Week 8

3.3 | 6, 22, 30

6 | Use Cramer's rule to solve

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ -x_1 \quad + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 &= -2 \end{aligned}, \quad \text{so } A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix},$$

$$\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}.$$

$$|A| = 1(3-1) - 2(2-3) = 2 + 2 = 4$$

$$|A_1(\vec{b})| = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = -1(6+4) - 1(8-2) = -16,$$

$$|A_2(\vec{b})| = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = 2(6+4) - 4(-3-6) + (2-6)$$

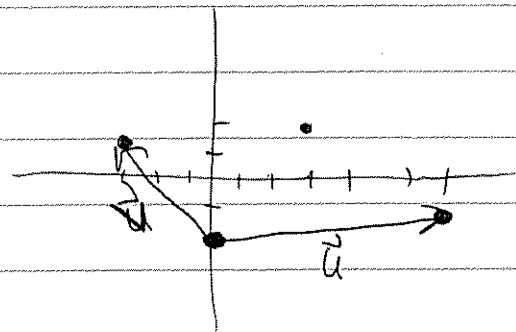
$$= 20 + 36 - 4 = 52$$

$$|A_3(\vec{b})| = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} = -(2-6) - (4+4) = 4 - 8 = -4.$$

Sol'n is $\vec{x} = \begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$

22] Find the area of the parallelogram with vertices

$(0,-2),\ (6,-1),\ (-3,1),\ (3,2)$



$$\vec{u} = (6,-1) - (0,-2) = (6,1)$$

$$\vec{v} = (-3,1) - (0,-2) = (-3,3).$$

$$\text{Area} = \left|\begin{matrix} 6 & -3 \\ 1 & 3 \end{matrix}\right| = 18 + 3 = 21.$$

30] Let R be the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). Note that this triangle has the same area as the triangle with vertices $(0,0)$, $(x_2 - x_1, y_2 - y_1)$, and $(x_3 - x_1, y_3 - y_1)$.

This triangle has half the area of the parallelogram spanned by

$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \text{ and } \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{bmatrix}, \quad \text{so is}$$

$$\frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2}\Big((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\Big)$$

$$= \frac{1}{2}\Big(x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 - (x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1)\Big).$$

Note also that

$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}\Big(x_2y_3 - x_3y_2 - (x_1y_3 - x_3y_1) + x_1y_2 - x_2y_1\Big).$$

These two expressions are the same,

so

$$\text{Area triangle} = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

Week 9

4.1 2, 12, 32
4.2 32

2) Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| xy \geq 0 \right\}$

(a) If $\vec{u} \in W$, $c \in \mathbb{R}$, is $c\vec{u} \in W$?

$c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$, so $(cx)(cy) = c^2 xy \geq 0$ since c^2 and $xy \geq 0$.

Yes!

(b) Find $\vec{u}, \vec{w} \in W$ so that $\vec{u} + \vec{w} \notin W$.

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $\vec{u} + \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $-1 \cdot 1 < 0$.

12) Let $W = \begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$. Note that $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix} \right\}$,

so is a subspace of $\mathbb{R}^4$.

32) Let H, K be subspaces of v.s. V. Show that $H \cap K$ is a subspace! Note that $H \cap K \subseteq V$.

(0) $\vec{0} \in H$ and $\vec{0} \in K$, so $\vec{0} \in H \cap K$.

(1) Suppose $\vec{u}, \vec{v} \in H \cap K$. Then $\vec{u}, \vec{v} \in H$ and $\vec{u}, \vec{v} \in K$, so $\vec{u} + \vec{v} \in H$ and $\vec{u} + \vec{v} \in K$. $\therefore$ $\vec{u} + \vec{v} \in H \cap K$.

(2) $\downarrow$

(2) If $\vec{u} \in H \cap K$, $c \in \mathbb{R}$, then since $\vec{u} \in H$, $c\vec{u} \in H$ and since $\vec{u} \in K$, $c\vec{u} \in K$, so $c\vec{u} \in H \cap K$.

Note that $H \cup K$ is not necessarily a subspace of V:

$$H = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \middle| \; x \in \mathbb{R} \right\}$$

$$K = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| \; y \in \mathbb{R} \right\}$$

gives an example.

4.2| 32| Let T be the linear transformation $T: \mathbb{P}_2 \to \mathbb{R}^2$ defined by

$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$. Find $p_1, p_2 \in \mathbb{P}_2$

Spanning $\operatorname{Ker}(T)$. Describe the range of T.

$\operatorname{Ker} T = \left\{ p \in \mathbb{P}_2 \middle| \begin{bmatrix} p(0) \\ p(0) \end{bmatrix} = 0 \right\}$. If $p(x) = ax^2 + bx + c$, then $p(0) = 0$ iff $c = 0$, so

$\operatorname{Ker} T = \{ p \in \mathbb{P}^2 \mid p(x) = ax^2 + bx \} = \operatorname{Span}\{ x^2, x \}$.

$\text{Range}(T) = \left\{ \begin{bmatrix} c \\ c \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$ since

$T(ax^2+bx+c) = c.$