Week 4

1.7 | 6, 12, 26

6) Do the columns form a lin. ind set?

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix} \sim \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 4 & 4 & -20 \\ 4 & 2 & -20 \end{bmatrix} \sim \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 1 & -20 \\ 0 & -1 & -20 \end{bmatrix}$$

$$\sim \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & -15 \\ 0 & 0 & -25 \end{bmatrix}$$

Yes. There are 3 pivot columns, so there are no free variables in corresponding homog. linear system. Hence it has only the trivial solution.

12) For which h are the following vectors independent? (we need to find h so that there is a free variable in corresponding homog. system).

$$\begin{bmatrix} 3 & -6 & 9 \\ -6 & 4 & h \\ 1 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -8 & 18+h \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -8 & 18+h \\ 0 & 0 & -\frac{(18+h)}{8} \end{bmatrix}$$

$\downarrow$

The vectors are dependent iff

$$\frac{-(18+h)}{8} = 0 \quad \text{iff} \quad \underline{h = -18}$$

26] What are the possible echelon forms of 4x3 matrices $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ st

$\{\vec{a}_1, \vec{a}_2\}$ is independent and

$\vec{a}_3$ is not in the span of $\vec{a}_1$ and $\vec{a}_2$.

If $\vec{a}_3 \notin \operatorname{Span}\{\vec{a}_1, \vec{a}_2\}$, then $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is independent.

$\therefore$ only echelon form is $\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$

with $a, b, c \neq 0$.

1.8 10, 12, 36

10) Find all $\vec{x}$ that are mapped to $\vec{0}$ by $\vec{x} \mapsto A\vec{x}$ where

$$A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$$

In other words, find solution set of $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so need $x_1 = 4x_4 - 2x_3$ and $x_2 = -3x_4 - 2x_3$, so

Sol' set is $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \middle| \; x_3, x_4 \in \mathbb{R} \right\}$

$$= \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

12) Let $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$ and let A as in

the previous exercise. Is $\vec{b} \in \text{Range}(\vec{x} \mapsto A\vec{x})$?

If so, then $A\vec{x} = \vec{b}$ is consistent.

let's see!

$$\begin{bmatrix} 3 & 2 & 10 & -6 & -1 \\ 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 1 & 4 & 10 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 2 & 4 & 6 & -10 \\ 0 & 4 & 8 & 12 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 2 & 4 & 6 & -10 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

This is inconsistent, so no, $\vec{b}$ is not in the range.

36 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Suppose that $\{\vec{u}, \vec{v}\}$ is independent, but $\{T(\vec{u}), T(\vec{v})\}$ is dependent.

Show that $T(\vec{x}) = \vec{0}$ has a nontrivial solution.

There are constants c_1, c_2, not both zero such that

$$c_1 T(\vec{u}) + c_2 T(\vec{v}) = \vec{0}, \text{ so}$$

$$T(c_1 \vec{u} + c_2 \vec{v}) = \vec{0}.$$

Since $\{\vec{u}, \vec{v}\}$ is independent, $c_1 \vec{u} + c_2 \vec{v}$ is non-zero, so is a nontrivial sol'n to $T(\vec{x}) = \vec{0}$.