D := f_d(C, F, \xi_d) \\ E := f_e(B, G, \xi_e) \\ F := f_f(C, G, \xi_f) \\ G := f_g(\xi_g) \\ H := f_h(G, \xi_h) \\ I := f_i(G, \xi_i) \end{cases}$$

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Interventional SCMs naturally helps us in computing counterfactual queries:

$$P(D_{C=c} \mid C = c_0, D = d_0)$$

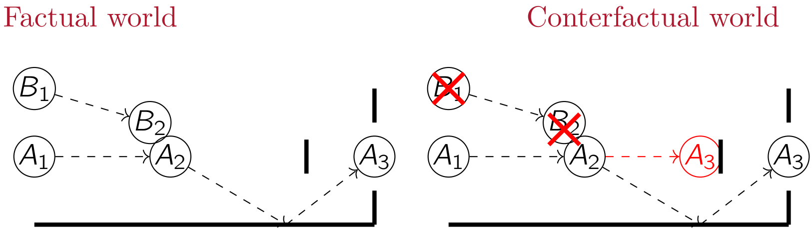
3 Counterfactuals

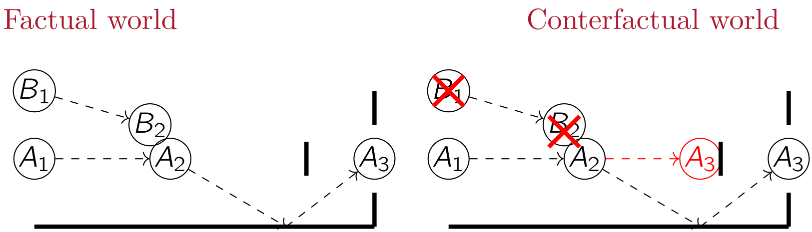
Computing counterfactuals using deterministic SCMs

1. Abduction: use the observations to determine the value of ξ_c
2. Action: modify the model M by removing the structural equations for the variable C and replacing it with the appropriate functions $C = c$, to obtain the interventional SCM M_c
3. Prediction: use M_c and the value of ξ_c to compute the value of $D_{C=c}$

If $D_{C=c} \neq d_0$ then c_0 is a cause of d_0

Computing counterfactuals using deterministic SCMs



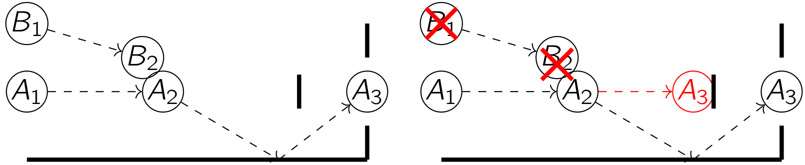


Toy example (without noise on C, W, and G):

$$M : \begin{cases} H \quad (B \text{ hits A}) := \xi_h \\ C \quad (A \text{ changes tragectory}) := H \\ W \quad (A \text{ hits the wall}) := 1 - C \\ G \quad (A \text{ reaches the goal}) := 1 - W \end{cases}$$

Factual world

Conterfactual world



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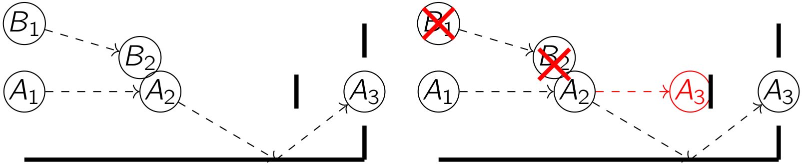
Computing counterfactuals using deterministic SCMs

Factual world

Conterfactual world



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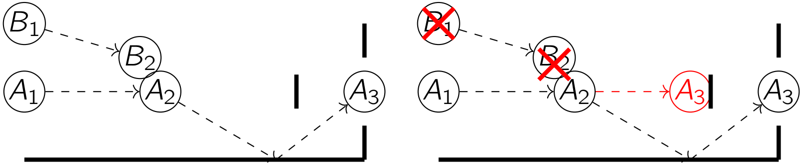
Computing counterfactuals using deterministic SCMs

Factual world | Conterfactual world



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$$G_{H=0} \neq 1 \implies B \text{ causes } A \text{ to reach the goal}$$

3

Counterfactuals

Structural causal models

Counterfactual reasoning

Mediation analysis

Difference with the potential outcome framework

Linear structural causal models

A linear structural causal model consists of a set of structural equations of the form:

$$y := \sum_{x \in Pa(y)} \alpha_{xy} x + \xi_y$$

where $Pa(y)$ are direct causes of y, ξ_y represent errors due to ommitted factors and α_{xy} which are known as a <u>structural coefficient</u> represents the strength of the causal relation.

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α_{xy} is also known as the direct effect of X on Y.

The direct effect is not necessarily equivalent to a regression coefficient.

Single-door criterion

Consider a causal graph $\mathcal{G}$ and a direct effect α_{xy} associated with link $X \rightarrow Y$. Let $\mathcal{G}_{\alpha_{xy}}$ denote the causal graph that results when $X \rightarrow Y$ is deleted from $\mathcal{G}$. A set of variables $\mathbb{Z}$ satisfies the single-door criterion iff:

- $\mathbb{Z}$ contains no descendant of Y;
- $\mathbb{Z}$ d-separates X from Y in $\mathcal{G}_{\alpha_{xy}}$.

Single-door adjustment [11]

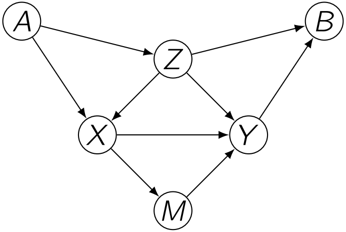
Theorem

If $\mathbb{Z}$ satisfies the single-door criterion relative to (X, Y), then the direct effect α_{xy} is identifiable and is given by

$$\alpha_{xy} = r_{XY.Z}$$

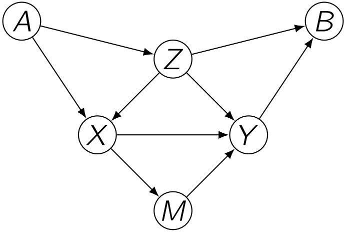
where $r_{XY.\mathbb{Z}}$ represents the residual correlation between Y and X after $\mathbb{Z}$ is "partialled out".

Single-door in action



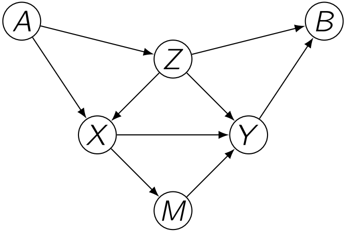
Does the following set satisfy the single-door criterion?

Single-door in action



Does the following set satisfy the single-door criterion?

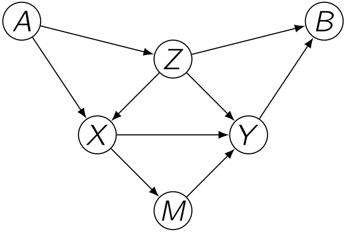
$$\{Z\} \text{ ?}$$



Does the following set satisfy the single-door criterion?

$\{Z\}$? No

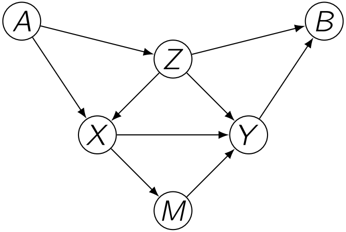
Single-door in action



Does the following set satisfy the single-door criterion?

$\{Z\}$? No
$\{A\}$?

Single-door in action



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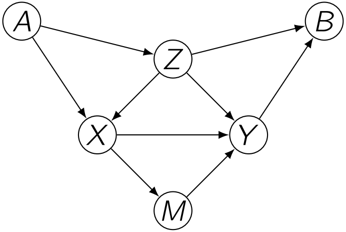
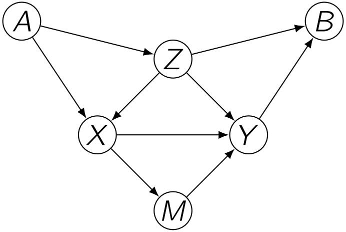
$\{Z\}$? No
$\{A\}$? No



Does the following set satisfy the single-door criterion?

$\{Z\}$? No
$\{A\}$? No
$\{B\}$?



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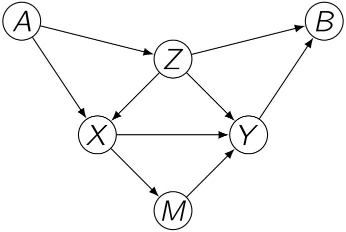
Single-door in action



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$\{M\}$?

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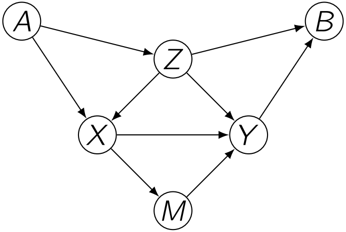

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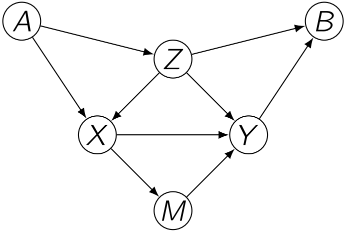

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$\{Z\}$? No

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$\{M\}$? No

$\{A, B\}$?

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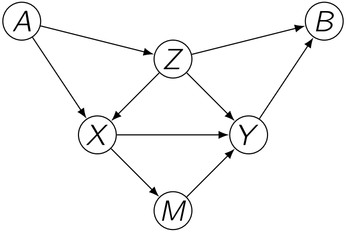

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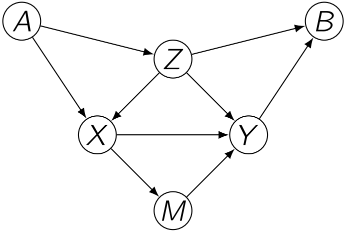

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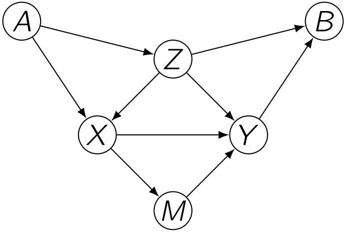

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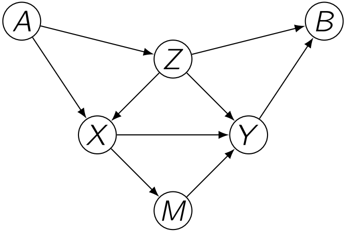

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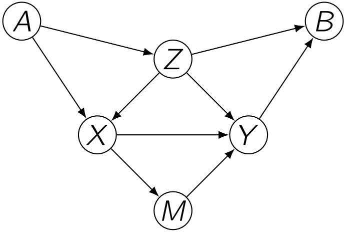
Single-door in action



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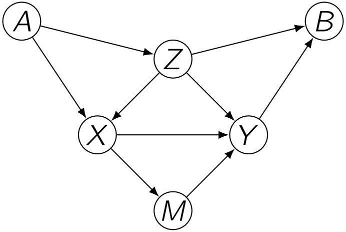

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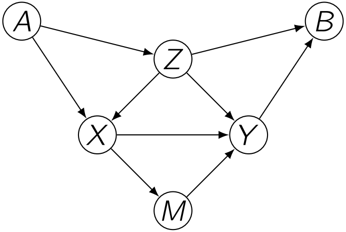
Single-door in action



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$\{Z, M, B\}$? No
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Indirect effect

Indirect effect = total effect − direct effect

It is possible to use the single-door criterion in a non-parametric model?

The single-door criterion in the non-parametric case

It is possible to use the single-door criterion in a non-parametric model? No!

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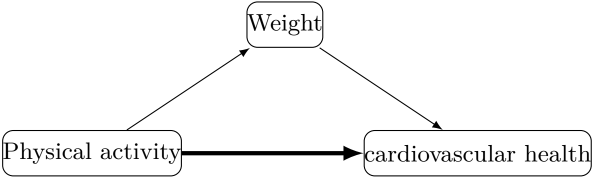
Example



$$Y = X \times M + \xi_y$$

The single-door criterion in the non-parametric case

It is possible to use the single-door criterion in a non-parametric model? No!

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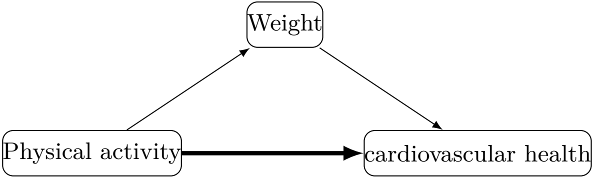

$$Y = X \times M + \xi_y$$

The question "What is the direct effect of X on Y?" in a non-parametric setting is not simple.

Natural direct effect (NDE)

Natural direct effect (NDE):

$$NDE(x, x'; Y) = \mathbb{E}(y \mid do(x, pa^{x'}_{Y \setminus X})) - \mathbb{E}(y \mid do(x', pa^{x'}_{Y \setminus X}))$$

where $pa^{x'}_{Y \setminus X}$ is the value of $Pa(Y) \setminus \{X\}$ when $X = x'$.

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Advantage:

- It allows for the complete decomposition of the total effect into the NDE and NIE

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Limitation:

- NDE cannot always be measured via experiments since it is a conterfactual expression

Identifiability of NDE from experiments [5]

Theorem

Let $\mathbb{M}$ stand for the set of all intermediate variables between X and Y. If there exists a set $\mathbb{Z}$ such that

- No member of $\mathbb{Z}$ is a descendant of X or $\mathbb{M}$
- $\mathbb{Z}$ blocks all back-door paths from $\mathbb{M}$ to Y not traversing X

then the natural average direct effect of X on Y is experimentally identifiable, and it is given by

$$NDE(x, x'; Y) = \sum_m \sum_z (\mathbb{E}(y \mid do(x, m), z) - \mathbb{E}(y \mid do(x', m), z)) \times \Pr(m, do(x'), z) \Pr(z)$$

Identifiability of NDE from observations [5]

Theorem

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- No member of $\mathbb{Z}$ is a descendant of of X or $\mathbb{M}$
- $\mathbb{Z}$ blocks all back-door paths from $\mathbb{M}$ to Y not traversing X
- $\Pr(m \mid do(x), z)$ is identifiable
- $\Pr(y \mid do(x, m), z)$ is identifiable

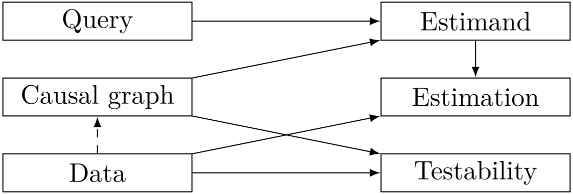
then the natural average direct effect of X on Y is identifiable, and it is given by

$$NDE(x, x'; Y) = \sum_m \sum_z (\mathbb{E}(y \mid do(x, m), z) - \mathbb{E}(y \mid do(x', m), z)) \\ \times \Pr(m, do(x'), z) \Pr(z)$$

3

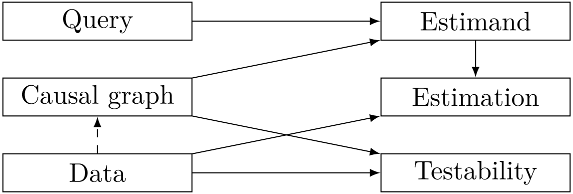
Counterfactuals

Structural causal models

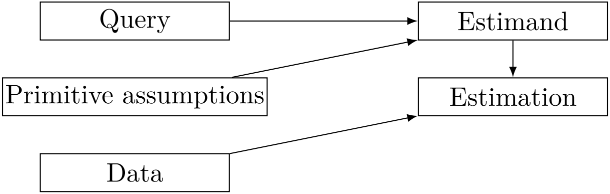


Structural causal models vs potential outcome

Structural causal models

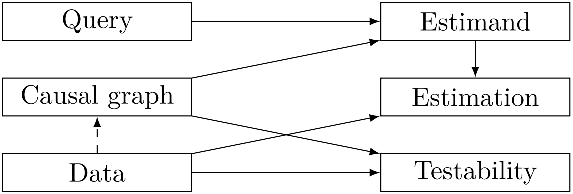


Potential outcome

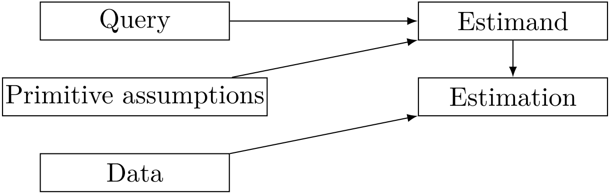


Structural causal models vs potential outcome

Structural causal models



Potential outcome



In the structural causal models framework, the primitive assumptions becomes theorems!

4

Conclusion

Counterfactual data

Experimental data

Observational data

SCMs
counterfactual reasoning
mediation analysis

Causal graphs
causal reasoning
causal discovery

Bayesian networks
prediction

5

Acknowledgements

Acknowledgements

- David Hajage for the template;
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- Simon Ferreira for proofreading

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6

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