EXTRA DYNAMIC PROGRAMMING PRACTICE PROBLEMS

1. Total variation of a finite sequence $\vec{x} = \{x_1, x_2, \ldots, x_n\}$ is defined as

$$V(\vec{x}) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|$$

for $n \geq 2$, and $V(\vec{x}) = 0$ if $n < 2$. Given a sequence $\vec{x}$ of length $n \geq 2$, partition it into two disjoint subsequences such that the sum of total variations of the two subsequences is minimal.

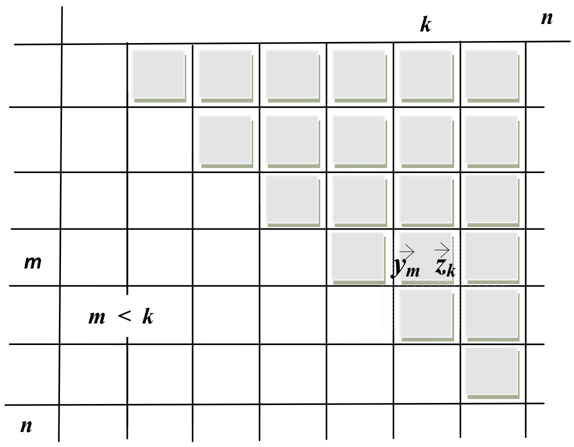
Solution. One might try to proceed by apply dynamic programming, finding recursively such splitting for all subsequences of $\vec{x}$. Thus, assume that we have a solution for the subsequence $\vec{x}_k = \{x_1, x_2, \ldots, x_k\}$, $k < n$, i.e., a partition of $\vec{x}_k$ into two disjoint subsequences: one is $\vec{y}_m$ and it ends with x_m for some $m < k$, and the other is $\vec{z}_k$ and it ends with x_k, and such that $V(\vec{y}_m) + V(\vec{z}_k)$ is minimal. Then one might hope that we can get such a partition for the sequence $\vec{x}_{k+1} = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ by either extending $\vec{y}_m$ with x_{k+1} or by extending $\vec{z}_k$ with x_{k+1}, depending on which one of $V(\vec{y}_m) + |x_{k+1} - x_m| + V(\vec{z}_k)$ and $V(\vec{y}_m) + V(\vec{z}_k) + |x_{k+1} - x_k|$ is smaller. This is because $V(\vec{y}_m) + |x_{k+1} - x_m|$ is the total variation of the sequence obtained by extending the sequence $\vec{y}_m$ with x_{k+1} (keeping sequence $\vec{z}_k$ unchanged), and also $V(\vec{z}_k) + |x_{k+1} - x_k|$ is the total variation of the sequence obtained by extending the sequence $\vec{z}_k$ by x_{k+1} (and keeping y_m unchanged). However, there is a problem with such reasoning, because the value of $V(\vec{y}_m) + |x_{k+1} - x_m| + V(\vec{z}_k)$ depends on the end point x_m of the sequence $\vec{y}_m$. It is conceivable that one can choose m' so that the value of the corresponding $V(\vec{y}_{m'})$ is suboptimal, but the end point $x_{m'}$ is much closer to x_{k+1} then the end point x_m of the optimal solution, and thus the total value of $V(\vec{y}_{m'}) + |x_{k+1} - x_{m'}| + V(\vec{z}_k)$

could be smaller than the total value of $V(\vec{y}_m) + |x_{k+1} - x_m| + V(\vec{z}_k)$ for m for which $V(\vec{y}_m) + V(\vec{z}_k)$ is the smallest.

For that reason we have to embed the problem into a more general one that requires a two dimensional table, but for which the recursion step is not problematic. We solve the following collection problems:

For every $k \leq n$ and every $m < k$ find a disjoint partition of the subsequence $\vec{x}_k = \{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_k\}$ into two disjoint subsequences, such that one subsequence ends with x_m and the other with x_k.

Thus, our table containing the pair of optimal subsequences $\vec{y}_m$ and $\vec{z}_k$ together with their total variations $V(\vec{y}_m)$ and $V(\vec{z}_k)$ will be triangular and without the values on the main diagonal, because $m < k$:



For all k, m table is filled row by row. Assume all rows with indices smaller than m have been filled. The start of the m^{th}-row is the entry for $k = m + 1$, i.e., $(m, m+1)$. To find disjoint partitioning of the sequence $\{x_1, \ldots, x_{m+1}$ into subsequences $\vec{y}_m$ ending with x_m and $\vec{z}_{m+1}$ ending with x_{m+1} such that $V(\vec{y}_m) + V(\vec{z}_{m+1})$ is minimal, we consider for all $j < m$ the sequences $\vec{y}_j \oplus x_{m+1}$ obtained by extending the sequence $\vec{y}_j$ with the extra last element x_{m+1} and set $v_j = V(\vec{y}_j \oplus x_{m+1}) + V(\vec{z}_m) = V(\vec{y}_j) + |x_{m+1} - x_j| + V(\vec{z}_m)$, where sequences $\vec{y}_j, \vec{z}_m$ and their corresponding variations can be found as the entries in the (j, m)-cell of the j^th row, because $j < m$. Finally, we consider the sequences $\vec{y}_0 = \{x_0, \ldots, x_m\}$ and $\vec{z}_0 = \{x_{m+1}\}$, and the corresponding sum of their variations $v_0 = V(\vec{y}_0) + V(\vec{z}_0) = \sum_{i<m} |x_{i+1} - x_i| + 0 = \sum_{i<m} |x_{i+1} - x_i|$. As the $(m, m+1)$-entry of the table we take the sequences whose corresponding value $v_j,\ 0 \leq j < m$ is smallest, together with their total variations. We can now fill the entire m^{th} row, because, since one of the sequences has to terminate at x_m, the corresponding sequence for $(m, k+1)$ is obtained by extending the sequence $\vec{z}_k$ found in the entry (m, k), keeping $\vec{y}_m$ that terminates with x_m unchanged.

After we complete the construction of all n rows of the table we simply chose among all pair of sequences appearing in the n^{th} column the pair of sequences with the smallest total variation, because the n^{th} row contains precisely the sequences that partition the entire sequence $\{x_0, \ldots, x_n\}$.

2. You are given a piece of wire of integer length L. You can cut out pieces of integer lengths $l_1, \ldots, l_k$. A piece of length l_j sells for a price p_j, $j \leq k$, not necessarily proportional to its length. Cut the wire so that you maximize your profit.

Try solving it yourself!