Week 8

3.5(a) 18, 28, 34

18) Find $\frac{dy}{dx}$ if $\tan(x-y) = \frac{y}{1+x^2}$

$$\frac{d}{dx}\left(\tan(x-y)\right) = \frac{d}{dx}\left(\frac{y}{1+x^2}\right)$$

$$\sec^2(x-y)\cdot\left(1-\frac{dy}{dx}\right) = \frac{\frac{dy}{dx}(1+x^2) - 2xy}{(1+x^2)^2}$$

$$(1+x^2)^2\sec^2(x-y)\left(1-\frac{dy}{dx}\right) = \frac{dy}{dx}(1+x^2) - 2xy$$

$$(1+x^2)^2\sec^2(x-y) + 2xy = \frac{dy}{dx}\left((1+x^2) + (1+x^2)^2\sec^2(x-y)\right)$$

$$\frac{dy}{dx} = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2) + (1+x^2)^2\sec^2(x-y)}.$$

28) Find an equation of tangent line to $x^{2/3} + y^{2/3} = 4$ at $(-3\sqrt{3}, 1)$.

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\sqrt[3]{\frac{y}{x}}$$

At $(-3\sqrt{3}, 1)$, $\frac{dy}{dx} = -\sqrt[3]{\frac{1}{-3\sqrt{3}}} = \frac{1}{\sqrt{3}}$, so TL: $y-1 = \frac{1}{\sqrt{3}}(x+3\sqrt{3})$.

34) Find y'' by implicit diff:

$$\sqrt{x}+\sqrt{y}=1$$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2}=\frac{-\frac{1}{2\sqrt{y}}\frac{dy}{dx}+\frac{\sqrt{y}}{2\sqrt{x}}}{x}, \text{ so}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{-1}{2\sqrt{y}}\cdot\frac{-\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{y}}{2\sqrt{x}}}{x}=\frac{\sqrt{y}+1}{2\sqrt{x}\cdot x}$$

3.5(D) 42, 50, 52

42) Show that the sum of the x- and y-intercepts of any TL to $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is c.

Let's find the TL at a generic point on the curve:

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0, \text{ so } \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}\ldots$$

If $x=a$, then

$$\sqrt{a}+\sqrt{y}=\sqrt{c}, \text{ so}$$

$$\sqrt{y}=\sqrt{c}-\sqrt{a} \text{ and}$$

$$y=(\sqrt{c}-\sqrt{a})^2, \text{ so}$$

Points on the curve have the form

$(a, (\sqrt{c}-\sqrt{a})^2)$. The tangent line at such a point is

$$y-(\sqrt{c}-\sqrt{a})^2=-\frac{\sqrt{c}-\sqrt{a}}{\sqrt{a}}(x-a).$$ The x intercept is when $y=0$, so . . .

$$-(\sqrt{c}-\sqrt{a})^2 = -\frac{\sqrt{c}-\sqrt{a}}{\sqrt{a}}(x-a)$$

$\sqrt{a}(\sqrt{c}-\sqrt{a}) = x-a$, so

$x = a+\sqrt{a}(\sqrt{c}-\sqrt{a})$ $\triangleright$ x intercept.

y-intercept is when $x=0$, so

$$y-(\sqrt{c}-\sqrt{a})^2 = -\frac{\sqrt{c}-\sqrt{a}}{\sqrt{a}}(-a)$$

$y = (\sqrt{c}-\sqrt{a})^2 + \sqrt{a}(\sqrt{c}-\sqrt{a})$ is y-intercept.

Let's add them together and hope for the best:

$$x+y = a+\sqrt{ca}-a + c-2\sqrt{ac}+a+\sqrt{ac}-a$$
$$= c$$

Hooray!

50) Find derivative:

$$\frac{d}{dx}\left(\arctan\left(x-\sqrt{1+x^2}\right)\right) = \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2}\left(1-\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= \frac{1}{1+x^2-2\cdot x\sqrt{1+x^2}+1+x^2}\cdot\frac{\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}}$$

$$= \frac{\left(\sqrt{1+x^2}-x\right)}{\left(2+2x^2-2x\sqrt{1+x^2}\right)\left(\sqrt{1+x^2}\right)}$$

52)

$$\frac{d}{dx}\left(\arcsin\sqrt{\sin\theta}\right) = \frac{1}{\sqrt{1-\sin\theta}}\cdot\frac{\cos\theta}{2\sqrt{\sin\theta}}$$

3.6 : 18, 40, 50

18) Find $H'(z)$:

$$\frac{d}{dz}\left(\ln\left(\sqrt{\frac{a^2-z^2}{a^2+z^2}}\right)\right)=\frac{d}{dz}\left(\frac{1}{2}\left(\ln(a^2-z^2)-\ln(a^2+z^2)\right)\right)$$

$$=\frac{1}{2}\left(\frac{-2z}{a^2-z^2}-\frac{2z}{a^2+z^2}\right)=\frac{-z(a^2+z^2)-z(a^2-z^2)}{a^4-z^4}$$

$$=\frac{-2a^2z}{a^4-z^4}.$$

40) Find $\frac{dy}{dx}$ where $y=\sqrt[4]{\frac{x^2+1}{x^2-1}}$

$$\ln y=\frac{1}{4}\left(\ln(x^2+1)-\ln(x^2-1)\right)$$

$$\frac{dy}{dx}=\sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{4}\left(\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right).$$

50) Find $\frac{dy}{dx}$ if $x^y=y^x$.

$$y\ln x=x\ln y$$

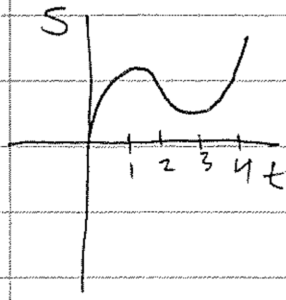
$$\frac{dy}{dx}\ln x+\frac{y}{x}=\ln y+\frac{x}{y}\frac{dy}{dx}.$$

$$\frac{dy}{dx}=\frac{\ln y-y/x}{\ln(x)-x/y}.$$

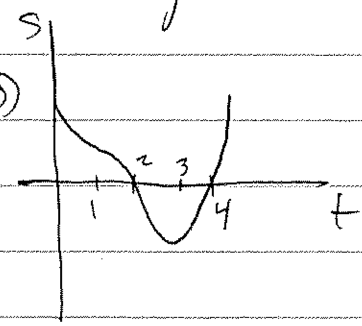
3.7 / 6, 12

(6) On what intervals are particles speeding up and slowing down?

(a)

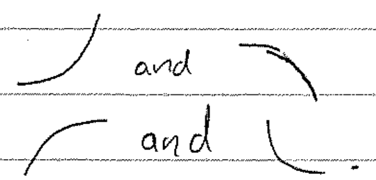


(b)



Speeding up

Slowing down



(a) $(0,1) \cup (2,3)$ Slowing down

$(1,2) \cup (3,4)$ speeding up

b) $(0,1) \cup (2,3)$ Slowing down

$(1,2) \cup (3,4)$ Speeding up.

12) (a) Certain crystals grow in a cube. If V is the volume of a cube of sidelength x, calculate $\frac{dV}{dx}$ when $x = 3mm$

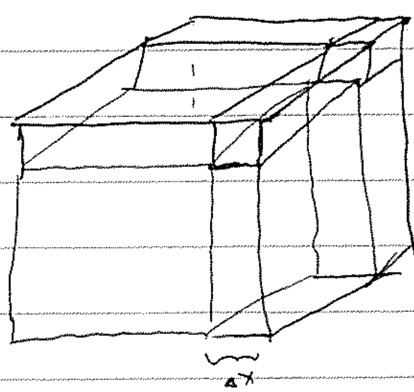
and explain its meaning.

$V = x^3$

$\frac{dV}{dx} = 3x^2$, so at $3mm$, $\frac{dV}{dx} = 27$

This means that as x is increased by a small amount, the volume of the cube increases by about 27 times that amount.

(b) Since the surface area of a cube is $6x^2$, we see that $\frac{dV}{dx}$ is half the surface area of the cube.



We can see that

$\Delta V = 3x^2\Delta x + 3(\Delta x)^2 x + (\Delta x)^3$.

So

$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2$

$= 3x^2$.