Week 10

$\boxed{4.3}$ 16, 32, 34

16) Find a basis for the set spanned by

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}.$$

By thm 6, if we form column matrix, then the pivot columns form a basis.

$$\begin{bmatrix}1 & -2 & 3 & 5 & 2\\0 & 0 & -1 & -3 & -1\\0 & 0 & 1 & 3 & 1\\1 & 2 & -1 & -4 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3 & 5 & 2\\0 & 0 & -1 & -3 & -1\\0 & 0 & 1 & 3 & 1\\0 & 4 & -4 & -9 & -2\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & -2 & 3 & 5 & 2\\0 & 4 & -4 & -9 & -2\\0 & 0 & 1 & 3 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$, so first 3 vectors

form a basis:

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix} \right\}.$$

32 Suppose T is one-to-one. Show that if $\{T(\vec{v}_1), \dots, T(\vec{v}_p)\}$ is dependant, then so is $\{\vec{v}_1, \dots, \vec{v}_p\}$.

If $\{T(\vec{v}_1), \dots T(\vec{v}_p)\}$ is dependent, then there exist $c_1, \dots, c_p$, not all zero, such that

$$\sum_{i=1}^{p} c_i T(\vec{v}_i) = \vec{0}$$ since T is linear,

$$\sum_{i=1}^{p} T(c_i v_i) = T\left(\sum_{i=1}^{p} c_i v_i\right) = \vec{0}.$$

Since T is one to one,

$$\sum_{i=1}^{p} c_i v_i = \vec{0},$$ so the v_i are also dependent.

$\underline{34)}$ Let $p_1(t) = 1+t$, $p_2(t) = 1-t$ and $p_3(t) = 2$.

Note that $p_1(t) + p_2(t) - p_3(t) = 0$.

$\mathcal{B} = \{p_1(t), p_2(t)\}$ is a basis for Span$\{p_1, p_2, p_3\}$ since it is clearly independent (If $c_1 p_1 + c_2 p_2 = 0$, then $c_1 + c_2 + c_1 - c_2 = 0$, so $c_1 = 0$ and $c_2 = 0$.).

4.4) 8, 22, 32

8) Find $[x]_B$ relative to the basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 8 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ so}$$

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

22) Let $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$.

Find an $n \times n$ matrix that implements the coordinate mapping $x \mapsto [x]_B$.

Let $P_B = [\vec{b}_1 \cdots \vec{b}_n]$. We know that $P_B[\vec{x}]_B = \vec{x}$. Hence $P_B^{-1}\vec{x} = [x]_B$, so

P_B^{-1} is the matrix we want.

32) Let $p_1(t) = 1+t^2$, $p_2(t) = t-3t^2$ and $p_3(t) = 1+t-3t^2$.

(a) Use coord vectors to see that these form a basis for $\mathbb{P}_2$.

Let $\mathcal{B} = \{1, t, t^2\}$. Then

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \text{ and}$$

$$[p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$ Since $[\mathcal{B}]_{\mathcal{B}}$ is an isomorphism, the p_i form a basis iff the coord vectors do.

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} = 1(-3+3) + 1(-1) \neq 0, \text{ so}$$

The p_i form a basis.

(b) Find $q(t) \in \mathbb{P}_2$ st. $[q]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

$q(t) = -(1+t^2) + t-3t^2 + 2+2t-6t^2 = 1+3t-10t^2.$