Week 12

4.4 | 18, 42, 52

18) $\lim_{x\to\infty} \frac{\ln(\ln x)}{x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}\cdot\frac{1}{\ln x}}{1} = 0$

42) $\lim_{x\to 0^+} \underset{0\cdot\infty}{\sin x \ln x} = \lim_{x\to 0^+} \frac{\ln x}{\csc x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x}$

$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} \underset{\frac{0}{0}}{=} \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = 0.$

52) $\lim_{x\to\infty} \underset{(\infty-\infty)}{(xe^{1/x} - x)}$.

$\left(\lim_{x\to\infty} xe^{1/x} = \lim_{x\to\infty} \frac{e^{1/x}}{1/x} = \lim_{x\to 0} \frac{e^x}{x} = \infty \right)$

$= \lim_{x\to\infty} \underset{\frac{0}{0}}{\frac{e^{1/x}-1}{1/x}} \underset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{-\frac{1}{x^2}e^{1/x}}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = 1.$

4.5] 16, 40, 48

16] **Sketch** $y = 1 + \frac{1}{x} + \frac{1}{x^2}$

Domain: $x \neq 0$. **Asymptotes:** $y = 1$ (Horiz), $x = 0$ (Vert).

Symmetry: None. **Intercepts:** No y intercept.

$$1 + \frac{1}{x} + \frac{1}{x^2} = 0$$

$$x^2 + x + 1 = 0$$

has no real solutions.

No x-intercept.

$$y' = \frac{-1}{x^2} - \frac{2}{x^3}$$

$x = 0$ is only point not in domain

$$\frac{-1}{x^2} - \frac{2}{x^3} = 0$$

$$-x - 2 = 0$$

$$x = -2$$

$$y'' = \frac{2}{x^3} + \frac{6}{x^4} = 0$$

$$2x + 6 = 0$$

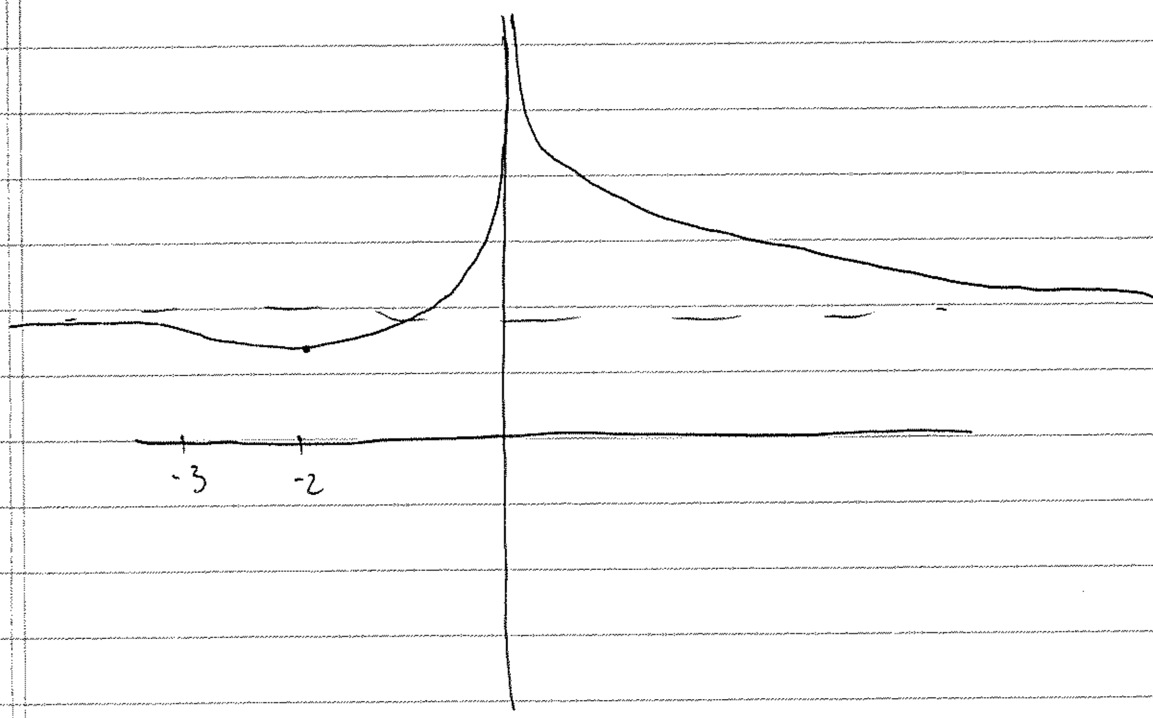
$$\boxed{x = -3}$$

		-2		0	
y'	$-$		$+$		$-$

-3		0	
$-$	$+$		$+$

$f(-2) = 1 - \frac{1}{2} + \frac{1}{4} = 3/4$ is local min.

$f(-3) = 7/9$ is inflection pt.



40 $y = e^{-x} \sin x \qquad 0 \le x \le 2\pi \leftarrow$ Domain

No asymptotes/symmetry. At $x=0$, $y=0$.

$e^{-x} \sin x = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi \leftarrow$ x-intercepts.

$$y' = -e^{-x}\sin x + e^{-x}\cos x = 0$$

$$-\sin x + \cos x = 0$$

$$\sin x = \cos x$$

$$\tan x = 1.$$

$$\boxed{x = \pi/4,\ 5\pi/4}$$

$$y'' = e^{-x}\sin x - e^{-x}\cos x - e^{-x}\cos x - e^{-x}\sin x$$

$$= -2e^{-x}\cos x = 0$$

$$\cos x = 0$$

$$\boxed{x = \pi/2,\ 3\pi/2}$$

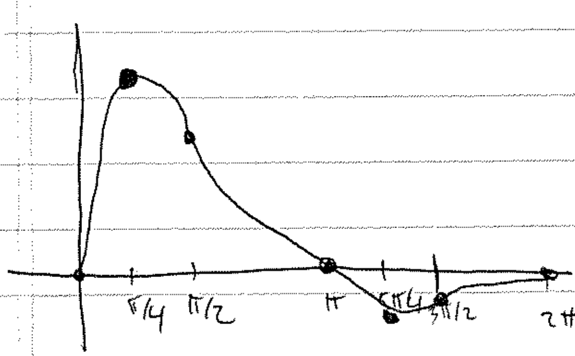
y'	+	−	+
		$\pi/2$	$3\pi/2$
	0 $\pi/4$	$5\pi/4$	2π
y''	−	+	−

$g(\pi/4) \approx 0.322$
local max

$g(5\pi/4) \approx -0.013$
local min.

$g(\pi/2) \approx 0.207$ inf.

$g(3\pi/2) \approx -0.008$ infl.



48] $y = \dfrac{\ln x}{x^2}$

Domain: $x > 0$

Asymptotes: Vert: $x = 0$

Horz: $y = 0$

(by L'Hôpital)

X-int: $x = 1$.

No symmetry.

$$y' = \frac{\frac{1}{x}\cdot x^2 - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{(1-2\ln x)}{x^3}$$

$y' = 0$ iff $1 - 2\ln x = 0$

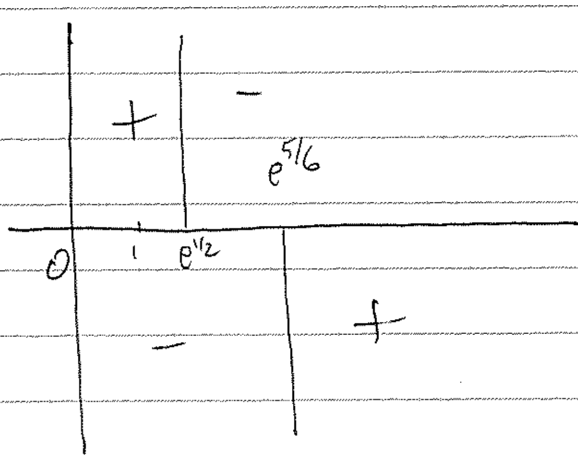
$\frac{1}{2} = \ln x$

$x = e^{1/2}$

$$y'' = \frac{-\frac{2}{x}x^3 - 3x^2(1-2\ln x)}{x^6} = \frac{-2-3+6\ln x}{x^4} = 0$$

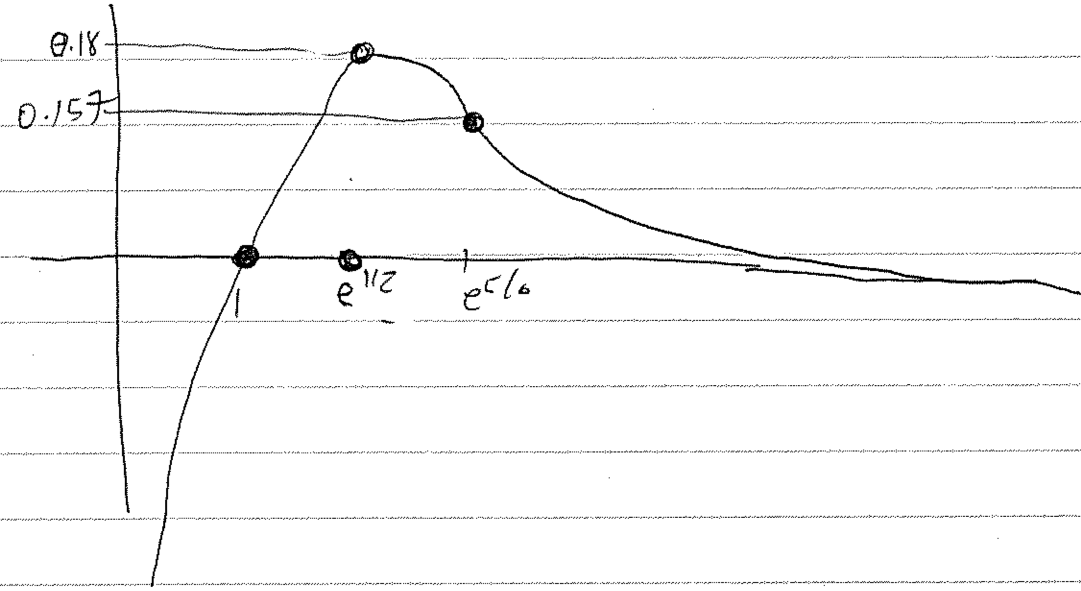
$\ln x = 5/6$

$\boxed{x = e^{5/6}}$



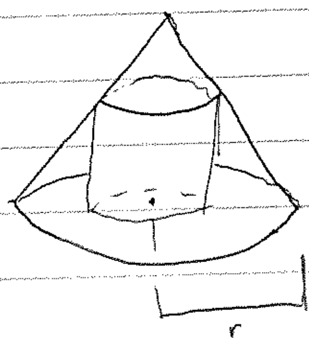
local max at $y(e^{1/2}) \approx 0.183\ldots$

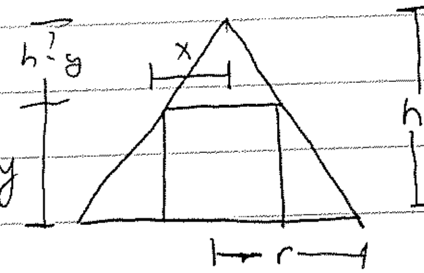
inf pt at $y(e^{5/6}) \approx 0.157$


0.18
0.157
1
$e^{1/2}$
$e^{5/6}$

4.7 | 28, 42

28 | A ~~right circular cone~~ cylinder is inscribed in a cone with height h and base radius r. Find largest possible volume of such a cylinder.





$$V(x,y) = \pi x^2 y \qquad \frac{x}{r} = \frac{h-y}{h}, \text{ so } y = h\left(1 - \frac{x}{r}\right)$$

$$0 \le x \le r \qquad V(x) = \pi x^2\left(h\left(1-\frac{x}{r}\right)\right) = \pi h\left(x^2 - \frac{x^3}{r}\right)$$

$$V'(x) = \pi h\left(2x - \frac{3x^2}{r}\right) = 0$$

$$x = \tfrac{2}{3} r.$$

$$V(0) = 0$$

$$V(r) = 0$$

$$V(2/3) = \frac{4\pi r^2 h}{27}$$

~~$V(0) = V(h) = 0$,~~

~~$V\left(\frac{2h}{3}\right) = \frac{\pi r^2}{h^2}\left(\frac{4}{9}h^2 - \frac{8}{27}h^3\right)^2$ is max volume.~~

42) For a fish swimming at speed v relative to the water, energy expenditure per unit time is proportional to v^3. Fish try to minimize total energy over a journey. If fish are swimming against a current u $(u<v)$, then time required to swim distance L is $L/(v-u)$ and energy required is

$$E(v) = av^3 \cdot \frac{L}{v-u} \quad \text{(for some constant a).}$$

(a) Find v mimizing E.

$$E'(v) = \frac{d}{dv}\frac{Lav^3}{v-u} = \frac{3Lav^2(v-u) - Lav^3}{(v-u)^2} = 0$$

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e[v_] := a v^3 L / (v - u)
a = 1; L = 1; u = 1;
Plot[e[v], {v, 0, 3}]
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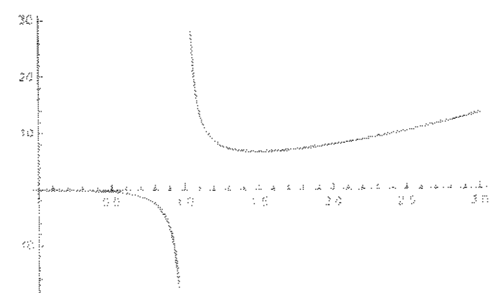

when $Lav^2(3(v-u) - v) = 0$

$2v - 3u = 0$

$v = \frac{3u}{2}$ ← That's it!