CS 97SI: INTRODUCTION TO PROGRAMMING CONTESTS

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Today's Lecture

- Stack/Queue
- Heap and Priority Queue
- Union-Find Structure
- Binary Search Tree (BST)
- Binary Indexed Tree (BIT)
- Lowest Common Ancestor (LCA)

Typical Quarter at Stanford

```
void quarter() {
    while(true) { // no break
        task x = GetNextTask(tasks);
        process(x);
        // new tasks may enter
    }
}
```

- `GetNextTask()` decides the order of the tasks

`GetNextTask()`

- Possible behaviors of `GetNextTask()`:
  - Returns the newest task (stack)
  - Returns the oldest task (queue)
  - Returns the most urgent task (priority queue)
  - Returns the easiest task (priority queue)
- We want `GetNextTask()` to run quickly
  - … by organizing the tasks in a clever way

Stack

- Last in, first out (LIFO)
- Supports three constant-time operations
  - `Push(x)`: inserts `x` into the stack
  - `Pop()`: removes the newest item
  - `Top()`: returns the newest item
- Very easy implementation using arrays

Stack Implementation

- Have a large enough array `s[]` and a counter `k`, which starts at zero
  - `Push(x)`: set `s[k] = x` and increment `k` by 1
  - `Pop()`: decrement `k` by 1
  - `Top()`: returns `s[k - 1]` (error if `k` is zero)
- C++ and Java have implementations of stack
  - `stack` (C++), `Stack` (Java)
- But you should be able to implement it from scratch

Queue

- First in, first out (FIFO)
- Supports three constant-time operations
  - `Enqueue(x)`: inserts `x` into the queue
  - `Dequeue()`: removes the oldest item
  - `Front()`: returns the oldest item

- Implementation is similar to that of stack
  - Just slightly trickier

Queue Implementation

- In many cases, you know the total number of elements that enter a queue
  - … which allows you to use an array for implementation
- Maintain two indices `head` and `tail`
  - `Dequeue()` increments `head`
  - `Enqueue()` increments `tail`
  - Use the value of `tail - head` to check emptiness
- You can use `queue` (C++) and `Queue` (Java)

Priority Queue

- Each element in a PQ has a priority value
- Three operations:
  - `Insert(x, p)`: inserts `x` into the PQ, whose priority is `p`
  - `RemoveTop()`: removes the element with the highest priority
  - `Top()`: returns the element with the highest priority
- All operations can be done quickly if implemented using a heap
- `priority_queue` (C++), `PriorityQueue` (Java)

Heap

- Complete binary tree with the heap property:
  - The value of a node $\geq$ values of its children
- The root node has the maximum value
  - Constant-time `top()` operation
- Inserting/removing a node can be done in $O(\log n)$ time without breaking the heap property
  - May need rearrangement of some nodes

Heap Example

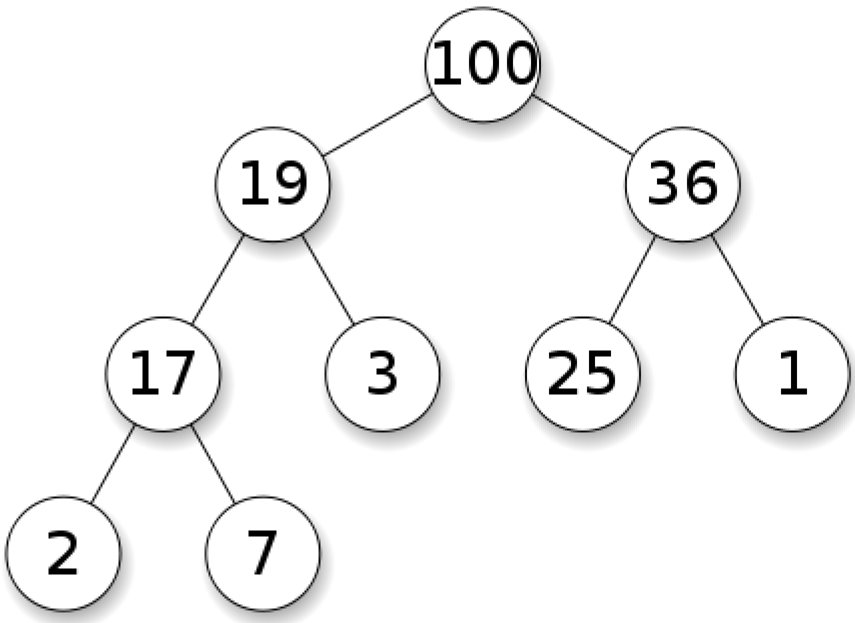
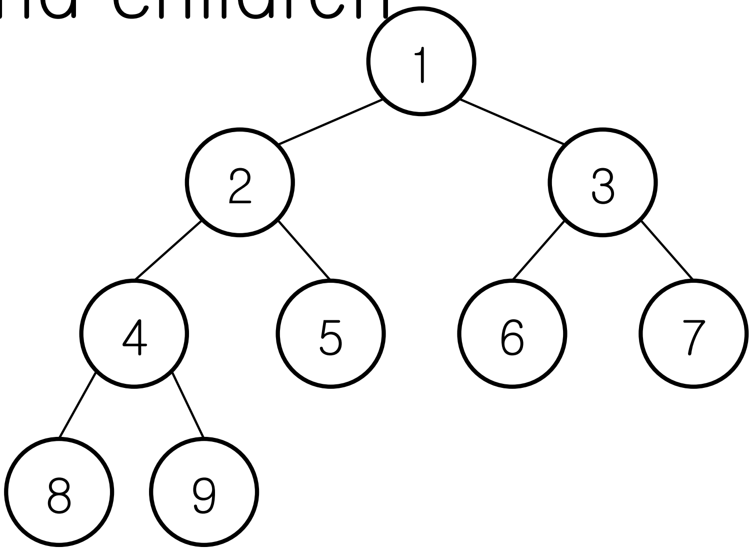


Figure from Wikipedia

Numbering the Nodes

- Start from the root, number the nodes 1, 2, ⋯ from left to right
- Given a node k, easy to compute the indices of its parent and children
  - Parent node: $\lfloor k/2 \rfloor$
  - Children: $2k$, $2k+1$



Inserting a Node

- 1. Make a new node in the last level, as far left as possible
  - If the last level is full, make a new one
- 2. If the new node breaks the heap property, swap with its parent node
  - The new node moves up the tree, which may introduce another conflict
- 3. Repeat 2 until all conflicts are resolved
- Running time = tree height = $O(\log n)$

Implementation: Node Insertion

- Inserting a new node with value v into a heap H

```
void InsertNode(int v) {
      H[++n] = v;
      for(int k = n; k > 1; k /= 2) {
            if(H[k] > H[k / 2])
                  swap(H[k], H[k / 2]);
            else break;
      }
}
```

Deleting the Root Node

- 1. Remove the root, and bring the last node (rightmost node in the last level) to the root
- 2. If the root breaks the heap property, look at its children and swap it with the larger one
  - Swapping can introduce another conflict
- 3. Repeat 2 until all conflicts are resolved
- Running time = $O(\log n)$
- Exercise: implementation
  - Just a few edge cases to consider

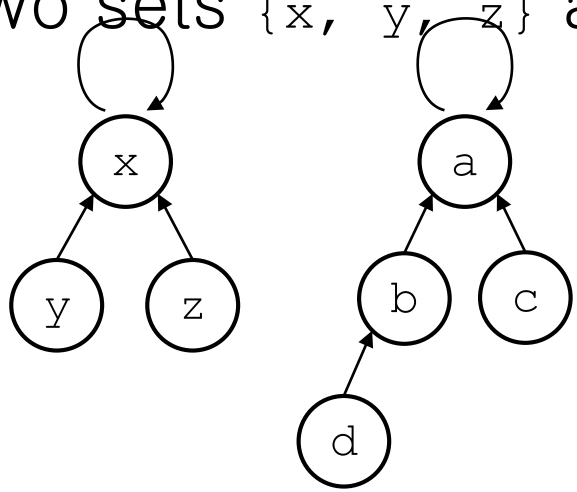
Union-Find Structure

- Can support two types of operations efficiently
  - `Find(x)`: returns the "representative" of the set that `x` belongs
  - `Union(x, y)`: merges two sets that contain `x` and `y`

- Both operations can be done in (essentially) constant time
- Super-short implementation!

Union-Find Structure

- Main idea: represent each set by a rooted tree
  - Every node maintains a link to its parent
  - A root node is the "representative" of the corresponding set
  - Example: two sets `{x, y, z}` and `{a, b, c, d}`



Implementation Idea

- `Find(x)`: follow the links from `x` until a node points itself
  - This can take $O(n)$ time but we will make it faster

- `Union(x, y)`: run `Find(x)` and `Find(y)` to find corresponding root nodes and direct one to the other

Implementation

- We will assume that the links are stored in `L[]`

```
int Find(int x) {
     while(x != L[x]) x = L[x];
     return x;
}
void Union(int x, int y) {
     L[Find(x)] = Find(y);
}
```

Path Compression

- In a bad case, the trees can become too deep
  - Which slows down the operations
- Path compression makes the trees shallower every time `Find()` is called
- We don't care how a tree looks like as long as the root stays the same
  - After `Find(x)` returns the root, backtrack to `x` and reroute all the links to the root

Path Compression Implementations

```
int Find(int x) {
  if(x == L[x]) return x;
  int root = Find(L[x]);
  L[x] = root;
  return root;
}

int Find(int x) {
  return x == L[x] ? x : L[x] = Find(L[x]);
}
```

Binary Search Tree (BST)

- A binary tree with the following property: for each node v,
  - value of $v \geq$ values in v's left subtree
  - value of $v \leq$ values in v's right subtree

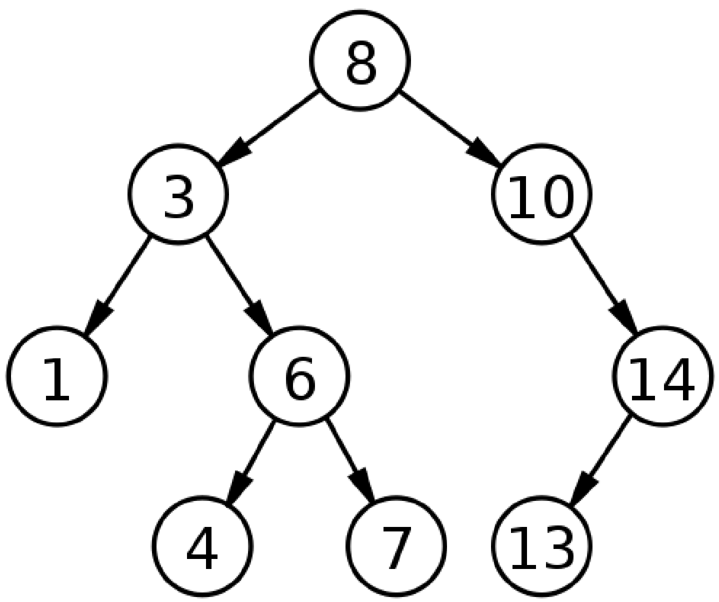


Figure from Wikipedia

What BSTs can do

- Supports three operations
  - `Insert(x)`: inserts a node with value `x`
  - `Delete(x)`: deletes a node with value `x`, if there is any
  - `Find(x)`: returns the node with value `x`, if there is any
- Many extensions are possible
  - `Count(x)`: counts the number of nodes with value less than or equal to `x`
  - `GetNext(x)`: returns the smallest node with value ≥ `x`

BSTs in Programming Contests

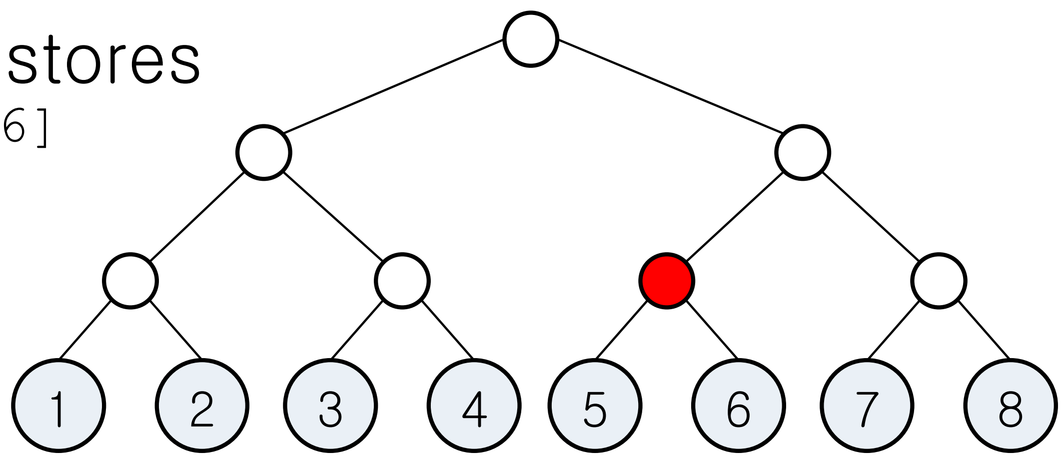
- Simple implementation cannot guarantee efficiency
  - In worst case, tree height becomes n (which makes BST useless)
  - Guaranteeing $O(\log n)$ running time per operation requires balancing of the tree (hard to implement)
  - We will skip the implementation details
- Use the standard library implementations
  - `set`, `map` (C++)
  - `TreeSet`, `TreeMap` (Java)

Binary Indexed Tree (BIT)

- A variant of segment trees
- Supports very useful interval operations
  - `Set(k, x)`: sets the value of `k`th item equal to `x`
  - `Sum(k)`: computes the sum of items `1…k`
    - Note: sum of items `i…j` = `Sum(j) - Sum(i - 1)`
- Both operations can be done in $O(\log n)$ time using $O(n)$ space

BIT Structure

- Full binary tree with at least n leaf nodes
  - We will use $n = 8$ for our example
- kth leaf node stores the value of item k
- Each internal node stores the sum of values of its children
  - e.g. Red node stores `item[5] + item[6]`



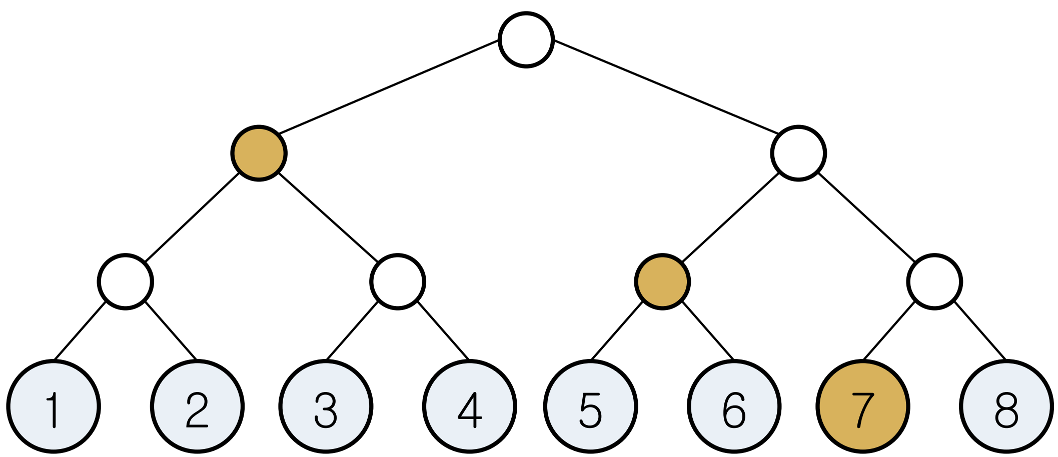
Summing Consecutive Values

- Main idea: choose the minimal set of nodes whose sum gives the desired value
- We will see that
  - at most 1 node is chosen at each level so that the total number of nodes we look at is $\log_2 n$
  - and this can be done in $O(\log n)$ time

- Let's start with some examples

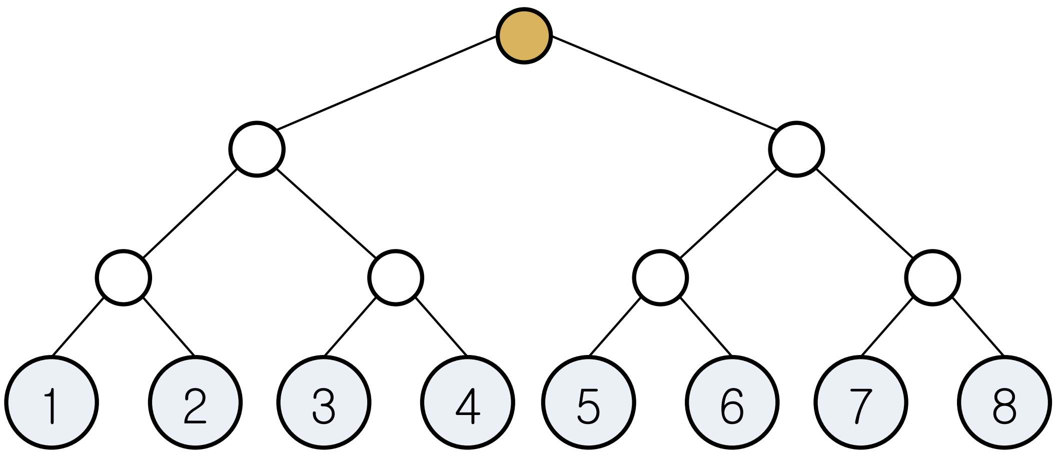
Summing: Example 1

- `Sum(7)` = sum of the values of gold-colored nodes



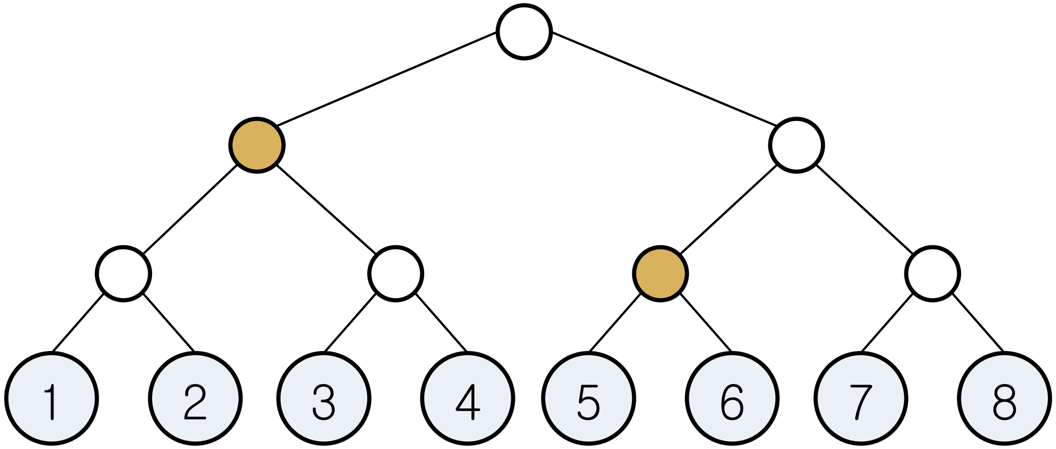
Summing: Example 2

- `Sum(8)`



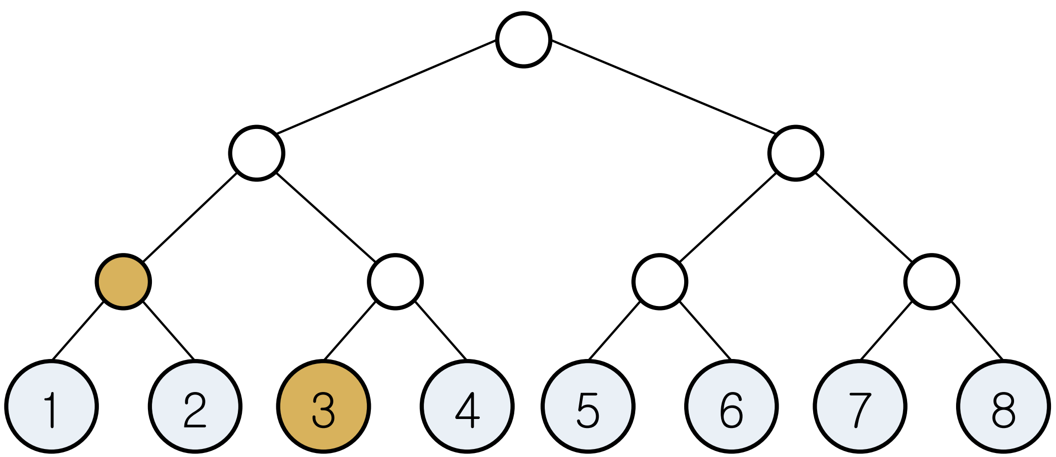
Summing: Example 3

- `Sum(6)`



Summing: Last Example

- `Sum(3)`



Implementing `Sum(k)`

- Maintain a pointer `P` which initially points at leaf `k`
- Climb the tree using the following procedure:
  - If `P` is pointing to a left child of some node:
    - Add the value of `P`
    - Set `P` to the parent node of `P`'s left neighbor
    - If `P` has no left neighbor, terminate
  - Otherwise:
    - Set `P` to the parent node of `P`
- Use an array to implement (review the heap section)

Updating a Value: `Set(k, x)`

- This part is a lot easier
- Only the values of leaf `k` and its ancestors change

- 1. Start at leaf `k`, change its value to `x`
- 2. Go to its parent, and recompute its value
- 3. Repeat 2 until the root

Extension

- Make the `Sum()` function work for any interval
  - … not just ones that start from item 1

- Can support more operations with the new `Sum()` function
  - `Min(i, j)`: Minimum element among items `i…j`
  - `Max(i, j)`: Maximum element among items `i…j`

Lowest Common Ancestor (LCA)

- Input: a rooted tree and a bunch of node pairs
- Output: lowest (deepest) common ancestors of the given pairs of nodes

- Goal: preprocessing the tree in $O(n \log n)$ time in order to answer each LCA query in $O(\log n)$ time

Preprocessing

- Each node stores its depth, as well as the links to every 2^kth ancestor
  - $O(\log n)$ additional storage per node
  - We will use `Anc[x][k]` to denote the 2^kth ancestor of node `x`
- Computing `Anc[x][k]`
  - `Anc[x][0]` = `x`'s parent
  - `Anc[x][k]` = `Anc[ Anc[x][k-1] ][ k-1 ]`

Answering a Query

- Given two node indices `x` and `y`
  - Without loss of generality, assume depth(`x`) ≤ depth(`y`)
- Maintain two pointers `p` and `q`, initially pointing at `x` and `y`
- If depth(`p`) < depth(`q`), bring `q` to the same depth as p
  - using `Anc` that we computed before
- Now we will assume that depth(`p`) = depth(`q`)

Answering a Query

- If `p` and `q` are the same, return `p`
- Otherwise, initialize `k` as $\lceil \log_2 n \rceil$ and repeat:
  - If `k` is 0, return `p`'s parent node
  - If `Anc[p][k]` is undefined, or if `Anc[p][k]` and `Anc[q][k]` point to the same node:
    - Decrease `k` by 1
  - Otherwise:
    - Set `p = Anc[p][k]` and `q = Anc[q][k]` to bring `p` and `q` up by 2^k levels

Conclusion

- We covered LOTS of stuff today
  - Try many small examples with pencil and paper to completely internalize the material
  - Review and solve relevant problems

- Discussion and collaboration are strongly recommended!