Shape and size in hyperbolic space

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Motivating questions

Question

How does the topology (shape) of a hyperbolic space influence the geometry (size) of the space?

Question

What are the smallest (in volume) examples of hyperbolic spaces of various types?

We will see that hyperbolic geometry gives a sort of rigidity that makes these questions meaningful.

Warning!

In Euclidean geometry, there is not a meaningful connection between shape and size.

Euclidean spaces and objects can be scaled while preserving shape.

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What is the area of a triangle having all angles measuring $\pi/3$ *radians?*

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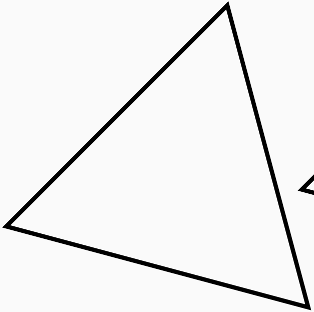
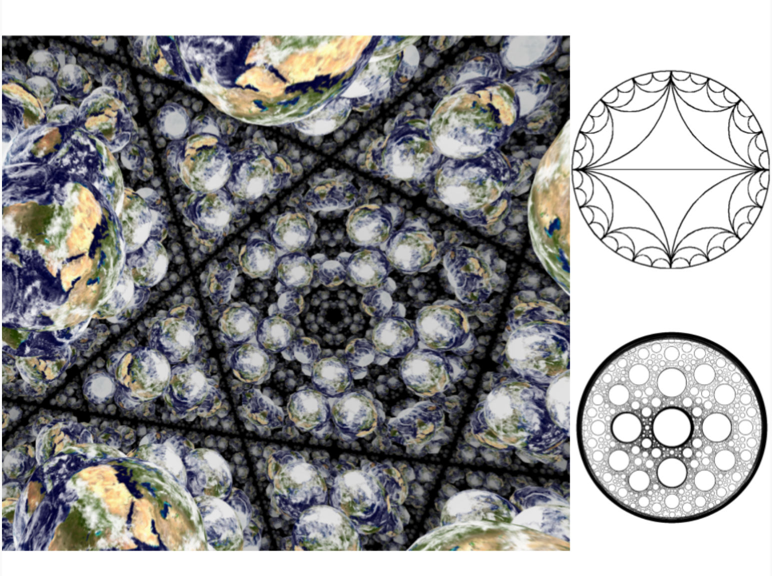


Basic Hyperbolic Geometry and Rigidity

Why study hyperbolic geometry?

It's beautiful!



Why study hyperbolic geometry?

In low-dimensional topology, Thurston discovered that hyperbolic geometry is ubiquitous (1970s).

In a certain sense, most geometric manifolds in 2 and 3 dimensions are hyperbolic. Those that are not hyperbolic are usually made up of hyperbolic pieces.

Understanding hyperbolic geometry informs understanding of 2- and 3-dimensional spaces (and more!).

The hyperbolic plane $\mathbb{H}^2$

The hyperbolic plane was discovered independently by Bolyai, Gauss, and Lobachevsky in 1820-30s. Its existence proved that the parallel postulate of Euclid was not a consequence of Euclid's other four axioms for Euclidean geometry.

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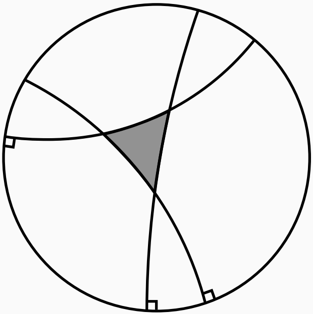
We use models of the hyperbolic plane to understand and reason about it.

Poincaré Disk model of $\mathbb{H}^2$

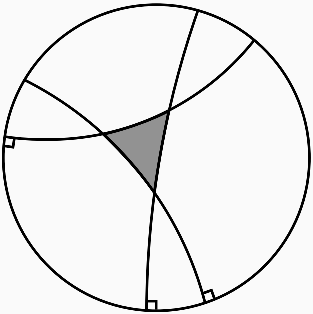
- $\mathbb{H}^2$ consists of the points of the open unit disk
- $\mathbb{H}^2$ has geodesic lines consisting of arcs of circles that meet the boundary at right angles



Poincaré Disk model of $\mathbb{H}^2$

- Model is conformal: the angles you see are the actual hyperbolic angles
- Distances are very distorted. To measure the arc length L of a path parameterized by $\mathbf{r}(t)$ for $0 \leq t \leq 1$, one computes

$$L = \int_0^1 \frac{2|\mathbf{r}'(t)|}{1 - |\mathbf{r}(t)|^2}\, dt$$



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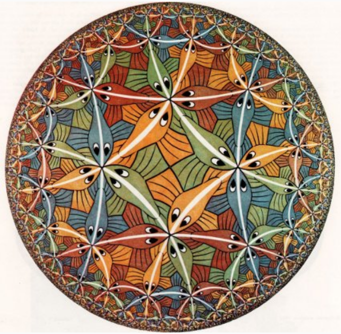


M.C. Escher's *Circle Limit III*

Shape $\Rightarrow$ size in $\mathbb{H}^2$

A first rigidity theorem:

Theorem (Lambert, 18th century)

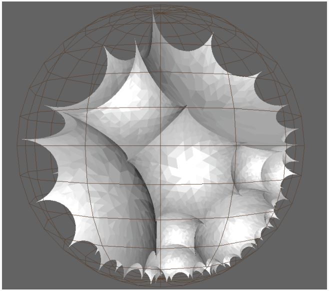
The area of a triangle in $\mathbb{H}^2$ with angles α, β, and γ is $\pi - (\alpha + \beta + \gamma)$.

One can prove that two hyperbolic triangles with the same angles are congruent (but the same is not true for polygons with more than three vertices).

Hyperbolic 3-space

Hyperbolic 3-space $\mathbb{H}^3$ is the 3-dimensional analogue of $\mathbb{H}^2$.

- $\mathbb{H}^3$ consists of the points of the open unit ball in $\mathbb{R}^3$
- $\mathbb{H}^3$ has geodesic lines consisting of arcs of circles that meet the boundary sphere orthogonally



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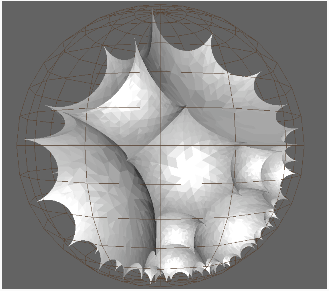
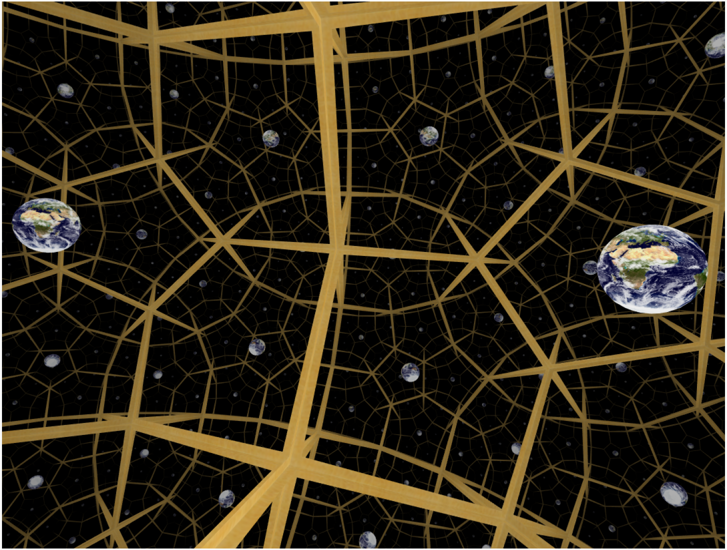
Hyperbolic 3-space $\mathbb{H}^2$ is the 3-dimensional analogue of $\mathbb{H}^2$.

- $\mathbb{H}^3$ has geodesic lines consisting of arcs of circles that meet the boundary sphere orthogonally
- Geodesic planes in $\mathbb{H}^3$ are portions of spheres that meet the bounding sphere orthogonally.



Hyperbolic 3-space

A tiling of $\mathbb{H}^3$ by right-angled dodecahedra



Shape $\Rightarrow$ size in $\mathbb{H}^3$

A second rigidity theorem: hyperbolic polyhedra are uniquely determined by their shape.

Theorem (Andreev 1970s, Rivin 1990s)
A hyperbolic polyhedron is uniquely determined (up to rigid motions) by its 1*-skeleton and labeling by dihedral angles.*

Both Andreev and Rivin's theorems give combinatorial conditions on the 1-skeleton and dihedral angles for hyperbolicity.

Shape $\Rightarrow$ size in $\mathbb{H}^3$

Question

What can we say about the geometry of a polyhedron in terms of its combinatorics?

Question

The volume of a polyhedron is a function of its combinatorics. What can we say about this function?

Shape $\Rightarrow$ size in $\mathbb{H}^3$

Exploring this question was the main goal of my thesis. Gave algorithmically computable upper and lower bounds on the volume of non-obtuse polyhedra. An example theorem:

Theorem (A 2009)

If $\mathcal{P}$ is a hyperbolic polyhedron with all right angles and N vertices, then

$$\frac{N-8}{32} \cdot V_8 \leq \mathrm{Vol}(\mathcal{P}) < \frac{5(N-10)}{8} \cdot V_3.$$

$(V_3 \approx 1.01494,\ V_8 = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 3.66386)$

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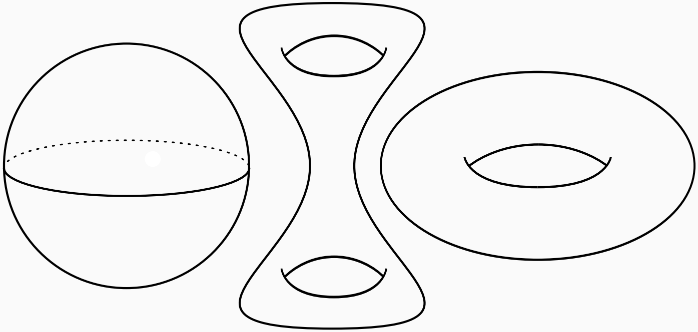
Proof of this theorem involves computations in hyperbolic geometry, tools from the geometric topology of orbifolds, and an application of the four-color map theorem.

Manifolds and orbifolds

Manifolds

An *n*-dimensional manifold is a space in which each point is contained in a neighborhood of points that "looks like" $\mathbb{R}^n$.

Some examples of 2–manifolds (also known as surfaces):



(The 2-sphere S^2, the genus 2 surface S_2, and the 2-torus T^2)

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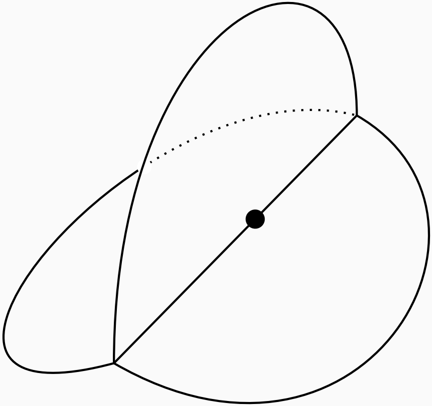
A point whose neighborhood does not look like $\mathbb{R}^2$.



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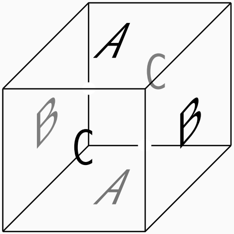

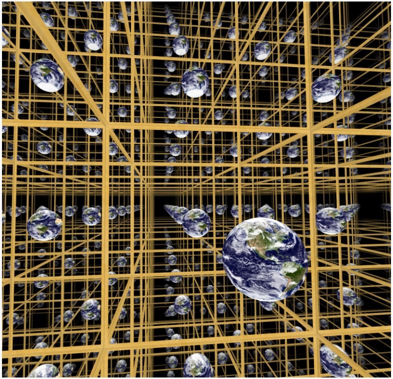

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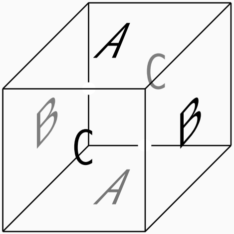
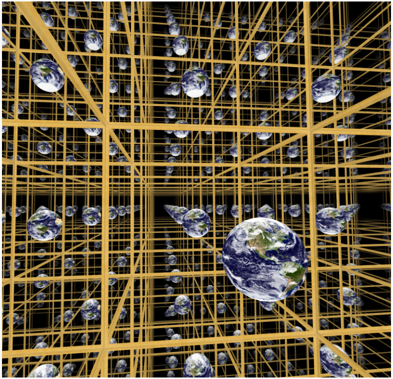




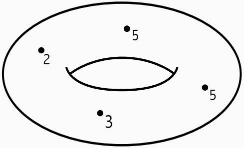
Note that different images of Earth differ by translations

Orbifolds

An orbifold is a generalization of the idea of manifold that allows for singular points. Points in an n–orbifold are contained in neighborhoods that either look like $\mathbb{R}^n$ or look like $\mathbb{R}^n$ modulo the action of a finite group.

2-**Orbifolds**

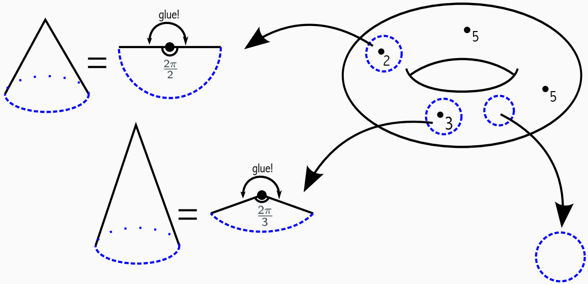
A 2–orbifold $\mathcal{O}$ can be specified by a pair $(X_{\mathcal{O}}, \Sigma_{\mathcal{O}})$ where $X_{\mathcal{O}}$ (the base space) is a 2-manifold and $\Sigma_{\mathcal{O}}$ (the singular locus) is a finite collection of points in $X_{\mathcal{O}}$ labeled by integers ≥ 2.



2-Orbifolds

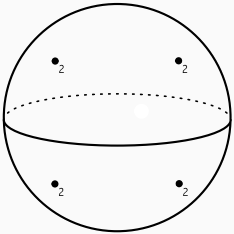
The points of $\Sigma_{\mathcal{O}}$ are suggestively called cone points.

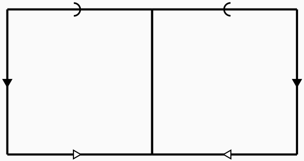
The cone points describe the local modeling conditions.

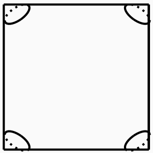


2-**Orbifolds**

This orbifold is called a pillowcase. We can see that it has a Euclidean structure. If one changes the labels on the cone points, we'll end up with a hyperbolic orbifold.







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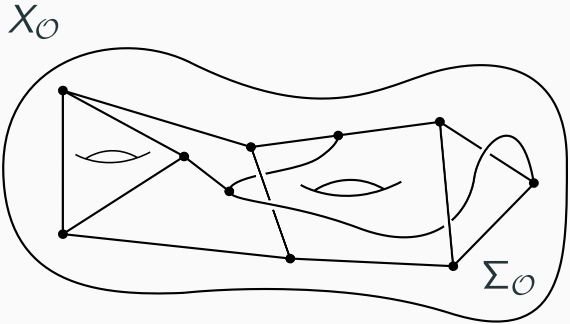
A 3–orbifold $\mathcal{O}$ can be specified by a pair $(X_{\mathcal{O}}, \Sigma_{\mathcal{O}})$ where $X_{\mathcal{O}}$ (the base space) is a 3-manifold and $\Sigma_{\mathcal{O}}$ (the singular locus) is an embedded trivalent graph with edges labeled by integers ≥ 2.



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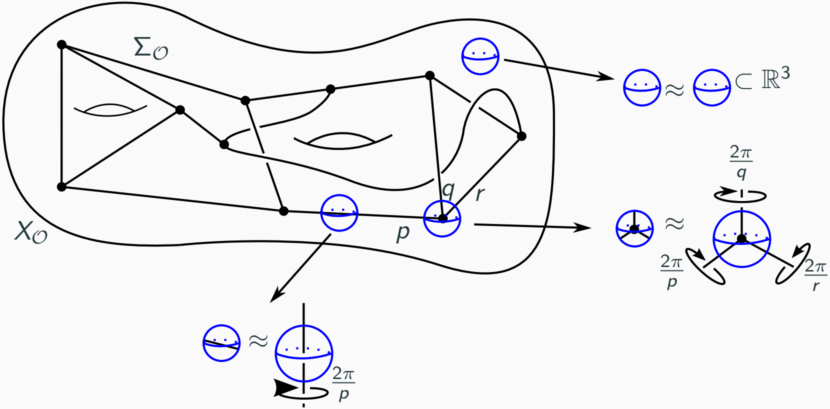
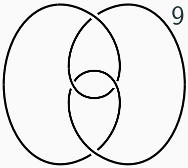


$(p, q, r) \in \{(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)\}$

What does this look like?

This is the orbifold $\mathtt{m004}(9,0)$ with base space S^3 and singular locus as displayed. It is a hyperbolic orbifold. This means that it can be obtained as $\mathbb{H}^3/\Gamma$ where Γ is a discrete group of isometries or rigid motions of $\mathbb{H}^3$.



Along an edge of the singular locus, there is a "hall of mirrors" effect.

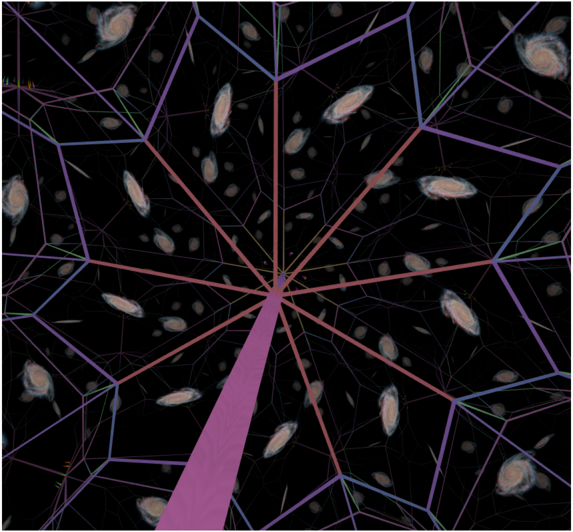


Figure 1: Looking along an edge labeled 9 in the hyperbolic structure on `m004(9, 0)`.

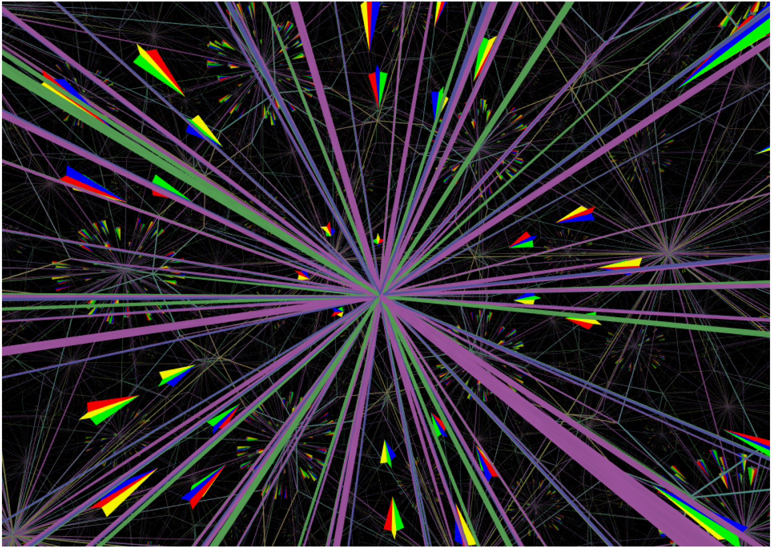


Figure 2: Looking towards a $(2, 3, 5)$ vertex in a doubled hyperbolic tetrahedron.

Mostow-Prasad rigidity

A theorem of Mostow and Prasad (1970s) says that hyperbolic orbifolds are rigid in dimensions at least 3. Their theorem implies the following:

Theorem

If $\mathcal{O}$ is a hyperbolic 3-orbifold, then the function

$$(X_{\mathcal{O}}, \Sigma_{\mathcal{O}}) \mapsto \mathrm{Vol}(\mathcal{O})$$

is well defined.

This means that it makes sense to try to understand the volume of a 3-orbifold in terms of the base space and singular locus.

Dunbar-Meyerhoff on volume structure

Dunbar and Meyerhoff (1990s) proved a theorem that describes the structure of the set of all volumes of hyperbolic orbifolds.

Theorem

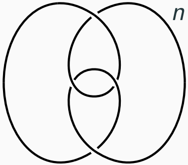
The volumes of hyperbolic 3*-orbifolds forms a closed, non-discrete, well-ordered subset of* $\mathbb{R}^+$*.*

Consequence: if one considers a family of hyperbolic orbifolds satisfying some condition, there is a smallest volume member of that family.

Small volume link orbifolds

Theorem (A-Futer, 2017)

For all $n \geq 4$, the unique lowest–volume orbifold with all torsion orders at least n, with base space S^3 and singular locus the figure–8 knot labeled n.



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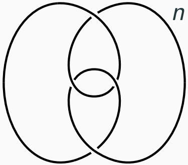

We also proved volume bounds in the $n = 3$ case and have a conjectured volume minimizer. The $n = 2$ case is the question of identifying the smallest hyperbolic orbifold with no conditions. Identified by Gehring-Marshall-Martin (2009, 2012).

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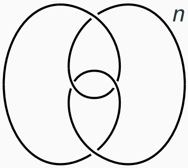

Proof was computer assisted and discovered via computer experimentation with the program SnapPy (Culler-Dunfield-Goerner-Weeks).

Ongoing and future work

- Small volume hyperbolic orbifolds (Joint with Rafalski): Large-scale project to identify low-volume orbifolds of certain types. Builds on some previous work with undergraduates.
- More on volumes of hyperbolic polyhedra: What are the possibilities for the limiting volume-per-vertex ratios for various families of hyperbolic polyhedra? How does volume change as polyhedra undergo degenerations and regenerations (i.e. changing combinatorics)? Numerical experimentation by undergraduates would be an interesting project.

Ongoing and future work

- Finishing $n = 3$ case of A-Futer: Proving that our conjectured volume minimizer is correct.
- Modernization of Orb: Rewrite portions of computer program Orb (Damien Heard) to fully integrate it with SnapPy. Will allow for more sofisticated experimentation with hyperbolic orbifolds. Student help would be appreciated!

Thank you!