Review sheet for final exam

As stated in the syllabus, the final exam is comprehensive. Listed below are the sections and topics that I consider to be fair game for the final. I've concatenated the previous review sheets and deleted topics that I will not ask you about.

The best way to study for this is to do many exercises. Start with the shortlist posted on the course webpage. See the course webpage for a long list of suggested problems. Here's one way to tackle this: Find a topic that seems confusing or difficult. Do enough problems to feel confident about the topic. Repeat. Also, when doing problems from "early" sections, think about your whole toolkit. Frequently there are easier ways to do problems if you use more advanced tools.

You may use a cheat sheet for this exam. **Your sheet must be written by hand by you**. No photocopies or printouts are allowed. There are two options for your cheat sheet:

1. You may use one side of standard sheet of 8.5in $\times$11in sheet of paper. The only thing that may be written on the other side of the sheet is your name. I'll collect your sheets with your exams.
2. You may use the three half–sheets that you used for previous exams (again with nothing written on the back). I'll collect your sheets with your exams. It's worth pointing out that the act of making the cheat sheet is a useful exercise on its own, so you should consider using option (1).

- 1.1: Systems of linear equations
  - What is a system of linear equations? What is a solution? What is the solution set? When are two systems equivalent?
  - What's the difference between the coefficient matrix of a system and an augmented matrix for the system?
  - Can you solve some systems? Can you do so using the elementary row operations? What does it mean for augmented matrices to be row equivalent?
- 1.2: Row reduction and echelon form
  - What is echelon form? What is reduced echelon form? Why isn't echelon form unique? Is reduced echelon form unique?
  - How do you use row reduction and reduced echelon form to solve a system? You should know how to systematically do this, but I won't ask you about the details of the formal algorithm for doing so.
  - What is a pivot position? Pivot column?

  - What are leading variables? Free variables? How do you use these to describe solution sets?
  - You should know what Theorem 2 means and how to use it to detect inconsistent systems.

- 1.3: Vector equations
  - What are vectors in $\mathbb{R}^n$? How does vector addition and scalar multiplication work?
  - What is a linear combination of vectors? What is the span of a set of vectors? What does it mean to say that a set of vectors spans $\mathbb{R}^m$?
  - How are vector equations and linear systems related? What does this have to do with span and linear combinations? You should be able to translate seamlessly between these interpretations to answer questions about vectors and linear systems.

- 1.4: The matrix equation
  - What does a matrix equation of the form $A\mathbf{x} = \mathbf{b}$ mean? Be able to translate seamlessly between systems of linear equations, vector equations, and matrix equations (this is essentially Theorem 3).
  - You should have a good feeling for why Theorem 4 is true. A useful exercise would be to prove the theorem. Doing so will likely cement the connections in your head.

- 1.5: Solution sets of linear systems
  - What is a homogeneous linear system? What is a trivial/nontrivial solution? How can you tell whether or not a homogeneous system has a nontrivial solution?
  - You should be able to describe the solution sets to nonhomogeneous systems. Can you identify the $\mathbf{p}$ and $\mathbf{v_h}$ parts? What does this even mean? (See Theorem 6 to see what I'm talking about here)

- 1.7: Linear independence
  - What is the definition of linear independence/dependence? Work through some examples. Note the connection with homogeneous systems discussed on page 57.
  - Build up some intuition for linear independence (Theorems 7, 8, 9).

- 1.8: Linear transformations

- What is a transformation? What makes a transformation a linear transformation? What's a matrix transformation? Is it linear?
- The properties discussed on page 66 are good to know.

• 1.9: The matrix of a linear transformation

  - Is every linear transformation represented by a matrix? See Theorem 10. How do you find such a matrix?
  - What does it mean for a transformation to be one-to-one? Onto? How can you tell if a linear transformation is one-to-one or onto? (Theorems 11 and 12).
  - Intuitively speaking, what do one-to-one and onto mean?

• 2.1: Matrix operations

  - When and how can you multiply two matrices? How do you do it? How does it correspond to composition of linear transformations?
  - What are the basic properties of matrix multiplication? (See Theorem 2)
  - Also, see the WARNINGS on page 98.
  - What is the transpose of a matrix? What are its properties? (Theorem 3)

• 2.2: Inverse matrices

  - What does it mean to say that a matrix is invertible? What about non-singular?
  - How can you find the inverse of a matrix (see algorithm on page 108)? What if the matrix is 2×2? What is the determinant of a 2×2 matrix?
  - How can you solve systems of linear equations using inverse matrices?
  - What are some of the basic properties of taking inverses? (See Theorem 6)

• 2.3: Characterizations of invertible matrices

  - How can you tell if a matrix is invertible (see Theorem 8 and the theorem on page 235)? **A useful exercise is to see how many equivalent conditions you can think of for a matrix to be invertible.**
  - What does it mean for a linear transformation to be invertible?
  - If you know the standard matrix for a linear transformation, how can you find the standard matrix for the inverse transformation?

• 3: Determinants

- Can you compute determinants? How can you compute the determinant of AB if you know the determinants of A and B? What about the determinant of A^{-1}?

- 4.1: Vector spaces and subspaces
  - What is a vector space? Can you name some examples? What are some sets that aren't vector spaces? See if you can come up with examples that fail for various reasons.
  - Know the basic properties ((1), (2), and (3) on page 191).
  - What is a subspace? Remember that it's easiest to check that something is a vector space by showing that it's a subspace of a known vector space. Play with some examples.
  - What is a linear combination of vectors? What is the span of a set of vectors? Remember that the span of a collection of vectors in a vector space is a subspace.
- 4.2: Null spaces, column spaces, and linear transformations
  - What is the null space of a matrix? If A is $m \times n$, then how does the null space relate to systems of equations? What is it a subspace of? Can you find the null space in examples?
  - What is the column space of a matrix? How does it relate to systems of equations? What is it a subspace of? Can you compute it in examples?
  - What are the kernel and range of a linear transformation? If you conclude that they're analogous to the null space and column space of a matrix, you're thinking the right thoughts.
- 4.3: Linearly independent sets; bases
  - What is a linearly independent set of vectors in a vector space? How can you detect independence (See Theorem 4)?
  - What is a basis for a vector space? What is the standard basis for $\mathbb{R}^n$? What is the standard basis for $\mathbb{P}^n$?
  - Think about the spanning set theorem (Theorem 5). Keep thinking about it until it is obvious to you.
  - How can you compute bases for the null and column spaces of a matrix?
- 4.4: Coordinate systems
  - Know Theorem 7.

- How do you compute the coordinates of a vector with respect to a basis? Try some examples. Think about what it means graphically. **What if the basis is orthogonal?**
- What is the coordinate mapping? How is it defined? What are it's properties? Think about this until you're convinced that all finite-dimensional vector spaces really look like $\mathbb{R}^n$ for some n.
- What is an isomorphism?

- 4.5: The dimension of a vector space
  - What is the dimension of a vector space? What does Theorem 9 say about dimension and collections of vectors?
  - Know the basis theorem (Theorem 12).
  - What is the dimension of the null space of a matrix? What about the column space?
- 4.6: Rank
  - What is the row space of a matrix? How do row operations affect the row space?
  - What is the rank of a matrix?
  - What does the Rank (-Nullity) Theorem say? Think about the intuitive picture we talked about in class (the author talks about this in Example 4).
  - See the Theorem on page 235. You can extend the characterizations of invertible matrices using the concepts introduced in chapter 4. I think it should be clear to you why these things are true.
- 4.7: Change of basis
  - How is the change of coordinates matrix defined? Can you compute it for examples in $\mathbb{R}^n$? Try a handful of examples.
- 5.1: Eigenvalues and eigenvectors
  - What is an eigenvector? What is an eigenvalue?
  - How can you find the eigenvalues of a triangular matrix?
  - What can you say about eigenvectors corresponding to distinct eigenvalues with respect to dependence?
- 5.2: The characteristic equation

- What is the characteristic equation? What is the characteristic polynomial?
- How can you use the characteristic equation to compute eigenvalues?
- What does it mean for two matrices to be similar?

- 5.3: Diagonalization
  - What does it mean for a matrix to be diagonalizable?
  - When can a matrix be diagonalized? How do you do it?
- 5.4: Eigenvectors and linear transformations
  - How can you compute the matrix for a linear transformation relative to two bases?
- 5.5: Complex eigenvalues
  - Can you do basic complex arithmetic?
  - What do complex eigenvalues tell you about a matrix? See Theorem 9.