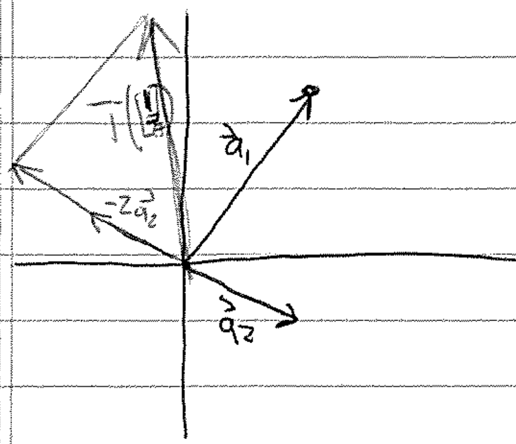
1.9 | 14, 18, 28

14) $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear w/ std. matrix $A = [\vec{a}_1\ \vec{a}_2]$



Draw $T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right)$.

Since $\vec{a}_i = T(\vec{e}_i)$,

$T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right) = \vec{a}_1 - 2\vec{a}_2$

18) Find std. matrix of

$T(x_1, x_2) = (x_1 + 4x_2,\ 0,\ x_1 - 3x_2,\ x_1)$

$T: \mathbb{R}^2 \to \mathbb{R}^4$.

$T(1,0) = (1, 0, 1, 1)$

$T(0,1) = (4, 0, -3, 0)$, so

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix}.$$

28) Is the transformation in 14 (a) 1-1?
(b) onto?

(a) By Thm 12, T is one-to-one iff $\{\vec{a}_1, \vec{a}_2\}$ are linearly independent. They clearly are (not scalar multiples), So yes. ~~They~~ T is one-to-one.

(b) By Thm 12, T is onto iff $\{\vec{a}_1, \vec{a}_2\}$ span $\mathbb{R}^2$. Is it the case that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$? Yes! The reason is that since $\{\vec{a}_1, \vec{a}_2\}$ are L.I, then system $A\vec{x} = \vec{0}$ has only trivial solution. Since A is 2x2, there is a pivot position in each row of A, so $A\vec{x} = \vec{b}$ is always consistent.

Week 6

2.1 | 6, 22, 26

6| Compute product in two ways as specified.

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}.$$

(a) $$AB = \left[\begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$$

(b) $$AB = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}.$$

22| Show that if columns of B are dependent, then so are those of AB.

Suppose $B = [\vec{b}_1 \cdots \vec{b}_n]$ and $\sum_{i=1}^{n} c_i b_i = \vec{0}$

is a dependence (so at least one $c_i \neq 0$).

AB, by definition, is $[A\vec{b}_1 \cdots A\vec{b}_n]$.

Note that using the same c_i's,

$$\sum_{i=1}^{n} c_i A\vec{b}_i = \sum_{i=1}^{n} A(c_i\vec{b}_i) = A\sum_{i=1}^{n} c_i\vec{b}_i$$
$$= A\cdot\vec{0} = \vec{0},$$

so $\{A\vec{b}_i\}$ are dependent.

26) Suppose $AD = I_m$. Show that for any $\vec{b}\in\mathbb{R}^m$, $A\vec{x}=\vec{b}$ has a solution.

Consider $\vec{x} = D\vec{b}$. Then

$A(D\vec{b}) = I\vec{b} = \vec{b}$, so $\vec{x}$ is a solution.

Why can't A have more rows than columns?

Since $AD = I_m$, A has m rows.

~~and D has m columns. Since $A(D\vec{b})$ is defined, the number of columns of A equals the number of rows of $D\vec{b}$.~~

I was overthinking this!

If A had fewe than m columns, the system $A\vec{x}=\vec{b}$ would have fewer than m pivots. In matrix form, A would de row equivalent to a matrix with a row of all zeros. This would contadict the fact that $A\vec{x}=b$ is always consistent.

2.2: 14, 20, 34

14) Suppose $(B-C)D=0$ where B, C are $m\times n$ and D is invertible. Show $B=C$.

$$(B-C)D=0$$
$$(B-C)D\cdot D^{-1}=0\cdot D^{-1}$$
$$(B-C)I=0$$
$$B-C=0$$
$$B=C.$$

20) Suppose A, B, X are $n\times n$, with $A, X, A-AX$ invertible. Suppose

$$(A-AX)^{-1}=X^{-1}B.$$

(a) Why is B invertible?

$B=X(A-AX)^{-1}$. X is invertible and

$(A-AX)^{-1}$ is invertible since $A-AX$ is.

Since B is the product of invertible matrices, it's invertible too

↓

(b) Solve

$$(A-AX)^{-1} = X^{-1}B \quad \text{for } X.$$

Invert both sides (all of $(A-AX)^{-1}$, X^{-1}, B are invertible)

$$A - AX = B^{-1}X$$

$$A = B^{-1}X + AX$$

$$A = (B^{-1}+A)X.$$ Since A and X are invertible, $B^{-1}+A$ must be too!

$$\underline{X = (B^{-1}+A)^{-1}A}.$$

34 Try some simple cases and look for a pattern to guess the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 3 & 3 & \cdots & 0 \\ \vdots & & & & \\ n & n & n & \cdots & n \end{bmatrix}.$$

↓

$$\left[A_2 \; I_2\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/2 \end{array}\right]$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1/2 \end{bmatrix}$$

$$\left[A_3 \; I_3\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 3 & 3 & -3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 3 & 0 & -3/2 & 1 \end{array}\right], \text{ so } B_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/3 \end{bmatrix}.$$

Is $B_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 0 & -1/2 & 1/3 & 0 \\ 0 & 0 & -1/3 & 1/4 \end{bmatrix}$?

Let's see!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 0 & -1/2 & 1/3 & 0 \\ 0 & 0 & -1/3 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $A^{-1} = B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1/2 & 0 & \cdots & 0 \\ 0 & -1/2 & 1/3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots -1/(n-1) & 1/n \end{bmatrix}.$