Week 7

2.3 | 20, 34, 38

20 | If A is 5×5 and $A\vec{x}=\vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^5$, is it possible that for some $\vec{b}$, $A\vec{x}=\vec{b}$ has more than one sol'n?

No! If $A\vec{x}=\vec{b}$ is always consistent, then the columns of A span $\mathbb{R}^5$. Since A is 5×5, IMT $\Rightarrow$ $x\mapsto A\vec{x}$ is one-to-one. This implies that $Ax=b$ has ≤ 1 sol'n.

34 | Let $T:\mathbb{R}^2\to\mathbb{R}^2$

$$T(x_1,x_2)=(2x_1-8x_2,\ -2x_1+7x_2).$$

T has standard matrix $A=\begin{bmatrix}2 & -8\\ -2 & 7\end{bmatrix}$.

A is invertible (since $A^{-1}=\frac{1}{-2}\begin{bmatrix}7 & 8\\ 2 & 2\end{bmatrix}$), so T is also invertible (Thm 9).

$$T^{-1}(x_1,x_2)=\left(-\tfrac{1}{2}(7x_1+8x_2),\ -\tfrac{1}{2}(2x_1+2x_2)\right).$$

$\downarrow$

38) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Let S and U be functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose

$$S(T(\vec{x})) = \vec{x} \text{ and } U(T(\vec{x})) = \vec{x} \text{ for}$$

all $\vec{x} \in \mathbb{R}^n$. Show $S(\vec{v}) = U(\vec{v})$ $\forall \vec{v} \in \mathbb{R}^n$.

By IMT, T is onto, so for any $\vec{v} \in \mathbb{R}^n$, $\exists\, \vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{v}$. Then

$$S(\vec{v}) = S(T(\vec{x})) = \vec{x} \quad \text{and}$$
$$U(\vec{v}) = U(T(\vec{x})) = \vec{x} \text{ , so}$$

$$S(\vec{v}) = U(\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^n$$

3.1 14, 38, 42

14 Compute:

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 3\left(3\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right) = 9(-8-3) + 6(2+16)$$

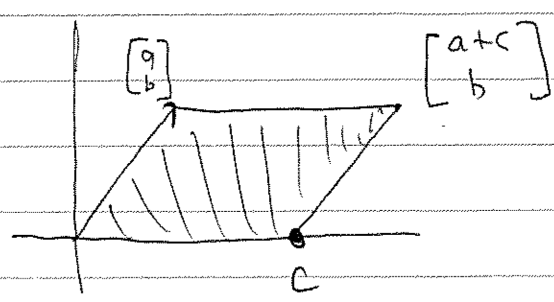
$$= -99 + 108 = 9.$$

38 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $k \in \mathbb{R}$.

$$|kA| = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 ad - k^2 bc = k^2 |A|.$$

42 Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, $a, b, c > 0$.

What is area of parallelogram spanned by $\vec{u}$ and $\vec{v}$?



has area bc.

↓

$$\left|\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right| = \begin{vmatrix} a & c \\ b & 0 \end{vmatrix} = -bc$$

$$\left|\begin{bmatrix}\vec{v} & \vec{u}\end{bmatrix}\right| = \left|\begin{bmatrix} c & a \\ 0 & b \end{bmatrix}\right| = bc.$$

It looks like $|\cdot|$ gives signed area!

3.2 | 26, 34, 42

26) Use determinants to decide if the vectors

$$\begin{bmatrix}3\\5\\-6\\4\end{bmatrix}, \begin{bmatrix}2\\-6\\0\\7\end{bmatrix}, \begin{bmatrix}-2\\-1\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-3\end{bmatrix}$$ are linearly independent.

If det is nonzero, then vectors are LI.

$$\begin{vmatrix}3 & 2 & -2 & 0\\5 & -6 & -1 & 0\\-6 & 0 & 3 & 0\\4 & 7 & 0 & -3\end{vmatrix} = -3\begin{vmatrix}3 & 2 & -2\\5 & -6 & -1\\-6 & 0 & 3\end{vmatrix} = -3\left(-6\left(-2-12\right)+3\left(-18-10\right)\right)$$

$$= 18(-14) - 9(-28)$$

$$= 9(-28) + 9(28) = 0.$$

The vectors are dependent.

34) Let A, P be square with P invertible.

$$\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \frac{\det(P)\det(A)}{\det(P)}$$

$$= \det(A).$$

42) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show

$\det(A+B) = \det(A) + \det(B)$ iff $a+d=0$.

$$\det(A+B) = \begin{vmatrix} a+1 & b \\ c & d+1 \end{vmatrix} = (a+1)(d+1) - bc.$$

$$\det(A) + \det(B) = 1 + ad - bc.$$

$(a+1)(d+1) - bc = 1 + ad - bc$ iff

$ad + a + d + 1 - bc = 1 + ad - bc$ iff

$a + d = 0.$

In[243]:=
```
ma = {{a, b}, {c, d}};
mx = {{u, v}, {x, y}};
MatrixForm[ma]
MatrixForm[mx]
Simplify[Det[ma + mx] == Det[ma] + Det[mx]]
```

Out[245]//MatrixForm=

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Out[246]//MatrixForm=

$$\begin{pmatrix} u & v \\ x & y \end{pmatrix}$$

Out[247]= $d\,u + a\,y == c\,v + b\,x$

So, the 2x2 matrices above have additive determinants if and only if du+ay = cv + bx.