

Review sheet for Final Exam

What follows is a slightly shortened version of the previous two review sheets with new material at the end. The final is comprehensive, so I couldn't justify removing most items from the list. You should note that I have removed a few of the less commonly used topics from the list.

Studying for a final is an opportunity to synthesize what we've learned. You should think about the various topics as being small pieces of a larger puzzle. Ideally, you should enter the exam with all of the puzzle pieces connected into a nice picture.

While studying, you should work on examples. Can you create an example that illustrates a theorem? Can you construct an example that illustrates how a theorem fails if all the hypotheses are not satisfied? Look through the exercises for problems that illustrate the ideas and work out the details. If you're stuck on something, feel free to ask me for some help.

These are **not** sample exam questions, but to be prepared, you should know the material enough to answer them. A useful exercise is to construct an example for as many of the questions as you can where doing so makes sense.

For this exam, you may use one side of a standard $8.5'' \times 11''$ sheet of paper as a cheat sheet.

1.1: Introduction

- What is a mathematical knot? When are two knots of the same knot type?
- What is a projection or diagram of a knot? What is a crossing in a diagram?
- Can you prove that knots with one or two crossing diagrams are trivial?
- What is the unknot recognition problem?

1.2: Composition

- How do you compose two knots? Is the composition unique? Can you construct an example showing that it's (potentially) not unique?
- What are the factor knots of a composite knot?
- What happens if you compose a knot with the unknot?
- What is a prime knot? Is the unknot prime?

1.3: Reidemeister moves

- What is an isotopy? What's the difference between an ambient and planar isotopy?
- What are the three types of Reidemeister moves (knowing this means that you can draw the six diagrams)?
- Can you actually perform some Reidemeister moves on an example diagram?
- What is the point of the Reidemeister moves? What does Reidemeister's theorem say?

1.4: Links and the linking number

- What is a link? Can you sketch a few examples?
- When is a link splittable?
- What is a knot or link invariant?
- What is the definition of the linking number of two components of a link? Can you compute it for some examples? Why is it an invariant? What happens if you change the orientation on a component?

1.5: Tricolorability

- What does it mean to say that a knot or link is tricolorable?
- Try tricoloring a bunch of knots. For example, which of the knots up through 5 crossings are tricolorable? 6 crossings?
- Why is tricolorability an invariant?

2.3: Tangles

- What is a tangle?
- What is an integral tangle? Can you draw them? What is a rational tangle? Can you draw them?
- Can you draw a tangle that is not rational?
- How can you tell if two rational tangles are equivalent? Try some examples.

- Do you know how to multiply and add tangles to get algebraic tangles? Try drawing some.
- How do you get a knot or link from a rational or algebraic tangle? Given two knots or links presented as rational knots, how can you tell if they are the same or not?

2.4: Knots and planar graphs

- Can you prove that knot and link diagrams are checkerboard colorable using smoothings of crossings?
- How do you generate the Tait graph from a knot or link diagram?
- Given a Tait graph, can you reconstruct the knot or link that it encodes?

3.2: Bridge number

- What is an overpass? A maximal overpass? What is the bridge number of a knot?
- What do you know about knots with bridge number 1?
- What do you know about knots with bridge number 2?

3.3: Crossing number

- What is the definition of the crossing number of a knot? Why is it an invariant?
- Can you compute the crossing number of the knot 5_2 (without using the fact that it has a “5” in its name)?
- Can you do the previous point without appealing to the Kauffman-Murasugi-Thistlethwaite theorem?
- What do you know about the crossing number of a composite knot?

4.1: Surfaces without boundary

- What is a surface without boundary? Do you know some examples? Can you build them by gluing up polygons?
- What is a triangulation of a surface? When are two surfaces homeomorphic?

- What is the Euler characteristic of a surface? How do you compute it? Does it depend on the triangulation? Is using a triangulation even the most efficient way to do it? Compute some examples.
- What is the connect sum of two surfaces? How is Euler characteristic affected by connect sum?
- What is the genus of a surface? How are genus and Euler characteristic related?

4.2: Surfaces with boundary

- What is a surface with boundary? Do you know some examples?
- What is the Euler characteristic of a surface with boundary? Compute some examples.
- What does it mean for a surface to be orientable?
- How can we determine if two surfaces are homeomorphic using orientation, Euler characteristic, and number of boundary components?
- What does the classification of surfaces say? Given a surface, can you identify it using the classification theorem? Try some examples.

4.3: Genus and Seifert surfaces

- What is the genus of a knot?
- How does Seifert's algorithm work? Try some examples. Compute the Euler characteristic of some Seifert surfaces. Compute their genus (and note that this might not be the genus of the knot; the genus could be less).
- Which knots have genus 0?
- Do you know a family of projections for which Seifert's algorithm always yields a minimum genus surface?
- How does genus behave under composition? Can you use this prove that the unknot is not a composite knot?

6.1: Jones/Bracket polynomials

- What is a Laurent polynomial?
- What is the definition of the bracket polynomial? Can you compute it? Try some examples. Why isn't the bracket polynomial a link invariant?
- How is the writhe of a projection computed?
- What's the definition of the X polynomial? Can you compute it?
- What's the definition of the Jones polynomial?
- How can you use the X polynomial to check for amphicheirality? What does amphicheirality even mean?

6.2: State sum

- What is an A -split? B -split? What is a state of a link projection?
- What is the state sum formulation of the bracket polynomial?
- Know that the number of crossings in a reduced, alternating projection of a link is an invariant of the link. What is this invariant? What does reduced mean?
- How can you use a checkerboard coloring to detect chirality of a knot?

5.1: Torus knots

- Thurston tells us that there are three non-overlapping types of knots. What are the three types?
- What is a torus knot? What is a meridian? A longitude?
- Can you draw a diagram for a (p, q) torus knot?
- Why are (p, q) and (q, p) the same?

5.2: Satellite knots

- What is a satellite knot? A companion knot?
- Draw some examples.
- Why is a composite knot a satellite knot? Illustrate.

5.3: Hyperbolic knots

- Qualitatively speaking, describe \mathbb{H}^3 . What are the geodesics? How does length behave as you approach $\partial_\infty \mathbb{H}^3$?
- What does it mean for a knot to be hyperbolic? Do you know an example of a hyperbolic knot (we talked about one for nearly a week!)?
- Describe what the volume of a hyperbolic knot is. I don't expect you to be able to compute it, but think that you should at least have an intuitive understanding.

5.4: Braids

- What is a braid? When are two braids equivalent? Can two braids with a different number of strands be equivalent?
- What is the closure of a braid?
- How can you describe braids using *words*? See figure 5.41 for the “letters” that can be used to form words.
- What does Markov's theorem say?

8.1: Links in graphs

- What is a graph? When are two graphs isomorphic? When are two realizations of graphs isotopic?
- What is the complete graph K_n ?
- What is a planar graph?
- Can you name a graph that is intrinsically linked?

8.3: Graph polynomials

- What is the definition of the dichromatic polynomial of a graph?
- What is a vertex q -coloring of a graph?
- What is the connection between the dichromatic polynomial and the number of q -colorings of a graph?